Motion of massive bodies: Testing the nonsymmetric gravitation theory

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We derive the equations of motion for massive extended bodies in the first post-Newtonian approximation to the nonsymmetric gravitation theory. The results are applied to the problem of the perihelion shift of Mercury and the periastron shifts of binary stars. We prove the equivalence of gravitational and inertial masses in the theory at the first post-Newtonian level. Hence, the theory predicts no Nordtvedt effect in the Moon's orbit about Earth. Other weak-field tests of the nonsymmetric gravitation theory are discussed.

I. INTRODUCTION

Until 1960, no serious challenge to the general theory of relativity existed. The philosophical beauty and simplicity of the theory, together with the empirical basis provided by the classical tests of gravitation theory, were compelling reasons not to consider any alternative theory of gravitation. Einstein, however, attempted to extend his theory to unify gravitation with electromagnetism.[1] He did this by introducing a nonsymmetric fundamental tensor $g_{\mu\nu}$. He did not succeed in unifying gravitation and electromagnetism, but the new nonsymmetric theories could be interpreted as purely gravitational theories. One such theory, proposed by Moffat,[2] has been studied in detail. The theory is known as the nonsymmetric gravitation theory (NGT) and its consequences and predictions are the subject of this work.

Nonsymmetric theories have nonsymmetric connections. As a result, they may contain other tensor fields, in addition to $g_{\mu\nu}$. These additional fields may couple to matter currents. Thus, not all matter may fall at the same rate under the influence of gravity. The equation of motion may not be the standard geodesic equation when charges coupling to the new fields are present. In NGT, there is a new coupling of the contraction of the torsion to a vector current denoted by $\mathbf{S}^{\mu}$. The charge associated with this current is denoted l^2 (though it need not be positive) and has dimensions of [length]^2. Nonsymmetric theories also contain more than one connection: e.g., the full connection that appears in the curvature tensor, the Levi-Civita connection which is compatible with $g_{(\mu\nu)}$, etc. It is not apparent which connection should be used to describe gravity in the equation of motion.[3] However, such theories are geometric in that they are based on a Riemann curvature formed from a specific nonsymmetric connection. They therefore contain Bianchi identities. If the coupling to matter is given, then application of the field equations to the contracted Bianchi identities gives rise to matter response equations. These equations are constraints on the motion of matter. If the theory under study possesses a conserved energy-momentum pseudotensor, the constraints may be integrated to give the laws of motion for material bodies.[4,5] These bodies may be of finite extent and may have non-negligible self-gravity ("ponderable matter").

If one can establish such laws of motion, then one can apply them to a large number of examples in order to test the underlying gravitation theory. In fact, theories that couple to matter only through the (symmetric) metric may be studied using the parametrized post-Newtonian formalism.[5] This formalism does not apply to theories with more general couplings to matter currents because it assumes the geodesic equation of motion in the case of a falling object with negligible self-gravity. However, we will find it useful to apply post-Newtonian techniques such as the weak-field, low-velocity (post-Newtonian or PN) expansion.[5,6]

In Sec. II the weak-field expansion[7] is outlined. Section III describes the solution to the first post-Newtonian (1PN) approximation to NGT (Ref. 6) and applies the results to the equations of Sec. II. Section IV contains the equations of motion of massive extended bodies at the 1PN level, which is our main result. It is also shown there that inertial and gravitational masses are equivalent in the 1PN approximation to NGT, unlike, for example, the Brans-Dicke theory.[8] Special cases of the equations of motion are presented. In Sec. V we discuss applications of the formulas developed in Sec. IV. We consider the perihelion shift of Mercury and the periastron shifts of several eclipsing binary stellar systems. Other tests of NGT are briefly discussed. Because of our weak-field assumption, we do not discuss the binary pulsar system PSR1913 + 16 (Ref. 9). The equations of NGT are outlined in Appendix A. Appendix B contains Newtonian virial theorems used to simplify the equations of motion.

This work corrects and extends the previous work of McDow and Moffat.[10] They assumed a geodesic equation of motion for negligibly self-gravitating objects and constructed massive bodies from large concentrations of such particles. They imposed restrictive symmetry conditions on the massive objects formed in this manner. We instead derive our equations as outlined above, removing the assumption of geodesic motion. We impose no symmetry conditions. We also correct our previously published periastron shift formula,[11] which was derived using the geodesic motion assumption. We use $G=c=1$, except where the constants have been explicitly included (in parts of Sec. V), and take $\eta_{\mu\nu}=\mathrm{diag}(+1,-1,-1,-1)$. Parentheses around tensor indices denote symmetrization and square brackets denote antisymmetrization. Greek indices run from 0 to 3. Roman indices run from 1 to 3.

II. WEAK-FIELD FORMALISM

NGT possesses a symmetric post-Newtonian conserved energy-momentum complex,[9,12] comparable to the Landau-Lifshitz pseudotensor[13] of general relativity (GR). The post-Newtonian limit of NGT, therefore, contains a full set of conservation laws for energy, momentum, and angular momentum, and Newton's first and third laws hold in NGT. We shall use the law of conservation of energy to define the inertial mass (conserved mass energy) of a body or system of bodies and use the law of conservation of momentum to generate Newton's second law for the motion of this body or system. The technique is described in Ref. 5. We begin by providing a brief derivation of the stress-energy complex in the weak-field approximation.

The mass energy of a gravitational field must be defined relative to some background metric since it is described by a pseudotensor. We make the following definitions:

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} , \tag{2.1a}$$

$$h=\eta^{\mu\nu}h_{\mu\nu} , \tag{2.1b}$$

$$\theta_{\mu\nu}=h_{(\mu\nu)}-\tfrac{1}{2}\eta_{\mu\nu}h , \tag{2.1c}$$

$$\phi_{\mu\nu}=h_{[\mu\nu]} , \tag{2.1d}$$

$$\theta^{\mu\nu}=\eta^{\mu\alpha}\eta^{\nu\beta}\theta_{\alpha\beta} , \tag{2.1e}$$

$$\phi^{\mu\nu}=\eta^{\mu\alpha}\eta^{\nu\beta}\phi_{\alpha\beta} . \tag{2.1f}$$

We then solve Eqs. (A23) and (A24) iteratively in terms of the metric potentials $\theta_{\mu\nu}$ and $\phi_{\mu\nu}$:

$$\Lambda^{\alpha}_{\mu\nu}={}^{1}\Lambda^{\alpha}_{\mu\nu}+{}^{2}\Lambda^{\alpha}_{\mu\nu}+\cdots , \tag{2.2a}$$

$${}^{1}\Lambda^{\alpha}_{\mu\nu}=\tfrac{1}{2}\eta^{\alpha\beta}(h_{\mu\beta,\nu}+h_{\beta\nu,\mu}-h_{\nu\mu,\beta}) , \tag{2.2b}$$

$${}^{2}\Lambda^{\alpha}_{\mu\nu}=-\eta^{\alpha\beta}(h_{(\lambda\beta)}\,{}^{1}\Lambda^{\lambda}_{\mu\nu}+h_{[\mu\lambda]}\,{}^{1}\Lambda^{\lambda}_{\nu\beta}+h_{[\lambda\nu]}\,{}^{1}\Lambda^{\lambda}_{\beta\mu}) , \tag{2.2c}$$

$$D^{\alpha}_{\mu\nu}={}^{1}D^{\alpha}_{\mu\nu}+{}^{2}D^{\alpha}_{\mu\nu}+\cdots , \tag{2.2d}$$

$${}^{1}D^{\alpha}_{\mu\nu}=\frac{4\pi}{3}S^{\beta}(\delta^{\alpha}_{\nu}\eta_{\mu\beta}-\delta^{\alpha}_{\mu}\eta_{\beta\nu}) , \tag{2.2e}$$

$$\begin{aligned}{}^{2}D^{\alpha}_{\mu\nu}=\frac{4\pi}{3}S^{\lambda}(\tfrac{3}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{[\beta\lambda]}-\tfrac{1}{2}\delta^{\alpha}_{\mu}h_{[\nu\lambda]}\\ -\tfrac{1}{2}\delta^{\alpha}_{\nu}h_{[\mu\lambda]}-\delta^{\alpha}_{\mu}h_{(\nu\lambda)}-\delta^{\alpha}_{\nu}h_{(\mu\lambda)}) .\end{aligned} \tag{2.2f}$$

These results can now be substituted into Eq. (A24) for Γ and the expressions so obtained may be used to write $R_{\mu\nu}(\Gamma)$ in terms of the variables $\theta_{\mu\nu}$ and $\phi_{\mu\nu}$ through Eq. (A10). If these results are then used in the field equations (A12) with the indices symmetrized, we obtain

$$\partial_{\rho}\partial_{\lambda}(\eta^{\rho\lambda}\theta^{\mu\nu}+\eta^{\mu\nu}\theta^{\rho\lambda}-\eta^{\rho\nu}\theta^{\mu\lambda}-\eta^{\rho\mu}\theta^{\nu\lambda})=-16\pi\tau^{(\mu\nu)} , \tag{2.3}$$

