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Quantum cosmology and the initial state of the Universe

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Cosmological wave functions are found in a minisuperspace model (1) with "tunneling" and (2) with Hartle-Hawking boundary conditions. The probability distributions for the initial states of the Universe corresponding to the two wave functions are calculated and compared. It is shown that the tunneling wave function predicts initial states that lead to inflation, while the Hartle-Hawking wave function does not. Small perturbations about the minisuperspace model are considered and it is argued that both wave functions predict that the Universe nucleates with quantum fields in de Sitter-invariant vacuum states.

I. INTRODUCTION

Two different approaches have been recently suggested to the problem of determining the quantum state of the Universe. The first approach[1–5] is based on the picture that the Universe spontaneously nucleates in a de Sitter space and then evolves along the lines of an inflationary scenario. The mathematical description of this cosmic nucleation is closely analogous to that of quantum tunneling through a potential barrier, and it is often referred to as "quantum tunneling from nothing" or "creation of the Universe from nothing." An alternative approach to quantum cosmology is being developed by Hawking and collaborators.[6–10] Their proposal is that the wave function of the Universe, ψ, is given by a path integral over compact Euclidean geometries. Both schools of thought claim that their respective wave functions "predict" a period of inflation, but a direct comparison of the wave functions is obscured by the fact that they are calculated for different models. For example, Refs. 8–10 consider a universe filled with a scalar field having a potential $V(\phi)=\frac{1}{2}m^2\phi^2$, while in Ref. 5 the potential is taken to be $V(\phi)=-\frac{1}{2}m^2\phi^2+\text{const}$ $(m^2>0)$.

In this paper I shall further discuss the tunneling approach and compare its cosmological predictions with those of the Hartle-Hawking approach. The paper is organized as follows. The next section reviews the basic formalism of quantum cosmology. The tunneling boundary condition for the wave function of the Universe is formulated in Sec. III. (This formulation is more detailed than the one previously given in Ref. 5.) In Sec. IV the tunneling wave function ψ_T and the Hartle-Hawking wave function ψ_H are calculated in a minisuperspace model with 2 degrees of freedom: the scale factor and a scalar field ϕ with an arbitrary but slowly varying potential $V(\phi)$. The cosmological predictions of the two wave functions are discussed in Sec. V. Perturbations around the minisuperspace model and the quantum states of the gravitational and scalar fields are considered in Sec. VI. The conclusions are summarized in Sec. VII.

II. THE BASIC FORMALISM

We shall consider a model defined by the Lagrangian

$$\mathcal{L}=l_P^{-2}R+\tfrac{1}{2}(\partial_\mu\phi)^2-V(\phi)\ , \tag{2.1}$$

where $l_P=(16\pi G)^{1/2}$ is the Planck length, G is Newton's constant, $\hbar=c=1$, R is the scalar curvature, and the matter fields are represented by a single scalar field ϕ. The wave function for this model is defined on the space of all three-metrics $h_{ij}(\mathbf{x})$ and three-dimensional scalar fields $\phi(\mathbf{x})$:

$$\psi(h_{ij}(\mathbf{x}),\phi(\mathbf{x}))\ .$$

This wave function satisfies the equations (assuming that the Universe is closed)[11]

$$H^i(\mathbf{x})\psi=0\ , \tag{2.2}$$

$$H^0(\mathbf{x})\psi=0\ , \tag{2.3}$$

where

$$H^i=2iD_j\frac{\delta}{\delta h_{ij}}-ih^{ij}\phi_{,j}\frac{\delta}{\delta\phi}\ , \tag{2.4}$$

$$\begin{aligned}H^0&=-l_P^2\nabla^2+h^{1/2}[-l_P^{2\ (3)}R+\tfrac{1}{2}h^{ij}\phi_{,i}\phi_{,j}+V(\phi)]\\&\equiv-l_P^2(\nabla^2-U)\ ,\end{aligned} \tag{2.5}$$

$$\nabla^2=G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}}+\gamma_{ij}\frac{\delta}{\delta h_{ij}}+\frac{1}{2}l_P^{-2}h^{-1/2}\frac{\delta^2}{\delta\phi^2}\ , \tag{2.6}$$

$h=\det(h_{ij})$, D_j is a covariant derivative in the metric $h_{ij}(\mathbf{x})$, and

$$G_{ijkl}=\tfrac{1}{2}h^{-1/2}(h_{ik}h_{jl}+h_{il}h_{jk}-h_{ij}h_{kl})\ .$$

The coefficients γ_{ij} in the second term of (2.6) depend on the choice of factor ordering in the first term. The correct choice is presently unknown and possibly does not exist, since it is likely that a consistent quantum theory based on Einstein's gravity cannot be formulated.

However, Eqs. (2.2)–(2.6) can still be adequate for calculating the wave function ψ in the semiclassical regime. The factor-ordering ambiguity, as well as higher-order corrections and renormalization issues, become important only at or above the Planck curvature, $R \gtrsim l_P^{-2}$. We will see that in some models the Universe can be treated semiclassically throughout its entire history.

Equations (2.2) and (2.4) imply that ψ is independent of the choice of coordinates on the three-space. This fact is often expressed by saying that ψ is defined on a space of all three-geometries and matter-field configurations (superspace) in which all sets of $\{h_{ij}(\mathbf{x}), \phi(\mathbf{x})\}$ that differ only by a coordinate transformation are identified.

Using the notation introduced in (2.5) and (2.6), we can write the Wheeler-DeWitt equation (2.3) in a form similar to the Klein-Gordon equation:

$$(\nabla^2 - U)\psi = 0 \; . \tag{2.7}$$

Just as in the Klein-Gordon case, we can now construct a conserved current:

$$J = \frac{i}{2}(\psi^* \nabla \psi - \psi \nabla \psi^*) \; , \tag{2.8}$$

$$\nabla \cdot J = 0 \; , \tag{2.9}$$

where the scalar product is in the metric defined by (2.6). This current can be identified with the probability flux in superspace,[11,5] but just as in the Klein-Gordon case one has to bear in mind potential problems with negative probabilities.

