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Berry's phase, locally inertial frames, and classical analogues

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Berry's phase that appears in adiabatically changed quantum systems can be derived, in many cases, by considering locally inertial coordinate frames in classical mechanics, without having to appeal to quantum mechanics. This derivation is applicable to light propagation in twisted optical fibers and to other systems where the perturbation can be reduced to a coordinate transformation. The classical nature of this derivation clarifies some nonquantum analogues of Berry's phase.

I. INTRODUCTION

Recently, Berry[1] has shown that a quantum system that is perturbed adiabatically and returned to its initial state can acquire a phase, the Berry phase, that is of a geometrical nature. This result has sparked several investigations[2] relating the Berry phase to other geometric phenomena in theoretical physics. Along with the theoretical interest there have been attempts to observe this phase experimentally.[3] Some of the observations have been carried out by measuring the rotation of polarization of light in a twisted optical fiber.[4–6]

In this paper we will show that in many cases the rotation by an angle equivalent to Berry's phase can be derived by using classical mechanics without appealing to quantum-mechanical considerations. This derivation is valid for cases where the adiabatic transformation can be reduced to a coordinate transformation, and applicable to both quantum and classical systems. As an example we will apply our method to the rotation of polarization in twisted optical fibers, an experiment which is essentially classical.[7]

Rather than considering, as Berry does, what happens to states in Hilbert space, we prefer to consider the coordinates on which the system depends. An appropriate choice of coordinates, those that undergo parallel transport, will simplify the equations of motion. Coordinates that undergo parallel transport define, at each point, a local inertial frame.[8] If such frames are locally but not globally inertial, the equations of motion will not be modified except for the appearance of inertial forces, such as tidal forces. When the problem is adiabatic, the coordinates that undergo parallel transport are close to being globally inertial and the tidal forces are small and may be neglected. Then, the main effect of noninertial motion is due to the parallel transport of the coordinates. Note that we discuss parallel transport of coordinates in configuration space, while the derivation of Berry makes use of the parallel transport of states in Hilbert space. We consider a classical effect, while Berry deals with quantum systems. When applicable to quantum systems, our methods yields the same results as Berry's. Many, but not all, results of Berry's analysis can be derived in by considering locally inertial frames.

Our derivation enables us to investigate several phenomena in a closely related form. It gives us a recipe for calculating corrections to the Berry phenomenon, and, being a classical derivation, allows us to find classical analogues to Berry's phase. One of the simplest examples is motion in two dimensions with a central potential when the plane of motion is slowly rotated. The Foucault pendulum is a specific case of such motion.

In Sec. II we will discuss a simple classical example, which will help us introduce the physical ideas in an easily understandable case, and establish notation. In Sec. III we consider the propagation of waves in a twisted waveguide. Using the ideas outlined here we will be able to investigate the problem in a general way, without having to consider the nature of the propagating wave and details of the waveguide. Conclusions will be presented in Sec. IV.

II. A SIMPLE CLASSICAL EXAMPLE

In this section we will present a simple classical example of our approach. It will serve a double purpose: (i) to present a transparent physical picture of the problem and (ii) to introduce a notation which will be useful in other problems to be treated later.

Consider a point particle constrained to move in a two-dimensional plane. We assume that the potential, in this plane, is cylindrically symmetric, $V=V(r)$. We denote the orientation of the plane by a unit vector $\mathbf{S}$ perpendicular to it. The problem to be investigated is the motion of the particle in that plane when an external force changes $\mathbf{S}$ slowly, along a closed path, so that after some time it returns to its initial value.

The time dependence of $\mathbf{S}$ is characterized by the quantities[9]

$$\frac{d\mathbf{S}}{dt}=\chi\mathbf{N} , \quad (2.1)$$

$$\frac{d\mathbf{N}}{dt}=-\chi\mathbf{S}+\tau\mathbf{B} , \quad (2.2)$$

$$\frac{d\mathbf{B}}{dt}=-\tau\mathbf{N} , \tag{2.3}$$

where the normal **N** and binormal **B** are unit vectors. The vectors **S**, **N**, and **B** are orthonormal triad. χ is the curvature and τ the torsion of a curve in differential geometry. An adiabatic change implies that χ and τ are small. The exact sense in which these parameters should be small will be clarified later.

When **S** is time independent the equation of motion for a particle with unit mass is

$$\ddot{\mathbf{x}}=-\hat{\mathbf{x}}\frac{dV(r)}{dr} , \tag{2.4}$$

where **x** is a two-dimensional vector in the plane, $\mathbf{x}\cdot\mathbf{S}=0$, and $\hat{\mathbf{x}}$ is a unit vector in the direction of **x**.