where

$$\tau^{(\mu\nu)}=T^{(\mu\nu)}+t^{(\mu\nu)} . \tag{2.4}$$

To bilinear order, the pseudotensor $t^{(\mu\nu)}$ is given by

$$\begin{aligned}-16\pi t^{(\mu\nu)}=&-\theta_{\lambda\beta}(\theta^{\mu\lambda,\nu\beta}+\theta^{\nu\lambda,\mu\beta}-\theta^{\mu\nu,\lambda\beta}-\theta^{\lambda\beta,\mu\nu})-\tfrac{1}{2}\theta^{\mu\nu}\theta_{,\rho}{}^{\rho}-\theta^{\alpha\nu}(\theta_{\alpha\lambda}{}^{,\lambda\mu}+\theta^{\mu\lambda}{}_{,\lambda\alpha}-\theta^{\mu}{}_{\alpha,\rho}{}^{\rho})-\theta^{\mu\alpha}(\theta^{\nu\lambda}{}_{,\lambda\alpha}+\theta_{\alpha\lambda}{}^{,\lambda\nu}-\theta^{\nu}{}_{\alpha,\rho}{}^{\rho})\\ &+\tfrac{1}{2}\theta^{\lambda\beta,\mu}\theta_{\lambda\beta}{}^{,\nu}-\theta^{\rho\nu,\lambda}\theta^{\mu}{}_{\lambda,\rho}+\theta^{\rho\nu,\lambda}\theta^{\mu}{}_{\rho,\lambda}+\theta^{\beta(\mu}\theta^{,\nu)}{}_{\beta}-\tfrac{1}{2}\theta^{,\mu\nu}\theta-\tfrac{1}{2}\theta\theta^{\mu\nu,\rho}{}_{\rho}-\tfrac{1}{4}\theta^{,\mu}\theta^{,\nu}-8\pi S^{\nu,\lambda}\phi^{\mu}{}_{\lambda}-8\pi S^{\mu,\lambda}\phi^{\nu}{}_{\lambda}\\ &-\phi_{\lambda\rho}\phi^{\lambda\rho,\mu\nu}-2\phi^{\mu\lambda,\rho}\phi^{\nu}{}_{(\lambda,\rho)}-2\phi_{\lambda\rho}{}^{,(\mu}\phi^{\nu)\lambda,\rho}-\tfrac{3}{2}\phi^{\lambda\rho,\nu}\phi_{\lambda\rho}{}^{,\mu}-\frac{32\pi^{2}}{3}S^{\mu}S^{\nu}\\ &-\eta^{\mu\nu}\left[\theta^{\rho\lambda}\theta_{\rho\lambda,\alpha}{}^{\alpha}+\tfrac{3}{4}\theta_{\rho\beta,\lambda}\theta^{\rho\beta,\lambda}-\tfrac{1}{2}\theta_{\rho\beta,\lambda}\theta^{\rho\lambda,\beta}-\tfrac{3}{8}\theta_{,\lambda}\theta^{,\lambda}-\tfrac{1}{2}\theta\theta_{,\lambda}{}^{\lambda}\right.\\ &\left.\quad+\tfrac{1}{2}\theta_{\lambda\beta}\theta^{,\lambda\beta}-\phi^{\rho\lambda}\phi_{\rho\lambda,\alpha}{}^{\alpha}-\tfrac{5}{4}\phi^{\beta\lambda,\rho}\phi_{\beta\lambda,\rho}-\tfrac{1}{2}\phi^{\beta\lambda,\rho}\phi_{\lambda\rho,\beta}+\frac{32\pi^{2}}{3}S_{\lambda}S^{\lambda}\right]\\ &+\tfrac{4}{3}(\eta^{\mu\beta}\phi^{\alpha\nu}A_{[\alpha,\beta]}+\eta^{\alpha\nu}\phi^{\mu\beta}A_{[\alpha,\beta]}-\tfrac{1}{2}\eta^{\mu\nu}\phi^{\alpha\beta}A_{[\alpha,\beta]})-\theta_{\lambda\beta}{}^{,\beta}(\theta^{\mu\lambda,\nu}+\theta^{\lambda\nu,\mu}-\theta^{\mu\nu,\lambda}+\eta^{\mu\nu}\theta^{,\lambda}-\eta^{\mu\nu}\theta^{\lambda\rho}{}_{,\rho})\\ &+\tfrac{1}{2}(\theta^{\mu\beta}{}_{,\beta}\theta^{,\nu}+\theta^{\nu\beta}{}_{,\beta}\theta^{,\mu}+2\theta^{\mu\nu}\theta^{\rho\lambda}{}_{,\rho\lambda})+\theta(3\theta^{\beta(\mu,\nu)}{}_{\beta}-\theta^{\mu\nu,\rho}{}_{\rho})-\tfrac{1}{2}\eta^{\mu\nu}(3\theta\theta_{\lambda\beta}{}^{,\lambda\beta}-4\theta_{\alpha\beta}\theta^{\alpha\lambda,\beta}{}_{\lambda}) .\end{aligned} \tag{2.5}$$

Indices are raised and lowered with η. Following Krisher,[9] we have split the torsion vector into a source term and a term A_{μ} which will be shown to be nonlinear in the sources and potentials:

$$W_{\mu}=A_{\mu}-8\pi S_{\mu} . \tag{2.6}$$

The left-hand side of Eq. (2.3) is explicitly divergence-free, implying

$$\tau^{(\mu\nu)}{}_{,\nu}=0 \ . \tag{2.7}$$

We now make the standard definitions of total momentum and center of mass for a post-Newtonian system:[5]

$$P^{\mu}_{A}=\int_{A}\tau^{(0\mu)}d^3x,\quad m_A=P^0_A,\quad \mathbf{x}_A=\frac{1}{m_A}\int_A \tau^{(00)}\mathbf{x}\,d^3x \ . \tag{2.8}$$

From (2.7) and (2.8), it is apparent that the center of mass of an isolated system is unaccelerated in the chosen coordinates (take the boundary of the region of integration to be static and to lie far outside matter). We will differentiate the center of mass twice with respect to time and obtain the equations of motion in the case that the system is not isolated.

It will first be necessary to expand the remaining NGT field equations in order to obtain solutions for $\phi^{\mu\nu}$ and A_μ. To bilinear order, Eqs. (A13) and (A27) produce

$$\phi^{\mu\rho}{}_{,\rho}=4\pi S^{\mu}-\tfrac{1}{2}\theta_{,\rho}\phi^{\mu\rho}+4\pi\theta^{\mu\rho}S_{\rho}-4\pi\theta S^{\mu}+\theta^{\alpha\mu,\rho}\phi_{\alpha\rho}+\phi^{\mu\alpha,\rho}\theta_{\alpha\rho} \ . \tag{2.9}$$

Then the skew-symmetric part of Eq. (A12), together with Eqs. (2.2) and (2.9), gives an equation for $\phi^{\mu\nu}$ to bilinear order:

$$\begin{aligned}\partial^{\rho}\partial_{\rho}\phi^{\mu\nu}=&-\tfrac{1}{2}\partial^{\rho}\partial_{\rho}(\theta\phi^{\mu\nu})+8\pi S^{[\mu,\nu]}-\tfrac{4}{3}A^{[\mu,\nu]}-16\pi T^{[\mu\nu]}\\&+\frac{8\pi}{3}(S^{\rho,[\mu}\theta^{\nu]}{}_{\rho}-S_{\rho}\theta^{\rho[\mu,\nu]}-S^{[\mu}\theta^{,\nu]}-S^{[\mu,\nu]}\theta)+16\pi(\theta^{\mu}{}_{\rho}T^{[\nu\rho]}-\theta^{\nu}{}_{\rho}T^{[\mu\rho]}+\theta T^{[\mu\nu]})\\&-2\theta^{\lambda[\mu,\nu]\rho}\phi_{\lambda\rho}-2\theta_{\lambda\rho}{}^{,[\mu}\phi^{\nu]\lambda,\rho}-2\theta^{[\mu}{}_{\lambda,\rho}\phi^{\nu]\lambda,\rho}+2\theta^{[\mu}{}_{\rho,\lambda}\phi^{\nu]\lambda,\rho}+\theta_{\lambda\rho}\phi^{\mu\nu,\lambda\rho}-\theta_{,\lambda}{}^{[\mu}\phi^{\nu]\lambda}-\theta_{,\lambda}\phi^{\lambda[\mu,\nu]}+4\pi S^{[\mu,\nu]}\theta \ .\end{aligned} \tag{2.10}$$

An equation for A^{μ} will be obtained later from the post-Newtonian approximations to Eqs. (2.9) and (2.10).

III. POST-NEWTONIAN FLUID APPROXIMATION

Equations (2.3) and (2.10) may be solved using standard post-Newtonian methods. We choose coordinates in which

$$\theta^{\mu\nu}{}_{,\nu}=0 \tag{3.1}$$

and adopt the standard post-Newtonian gauge.[5] We now assume that material objects are comprised of fluids described by the tensor (A17) and current (A21). To necessary order we obtain the metric[6]

$$\begin{aligned}g_{00}=&1-2U+2(U^2-\tilde{\Phi})+\nabla\lambda\cdot\nabla\lambda\\&-\int d^3x'S^0(\mathbf{x}')\nabla'\lambda'\cdot\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \ ,\end{aligned} \tag{3.2a}$$

$$g_{(0i)}=\tfrac{7}{2}V^i+\tfrac{1}{2}\mathcal{W}^i \ , \tag{3.2b}$$

$$g_{(ij)}=-\delta_{ij}(1+2U) \ , \tag{3.2c}$$

$$g_{[0i]}=\lambda_{,i} \ , \tag{3.2d}$$

$$g_{[ij]}=2\alpha_{[i,j]} \ . \tag{3.2e}$$

We define the potentials as

$$\nabla^2 U=-4\pi\rho_0,\quad U(\mathbf{x})=\int d^3x'\frac{\rho_0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ , \tag{3.3a}$$

$$\nabla^2 V^k=-4\pi\rho_0 v^k,\quad V^k(\mathbf{x})=\int d^3x'\frac{\rho_0(\mathbf{x}')v^k(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ , \tag{3.3b}$$

$$\nabla^2\lambda=-4\pi S^0,\quad \lambda(\mathbf{x})=\int d^3x'\frac{S^0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ , \tag{3.3c}$$

$$\nabla^2\alpha^k=-4\pi S^0 v^k,\quad \alpha^k(\mathbf{x})=\int d^3x'\frac{S^0(\mathbf{x}')v^k(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ , \tag{3.3d}$$

$$\begin{aligned}\Phi_1(\mathbf{x})=&\int d^3x'\frac{\rho_0(\mathbf{x}')v^2(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ ,\\ \Phi_2(\mathbf{x})=&\int d^3x'\frac{\rho_0(\mathbf{x}')U(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ ,\end{aligned} \tag{3.3e}$$

$$\begin{aligned}\tilde{\Phi}_3(\mathbf{x})=&\int d^3x'\frac{\rho_0(\mathbf{x}')\tilde{\Pi}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ ,\\ \tilde{\Phi}_4(\mathbf{x})=&\int d^3x'\frac{\tilde{p}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ ,\end{aligned} \tag{3.3f}$$

$$\tilde{\Phi}=2\Phi_1+2\Phi_2+\tilde{\Phi}_3+3\tilde{\Phi}_4 \ , \tag{3.3g}$$

$$\tilde{\Pi}=\Pi-\frac{2\pi}{3}\frac{S^0S^0}{\rho_0},\quad \tilde{p}=p-\frac{2\pi}{3}S^0S^0 \ , \tag{3.3h}$$

$$\mathcal{W}^k(\mathbf{x})=\int d^3x'\frac{\rho_0(\mathbf{x}')\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')(x^k-x'^k)}{|\mathbf{x}-\mathbf{x}'|^3} \ . \tag{3.3i}$$

Iteration was required to obtain the solution (3.2a). Note that the metric has the same values for the post-Newtonian parameters as the general-relativity metric ($\alpha=\gamma=1$, etc.), but that it also reflects the new coupling to the S^{μ} current, e.g., in (3.2a).