III. BOUNDARY CONDITIONS

In the quantum tunneling approach, the nucleation of the Universe is a nonsingular event. Semiclassically, the underbarrier propagation corresponds to evolution in imaginary time, and so the tunneling is described by a regular solution of the Euclidean field equations (an instanton) which is matched to a Lorentzian solution at the nucleation point. Even though the Universe begins in a nonsingular way, it will develop singularities in the future (e.g., black holes or big crunch). The boundary condition for ψ corresponding to this picture of a nonsingular beginning of the Universe was formulated in Ref. 5. Here I shall give a somewhat more detailed formulation.

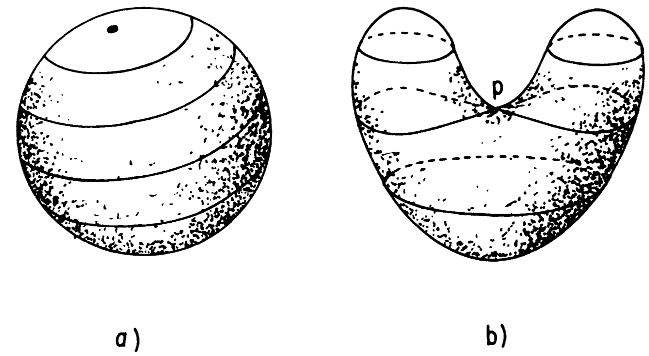

FIG. 1. Singular three-geometries can be obtained by the slicing of regular four-geometries. Here this is illustrated in a two-dimensional analogy.

The boundary of superspace can be thought of as consisting of singular configurations which have some points or regions with infinite three-curvature or with infinite ϕ or $(\partial_i \phi)^2$, as well as configurations of infinite three-volume. It is important to note, however, that singular three-geometries do not necessarily represent singular four-geometries. For example, if one slices a four-sphere as shown in Fig. 1(a), one would get three-spheres of vanishing radius and infinite curvature near the poles, although the four-geometry is perfectly regular there. More generally, if we write the three-metric as $h_{ij} = \Omega^2 \tilde{h}_{ij}$, where $\tilde{h}_{ij}$ has a unit determinant, then the configurations with $\Omega \to 0$ but $\tilde{h}_{ij}$ and ϕ nonsingular do not necessarily correspond to four-dimensional singularities. Another example of this sort is shown in Fig. 1(b) where the three-space develops a singularity at point P. Such configurations can be important in tunneling transitions with a topology change. It would be interesting to classify all possible singularities that can arise due to slicing of regular four-geometries. This will not be attempted here; we shall simply assume that it can be done. Then we can divide the boundary of superspace into two parts. The first part includes three-geometries which have only singularities which can be attributed to slicing of regular four-geometries; we call this the nonsingular boundary of superspace. The second part includes the rest of the boundary and is called the singular boundary of superspace. The "tunneling" boundary condition for ψ can be now formulated as follows.

At singular boundaries of superspace, ψ includes only outgoing modes (carrying flux out of superspace). (3.1)

Note that this boundary condition is formally similar to the causal boundary condition for the Feynman propagator.

The definition of ingoing and outgoing modes is similar to that of positive- and negative-frequency modes, with the direction toward the boundary playing the role of "time" direction. It is not obvious that these modes can be unambiguously defined in the general case, but such a definition is possible in the semiclassical regime. The semiclassical wave function can be written as a superposition

$$\psi = \sum_n C_n e^{iS_n} \; , \tag{3.2}$$

where S_n are rapidly varying functions satisfying the Hamilton-Jacobi equation in superspace[11]

$$(\nabla S_n)^2 + U = 0 \ . \tag{3.3}$$

The current J_n for the nth term of (3.2) is

$$J_n = -\mid C_n \mid^2 \nabla S_n \tag{3.4}$$

and the boundary condition (3.1) requires that the vectors $-\nabla S_n$ should point out of superspace at the boundaries. To put it differently we note that each function S_n defines a congruence of classical paths in superspace. The boundary condition (3.1) states that these paths can end at the singular boundary of superspace, but none of the paths is allowed to begin there.

In addition to (3.1) we shall impose a regularity condition

$$|\psi| < \infty \ . \tag{3.5}$$

It is not clear whether it should be supplemented by some sort of normalizability condition. (Even if ψ is not normalizable, it may still be possible to calculate some conditional probabilities.[9,12]) It is also not clear whether or not the wave function specified by conditions (3.1) and (3.5) is unique.

The Hartle-Hawking proposal for ψ states that the wave function for a certain three-dimensional configuration is given by a path integral

$$\psi_H = \int [dg_{\mu\nu}][d\phi] \exp[-S_E(g_{\mu\nu}, \phi)] \tag{3.6}$$

which is taken over all compact Euclidean histories terminating at this configuration.[6,7] Compact four-geometries can be thought of as interpolating between a point ("nothing") and a finite three-geometry. In this sense the proposal (3.6) is similar to "tunneling from nothing." An important difference between the two wave functions is that ψ_H is real, and so the current (2.8) is identically zero, while the tunneling wave function ψ_T specified by (3.1) is necessarily complex.

The gravitational part of the Euclidean action S_E is unbounded from below, and the integral (3.6) is badly divergent. It has been suggested that S_E can be made positive definite by analytically continuing to complex scale factors.[13] This prescription works for pure gravity, but not in the general case, and at present it is not clear whether one can meaningfully define an integral such as (3.6). In practice, all calculations of ψ_H have been performed not by evaluating the path integral, but by solving the Wheeler-DeWitt equation with a boundary condition deduced from the formal expression (3.6). Such a boundary condition has been specified in some simple cases using the semiclassical approximation, but its general formulation has not yet been given.