Once we consider **S** to be time dependent we have to pick a set of transverse coordinates. **N** and **B** do not undergo parallel transport, since the vector $\mathbf{N}(t+\delta t)$ has a component parallel to $\mathbf{B}(t)$, and a similar statement holds for **B**. This is obvious from Eqs. (2.2) and (2.3) since

$$\mathbf{B}\cdot\frac{d\mathbf{N}}{dt}=\tau \quad\text{and}\quad \mathbf{N}\cdot\frac{d\mathbf{B}}{dt}=-\tau . \tag{2.5}$$

Since **N** and **B** do not undergo parallel transport, they do not constitute a locally inertial frame and the equations of motion in such a basis will be complicated. Instead, we prefer two basis vectors defined by

$$\begin{bmatrix}\mathbf{U}_1\\ \mathbf{U}_2\end{bmatrix}=\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\mathbf{N}\\ \mathbf{B}\end{bmatrix} \tag{2.6}$$

and take

$$\frac{d\phi}{dt}=\tau(t) \tag{2.7}$$

so that

$$\frac{d\mathbf{U}_1}{dt}=-\chi\cos\phi\,\mathbf{S},\quad \frac{d\mathbf{U}_2}{dt}=-\chi\sin\phi\,\mathbf{S} . \tag{2.8}$$

The vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ undergo parallel transport and define a locally inertial frame since

$$\mathbf{U}_i\cdot\frac{d\mathbf{U}_j}{dt}=0 . \tag{2.9}$$

In this frame, the position **x** is given by

$$\mathbf{x}=u_1\mathbf{U}_1+u_2\mathbf{U}_2 \tag{2.10}$$

and the equations of motion are

$$\begin{aligned}\ddot{u}_1&=-\frac{d}{du_1}V(r)-\chi^2(u_1\cos^2\phi+u_2\sin\phi\cos\phi) ,\\ \ddot{u}_2&=-\frac{d}{du_2}V(r)-\chi^2(u_1\cos\phi\sin\phi+u_2\sin^2\phi) ,\end{aligned} \tag{2.11}$$

where $r^2=u_1^2+u_2^2$. The transformation is adiabatic when **S** changes slowly and the curvature χ is sufficiently small so that terms proportional to χ^2 in Eq. (2.11) are negligible. These terms are the inertial forces which we discussed in Sec. I. When the inertial forces are neglected, Eq. (2.11) becomes identical to Eq. (2.4) when the coordinates x_i are replaced by u_i.

When the direction of the plane **S** is transformed into itself along a closed smooth curve the basis vectors $\mathbf{U}_1\mathbf{U}_2$ at the end of the curve differ from the vectors **N**,**B** by a rotation angle ϕ, which by virtue of Eq. (2.7) is given by

$$\phi=\int\tau(t)dt . \tag{2.12}$$

It is this phase which is the classical analogue of Berry's phase.[10]

The result given here can be used in a special case: a two-dimensional harmonic oscillator. The best known case of a two-dimensional harmonic oscillator whose plane is slowly rotated is the Foucault pendulum. From the derivation here it is obvious that the phase does not depend on the rate of change of **S** but only on the geometry of the curve traced by **S**. Since **S** is a unit vector, its tip traces a curve on surface of a sphere and the angle ϕ is equal to the solid angle of the section of the sphere traced by that curve.

Whenever dealing with a quantum-mechanical system where the adiabatic perturbation is reducible to a coordinate transformation, one can repeat the above procedure. There are many systems where this can be done. As an example we prefer to discuss the case of the twisted wave guides. Although this system is classical, the derivation presented here will work for quantum systems as well.

III. WAVE PROPAGATION IN A TWISTED WAVE GUIDE

A phase similar to the one discussed in Sec. II appears when we discuss the propagation of waves through a twisted waveguide, provided (1) the waveguide has circular symmetry, (2) it contains an isotropic medium, and (3) the radii of curvature and twist are large compared to the radius of the cross section and the wavelength. As we shall see, the phase depends only on the parallel transport of the transverse coordinates. It is therefore independent of the nature of the wave, be it sound waves, electromagnetic waves or others, or the mode propagating along the waveguide.

Consider a waveguide with circular cross section, its center defines a curve $\mathbf{x}_0(s)$. We choose s to correspond to the length of the curve so that $d\mathbf{x}_0/ds=\mathbf{S}$ is a unit vector. In addition to **S** we can define two unit vectors **N**,**B** as in Eqs. (2.1)–(2.3) except that time derivatives are replaced by derivatives with respect to s. The three vectors **S**, **N**, and **B** form a triad on which we can base our calculation. As in Sec. II this is not an ideal triad, and we choose, instead, to define the transverse basis vectors $\mathbf{U}_1, \mathbf{U}_2$ in terms of $\mathbf{N}$ and $\mathbf{B}$ as in Eqs. (2.6) and (2.7) where again a derivative with respect to s replaces the time derivative in Eq. (2.7). The fact the vectors $\mathbf{U}_i$ undergo parallel transport will play a crucial role in this case as well.