We may now calculate τ^{00} to post-Newtonian order:

$$\tau^{00}=\rho^*(1+\tilde{\Pi}+\tfrac{1}{2}v^2-\tfrac{1}{2}U)-\frac{3}{8\pi}\nabla\cdot(U\nabla U)-\tfrac{1}{2}\nabla\cdot(S^*\nabla\lambda)\ , \tag{3.4}$$

where we define (to this order)

$$\rho^*=\sqrt{-g}\,\rho_0u^0=\rho_0(1+3U+\tfrac{1}{2}v^2)\ , \tag{3.5a}$$

$$S^*=\sqrt{-g}\,S^0=(1+2U)S^0\ . \tag{3.5b}$$

The fluid variables (p, ρ_0, Π, etc.) are defined in Appendix A. The densities ρ^* and S^* are called conserved densities because they obey the continuity equations

$$\frac{\partial\rho^*}{\partial t}+\nabla\cdot(\mathbf{v}\rho^*)=0,\quad \frac{\partial S^*}{\partial t}+\nabla\cdot(\mathbf{v}S^*)=0\ . \tag{3.6}$$

Using (3.4) in (2.8) and differentiating twice, we obtain an expression for the acceleration of the center of mass of an extended object relative to the center of mass of the system

$$\mathbf{v}_A=\frac{d\mathbf{x}_A}{dt},\quad \mathbf{a}_A=\frac{d\mathbf{v}_A}{dt}\ , \tag{3.7a}$$

$$\begin{aligned}\mathbf{a}_A=&\frac{1}{m_A}\int_A\rho^*(1+\tilde{\Pi}+\tfrac{1}{2}\bar{v}^2-\tfrac{1}{2}\bar{U})\frac{d^2\mathbf{x}}{dt^2}d^3x\\&+\frac{1}{m_A}\left[v_A^j\int_A\tilde{p}_{,j}\bar{\mathbf{v}}\,d^3x+\int_A\tilde{p}_{,0}\bar{\mathbf{v}}\,d^3x-\int_A\frac{\tilde{p}}{\rho^*}\nabla\tilde{p}\,d^3x\right]\\&+\frac{1}{m_A}(\mathcal{T}_A-\mathcal{T}_A^*-\tfrac{3}{2}\mathcal{T}_A^{**}+\tfrac{3}{2}\tilde{\mathbf{P}}_A+\tfrac{1}{2}\mathbf{\Omega}_A^*+\tfrac{1}{2}\mathbf{t}_A)\ ,\end{aligned} \tag{3.7b}$$

$$m_A=\int_A\rho^*(1+\tilde{\Pi}+\tfrac{1}{2}\bar{v}^2-\tfrac{1}{2}\bar{U})d^3x\ . \tag{3.7c}$$

From (2.7) and (2.8), we have

$$\frac{dm_A}{d\bar{t}}=0\ . \tag{3.8}$$

We choose to call m_A the *inertial mass* of body A. The integrals $\mathcal{T}$, $\tilde{\mathbf{P}}$, etc., are defined in Appendix B. Virial theorems listed in Appendix B have been used to simplify the integrals. We also define

$$\bar{\mathbf{v}}=\mathbf{v}(t,\mathbf{x})-\mathbf{v}_A(t),\quad \mathbf{x}\in A\ , \tag{3.9}$$

and split any potentials (generically denoted Ψ) into self and external parts

$$\Psi=\begin{cases}\bar{\Psi}(t,\mathbf{x})+\sum_{B\neq A}\Psi_B & \text{for } \mathbf{x}\in A\ ,\\ \sum_B\Psi_B & \text{otherwise}\ ,\end{cases} \tag{3.10}$$

where Ψ_B denotes the potential generated by sources with support contained within volume B.

We cannot yet evaluate (3.7b) because we do not as yet have an expression for the acceleration $d^2\mathbf{x}/dt^2$ of the fluid element at $(t,\mathbf{x})$. This may be obtained by substituting the form of the fluid tensor into the matter response equation (A19) and applying the post-Newtonian approximation. The result is

$$\begin{aligned}\rho^*\frac{d^2x^k}{dt^2}=&\rho^*U_{,k}-\tilde{p}_{,k}\left[1+3U-\tilde{\Pi}-\tfrac{1}{2}v^2-\frac{\tilde{p}}{\rho^*}\right]+v^k(\rho^*\dot{U}-\dot{\tilde{p}})-\rho^*\frac{d}{dt}(4Uv^k-\tfrac{7}{2}V^k-\tfrac{1}{2}\mathcal{W}^k)\\&-\tfrac{7}{2}\rho^*v^jV^j{}_{,k}-\tfrac{1}{2}v^j\mathcal{W}^j{}_{,k}+\rho^*v^2U_{,k}+\rho^*\tilde{\Phi}_{,k}-\partial_k\left[\frac{2\pi}{3}S^*S^*v^2+4\pi S^*S^*U\right]\\&-\tfrac{1}{2}\rho^*\partial_k(\nabla\lambda\cdot\nabla\lambda)+\tfrac{1}{2}S^*\partial_k(\nabla U\cdot\nabla\lambda)-\tfrac{1}{2}\rho^*\partial_j\partial_k\int\frac{S^{*\prime}\partial_j'\lambda'}{|\mathbf{x}-\mathbf{x}'|}d^3x'-S^*\partial_j\partial_k\int\frac{S^{*\prime}\partial_j'U'-2\rho^{*\prime}\partial_j'\lambda'}{|\mathbf{x}-\mathbf{x}'|}d^3x'\ .\end{aligned} \tag{3.11}$$

In order to evaluate the matter response equation (A19), we had to determine the field $W_{[\mu,\nu]}$. To do this, we first used Eq. (2.6) to replace $W_{[\mu,\nu]}$ by $A_{[\mu,\nu]}$ in (A19). We solved for $A_{[\mu,\nu]}$ by taking another gradient on the post-Newtonian approximation to Eq. (2.10) and antisymmetrizing to eliminate A_μ. The resulting equation was integrated to find a solution for the totally antisymmetric combination $\phi^{[\mu\nu,\alpha]}$. From this combination and the divergence $\phi^{\mu\nu}{}_{,\nu}$ given by Eq. (2.9), the d'Alembertian of $\phi^{\mu\nu}$ was constructed by differentiation. Thus, (2.10) becomes an algebraic equation for $A^{[\mu,\nu]}$. This is the standard technique of first solving what is called the "weak system" of equations for a nonsymmetric theory, meaning the system in which the antisymmetric derivative $A^{[\mu,\nu]}$ is eliminated. To required order, we need only $A^{[0,i]}$, which actually reduces to $A^{0,i}$ to necessary order. We obtain

$$\nabla A^0=\nabla\left[16\pi US^0-3\nabla U\cdot\nabla\lambda+3\nabla\cdot\int\frac{S^{0\prime}\nabla'U'-2\rho'\nabla'\lambda'}{|\mathbf{x}-\mathbf{x}'|}d^3x'\right]\ . \tag{3.12}$$

This result was used in (A19) to produce (3.11). Finally, substitution of (3.11) into (3.7b) yields the equation of motion for the center of mass of object A:

$$\frac{d^2x'^k_A}{dt^2}=\frac{1}{m_A}\int_A\Big[\rho^* U_{,k}(1+\tfrac{1}{2}\bar{v}^2-\tfrac{1}{2}\bar{U}+\tilde{\Pi})+\rho^* U_{,k}v^2+\tilde{p}(3U_{,k}-\tfrac{1}{2}\bar{U}_{,k})-\rho^* U\frac{dv^k}{dt}$$
$$-\rho^*(\tfrac{7}{2}v^jV^j{}_{,k}+\tfrac{1}{2}v^j\mathcal{W}^j{}_{,k}+v^k\mathbf{v}\cdot\nabla U-2\Phi_{1,k}-2\Phi_{2,k}-\tilde{\Phi}_{3,k}-3\tilde{\Phi}_{4,k})$$
$$-\rho^*\frac{d}{dt}(3Uv^k-\tfrac{7}{2}V^k-\tfrac{1}{2}\mathcal{W}^k)\Big]d^3x$$
$$+\frac{1}{m_A}\int_A\Big[-\tfrac{1}{2}\rho^*\partial_k(\nabla\lambda\cdot\nabla\lambda)+\tfrac{1}{2}S^*\partial_k(\nabla U\cdot\nabla\lambda)-\tfrac{1}{2}\rho^*\partial_j\partial_k\int\frac{S^{*\prime}\partial'_j\lambda'}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$
$$-\tfrac{1}{2}S^*\partial_j\partial_k\int\frac{S^{*\prime}\partial'_jU'-2\rho^{*\prime}\partial'_j\lambda'}{|\mathbf{x}-\mathbf{x}'|}d^3x'\Big]d^3x$$
$$+\frac{1}{m_A}\Big[v^j_A\int_A\bar{v}^k\partial_j\tilde{p}\,d^3x+v^j_A\int_A\bar{v}^j\partial_k\tilde{p}\,d^3x-v^k_A\frac{d\tilde{P}_A}{dt}+\mathcal{T}^k_A-\mathcal{T}^{*k}_A-\tfrac{3}{2}\mathcal{T}^{**k}_A+\tfrac{3}{2}\tilde{P}^k_A+\tfrac{1}{2}\Omega^{*k}_A+\tfrac{1}{2}t^k_A\Big]. \tag{3.13}$$

IV. EQUATIONS OF MOTION

We now have to simplify this equation. This is done by splitting all velocities and potentials into their internal and external parts using (3.9) and (3.10). Then the virial theorems (Appendix B) are used to simplify integrals over purely internal quantities. These integrals would lead to self-accelerations if it were not for the virial theorems, in violation of the conclusions drawn from (2.8). In fact, the occurrence of self-acceleration terms in exactly the combinations required for cancellation through use of the virial theorems is a useful check of (3.13).