IV. MINISUPERSPACE WAVE FUNCTIONS

A. de Sitter space

To illustrate the difference between the wave functions obtained in the two approaches, we first consider a simple minisuperspace model

$$S = \int d^4x \sqrt{-g}\, (l_P^{-2} R - \rho_v) \ , \tag{4.1}$$

where ρ_v is a constant vacuum energy and the Universe is assumed to be homogeneous, isotropic, and closed:

$$ds^2 = \sigma^2[N^2(t)dt^2 - a^2(t)d\Omega_3^2] \ . \tag{4.2}$$

Here, $N(t)$ is an arbitrary lapse function, $d\Omega_3^2$ is the metric on a unit three-sphere, and $\sigma^2 = 2G/3\pi$ is a normalizing factor chosen for later convenience. This model has a single degree of freedom, the scale factor a, and so $\psi = \psi(a)$. The classical solution of the model [for $N(t) = 1$] is the de Sitter space

$$a(t) = H^{-1}\cosh(Ht) \ , \tag{4.3}$$

where

$$H = \tfrac{4}{3} G \rho_v^{1/2} \ . \tag{4.4}$$

The Wheeler-DeWitt equation for $\psi(a)$ is

$$\left[a^{-p} \frac{\partial}{\partial a} a^p \frac{\partial}{\partial a} - U(a) \right] \psi = 0 \ , \tag{4.5}$$

where the parameter p represents the factor-ordering ambiguity and

$$U(a) = a^2(1 - H^2 a^2) \ . \tag{4.6}$$

Equation (4.5) has the form of a one-dimensional Schrödinger equation for a "particle" described by a coordinate $a(t)$, having zero energy and moving in a potential $U(a)$. The classically allowed region is $U(a) \le 0$ or $a \ge H^{-1}$. The WKB solutions of Eq. (4.5) in this region are (disregarding the preexponential factor)

$$\psi_\pm^{(1)}(a) = \exp\left[\pm i \int_{H^{-1}}^{a} p(a')da' \mp \frac{i\pi}{4} \right] \tag{4.7}$$

and the underbarrier ($a < H^{-1}$) solutions are

$$\psi_\pm^{(2)}(a) = \exp\left[\pm \int_a^{H^{-1}} |p(a')|\, da' \right] \ , \tag{4.8}$$

where $p(a) \equiv [-U(a)]^{1/2}$. Equation (4.7) with the upper (lower) sign describes a contracting (expanding) universe. Tunneling through the barrier corresponds to the choice of the "outgoing" wave for $a > H^{-1}$:

$$\psi_T(a > H^{-1}) = \psi_-^{(1)}(a) \ . \tag{4.9}$$

Then the WKB connection formula gives the underbarrier wave function of the form

$$\psi_T(a < H^{-1}) = \psi_+^{(2)}(a) - \frac{i}{2}\psi_-^{(2)}(a) \ . \tag{4.10}$$

Except in the immediate vicinity of $a = H^{-1}$, the second term in (4.10) is negligible, and $\psi_T \approx \psi_+^{(2)}(a)$. The wave function grows exponentially toward $a = 0$ [see Fig. 2(a)]. The "tunneling amplitude" is proportional to[2,3]

$$\begin{aligned} \psi_T(H^{-1})/\psi_T(0) &= \exp\left[-\int_0^{H^{-1}} |p(a')|\, da' \right] \\ &= \exp\left[-\frac{3}{16G^2\rho_v} \right] \ . \end{aligned} \tag{4.11}$$

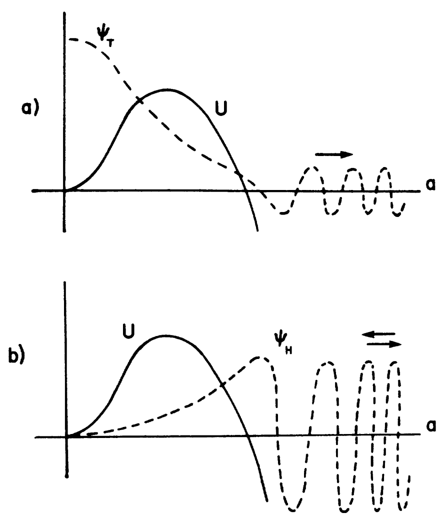

FIG. 2. (a) Tunneling and (b) Hartle-Hawking wave functions for the one-dimensional minisuperspace model describing a de Sitter space. The "potential" $U(a)$ is shown by a solid line and the wave functions by dashed lines.

The Hartle-Hawking proposal gives[7,8]

$$\psi_H(a<H^{-1})=\psi_-^{(2)}(a) \tag{4.12}$$

for the underbarrier wave function and

$$\psi_H(a>H^{-1})=\psi_+^{(1)}(a)+\psi_-^{(1)}(a) \tag{4.13}$$

in the classically allowed range. This wave function describes a contracting and reexpanding universe; under the barrier $\psi_H(a)$ is exponentially suppressed [see Fig. 2(b)].

B. Model with a scalar field

We now turn to a more realistic model defined by the action

$$S=\int d^4x\sqrt{-g}\,[l_P^{-2}R+\tfrac{1}{2}(\partial_\mu\tilde{\phi})^2-V(\tilde{\phi})] \ , \tag{4.14}$$

where the scalar field $\tilde{\phi}$ is assumed to be homogeneous and isotropic and the metric is restricted to the form (4.2). It is convenient to introduce dimensionless quantities

$$\phi=(4\pi G/3)^{1/2}\tilde{\phi}, \quad V=(4G/3)^2\tilde{V} \ . \tag{4.15}$$

Then the Wheeler-DeWitt equation takes the form

$$\left[\frac{\partial^2}{\partial a^2}+\frac{p}{a}\frac{\partial}{\partial a}-\frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}-U(a,\phi)\right]\psi=0 \ , \tag{4.16}$$

where

$$U(a,\phi)=a^2[1-a^2V(\phi)] \ . \tag{4.17}$$

The minisuperspace of this model is a two-dimensional manifold $0<a<\infty$, $-\infty<\phi<\infty$. Its nonsingular boundary is the line $a=0$ with $|\phi|<\infty$, while at singular boundaries at least one of the two variables is infinite. Introducing a new variable $\alpha=\ln a$, we can represent the minisuperspace and its boundaries by a conformal diagram (see Fig. 3). This diagram is the same as for Minkowski space with α playing the role of time and ϕ the role of a spatial coordinate.[5] The nonsingular boundary is represented by a single point i^- at past timelike infinity.