For simplicity consider a first scalar wave equation in a straight tube with cylindrical cross section. Any point in the tube is defined by a vector

$$\mathbf{x}=\hat{\mathbf{x}}r\cos\theta+\hat{\mathbf{y}}r\sin\theta+\hat{\mathbf{z}}z \ , \tag{3.1}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors. The wave equation for a scalar field ψ is

$$-\left[\frac{\partial^2}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\psi=E\psi \ . \tag{3.2}$$

We could have added a potential with arbitrary dependence on r to the problem but this would complicate the discussion without adding new insight. This equation can be solved introducing various m states so that

$$\psi=e^{im\theta}f_m(r,z) \ . \tag{3.3}$$

In case of the twisted waveguide we could use the coordinates

$$\mathbf{x}=\mathbf{N}r\cos\theta+\mathbf{B}r\sin\theta+\mathbf{x}_0(s) \ . \tag{3.4}$$

We prefer to simplify the equations by using a locally inertial frame:

$$\mathbf{x}=\mathbf{U}_1r\cos\beta+\mathbf{U}_2r\sin\beta+\mathbf{x}_0(s) \ , \tag{3.5}$$

where

$$\beta=\theta+\phi \tag{3.6}$$

and

$$\frac{d\phi}{ds}=\tau(s) \ . \tag{3.7}$$

In terms of these coordinates Eq. (3.2) can be rewritten as

$$-\left[\frac{\partial^2}{\partial s^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\beta^2}+\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\psi+D\psi=E\psi \ . \tag{3.8}$$

For constant χ and τ D is given by

$$D=\left[\frac{-2\chi r\cos\theta+\chi^2r^2\cos^2\theta}{d^2}\right]\frac{\partial^2}{\partial s^2}+\frac{\chi r\tau\sin\theta}{d^3}\frac{\partial}{\partial s}+\frac{1}{r^2}\frac{\chi r\sin\theta}{d}\frac{\partial}{\partial\beta}+\frac{\chi\cos\theta}{d}\frac{\partial}{\partial r} \tag{3.9}$$

and

$$d=1-\chi r\cos\theta \ . \tag{3.10}$$

Note that the set of coordinates we have used is singular when $\chi r=1$ as can be seen from the vanishing of d in the denominator of Eq. (3.9). This is of no importance when we are dealing with an adiabatic case where χ is very small and only regions where $\chi r \ll 1$ need be considered.

In the adiabatic case the contributions of D are negligible. The best way to see this is to use perturbation theory as in the Schrödinger equation. Note that perturbation theory does not imply quantum mechanics, it holds for any wave equation. We note that some terms in D are of order χ^2 or $\chi\tau$. Other terms are of order χ but they always contain terms such as $\chi\cos(\beta-\phi)$. When these terms are taken as first-order perturbations between states of well defined m they have only contributions with $\Delta m=\pm 1$ and the diagonal terms vanish. Thus D contributes only to second order in perturbation theory. Some care should be exercised to check that no small energy denominators appear to promote the second-order perturbation contributions but this is indeed the case.

We have thus shown that when written in terms of the triad $\mathbf{U}_1$, $\mathbf{U}_2$, and $\mathbf{S}$, the propagation of waves in a twisted waveguide, is identical, to first order in χ,τ, to the propagation along a straight waveguide. Thus we have no need to solve the twisted waveguide case, all we do is read off standard solutions of a straight waveguide. When we transform back to the coordinates based on the triad $\mathbf{S}$, $\mathbf{N}$, and $\mathbf{B}$ the only difference is the appearance of a phase ϕ:

$$\phi=\int_{s_1}^{s_2}ds\,\tau(s) \ . \tag{3.11}$$

It is now immediate to read off the difference between the propagation of different modes when we work in a coordinate system defined by $\mathbf{N}$ and $\mathbf{B}$. A wave with azimuthal quantum number m will have phase $e^{im\beta}$ rather than $e^{im\theta}$ and thus a phase difference $e^{im(\theta-\beta)}=e^{im\phi}$ will appear.

Our derivation was carried out for a scalar field ψ but it is simple to get the same results for a vector field $\mathbf{A}$ as when dealing with electromagnetic waves. The angle ϕ depends only on the parallel transport and not on the nature of the propagating wave. Neither does the phase ϕ depend on the exact boundary conditions of the waveguide. All we have assumed that in a straight waveguide polarization is preserved. The twisted waveguide results follow from this assumption only.

The system we have discussed above is a classical system, it is however clear that the identical method could have been applied to quantum systems such as the one discussed in Ref. 3. The adiabatic perturbation is, in this case, reducible to a rotation of coordinates. The discussion of such cases is almost identical to the one we presented here, and since no special insight is to be gained from further examples of this kind we will not present them here.

IV. CONCLUSIONS

The extra phase we have considered here in dealing with twisted waveguides can be derived in a method analogous to that of Berry's.[5] The derivation is not immediate, since Berry considers slow time changes in a physical system whereas here we have to treat slow spatial changes.

The derivation offered here has the advantages that (i) the derivation is valid for many systems, and independent of the details of the system, (ii) it is clarified that the underlying source of the effect is the use of locally inertial frames, (iii) classical analogues of Berry's phase can be found, and (iv) the method is useful in finding corrections to the adiabatic approximation which gives rise to Berry's phase, by using D in perturbation theory.

The method outlined here is not valid for the study of all systems perturbed adiabatically. Adiabatic changes in a potential which are not reducible to a coordinate transformation are an obvious exception. Many systems however, have as a source of perturbation a transformation of coordinates. These are, of course, amenable to our treatment.

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