Let us assume the separations between objects to be large so that potentials due to one object but evaluated at the position of some other object may be expanded in inverse powers of the separations (i.e., a multipole expansion). Self-accelerations will be independent of the separations (generically denoted by r), while Keplerian terms will go as $1/r^2$. We now define the gravitational mass tensor in terms of the coefficient of the $1/r^2$ term in the force law when the centers of mass of the objects comprising the system are all taken to be (momentarily) at rest. Precisely, we assume that the force law computed from (3.13) takes the form

$$m_A a^k_A=-\sum_{B\neq A}\left[\alpha^{jk}_{AB}+\frac{\gamma^{jk}_{AB}}{r^2_{AB}}+O(1/r^3_{AB})\right]\frac{r^j_{AB}}{r_{AB}} \tag{4.1}$$

and check to see that $\alpha^{ij}_{AB}=0$. Since NGT possesses a full set of post-Newtonian conservation laws, γ^{jk}_{AB} will be symmetric in A and B, in order that the sum of all the forces on the system vanish.

An example serves to illustrate the point. In earlier studies concerning motion in NGT (Ref. 10), it was thought that "test particles" should follow geodesics of the symmetric metric $g_{(\mu\nu)}$. If one imposes this additional constraint on (3.13), the NGT contribution reduces to

$$\frac{1}{m_A}\int_A\Big[-\tfrac{1}{2}\rho^*\partial_k(\nabla\lambda\cdot\nabla\lambda)-\tfrac{1}{2}\rho^*\partial_j\partial_k\int\frac{S^{*\prime}\partial'_j\lambda'}{|\mathbf{x}-\mathbf{x}'|}d^3x'\Big]d^3x\,. \tag{4.2}$$

In this case, the integrated equations of motion would yield

$$\alpha^{jk}_{AB}=\frac{1}{2}\frac{r^j_{AB}}{r_{AB}}\int_A d^3x\,\rho^*\partial_k\left[\int_A d^3x' S^{*\prime}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\cdot\int_A d^3x' S^{*\prime\prime}\frac{\mathbf{x}-\mathbf{x}''}{|\mathbf{x}-\mathbf{x}''|^3}\right]$$
$$+\frac{1}{2}\frac{r^j_{AB}}{r_{AB}}\int_A d^3x\,S^*\left[\partial_j\int_A\frac{S^{*\prime}d^3x'}{|\mathbf{x}-\mathbf{x}'|}\right]\left[\partial_j\partial_k\int_A\frac{\rho^{*\prime\prime}d^3x''}{|\mathbf{x}-\mathbf{x}''|}\right], \tag{4.3a}$$

$$\gamma^{jk}_{AB}=m_Am_B\delta^{jk}+\tfrac{3}{2}l^2_B\int_A d^3x\,d^3x'\frac{\rho^*S^{*\prime}}{|\mathbf{x}-\mathbf{x}'|^3}\left[\delta^{jk}-3\frac{(x^j-x'^j)(x^k-x'^k)}{|\mathbf{x}-\mathbf{x}'|^2}\right], \tag{4.3b}$$

where m_A and m_B are the inertial masses defined by Eq. (3.7c). Thus, unless the integrals in (4.3a) vanished due to symmetry of object A, the object would suffer a self-acceleration. In addition, the parameter appearing in front of the Keplerian term in the force law is not $m_Am_B\delta^{jk}$ but γ^{jk}_{AB} as given in (4.3b). We may interpret

γ^{jk}_{AB} as the product of the *active* gravitational mass tensor[5,14] of object A with the *passive* gravitational mass tensor of object B and thus (incorrectly) conclude that NGT predicted self-accelerations and violations of the equality of inertial and gravitational mass.

In a previous study of this problem,[10] the massive-body equivalence principle was found to hold but only under the assumption of spherical symmetry of the accelerated object. That calculation also used a model for matter that constructed objects from a gas of infinite-density point masses. Self-interactions of the point masses were ignored. In the present calculation, we do not make these assumptions and observe that inertial and gravitational masses are equivalent in NGT because the *full conservation laws ensure cancellation* of equivalence violating terms at 1PN order. The interesting point is that metric theories guarantee the validity of Dicke's weak equivalence principle (WEP) when self-gravity is not important but appear to violate the equivalence of inertial and gravitational mass at 1PN order unless their post-Newtonian expansions are equivalent to (or differ only very slightly from) that of general relativity theory. Here we have an example of a theory with a nonmetric coupling which in general will not be expected to obey WEP. However, we will see that it does preserve the equivalence of mass at 1PN order.

We now proceed to write the equation of motion of the object under consideration by integrating (3.13). We do not require new virial theorems to simplify the NGT parts of the full equation of motion. We do however find that the fluid variables p and Π are replaced by $\tilde{p}$ and $\tilde{\Pi}$ in the existing virial theorems. We carry over from (3.13) all terms to $O(1/r^2)$ inclusive, as we must do in order to check the equivalence of inertial and gravitational mass. However, our formalism allows us to calculate higher-order terms and this we shall do. At $O(1/r^3)$, one obtains three different classes of terms. We classify terms at a given order in $1/r$ according to their order in the post-Newtonian and multipole expansions. For a bound system, $U_B(\mathbf{x}_A)\sim v_A^2$ for $B\neq A$ so we may define an expansion parameter $\epsilon^2=\mathrm{Sup}(v_A^2, U_B(\mathbf{x}_A))$. The first class of $O(1/r^3)$ terms is the EIH [Einstein-Infeld-Hoffmann (Ref. 15)] terms and is post-Newtonian, since their coefficients are reduced in size from the coefficient of the $O(1/r^2)$ term by ϵ^2. These terms depend only on the masses of the objects and not on their higher mass multipole moments. The second class of terms is obtained by multipole expansion of terms such as $\int_A d^3x\,\rho^* U_B \partial_k \overline{U}$. In other words, they are post-Newtonian but the parameter is not ϵ^2. It is instead $\delta^2\sim\overline{U}$. Now $\delta^2>\epsilon^2$, but these terms are also reduced by some parameter of the multipole expansion. We expect the only terms of this class that do not vanish ("efface")[16] will be "relativistic tidal force" terms. Such terms are usually taken to be small in any case and are ignored since they contain both post-Newtonian and multipole parameters. Since we are interested specifically in NGT, we will not study these terms here. The third type of term is similar to the latter type, though it consists of NGT contributions. We keep them because they are the leading NGT terms in the $1/r$ expansion. The first terms depending on monopole moments of the NGT l charge parameter do not occur until $O(1/r^5)$. We include both the above-mentioned NGT contributions because it may be difficult in some situations to determine which type of term should dominate. This situation arises because NGT contributions to physically measurable quantities must come from terms bilinear in the l^2 parameters. This condition is a consequence of transposition invariance,[1] which is a generalization of the condition in GR that $g_{\mu\nu}$ and the connection coefficients be symmetric. Hence, a monopole field such as $\nabla\lambda=-l^2\mathbf{r}/r^3$ must be squared to contribute. After squaring, its gradient must be taken to produce a direction. Only then can it contribute to a force. However, a product such as $\partial_j\partial_k\lambda_B \int_A d^3x\, S^*\partial_j\overline{U}$ has direction, is bilinear in the NGT current, and is $O(1/r^3)$. Terms such as $\int_A d^3x\,\rho^*\lambda_B\partial_k\lambda_B$ would produce $O(1/r^3)$ monopole forces if they existed in the metric (say, if $g_{[ij]}\sim\lambda$), but they are not present there (and have the wrong dimension).

Our integration of (3.13) yields, after application of the virial theorems,

$$a^k_A = a^k_{A_{\rm GR}} + a^k_{A_{\rm NGT}}\ , \tag{4.4a}$$

where

$$\begin{aligned}
a^k_{A_{\rm GR}} = \frac{1}{m_A}\sum_{B\neq A}\Bigg[& -\frac{r^j_{AB}}{r^3_{AB}}\left\{ m_A m_B\delta^{jk} - \frac{3m_B Q^{jk}_A}{r^2_{AB}} - \frac{3m_A Q^{jk}_B}{r^2_{AB}} + \frac{15}{2}\frac{r^i_{AB}r^l_{AB}}{r^4_{AB}}\delta^{jk}(m_A Q^{il}_B + m_B Q^{il}_A)\right\} \\
& + 4m_A m_B^2\frac{r^k_{AB}}{r^4_{AB}} + 5m_A^2 m_B\frac{r^k_{AB}}{r^4_{AB}} - 2m_A m_B v_B^2\frac{r^k_{AB}}{r^3_{AB}} - m_A m_B v_A^2\frac{r^2_{AB}}{r^3_{AB}} \\
& + 4m_A m_B \mathbf{v}_A\cdot\mathbf{v}_B\frac{r^k_{AB}}{r^3_{AB}} + \tfrac{3}{2}m_A m_B(\mathbf{r}_{AB}\cdot\mathbf{v}_B)^2\frac{r^k_{AB}}{r^5_{AB}} + 4m_A m_B\mathbf{r}_{AB}\cdot\mathbf{v}_A\frac{v^k_A}{r^3_{AB}} \\
& - 3m_A m_B\mathbf{r}_{AB}\cdot\mathbf{v}_B\frac{v^k_A}{r^3_{AB}} - 4m_A m_B\mathbf{r}_{AB}\cdot\mathbf{v}_A\frac{v^k_B}{r^3_{AB}} + 3m_A m_B\mathbf{r}_{AB}\cdot\mathbf{v}_B\frac{v^k_B}{r^3_{AB}} \\
& + \sum_{C\neq A,B} m_A m_B m_C\left\{4\frac{r^k_{AB}}{r^3_{AB}r_{AC}} + \frac{r^k_{AB}}{r^3_{AB}r_{BC}} - \frac{7}{2}\frac{r^k_{BC}}{r_{AB}r^3_{BC}} - \frac{1}{2}\frac{\mathbf{r}_{AB}\cdot\mathbf{r}_{BC}r^k_{AB}}{r^3_{AB}r^3_{BC}}\right\}\Bigg]\ ,
\end{aligned} \tag{4.4b}$$