In the WKB approximation we represent the wave function in the form (3.2) and the Hamilton-Jacobi equation (3.3) takes the form

$$\left[\frac{\partial S_n}{\partial\alpha}\right]^2-\left[\frac{\partial S_n}{\partial\phi}\right]^2+U=0 \ . \tag{4.18}$$

For small values of a ($\alpha\to-\infty$), the potential $U(a,\phi)$ approaches zero, and ψ is a superposition of terms of the form

$$\psi_k=e^{ik(\alpha\mp\phi)} \ . \tag{4.19}$$

Terms with $k>0$ describe universes collapsing to a singularity. The corresponding classical solutions of Einstein's equations with a free massless scalar field are

$$a\propto(t_0-t)^{1/3}, \quad |\phi|\approx\tfrac{1}{3}\ln(t_0-t) \ ,$$

with an arbitrary t_0. The paths representing these solutions on the conformal diagram cross the boundary at the past null infinity, $\mathcal{J}^-$. Terms with $k<0$ describe universes expanding out of singularity. The "tunneling" boundary condition (3.1) requires that only the positive-k modes should be present. At the nonsingular boundary i^- the wave function approaches a constant (corresponding to the $k=0$ mode):[14]

$$\psi(a=0,\phi)=\text{const} \ . \tag{4.20}$$

In this paper we shall be interested in the behavior of ψ in the neighborhood of i^-. This region corresponds to the very early stages in the evolution of the Universe (nucleation and the beginning of inflation), and it is in this region that quantum cosmological effects can be important.

Semiclassical solutions of Eqs. (4.16) and (4.17) have been studied in Ref. 5 for the case of a quadratic poten-

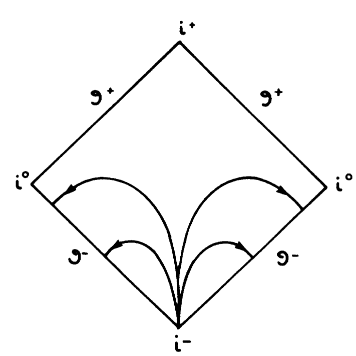

FIG. 3. The conformal diagram for the minisuperspace model (4.16) is the same as for a two-dimensional Minkowski space with $\alpha=\ln a$ playing the role of time and ϕ the role of spatial coordinate. Directed lines in the diagram represent semiclassical histories of the Universe originating at the nonsingular boundary i^- and collapsing to a singularity at $\mathcal{J}^-$.

tial $V(\phi)$. Here, we shall consider the general case, assuming that $V(\phi)$ is a slowly varying function of ϕ:

$$|V^{-1}dV/d\phi| \ll 1 . \tag{4.21}$$

[This is a preliminary form of a constraint which will be more rigorously formulated below; see Eq. (4.32).] We shall assume also that $\tilde{V}(\phi)$ is small compared to the Planck scale:

$$|V| \ll 1 . \tag{4.22}$$

When this condition is violated, the semiclassical approximation cannot be used and higher orders of quantum gravity become important. Equations (4.20) and (4.21) suggest that, for sufficiently small a, the wave function ψ is also a slowly varying function of ϕ. Then we can neglect derivatives with respect to ϕ, and Eq. (4.16) takes the form

$$\left[\frac{\partial^2}{\partial a^2}+\frac{p}{a}\frac{\partial}{\partial a}-U(a,\phi)\right]\psi=0 . \tag{4.23}$$

Now ϕ is just a parameter, and the problem is essentially identical to the one-dimensional minisuperspace model studied in the previous subsection. The minisuperspace (a,ϕ) can be divided into regions where $U>0$ and $U<0$. As long as Eq. (4.23) is accurate, this division coincides with the division into a classically forbidden region, where the behavior of ψ is exponential, and a classically allowed region, where ψ is oscillatory. The boundary between the two regions is $U=0$ or

$$a^2=1/V(\phi) . \tag{4.24}$$

To find the solutions of Eq. (4.23), we note that (i) the factor-ordering parameter p does not affect semiclassical probabilities and that (ii) with the choice of $p=-1$ Eq. (4.23) can be solved exactly.[3,5] Introducing a new variable

$$z=-(2V)^{-2/3}(1-a^2V) \tag{4.25}$$

and setting $p=-1$ we have

$$\left[\frac{\partial^2}{\partial z^2}+z\right]\psi=0 .$$

The general solution of this equation is a linear combination of Airy functions $\mathrm{Ai}(-z)$ and $\mathrm{Bi}(-z)$ with coefficients which are arbitrary functions of ϕ. For the convenience of the reader, I give the asymptotic forms[15] of the Airy functions at large values of $z(z\to+\infty)$:

$$\begin{aligned}
&\mathrm{Ai}(z)\sim\frac{1}{2\sqrt{\pi}}z^{-1/4}e^{-\zeta} ,\\
&\mathrm{Bi}(z)\sim\frac{1}{\sqrt{\pi}}z^{-1/4}e^{\zeta} ,\\
&\mathrm{Ai}(-z)\sim\frac{1}{\sqrt{\pi}}z^{-1/4}\sin\left[\zeta+\frac{\pi}{4}\right] ,\\
&\mathrm{Bi}(-z)\sim\frac{1}{\sqrt{\pi}}z^{-1/4}\cos\left[\zeta+\frac{\pi}{4}\right] ,
\end{aligned} \tag{4.26}$$

where $\zeta\equiv\frac{2}{3}z^{3/2}$.

The tunneling wave function ψ_T is found from the requirement that only an outgoing wave should be present in the classically allowed region ($i\psi^{-1}\partial\psi/\partial a>0$ for $a^2>V^{-1}$). With the aid of Eqs. (4.26) and the condition (4.21) we find, up to a numerical coefficient,

$$\psi_T=\frac{\mathrm{Ai}(-z)+i\,\mathrm{Bi}(-z)}{\mathrm{Ai}(-z_0)+i\,\mathrm{Bi}(-z_0)} , \tag{4.27}$$

where $z_0=z(a=0)=-(2V)^{-2/3}$. For negative values of $V(\phi)$, z and z_0 are complex. The correct analytic continuation which matches Eq. (4.27) at $V(\phi)=+0$ is obtained by setting

$$V(\phi)=e^{-i\pi}|V(\phi)| , \quad -z=e^{2\pi i/3}|z| ,$$
$$-z_0=e^{2\pi i/3}|z_0| .$$

Using the relation[15]

$$\mathrm{Ai}(e^{2\pi i/3}z)+i\,\mathrm{Bi}(e^{2\pi i/3}z)=2e^{\pi i/3}\mathrm{Ai}(z) , \tag{4.28}$$

we can write the wave function for $V(\phi)<0$ as

$$\psi_T=\frac{\mathrm{Ai}(|z|)}{\mathrm{Ai}(|z_0|)} \quad [V(\phi)<0] . \tag{4.29}$$

We note that in this range the wave function is real.