$$a^k_{A_{\rm NGT}} = \frac{1}{m_A}\sum_{B\neq A}[2m_A l_B^4 + 2m_B l_A^4 - 2(m_A+m_B)l_A^2 l_B^2]\frac{r^k_{AB}}{r^6_{AB}}$$
$$+\frac{1}{m_A}\sum_{B\neq A}\sum_{C\neq B,A}\tfrac{1}{2}(m_B l_A^2 l_C^2 - m_A l_B^2 l_C^2)\left[\frac{r^j_{AC}}{r^3_{AC}r^3_{AB}}\left[\delta^{jk}-3\frac{r^j_{AB}r^k_{AB}}{r^2_{AB}}\right]\right.$$
$$\left.+\frac{r^j_{AB}}{r^3_{AB}r^3_{AC}}\left[\delta^{jk}-3\frac{r^j_{AC}r^k_{AC}}{r^2_{AC}}\right]+\frac{r^j_{BC}}{r^3_{BC}r^3_{AB}}\left[\delta^{jk}-3\frac{r^j_{AB}r^k_{AB}}{r^2_{AB}}\right]\right]$$
$$+\frac{1}{m_A}\sum_{B\neq A}\sum_{C\neq A,B}\tfrac{1}{2}(m_B l_A^2 l_C^2 - m_C l_A^2 l_B^2)\frac{r^j_{BC}}{r^3_{BC}r^3_{AB}}\left[\delta^{jk}-3\frac{r^j_{AB}r^k_{AB}}{r^2_{AB}}\right]$$
$$+\frac{1}{m_A}\frac{3}{2}\sum_{B\neq A}\frac{1}{r^3_{AB}}\left[\delta^{jk}-3\frac{r^j_{AB}r^k_{AB}}{r^2_{AB}}\right](l_A^2\Sigma^j_B - l_B^2\Sigma^j_A)\ . \qquad (4.4c)$$

From (4.4), we see that $\gamma^{jk}_{AB}=\delta^{jk}m_A m_B$ so that the equivalence of inertial and gravitational masses is obtained at the order of this calculation. We define the mass quadrupole moment Q^{jk}_A and mass-NGT dipole interaction term Σ^j_A of some body A to be

$$Q^{jk}_A = I^{jk}_A - \tfrac{1}{3}\delta^{jk}I_A\ ,$$
$$I^{jk}_A = \int_A d^3x\,\rho^*(\mathbf{x})(x^j - x^j_A)(x^k - x^k_A)\ , \qquad (4.5a)$$

$$\Sigma^k_A = \int_A d^3x\, d^3x'\rho^* S^{*\prime}\frac{x^k - x'^k}{|\mathbf{x}-\mathbf{x}'|^3}\ . \qquad (4.5b)$$

We have kept Σ-type NGT terms only to leading $(1/r^3)$ order, though we have written the leading monopole NGT terms, which are $O(1/r^5)$.

There are several relevant special cases. In particular, we choose to concentrate on periastron motion in binary stellar systems (excluding systems with sufficiently strong gravity that the 1PN approximation is invalid), perihelion motion of a planet in solar orbit, and test-particle motion near Earth's surface. The two-body NGT acceleration, expressed in terms of the relative coordinates $\mathbf{x}_{12}=\mathbf{x}_1-\mathbf{x}_2$, is given at the 1PN level by

$$a^k_{12_{\rm NGT}} = 2m(l_1^2 - l_2^2)d_{12}\frac{r^k_{12}}{r^6_{12}} + \frac{3}{2}\frac{P^{jk}_{12}}{r^3_{12}}\frac{l_1^2\Sigma^j_2 - l_2^2\Sigma^j_1}{\mu}\ . \qquad (4.6)$$

We use the definitions

$$m = m_1 + m_2,\quad \mu = m_1 m_2/m\ ,$$
$$P^{jk}_{12} = \delta^{jk} - \frac{(x^j_1 - x^j_2)(x^k_1 - x^k_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^2}\ , \qquad (4.7)$$
$$d_{12} = \frac{l_1^2}{m_1} - \frac{l_2^2}{m_2}\ .$$

Perhaps more useful for the analysis of the motion of objects in terrestrial laboratories is the "test-particle limit." This result may be obtained from the above by putting both m_1 and l_1 to zero, though not the ratio l_1^2/m_1. The ratio Σ^j_1/m_1 probably can be ignored in this limit since it scales as $m_1\times(l_1^2/m_1)=l_1^2$. We obtain

$$a^k_{st_{\rm NGT}} = 2m_s l_s^2 d_{st}\frac{r^k_{st}}{r^6_{st}}\ , \qquad (4.8)$$

where s denotes the source and t denotes the test body. This is in agreement with previous results.[17]

We now derive the periastron shift formula for objects in binary orbits. In fact, because of the presence of Σ^j and Q^{ij} terms in the equations of motion, the acceleration of an object will, in general, not be confined to the orbital plane. In this case, not only will the periastron position change in time, but so will the eccentricity of the orbit, its inclination angle, its semimajor axis, and the position of the line of nodes. The formulas for the changes in these orbital elements have been given in terms of the components of the acceleration by Smart.[18]

Let us consider the case of two bodies, each with cylindrical symmetry about an axis normal to the orbital plane and each symmetric with respect to reflections in the orbital plane. Then, if the objects' centers of mass are $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, we have

$$\rho^* = \rho_1^*((x-x_1)^2+(y-y_1)^2,(z-z_1)^2)$$
$$+\rho_2^*((x-x_2)^2+(y-y_2)^2,(z-z_2)^2)\ ,$$
$$S^* = S_1^*((x-x_1)^2+(y-y_1)^2,(z-z_1)^2) \qquad (4.9)$$
$$+S_2^*((x-x_2)^2+(y-y_2)^2,(z-z_2)^2)\ .$$

Therefore, $\Sigma^j=0$ for each body and the quadrupole moment of each body takes the form

$$Q^{ij} = \tfrac{1}{3}(C-A)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}\ , \qquad (4.10)$$

where C is the moment of inertia of the body about the symmetry axis and A is the moment of inertia about an orthogonal axis. The calculation of the periastron shift is straightforward[19] and yields

$$\Delta\omega_0 = 6\pi\frac{m}{p}\left[1+\frac{p}{2m}\left[J_{2_{(1)}}\frac{R_1^2}{p^2}+J_{2_{(2)}}\frac{R_2^2}{p^2}\right] - \frac{(l_1^2-l_2^2)d_{12}}{mp^2}(1+e^2/4)\right]\ , \qquad (4.11)$$

where the relative orbit is approximated by the ellipse

$$r = \frac{p}{1+e\cos(\omega-\omega_0)} \ . \tag{4.12}$$

The $J_{2_{(a)}}$ parameter is the dimensionless measure of the quadrupole moment of body a and is given by $J_{2_{(a)}} = (C_a - A_a)/m_a R_a^2$, R_a being the radius of the body. The quadrupolar deformation in a star is attributable to tidal and rotational effects. Therefore, one often sees the classical part of the periastron shift formula rewritten to exhibit explicitly the dependence on rotational velocity and chemical composition (see Ref. 20 and citations therein). We note that this corrects our previously published periastron shift formula[11] and reduces to the GR result if l^2 is strictly proportional to mass. We comment qualitatively on this in the conclusions.

V. ANALYSIS

Let us begin with an analysis of the periastron shift formula (4.11). There are three arenas in which we shall work: the Sun-Mercury system, low-mass eclipsing binary stellar systems, and high-mass eclipsing binary stellar systems.

The anomalous perihelion shift of Mercury was initially considered to be irrefutable evidence for the general theory of relativity. However, Dicke[21] pointed out that, if the Sun has a nonzero quadrupole moment, then this will induce a classical perihelion shift. The observed shift may then be less than the sum of the calculated classical and relativistic contributions. We write

$$\Delta\omega = 6\pi \frac{m}{p} \lambda_{\odot\,\mathrm{Merc}} \ ,$$
$$\lambda_{\odot\,\mathrm{Merc}} = 1 + \frac{J_{2\odot} R_\odot^2}{2 m_\odot p} - \frac{K_{\odot\,\mathrm{Merc}}}{m_\odot^2 p^2}(1+e^2/4) \ . \tag{5.1}$$

We cannot extract l values from this equation so we define

$$K_{AB} = (m_A + m_B)(l_A^2 - l_B^2) d_{AB}$$

and place limits on this parameter. Shapiro, Councelman, and King[22] quote

$$\lambda_{\odot\,\mathrm{Merc}} = 1.003 \pm 0.005 \tag{5.2}$$

while Anderson *et al.*[23] quote

$$\lambda_{\odot\,\mathrm{Merc}} = 1.007 \pm 0.005 \ . \tag{5.3}$$

We adopt the mean of these values. The value of the coefficient $J_{2\odot}$ is controversial. Hill and Rosenwald[24] list a table of published values in a recent review. An equally weighted average of these values gives $J_{2\odot} = (5.7 \pm 6.5) \times 10^{-6}$. If we weight the average to exclude those points which are not consistent with any other results, we obtain

$$J_{2\odot} = (5 \pm 6) \times 10^{-6} \ . \tag{5.4}$$

The standard error quoted is much higher than most measurement errors quoted in the table of Hill and Rosenwald, indicating the controversy existing in this field. From these numbers, we obtain

$$K_{\odot\,\mathrm{Merc}} = (1.2 \pm 1.3) \times 10^{14}\ \mathrm{km}^4 \ . \tag{5.5}$$

This value is marginally consistent with GR ($K_{\odot\,\mathrm{Merc}} = 0$) at the 1σ (one-standard-deviation) level, though no conclusions can be drawn without a resolution of the $J_{2\odot}$ controversy.