In the classically allowed range, $a^2V>1$, but not too close to the barrier (4.24), z is large and positive, while z_0 is large and negative [see Eq. (4.22)]. Then, using the asymptotic forms (4.26) we can write

$$\psi_T=e^{i\pi/4}(a^2V-1)^{-1/4}\exp\left[-\frac{1+i(a^2V-1)^{3/2}}{3V}\right] \quad (a^2V>1) . \tag{4.30}$$

Similarly, in the classically forbidden range we find

$$\psi_T=(1-a^2V)^{-1/4}\exp\left[\frac{(1-a^2V)^{3/2}-1}{3V}\right] \quad (a^2V<1) . \tag{4.31}$$

Note that the last expression is not singular at $V=0$ and applies both for positive and negative values of $V(\phi)$. Using Eqs. (4.30) and (4.31), it is now easily verified that omitting the ϕ-derivative term in Eq. (4.16) is justified if

$$\left|\frac{dV}{d\phi}\right| \ll \max\{|V|,1/a^2\} . \tag{4.32}$$

This is the final form of the condition for $V(\phi)$, which now replaces the preliminary form (4.21).

Let us now find the Hartle-Hawking wave function ψ_H for our model. This wave function is specified by the requirement that it should be an exponentially growing function of a in the classically forbidden range. Together with Eq. (4.20) this condition fixes ψ_H up to a numerical factor, and we find

$$\psi_H=\frac{\mathrm{Ai}(-z)}{\mathrm{Ai}(-z_0)} . \tag{4.33}$$

Using the asymptotic forms (4.26) we obtain the WKB approximations for ψ_H in the classically allowed range,

$$\psi_H = 2(a^2V-1)^{-1/4}\exp\left[\frac{1}{3V}\right]\cos\left[\frac{(a^2V-1)^{3/2}}{3V}-\frac{\pi}{4}\right] \quad (a^2V>1)\ , \tag{4.34}$$

and in the classically forbidden range,

$$\psi_H = (1-a^2V)^{-1/4}\exp\left[\frac{1-(1-a^2V)^{3/2}}{3V}\right] \quad (a^2V<1)\ . \tag{4.35}$$

It is interesting to note that ψ_H given by Eq. (4.33) can be obtained from ψ_T of Eq. (4.27) by an analytic continuation. This is easily seen from Eq. (4.28). If we change a to $e^{i\pi/2}a$ and $V(\phi)$ to $e^{-i\pi}V(\phi)$, then $z\to e^{2\pi i/3}z$, $z_0\to e^{2\pi i/3}z_0$, and ψ_T goes into ψ_H:

$$\psi_H = \psi_T(V\to e^{-i\pi}V, a\to e^{i\pi/2}a)\ . \tag{4.36}$$

This raises an intriguing possibility that ψ_T and ψ_H may be related by an analytic continuation even in the general case. As a generalization of the transformation (4.36) one can propose

$$h_{ij}\to e^{i\pi}h_{ij},\quad V(\phi)\to e^{-i\pi}V(\phi)\ . \tag{4.37}$$

It is easily verified that this transformation leaves the Wheeler-DeWitt equation (2.7) invariant.

V. COSMOLOGICAL PREDICTIONS FROM ψ_T AND ψ_H

Soon after the nucleation, the evolution of the Universe as a whole becomes classical with a very high accuracy, and the cosmological wave functions (4.30) and (4.34) describe ensembles of classical universes. The role of quantum cosmology is to determine the probability distribution for the initial states of the Universe. In our simple model the initial state is characterized by the value of the scalar field ϕ. [The initial value of a is found from $a^2=1/V(\phi)$, and the initial values of $\dot{a}$ and $\dot{\phi}$ are both equal to zero.]

In the quantum tunneling approach, the probability distribution for ϕ can be found using the conserved current (2.8). In the minisuperspace model (4.16) the current has two components:

$$j^a = \frac{i}{2}a^p(\psi^*\partial_a\psi - \psi\partial_a\psi^*)\ , \tag{5.1}$$

$$j^\phi = -\frac{i}{2}a^{p-2}(\psi^*\partial_\phi\psi - \psi\partial_\phi\psi^*)\ , \tag{5.2}$$

and the continuity equation takes the form

$$\partial_a j^a + \partial_\phi j^\phi = 0\ . \tag{5.3}$$

With a proper normalization, the component j^a can be interpreted as the probability density for ϕ at a given value of a. The relation

$$\partial_a \int j^a d\phi = 0 \tag{5.4}$$

expresses the conservation of probability. The classical paths represented by the wave function (4.30) include only expanding universes,

$$a\approx V^{-1/2}\cosh(V^{1/2}t),\quad \phi\approx\text{const}\ , \tag{5.5}$$

and problems with negative probabilities do not arise. (In other words, the scale factor a is a good time variable for our model. See Ref. 5.)

The probability density $\rho(a,\phi)$ is defined so that $\rho(a,\phi)d\phi$ is the probability for the scalar field to be between ϕ and $\phi+d\phi$ when the scale factor is equal to a. Substituting (4.30) in (5.1) with $p=-1$, we obtain $\rho(a,\phi)$ corresponding to the tunneling wave function ψ_T,

$$\rho_T(a,\phi) = C_T\exp\left[-\frac{2}{3V(\phi)}\right]\ . \tag{5.6}$$

Note that ρ_T is independent of a. This is due to the fact that ϕ is approximately constant on classical trajectories, which is in turn due to the slow variation of $V(\phi)$. Equation (5.6) applies only in the region where $V(\phi)>0$. For $V(\phi)<0$, ψ_T is real and $\rho_T=0$. The normalization constant C_T is found from

$$C_T^{-1} = \int_{[V(\phi)>0]} d\phi\,\exp\left[-\frac{2}{3V(\phi)}\right]\ . \tag{5.7}$$

This integral converges if $V(\phi)<0$ as $\phi\to\pm\infty$ or if $V(\phi)\to 0$ faster than $2/3\ln|\phi|$. The integral is also convergent if ϕ is a cyclic variable defined in a finite range, $0<\phi<\phi_0$, with points $\phi=0$ and $\phi=\phi_0$ identified. If none of these conditions is satisfied, then the distribution (5.6) is not normalizable. It is not clear how serious a problem this is, since, as it was emphasized in Refs. 9 and 12, conditional probabilities can still be calculated, even for non-normalizable distributions. An alternative approach would be to require that the integral (5.7)