We now turn to low-mass binary stars. The periastron shifts of many such systems are dominated by classical effects and depend upon the density distributions of the stars. From studying such systems, a good understanding of stellar structure has been developed, allowing us to calculate with confidence the classical part of the periastron shift in systems where relativistic effects are important. We choose two such systems for further study, V1143 Cyg and EK Cep, which are 2.62 and 3.15 solar masses, respectively. Gimenez and Margrave[25] quote both the observed periastron shifts for these systems and the calculated (GR + classical) values. For EK Cep, they quote

$$\Delta\omega_{\mathrm{obs}} = (8^\circ.8 \pm 2^\circ.6)/100\ \mathrm{yr} \ ,$$
$$\Delta\omega_{\mathrm{GR+cl}} = (7^\circ.9 + 3^\circ.0)/100\ \mathrm{yr} \ , \tag{5.6}$$

while for V1143 Cyg they quote

$$\Delta\omega_{\mathrm{obs}} = (3^\circ.37 \pm 0^\circ.20)/100\ \mathrm{yr} \ ,$$
$$\Delta\omega_{\mathrm{GR+cl}} = (4^\circ.2 \pm 1^\circ.4)/100\ \mathrm{yr} \ . \tag{5.7}$$

From these data, we obtain

$$K_{\mathrm{EK\ Cep}} = (-7 \pm 32) \times 10^{14}\ \mathrm{km}^4 \ ,$$
$$K_{\mathrm{V1143\ Cyg}} = (8 \pm 13) \times 10^{14}\ \mathrm{km}^4 \ . \tag{5.8}$$

These are consistent with $K = 0$ (general relativity).

The last application of the periastron shift formula is to the more massive systems AS Cam ($5.8 m_\odot$) and DI Her ($9.7 m_\odot$). For the former system, Khaliullin and Kosyreva[26,27] quote

$$\Delta\omega_{\mathrm{obs}} = (14^\circ.6 \pm 1^\circ.2)/100\ \mathrm{yr} \ ,$$
$$\Delta\omega_{\mathrm{GR+cl}} = (43^\circ.6 \pm 3^\circ.5)/100\ \mathrm{yr} \ , \tag{5.9}$$

while, for the latter, Guinan and Maloney[28] quote

$$\Delta\omega_{\mathrm{obs}} = (0^\circ.65 \pm 0^\circ.18)/100\ \mathrm{yr} \ ,$$
$$\Delta\omega_{\mathrm{GR+cl}} = (4^\circ.27 \pm 0^\circ.30)/100\ \mathrm{yr} \ . \tag{5.10}$$

If they limit themselves to higher accuracy photoelectric observations alone, they obtain

$$\Delta\omega_{\mathrm{obs}} = (1^\circ.75 \pm 0^\circ.11)/100\ \mathrm{yr} \tag{5.11}$$

for DI Her. From this data, we obtain

$$K_{\mathrm{AS\ Cam}} = (3.37 \pm 0.43) \times 10^{16}\ \mathrm{km}^4 \ . \tag{5.12}$$

If we use Eq. (5.10), we obtain

$$K_{\mathrm{DI\ Her}} = (1.59 \pm 0.33) \times 10^{17}\ \mathrm{km}^4 \ , \tag{5.13}$$

while Eq. (5.11) yields

$$K_{\mathrm{DI\ Her}}=(1.10\pm0.14)\times10^{17}\ \mathrm{km}^4\ . \qquad (5.14)$$

These results are not consistent with zero. It has been pointed out that retrograde contributions to the classical periastron shift can arise if the spin axes of the binary components lie in the orbital plane.[29] This would then reconcile the observed and calculated periastron shifts and make K consistent with zero. However, this is a highly unstable configuration. It is likely that the stars would not remain long in this state. A retrograde classical shift may also be induced by a third body in the system, but there is no evidence that either system has a third companion.[28] Clearly this is an area that requires further study, both observationally and theoretically. It would be very interesting to know if other high-mass eclipsing binaries exhibit anomalously small periastron shifts. For some other examples, see Ref. 20, but note that it assumes geodesic motion in NGT.

As we have established above, there is no Nordtvedt effect at the 1PN level in NGT (meaning there is no effect in the lunar orbit caused by an inequivalence of m_I and m_G). However, we need to know how large the error terms are in this calculation. Therefore, we need an upper bound on the l value of Earth. Let us consider the LAGEOS artificial satellite and lunar laser ranging experiment (LURE) data. From our knowledge of the orbit of LAGEOS about Earth, a value for the mass of Earth times Newton's constant can be deduced:[30]

$$Gm_\oplus=3.986\,004\,34(2)\times10^{14}\ \mathrm{m^3s^{-2}}\ . \qquad (5.15)$$

The LURE gives

$$Gm_\oplus=3.986\,004\,44(10)\times10^{14}\ \mathrm{m^3s^{-2}}\ . \qquad (5.16)$$

This allows us to calculate the gradient

$$\frac{\Delta(Gm_\oplus/c^2)}{\Delta r}=(3.0\pm3.0)\times10^{-19}\ . \qquad (5.17)$$

Now, from Eqs. (4.4) and (4.6), we see that the NGT terms in the acceleration of a body (the Moon or LAGEOS) toward Earth may be absorbed into an effective definition of Newton's constant

$$\mathbf{a}_A=-G_{\mathrm{eff}}m_A m_\oplus\frac{\mathbf{r}}{r^3}+\text{the PN terms in (4.4b)}\ . \qquad (5.18)$$

Ignoring the Σ terms, we obtain

$$G_{\mathrm{eff}}=G_\infty\left[1-\frac{2}{r^3}(l_\oplus^2-l_A^2)d_{\oplus A}\right], \qquad (5.19)$$

where r is the distance from the object A to the center of Earth. We now compute the gradient of $G_{\mathrm{eff}}m_\oplus$ and compare it with Eq. (5.19), assuming any disagreement in the two determinations of $Gm_\oplus$ can be attributed to NGT. We can probably safely assume the Moon experiences little acceleration due to the l terms in G_{eff} because they fall off rapidly with distance. However, there is a question as to what to do about the $l^2_{\mathrm{LAGEOS}}/m_{\mathrm{LAGEOS}}$ term in $d_{\oplus\ \mathrm{LAGEOS}}$. If we set it to zero, we obtain $l_\oplus=(0.1\pm0.1)$ km. This is a very stringent limit. If, instead, we take $l^2\sim$baryon number, then the difference $d_{\oplus\ \mathrm{LAGEOS}}$ becomes small ($<10^{-2}$) and we obtain $l_\oplus=(3\pm3)$ km. However, the baryon-number model causes observable effects in the Eotvos experiment. A separate analysis[31] of this data in the light of last year's controversy has yielded $l_\oplus=(1.35\pm0.17)$ km using the baryon-number model of l^2, if one follows the reanalysis of the original Eotvos experiment by Fischbach *et al.*[32] Since baryon number is the conserved charge that most resembles mass, we consider this the model that would give the highest upper bound on $l_\oplus$, giving us the least confidence in the relation $m_I=m_G$. (For this model, the error induced by dropping 2PN terms in the Nordtvedt calculation will be the largest.) For l^2 precisely proportional to mass, $d_{\oplus\ \mathrm{LAGEOS}}$ would be zero and no limit on $l_\oplus$ could be obtained. We hasten to add that the baryon-number model does not fit very well the above periastron shift results. We do not presently think it provides a viable interpretation for l, but it is a worst case. We choose $l_\oplus<2$ km.

We calculate the error in the equality $m_I=m_G$ by looking at the terms in Eqs. (4.2) and (4.3b). These are typical 1PN terms. The largest 2PN terms have an additional factor of $\bar{U}$ inserted ($\bar{U}_\oplus\sim10^{-9}$). We also assume $l_\odot\sim10^3$ km. Now we may examine the 1PN and 2PN terms in the equation for Earth falling in the field of the Sun. By referring to Eqs. (4.2) and (4.3b) and inserting the extra $\bar{U}$ to get a typical 2PN term, we obtain

$$(m_G/m_I)_\oplus=1+O(1\mathrm{PN})+O(2\mathrm{PN})+\cdots$$
$$=1+O(10^{-4})+O(10^{-13})+\cdots\ . \qquad (5.20)$$

We have estimated

$$\int_\oplus d^3x\,d^3x'\frac{\rho^*S^{*\prime}}{|\mathbf{x}-\mathbf{x}'|}\sim\frac{m_\oplus l_\oplus^2}{R_\oplus^3}\sim7\times10^{-17}\ , \qquad (5.21)$$

where $R_\oplus$ is the radius of Earth. Currently, the LURE experiment can detect a Nordtvedt effect produced by a difference between m_I and m_G of 7 parts in 10^{-12} (see Ref. 5). We therefore expect that, if NGT predicts a difference between m_I and m_G at the 2PN level, this will not violate the null result of the LURE. However, our error analysis was, by necessity, crude.

LURE and LAGEOS measurements may have another interesting application to the testing of NGT. Expansion of the three-body version of Eqs. (4.4) about the Earth-Sun distance indicates that NGT may influence the relative motion of Earth and a satellite, causing oscillations similar to those caused by a difference in m_I and m_G but originating in three-body point-particle terms in (4.4c). Whether or not these terms are observable in a Fermi coordinate system at Earth's surface is not yet known and is the subject of current investigation. However, the calculation in our post-Newtonian coordinate system is straightforward and indicates that oscillatory terms in this system have divisors involving the Earth-satellite distance. Therefore, the laser tracking of LAGEOS may place rigid bounds on $l_\oplus$.

We emphasize that our analysis does not apply to compact objects such as the binary pulsar PSR1913 + 16. It appears impossible to build pulsars[33] with l values greater than about 10 km but this still gives $l^2/R^2 \sim 1$, where R is the radius of the pulsar. Therefore the perturbative PN expansion cannot be relied upon. Perhaps a matching procedure such as that of D'Eath[34] is more appropriate for the pulsar problem (see, however, Ref. 9).

Lastly, we point out that the appearance of the $d_{AB} = l_A^2/m_A - l_B^2/m_B$ term in the equation of motion means that laboratory-sized objects will fall at different rates if they carry different amounts of l charge. This is to be expected since S^μ is not coupled via the metric (not "universally coupled"). We have not analyzed this prediction in detail here because it necessitates a microscopic model for l charge that is imposed from outside the theory. We have studied the effects of this term if l is baryon number and have fit the model to the anomaly in the Eötvös data.[31] However, the model does not extrapolate well to objects of stellar size. Nonetheless, the recent Eötvös experiments may provide useful bounds on l^2/m for laboratory-sized objects.

VI. CONCLUSIONS

We have obtained the 1PN approximation to the nonsymmetric gravitation theory in the context of a post-Newtonian perfect fluid. Using a variation of a method discussed by Fock,[4] Nordtvedt,[8] and Will,[5] we obtained the 1PN equations of motion for an object consisting of this perfect fluid in the field of itself and an arbitrary number of other such objects, obtaining the equality of inertial and gravitational masses in NGT at 1PN order. We have derived some special cases of the equation of motion that are particularly useful for testing NGT. We have discussed the consequences of this formula for periastron shifts of binary stars and the perihelion shift of Mercury.

While we have illustrated certain cases where our results are amenable to observational tests of gravitation theory, we have not performed a detailed comparison of NGT with observational data, preferring here mostly to obtain the formalism for later application. One primary reason for this is that conclusive testing of NGT will occur only when the nature of the current S^μ is known or when model-independent tests can be developed, the latter meaning a combination of tests such that the l values of all objects in the test system are determined. We can, however, currently rule out models for l and hope to constrain the theory by limiting the parameter space of couplings to known currents such as baryon number and lepton number. For example, it seems unlikely in view of the new periastron formula that l^2 could be taken to be proportional to baryon number, owing to the fact that planets have greater baryon number per unit mass than stars. In this case, NGT effects would be presumably greater in planetary terrestrial systems than in stellar systems. As we have seen, the greatest deviations from GR predictions occur, however, in stellar systems. We believe we do not currently have a successful identification of the l^2 charge in terms of known charges.

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APPENDIX A

The fundamental tensor of NGT is

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]} \tag{A1}$$

and has inverse $g^{\mu\nu}$ given by

$$g^{\mu\alpha} g_{\nu\alpha} = g^{\alpha\mu} g_{\alpha\nu} = \delta^\mu_\nu \ . \tag{A2}$$

To form the Riemann curvature tensor, we introduce an unconstrained nonsymmetric connection $W^\alpha_{\mu\nu}$. Covariant differentiation on vectors is given in a basis $\{\underline{e}_\nu\}$ as

$$\underline{\nabla}_{\underline{Z}} \underline{X} = Z^\mu (\partial_\mu X^\nu + X^\rho W^\nu_{\mu\rho}) \underline{e}_\nu \ . \tag{A3}$$

A generalized Riemann curvature may be defined in the standard way by

$$\begin{aligned} \underline{R}(\underline{X},\underline{Y})\underline{Z} &= (\underline{\nabla}_{\underline{X}} \underline{\nabla}_{\underline{Y}} - \underline{\nabla}_{\underline{Y}} \underline{\nabla}_{\underline{X}} - \underline{\nabla}_{[\underline{X},\underline{Y}]})\underline{Z} \\ &= R^\alpha{}_{\beta\mu\nu} X^\mu Y^\nu Z^\beta \underline{e}_\alpha \ , \end{aligned} \tag{A4}$$

where

$$R^\alpha{}_{\beta\mu\nu} = W^\alpha_{\beta\nu,\mu} - W^\alpha_{\beta\mu,\nu} + W^\alpha_{\rho\mu} W^\rho_{\beta\nu} - W^\alpha_{\rho\nu} W^\rho_{\beta\mu} \ . \tag{A5}$$

There are two independent natural contractions of this tensor. The tensor $R_{\mu\nu}$ is formed from a linear combination of these two independent contractions:

$$\begin{aligned} R_{\mu\nu}(W) &= W^\beta_{\mu\nu,\beta} - \tfrac{1}{2}(W^\beta_{\mu\beta,\nu} + W^\beta_{\nu\beta,\mu}) \\ &\quad - W^\beta_{\alpha\nu} W^\alpha_{\mu\beta} + W^\beta_{\alpha\beta} W^\alpha_{\mu\nu} \ . \end{aligned} \tag{A6}$$

Often it is useful to work with the constrained connection (constrained to have zero torsion vector) defined by

$$\Gamma^\alpha_{\mu\nu} = W^\alpha_{\mu\nu} + \tfrac{2}{3}\delta^\alpha_\mu W_\nu \ , \tag{A7}$$

where the torsion vector is defined as

$$W_\nu = W^\beta_{[\nu\beta]} \ . \tag{A8}$$

Then we have

$$R_{\mu\nu}(W) = R_{\mu\nu}(\Gamma) + \tfrac{2}{3} W_{[\mu,\nu]} \ , \tag{A9}$$

where

$$\begin{aligned} R_{\mu\nu}(\Gamma) &= \Gamma^\beta_{\mu\nu,\beta} - \tfrac{1}{2}(\Gamma^\beta_{(\mu\beta),\nu} + \Gamma^\beta_{(\nu\beta),\mu}) \\ &\quad - \Gamma^\beta_{\alpha\nu}\Gamma^\alpha_{\mu\beta} + \Gamma^\beta_{(\alpha\beta)}\Gamma^\alpha_{\mu\nu} \ . \end{aligned} \tag{A10}$$

The NGT field equations may be derived from a Palatini variational principle

$$\delta \int \left[\mathbf{g}^{\mu\nu} R_{\mu\nu}(W) - 8\pi g^{\mu\nu} \mathbf{T}_{\mu\nu} - \frac{8\pi}{3} W_\mu \mathbf{S}^\mu \right] d^3x = 0 \tag{A11}$$

under unconstrained variations of $g_{\mu\nu}$ and $W^\alpha_{\beta\lambda}$. This gives

$$G_{\mu\nu}(W)=8\pi T_{\mu\nu}\ , \tag{A12}$$

$$\mathbf{g}^{[\mu\nu]}{}_{,\nu}=4\pi\mathbf{S}^{\mu}\ , \tag{A13}$$

where

$$G_{\mu\nu}(W)=R_{\mu\nu}(W)-\tfrac{1}{2}g_{\mu\nu}R(W)\ , \tag{A14}$$

$$R(W)=g^{\mu\nu}R_{\mu\nu}(W)\ . \tag{A15}$$

Boldface type denotes tensor densities

$$\sqrt{-g}\,g^{\mu\alpha}=\mathbf{g}^{\mu\alpha}\ . \tag{A16}$$

Vincent[35] has derived a form for the contravariant (nonsymmetric) stress tensor of a perfect fluid,

$$T^{\mu\nu}=(\rho+p)u^{\mu}u^{\nu}-pg^{\mu\nu}\ , \tag{A17}$$

from a variational principle. Here p is the pressure and $u^{\mu}=dx^{\mu}/d\tau$ is the four-velocity of an element of fluid. The mass-energy density is

$$\rho=\rho_0(1+\Pi)\ , \tag{A18}$$

where ρ_0 is the fluid rest-mass density and Π is the rest specific binding energy density. The symbol $T_{\mu\nu}$ denotes $g_{\mu\beta}g_{\alpha\nu}T^{\alpha\beta}$.

General covariance implies the existence of four Bianchi identities and the associated matter response equations

$$g_{(\mu\gamma)}\mathbf{T}^{(\mu\beta)}{}_{,\beta}+g_{[\mu\gamma]}\mathbf{T}^{[\mu\beta]}{}_{,\beta}+\tfrac{1}{2}(g_{\mu\gamma,\nu}+g_{\gamma\nu,\mu}-g_{\mu\nu,\gamma})\mathbf{T}^{\mu\nu}=\tfrac{1}{3}W_{[\alpha,\gamma]}\mathbf{S}^{\alpha}\ . \tag{A19}$$

In addition to $T^{\mu\nu}$, the theory contains a conserved current $\mathbf{S}^{\mu}$ such that

$$\mathbf{S}^{\mu}{}_{,\mu}=0\ . \tag{A20}$$

This current is taken to be a linear combination of number currents associated with an inhomogeneous fluid

$$S^{\mu}=\sum_i f_i^2 n_i(t,\mathbf{x})u^{\mu}(t,\mathbf{x})\ , \tag{A21}$$

though all that is required is that it be a conserved vector current. The f_i^2 are the couplings to the different number currents n_i present in the fluid. The charge associated with this current is

$$l^2=\int \mathbf{S}^0 d^3x\ . \tag{A22}$$

Variation of the Lagrangian with respect to the unconstrained connection $W^{\alpha}_{\mu\nu}$ produces the relation

$$\mathbf{g}^{\mu\nu}{}_{,\alpha}-\delta^{\nu}_{\alpha}\mathbf{g}^{(\mu\beta)}{}_{,\beta}+\mathbf{g}^{\beta\nu}W^{\mu}_{\beta\alpha}+\mathbf{g}^{\mu\beta}W^{\nu}_{\alpha\beta}-\delta^{\nu}_{\alpha}\mathbf{g}^{\beta\lambda}W^{\mu}_{\beta\lambda}-\mathbf{g}^{\mu\nu}W^{\beta}_{\alpha\beta}=\frac{4\pi}{3}(\mathbf{S}^{\mu}\delta^{\nu}_{\alpha}-\mathbf{S}^{\nu}\delta^{\mu}_{\alpha})\ . \tag{A23}$$

Contraction on μ and λ yields Eq. (A13). Equations (A23) are usually expressed in terms of the variables W_{μ} and $\Gamma^{\alpha}_{\mu\nu}$ (recall $\Gamma_{\mu}=0$). In turn, the Γ connection is often expressed as a tensor homogeneous in S^{μ}, denoted $D^{\alpha}_{\mu\nu}$, and a connection satisfying the usual form of the compatibility equation, denoted $\Lambda^{\alpha}_{\mu\nu}$. That is, we define

$$\Gamma^{\alpha}_{\mu\nu}=\Lambda^{\alpha}_{\mu\nu}-D^{\alpha}_{\mu\nu}\ , \tag{A24}$$

$$g_{\beta\nu}D^{\beta}_{\mu\alpha}+g_{\mu\beta}D^{\beta}_{\alpha\nu}=-\frac{4\pi}{3}S^{\beta}(g_{\mu\alpha}g_{\beta\nu}-g_{\mu\beta}g_{\alpha\nu}+g_{\mu\nu}g_{[\alpha\beta]})\ , \tag{A25}$$

and thus obtain

$$g_{\mu\nu,\alpha}-g_{\beta\nu}\Lambda^{\beta}_{\mu\alpha}-g_{\mu\beta}\Lambda^{\beta}_{\alpha\nu}=0\ , \tag{A26}$$

$$\sqrt{-g}_{\ ,\mu}=\sqrt{-g}\,\Gamma^{\beta}_{(\mu\beta)}+\frac{4\pi}{3}\mathbf{S}^{\beta}g_{[\beta\mu]}\ . \tag{A27}$$

APPENDIX B

We list here the Newtonian-order virial relations used to show explicit cancellation of self-terms and of composition-dependent $1/r^2$ terms in the equations of motion. Most of these relations may be found in the book by Will,[5] though there are minor numerical errors in some of the equations published there. The only differences between the usual form of these relations and the NGT form are the replacement of p by $\tilde{p}=p-(2\pi/3)S^0S^0$ and Π by $\tilde{\Pi}=\Pi-(2\pi/3)S^0S^0/\rho_0$.