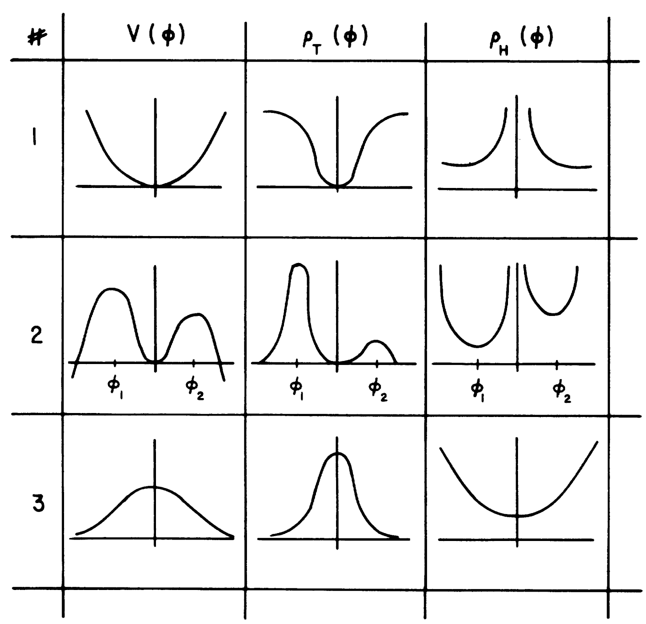

FIG. 4. Probability distributions $\rho_T(\phi)$ and $\rho_H(\phi)$ for the field ϕ at nucleation obtained from tunneling and Hartle-Hawking wave functions, respectively, for different types of the scalar field potential $V(\phi)$.

should converge and to consider this as a constraint on particle-physics models.

The probability distributions $\rho_T(\phi)$ for different types of the potential $V(\phi)$ are illustrated in Fig. 4. In the first example $V(\phi)\to\infty$ and $\rho_T(\phi)\to$const as $|\phi|\to\infty$. The largest values of $\rho_T(\phi)$ correspond to over-Planckian values of the potential, $V(\phi)>1$. Although our semiclassical approach should not be trusted in this range, the behavior of ρ_T for $V(\phi)<1$ indicates that in this type of model the Universe is most likely to nucleate at $V(\phi)>1$. If the growth of the potential at large ϕ is sufficiently slow, this initial state leads to "chaotic" inflation.[2] In the second example the potential is unbounded from below. This is not a problem, as long as the lifetime of the metastable state at $\phi=0$ is greater than the present age of the Universe. The largest nucleation probability is at the highest maximum of $V(\phi)$. This type of initial condition is required in the new inflationary scenario. A similar probability distribution is obtained if $V(\phi)\to 0$ at large $|\phi|$ (example 3 in Fig. 4). If $\max[V(\phi)]\ll 1$, then the initial density of the Universe is much smaller than Planckian, and the Universe can be treated semiclassically throughout its entire history.

In the Hartle-Hawking approach the wave function ψ_H is real and the current (5.1) is identically zero. To determine the probability distribution for ϕ, we rewrite Eq. (4.34) as

$$\psi_H=(a^2V-1)^{-1/4}\exp(1/3V)\exp\left[-\frac{i}{3V}(a^2V-1)^{3/2}+\frac{i\pi}{4}\right]+\text{c.c.} \tag{5.8}$$

The first term in (5.8) describes an ensemble of expanding universes, while the second term describes a time-reversed ensemble of contracting universes. Disregarding the second term, we can now use (5.1) to find the probability distribution

$$\rho_H(a,\phi)=C_H\exp\left[\frac{2}{3V(\phi)}\right]. \tag{5.9}$$

The normalization condition for C_H is

$$C_H^{-1}=\int_{[V(\phi)>0]}d\phi\exp\left[\frac{2}{3V(\phi)}\right]. \tag{5.10}$$

This integral is divergent if $V(\phi)=0$ for some value of ϕ. Since there should be a range of ϕ where $V(\phi)>0$ (otherwise, there is no classically allowed region), we conclude that the distribution (5.9) is normalizable only if (i) ϕ has a finite range *and* (ii) $V(\phi)$ is strictly positive. In such a case the maximum nucleation probability would correspond to the true minimum of the potential $V(\phi)$. This initial condition does not lead to inflation.

The claim that the Hartle-Hawking wave function predicts inflation is based on the models with $V(\phi)$ unbounded from above, as in the first example in Fig. 4. In this example, the distribution $\rho_H(\phi)$ diverges at $\phi=0$, where $V(\phi)=0$, and approaches a constant at $|\phi|\to\infty$; it is obviously not normalizable. Small values of ϕ correspond to large initial values of a and small initial densities. Hawking and Page[9] argue that values of ϕ for which the initial density of the Universe is too small (say, smaller than the present density) should be excluded and suggest that one should calculate conditional probabilities with the condition that the density of the Universe is in a given range (for an arbitrary a). If small values of ϕ are cut off, then the ensemble described by the distribution (5.9) is dominated by universes with arbitrarily large values of ϕ. Hawking and Page interpret this fact as indicating that the initial state of the Universe is, with probability equal to one, a state with $|\phi|\to\infty$ and $a\to 0$. The problem with this interpretation is that, in order to outweigh the exponentially large values of $\rho_H(\phi)$ at small ϕ, one has to go to extremely large values of ϕ, for which the potential $V(\phi)$ will far exceed the Planck energy density [except, perhaps, for a very special shape of $V(\phi)$]. The semiclassical approximation, on which the derivation of Eq. (5.9) was based, cannot be trusted in this regime. If the distribution (5.9) is cut off at $V(\phi)\sim 1$, then it would predict nucleation at very small ϕ with a probability close to one. The Hartle-Hawking probability distribution for the potentials of examples 2 and 3 is even more pathological. My conclusion is that at this stage inflation cannot be claimed as one of the predictions of the Hartle-Hawking approach.