We begin with the Newtonian approximation to the matter response equation (3.11). This is Euler's equation:

$$\frac{dv^k}{dt}=-\frac{1}{\rho^*}\partial_k\tilde{p}+\partial_k U\ . \tag{B1}$$

The conservation laws (3.6) and the time component of the matter response equation give

$$\rho_0\frac{d\Pi}{dt}=\left[\frac{p}{\rho_0}\right]\frac{d\rho_0}{dt}=-p\nabla\cdot\mathbf{v}\ . \tag{B2}$$

We define the following integrals:

$$\Omega^j_A=\int_A\frac{\rho^*\rho^{*\prime}\rho^{*\prime\prime}(x^j-x'^j)}{|\mathbf{x}'-\mathbf{x}''|\,|\mathbf{x}-\mathbf{x}'|^3}d^3x\,d^3x'd^3x''\ ,$$

$$\Omega^{*j}_A=\int_A\frac{\rho^*\rho^{*\prime}\rho^{*\prime\prime}(\mathbf{x}'-\mathbf{x}'')\cdot(\mathbf{x}-\mathbf{x}')(x^j-x'^j)}{|\mathbf{x}'-\mathbf{x}''|^3\,|\mathbf{x}-\mathbf{x}'|^3}d^3x\,d^3x'd^3x''\ ,$$

$$t^j_A=\int_A\frac{\rho^*\rho^{*\prime}\bar{v}'^2(x^j-x'^j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3x\,d^3x'd^3x'',\quad \mathcal{T}^j_A=\int_A\frac{\rho^*\rho^{*\prime}\bar{v}'^j\bar{\mathbf{v}}'\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x\,d^3x'd^3x''\ ,$$

$$\mathcal{T}_A^{*j}=\int_A \frac{\rho^*\rho^{*\prime}\bar{v}^j\bar{\mathbf{v}}'\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x\,d^3x'd^3x'',\quad \mathcal{T}_A^{**j}=\int_A \frac{\rho^*\rho^{*\prime}[\bar{\mathbf{v}}'\cdot(\mathbf{x}-\mathbf{x}')]^2(x^j-x'^j)}{|\mathbf{x}-\mathbf{x}'|^5}d^3x\,d^3x'd^3x'' ,$$

$$\tilde{\mathcal{P}}^j_A=\int_A \frac{\rho^*\tilde{p}'(x^j-x'^j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3x\,d^3x',\quad \tilde{\mathcal{E}}=\int_A \frac{\rho^*\rho^{*\prime}\tilde{\Pi}'(x^j-x'^j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3x\,d^3x' ,$$

$$T_A^{ij}=\tfrac{1}{2}\int_A \rho^*\bar{v}^i\bar{v}^j d^3x,\quad T_A=\tfrac{1}{2}\int_A \rho^*\bar{v}^2 d^3x , \tag{B3}$$

$$\Omega_A^{ij}=-\frac{1}{2}\int_A \frac{\rho^*\rho^{*\prime}(x^i-x'^i)(x^j-x'^j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3x\,d^3x',\quad \Omega_A=-\frac{1}{2}\int_A \frac{\rho^*\rho^{*\prime}}{|\mathbf{x}-\mathbf{x}'|}d^3x\,d^3x' ,$$

$$I_A^{ij}=\int_A \rho^*(x^i-x'^i)(x^j-x'^j)d^3x,\quad I_A=\int_A \rho^*|\mathbf{x}-\mathbf{x}'|^2d^3x ,$$

$$\tilde{P}_A=\int_a \tilde{p}\,d^3x,\quad \tilde{E}_A=\int_A \rho^*\tilde{\Pi}\,d^3x,\quad H_A^{ij}=\int_A \frac{\rho^*\rho^{*\prime}\bar{v}'^i(x^j-x'^j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3x\,d^3x'd^3x'' ,$$

$$K_A^{ij}=\int_A \frac{\rho^*\rho^{*\prime}\bar{\mathbf{v}}'\cdot(\mathbf{x}-\mathbf{x}')(x^i-x'^i)(x^j-x'^j)}{|\mathbf{x}-\mathbf{x}'|^5}d^3x\,d^3x'd^3x'' .$$

We also need to define $\bar{V}^j$ and $\bar{\mathcal{W}}^j$. These are given by Eqs. (3.3b), (3.3i), and (3.10).

We may derive several relations between the above quantities:

$$\frac{d^2I^{ij}}{dt^2}=4T^{ij}+2\tilde{P}\delta^{ij}+2\Omega^{ij},\quad \frac{d^2I}{dt^2}=4T+6\tilde{P}+2\Omega ,$$

$$\frac{dT^{ij}}{dt}=H^{(ij)}+\int \bar{v}_{(i}\tilde{p}_{,j)}d^3x,\quad \frac{dT}{dt}=H+\int \bar{\mathbf{v}}\cdot\tilde{\mathbf{p}}\,d^3x ,$$

$$\frac{d\Omega^{ij}}{dt}=2H^{(ij)}-3K^{ij},\quad \frac{d\Omega}{dt}=-K=H,\quad \frac{d\tilde{P}}{dt}=\int \frac{\partial\tilde{p}}{\partial t}d^3x , \tag{B4}$$

$$\frac{d}{dt}\int \rho^*\bar{\mathcal{W}}^j d^3x=-\tilde{\mathcal{P}}^j-\Omega^{*j}-t^j-\mathcal{T}^j+\mathcal{T}^{*j}+3\mathcal{T}^{**j} ,$$

$$\frac{d}{dt}\int \rho^*\bar{V}^j d^3x=\tilde{\mathcal{P}}^j+\Omega^j+\mathcal{T}^j+\mathcal{T}^{*j} ,$$

$$\frac{d}{dt}\int \rho^*\tilde{\Pi}\bar{v}^k d^3x=\int \rho^*\frac{d\tilde{\Pi}}{dt}\bar{v}^k d^3x-\int \tilde{\Pi}\partial_k\tilde{p}d^3x-\tilde{\mathcal{E}}^k .$$

These quantities appear in the equations of motion, both as self-acceleration terms and as contributions to the gravitational mass. However, if we average these quantities over time and assume that any secular effects due to these terms do not become significant over the time scales of interest, then the left-hand sides of Eqs. (B4) may all be replaced by zero. These averaged versions of (B4) are the virial theorems.

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[1]A. Einstein, *The Meaning of Relativity* (Princeton University, Princeton, NJ, 1956).

[2]J. W. Moffat, Phys. Rev. D **19**, 3554 (1979).

[3]R. B. Mann and J. W. Moffat, Can. J. Phys. **59**, 1723 (1981); **59**, 656(E) (1983).

[4]V. Fock, *The Theory of Space, Time, and Gravitational Physics* (MacMillan, New York, 1964).

[5]C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University, Cambridge, England, 1981).

[6]R. B. Mann and J. W. Moffat, Can. J. Phys. **59**, 1592 (1981).

[7]R. B. Mann and J. W. Moffat, J. Phys. A **14**, 2367 (1981).

[8]K. Nordtvedt, Jr., Phys. Rev. **169**, 1017 (1968); C. M. Will, Astrophys. J. **163**, 611 (1971).

[9]T. P. Krisher, Phys. Rev. D **32**, 329 (1985).

[10]J. C. McDow and J. W. Moffat, Can. J. Phys. **60**, 1545 (1982).

[11]J. W. Moffat and E. Woolgar, Can. J. Phys. **63**, 428 (1985).

[12]G. Kunstatter and J. W. Moffat, Phys. Rev. D **19**, 1084 (1979).

[13]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields,* translated by M. Hamermesh (Addison-Wesley, Reading, MA, 1974).

[14]H. Bondi, Rev. Mod. Phys. **29**, 423 (1957).

[15]A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938).

[16]T. Damour, in 300 Years of Gravitation, edited by S. W. Hawking and W. Isreal (Cambridge University, Cambridge, England, to be published).

[17]J. W. Moffat, Phys. Rev. D **35**, 3733 (1987).

[18]W. M. Smart, *Celestial Mechanics* (Longman Green, London, 1953).

[19]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[20]J. W. Moffat, Astrophys. J. **287**, L77 (1984).

[21]R. H. Dicke, Nature (London) **202**, 432 (1964).

[22]I. I. Shapiro, C. C. Councelman III, and R. W. King, Phys. Rev. Lett. **36**, 555 (1976).

[23]J. D. Anderson *et al.*, Acta Astron. **5**, 43 (1978).

[24]H. A. Hill and R. D. Rosenwald, in *Mathematical Aspects of*

Gravity and Supergravity (NATO Advanced Study Institute) (Plenum, New York, 1988).
[25]A. Giménez and T. E. Margrave, Astron. J. **90**, 358 (1985).
[26]Kh.F. Khalliullan and V. S. Kosyreva, Astrophys. Space Sci. **94**, 115 (1983).
[27]E. F. Guinan, F. P. Maloney, and P. T. Boyd have recently obtained $\Delta\omega_{\rm obs}=13^\circ.7/100$ yr$\pm1^\circ.5/100$ yr [E. F. Guinan (private communication)].
[28]E. F. Guinan and F. P. Maloney, Astron. J. **90**, 1519 (1985).
[29]N. I. Shakura, Pis'ma Astron. Eksp. **11**, 536 (1985) [Sov. Astron. Lett. **11**, 224 (1985)].
[30]F. D. Stacey *et al.*, Rev. Mod. Phys. **59**, 157 (1987).
[31]J. W. Moffat, P. Savaria, and E. Woolgar, University of Toronto Report No. UTPT-86-02 (unpublished).
[32]E. Fischbach *et al.*, Phys. Rev. Lett. **56**, 3 (1986).
[33]L. Campbell, J. W. Moffat, and P. Savaria (in preparation); L. Campbell, Ph.D. thesis, University of Toronto, 1988.
[34]P. D. D'Eath, Phys. Rev. D **12**, 2183 (1975).
[35]D. E. Vincent, Class. Quantum. Gravit. **2**, 409 (1985).