Of course, quantum cosmology can only give a probability distribution for the initial states of the Universe. Unfortunately, we have a single copy of the Universe, and our best guess seems to be that it is a "typical" universe which has started somewhere near the maximum of the probability distribution. It may happen that the most probable initial conditions result in a universe which is not suitable for life. Then we will have to invoke the anthropic principle and look for the most probable initial configuration consistent with the existence of intelligent life. It may be argued that inflation is necessary to provide certain amount of homogeneity and isotropy needed for our existence. This argument, however, does not "save" the Hartle-Hawking wave function, since it would predict *minimal* inflation in which homogeneity and isotropy extend to scales much smaller than the present horizon and the expected quadrupole anisotropy is $\delta T/T\sim 1$.

VI. PERTURBATIVE SUPERSPACE

In this section we shall consider linear perturbations about our minisuperspace model. The theory of such perturbations is equivalent to quantum field theory in de Sitter space, and the boundary conditions for the cosmological wave function should specify the quantum states of the gravitational and scalar fields.

In the Hartle-Hawking approach this problem was studied by Halliwell and Hawking[10] and then by Wada[16] and others[17,18] (see also an earlier treatment in Ref. 19). They concluded that the initial state of the quantum fields is the de Sitter-invariant vacuum. Here we shall discuss the quantum tunneling approach, with a similar conclusion.

We shall assume that the scalar field potential is bounded from above. Then our minisuperspace analysis suggests that the tunneling wave function ψ_T is peaked near the maximum of $V(\phi)$. In the vicinity of the maximum (which we take to be at $\phi=0$)

$$V(\phi)=H^2-\mu^2\phi^2+O(\phi^3) . \tag{6.1}$$

We shall assume that $\mu \ll H$, so that the maximum is sufficiently flat for inflation. Small perturbations of the scalar field around $\phi=0$ can be expanded in spherical harmonics:

$$\phi(x)=(2\pi^2)^{1/2}\sum_{n,l,m} f_{nlm}(t)Q^n_{lm}(x^i) , \tag{6.2}$$

where $n=1,2,3,\ldots$; $l=0,1,\ldots,n-1$; $m=-l,-l+1,\ldots,l$, and the factor $(2\pi^2)^{1/2}$ is introduced for the proper normalization of Q^n_{lm}. Hereafter, the labels $\{n,l,m\}$ will be denoted simply by n. With an appropriate choice of gauge, gravitational perturbations are formally equivalent to a pair of minimally coupled massless scalar fields,[20] and so it is sufficient to consider the scalar field only.

With all modes of the scalar field included, the wave function ψ becomes a function of an infinite number of variables, $\psi(a,f_1,f_2,\ldots)$, and the Wheeler-DeWitt equation takes the form

$$\left[a^2\frac{\partial^2}{\partial a^2}+pa\frac{\partial}{\partial a}-a^4(1-H^2a^2) -\sum_n\left[\frac{\partial^2}{\partial f_n^2}-(n^2-1)a^4f_n^2+\mu^2a^6f_n^2\right]\right]\psi=0 . \tag{6.3}$$

This equation can be analyzed using the method first introduced by Banks, Bender, and Wu.[21] Here, I follow the treatment of Wada.[16]

Representing the wave function as

$$\psi=e^{iS} \tag{6.4}$$

with

$$S(a,\{f_n\})=S_0(a)+\tfrac{1}{2}\sum_n S_n(a)f_n^2+O(f_n^3) , \tag{6.5}$$

we shall treat the scale factor a as a semiclassical variable and neglect terms higher than quadratic in f_n. This leads to the following equations for S_0 and S_n:

$$S_0'^2+a^2(1-H^2a^2)=0 , \tag{6.6}$$

$$a^2S_0'S_n'-S_n^2-(n^2-1)a^4+\mu^2a^6=0 , \tag{6.7}$$

where primes stand for derivatives with respect to a. Equation (6.6) is the semiclassical version of the one-dimensional minisuperspace model describing de Sitter space (see Sec. IV). In the classically allowed range, $a>H^{-1}$, its solution is

$$S_0(a)=-\frac{1}{3H^2}(H^2a^2-1)^{3/2} , \tag{6.8}$$

where the choice of sign corresponds to an expanding universe (outgoing wave).

Since $\mu \ll H$ and the characteristic scale of the tunneling problem is $a\sim H^{-1}$, the μ^2a^6 term in Eq. (6.7) can be neglected compared to the $(n^2-1)a^4$ term. The only exception is the homogeneous mode with $n=1$. The wave function for the homogeneous mode f_1 has been studied in the two-dimensional minisuperspace model of Sec. IV and in Ref. 5. We can find $S_1(a)$ by substituting $V(\phi)=H^2-\mu^2f_1^2$ in the exponent of Eq. (4.30) and then expanding it in powers of f_1^2. This gives

$$S_1(a)=\frac{2i\mu^2}{3H^4}+\frac{\mu^2}{3H^4}(H^2a^2-1)^{1/2}(H^2a^2+2) . \tag{6.9}$$

Because of the first term in this equation, the wave function ψ_T is an exponentially decreasing function of f_1. It is localized near $f_1=0$ within a tube of width $\Delta f_1\sim H^2/\mu$. In the rest of this section we concentrate on inhomogeneous modes with $n>1$ and neglect the last term in Eq. (6.7).

At this point it is convenient to introduce a time variable t through the relation

$$S_0'=-\frac{a\dot a}{N(a)} ,$$

where $\dot a\equiv da/dt$ and $N(a)$ is the lapse function [see Eq. (4.2)]. Following Wada, we choose $N(a)=a$, which corresponds to "conformal time"; then

$$a=(H\cos t)^{-1} . \tag{6.10}$$

We shall use t instead of a as an independent variable.

Equation (6.7) is a Riccatti equation, it can be linearized by a substitution

$$S_n(t)=a^2\dot\nu_n/\nu_n , \tag{6.11}$$

where a is from Eq. (6.10). The resulting equation for $\nu_n(t)$ is

$$\ddot\nu_n+2(\dot a/a)\dot\nu_n+(n^2-1)\nu_n=0 . \tag{6.12}$$

It coincides with the equation for scalar mode functions in de Sitter space (6.10). Hence, $\nu_n(t)$ play the role of mode functions for the scalar field ϕ. The general solution of Eq. (6.12) is

$$\nu_n(t)=\nu_n^{(1)}(-i\tan t)+B_n\nu_n^{(2)}(-i\tan t) , \tag{6.13}$$

where

$$\nu_n^{(1)}(y)=(y-1)^{(n-1)/2}(y+1)^{-(n+1)/2}(1+y/n) , \tag{6.14a}$$

$$\nu_n^{(2)}(y)=(y+1)^{(n-1)/2}(y-1)^{-(n+1)/2}(1-y/n) . \tag{6.14b}$$

The overall normalization of $\nu_n(t)$ is unimportant, since it cancels out in Eq. (6.11). The quantum state of the field ϕ is determined by the choice of the coefficients B_n.

In the classically forbidden range, $a<H^{-1}$, the unperturbed wave function is a superposition of two terms with

$$S_0(a)=\pm\frac{i}{3}(1-H^2a^2)^{3/2} \tag{6.15}$$

[see Eq. (4.10)]. The corresponding mode functions can be obtained by changing $t \rightarrow \pm it$ in Eqs. (6.13) and (6.14). The three branches of the wave function corresponding to growing exponential (ψ_{I}), decreasing exponential (ψ_{II}), and outgoing wave (ψ_{III}) are illustrated in Fig. 5. ψ_{I} and ψ_{II} have comparable magnitude near the nucleation point ($a=H^{-1}$), but ψ_{II} dominates in most of the forbidden region.

The coefficients B_n in Eq. (6.13) should be chosen s that

$$\mathrm{Im}S_n(a)\geq 0 \tag{6.16}$$

for all values of a. Otherwise, ψ would grow exponentially as a function of f_n and the finiteness condition (3.5) would be violated. (This argument is not quite rigorous, since our approximations break down at large f_n, but this seems the best one can do in perturbative superspace approach.) Using the explicit forms of the mode functions (6.14) it is easily shown[16] that for ψ_{II} and ψ_{III} the condition (6.16) is satisfied if $|B_n|<1$, while for ψ_{I} the only possible choice is $B_n=0$.

To find the relation between the values of B_n in different branches of the wave function, one has to analyze the neighborhood of the nucleation point, $a=H^{-1}$, where the semiclassical approximation breaks down. This problem, which requires a rather intricate analysis, should be addressed in both quantum tunneling and Hartle-Hawking approaches. Here we shall *assume* that the three branches of the wave function are analytic continuation of one another, so that B_n have the same values on ψ_{I}, ψ_{II}, and ψ_{III}. [A similar assumption is also made (implicitly) in Refs. 10, 17, and 18.] Then $B_n=0$ in all three branches.

For gravitons, the choice of mode functions (6.14a) corresponds to a de Sitter-invariant vacuum state.[22] A scalar field with a potential (6.1) is unstable (the expectation value of ϕ grows with time), and does not have de Sitter-invariant states. The same is true for a minimally coupled massless scalar fields (in which case the expectation value of ϕ^2 grows with time[23]). A massive scalar field does have a de Sitter-invariant state, and the corresponding mode functions go over into the functions (6.14a) in the zero-mass limit. In this sense the quantum state predicted by quantum cosmological models is as close as one can get to a de Sitter-invariant vacuum. The state predicted in the quantum tunneling approach is the same as that deduced from the Hartle-Hawking approach.[24] They only differ in the behavior of the homogeneous mode.

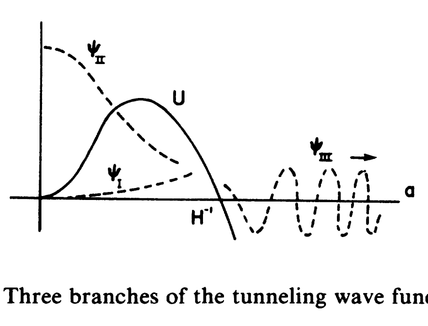

FIG. 5. Three branches of the tunneling wave function.

VII. CONCLUSIONS

In this paper we have compared the cosmological implications of the "tunneling" and Hartle-Hawking wave functions of the Universe (denoted by ψ_T and ψ_H, respectively). The results of this comparison are summarized in Fig. 4 which shows the probability distributions for the scalar field ϕ obtained from ψ_T and ψ_H for different types of the scalar potential $V(\phi)$. In the tunneling approach, the most probable initial states have the highest values of $V(\phi)$, while in the Hartle-Hawking approach the highest probability is near $V(\phi)=0$. Hence, ψ_T naturally predicts initial states leading to inflation, while ψ_H does not. (The claim that ψ_H predicts inflation made in Refs. 9 and 10 is not justified, since it is based on the behavior of the wave function at densities much greater than Planckian, where Einstein's gravity and the minisuperspace approximation cannot be trusted.) Despite this difference, both wave functions predict that the Universe nucleates with quantum fields in the de Sitter-invariant vacuum state (pending the proof of the assumption concerning the continuation of mode functions through the nucleation point, see Sec. VI). Finally, we have found, in a minisuperspace model, that the two wave functions can be obtained from one another by an analytic continuation (4.36). It would be interesting to know whether or not this relation holds in the general case.

I would like to conclude with a remark about the beginning and the end of the Universe. It is often said that a closed universe necessarily recollapses. If this were true, then all the inflating universes described by the wave functions ψ_T and ψ_H would thermalize and reach a big crunch in a finite time. However, this conclusion is drastically changed if quantum fluctuations of the scalar field responsible for inflation are taken into account. It has been shown[25,26] that, once inflation has started, it never ends completely. The total volume of inflating regions grows exponentially with time, and they form a self-similar fractal of dimension slightly less that 3. Hence, the Universe has no end. On the other hand, it appears that the Universe must have a beginning. The reason is that the full de Sitter spacetime (4.3), from which the inflating fractal is carved, corresponds to a bouncing universe and inflation is possible only in the expanding phase.[27] Thus, we are led to the conclusion that the Universe had a beginning, but it will have no end. We live in a region which thermalized about 10^{10} yr ago, but the Universe itself is probably much older than that.

Note added in proof. The assumption concerning the continuation of mode functions through the nucleation point, which is stated in Sec. VI, has now been proved [T. Vachaspati and A. Vilenkin, Phys. Rev. D **37**, 898 (1988)].

ACKNOWLEDGMENTS

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