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Internal structure and the spacetime of superconducting bosonic strings

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The set of coupled field equations for the standard model of a bosonic superconducting cosmic string is solved numerically. Only a straight, infinitely long string is considered. The regular solutions exist for a limited range of the constants in the potential. We compute the solutions of the field equations, the linear energy density, maximal current, and the deficit angle for several different choices of the constants. The maximal current occurs when the total energy is smaller than the energy of the string in the nonsuperconducting phase. We argue that the transition from the superconducting phase to the ordinary one is smooth.

I. INTRODUCTION

Recently there has been a major escalation of interest in superconducting cosmic strings,[1] even though they have nothing to do with ceramics and are not easy to manufacture. This popularity is not surprising because the ability of strings to conduct enormous currents, up to 10^{21} A, leads to many exotic phenomena.[2-4] In the paper that started the industry of superconducting strings Witten[1] showed that they can exist in two forms: with bosonic or fermionic charge carriers. This paper deals with bosonic strings exclusively.

The standard example of the theory describing bosonic strings is a gauge theory of two scalar fields ϕ and σ, with a gauge group $U(1)\times\tilde{U}(1)$. The field ϕ is coupled to the gauge field R_a which corresponds to the $\tilde{U}(1)$ group. These two fields give rise to strings. The field σ is coupled to the gauge field A_a, which is identified with the electromagnetic potential. The Lagrangian of the model is

$$\mathcal{L}=-\tfrac{1}{4}F_{ab}F^{ab}-\tfrac{1}{4}B_{ab}B^{ab}-\tfrac{1}{2}(D_a\sigma)^*D^a\sigma -\tfrac{1}{2}(D_a\phi)^*D^a\phi-V(\phi,\sigma)\ , \tag{1.1}$$

where $F_{ab}=\partial_a A_b-\partial_b A_a$, $B_{ab}=\partial_a R_b-\partial_b R_a$, $D_a\sigma=(\nabla_a+ieA_a)\sigma$, and $D_a\phi=(\nabla_a+igR_a)\phi$. The potential of the scalar fields is

$$V(\phi,\sigma)=\tfrac{1}{8}\lambda_\phi(\,|\,\phi\,|^2-\eta^2)^2+\tfrac{1}{4}\lambda_\sigma\,|\,\sigma\,|^4 +\tfrac{1}{2}f\,|\,\phi\,|^2\,|\,\sigma\,|^2-\tfrac{1}{2}m^2\,|\,\sigma\,|^2\ . \tag{1.2}$$

If the constants in the potential are chosen appropriately, then the symmetry $\tilde{U}(1)$ is broken and the fields ϕ and R_a form a vortex. Outside the vortex, where $|\,\phi\,|=\eta$, the expectation value of the field σ vanishes and the gauge symmetry of electromagnetism survives. However, inside the vortex the field ϕ vanishes and the potential develops a minimum at $\sigma\neq 0$. This breaks the gauge symmetry of electrodynamics and the resulting charged Goldstone boson as the charge carrier.

In his paper, Witten gives an elegant argument showing that indeed the solution $\sigma=0$ is unstable at the string core, but he does not discuss the internal structure of the string. Lately, because general interest has been focused mainly on the astrophysical applications of strings, people usually have been approximating the string by an infinitely thin Nambu string and neglecting its internal details.[3-5] Recently, Hill, Hodges, and Turner[6] discussed the internal structure of a bosonic superconducting string using a variational approach with a specific ansatz characterized by a few parameters. This method works very well for ordinary cosmic strings where we have a good idea how the fields behave. However, this procedure assumes that the solution for σ exists and is well approximated by the ansatz; without knowing anything about the behavior of the solutions it is risky to use such an approach. In particular, the question of existence and regularity of the solutions cannot be addressed in this way at all.

Here we report the results of a direct numerical study of the quantum fields and geometry of a straight infinite bosonic string. In Sec. II we derive the set of coupled field equations for all the fields present in the Lagrangian, calculate the stress-energy tensor, and write down the Einstein equation. In Sec. III we solve the field equations in flat spacetime. Instead of using a parametrized variational method we directly integrate all field equations. Our results are similar to the results of Hill, Hodges, and Turner.[6] The regular solutions exist for a rather limited range of the constants in the potential, particularly if their values are "natural." In this section we also calculate the electromagnetic energy and the maximal current. Surprisingly, the maximal current occurs when the total energy is significantly smaller than the energy of the solution with $\sigma=0$. In fact, when "binding energy" $E_T(\sigma=0)-E_T(\sigma)$ vanishes the current goes to zero as well (for simplicity we frequently refer to the linear energy density by energy). Finally, in Sec. IV the whole problem is considered again, but now we solve the full set of coupled Einstein and field equations. The metric and the bending angle are calculated for several choices of the constants in the potential and different values of the electric current in the string. The gravitational effects are rather small. For a large current the metric approaches the Kasner solution far away from the string.

II. FIELD EQUATIONS

In order to derive the field equations, it is convenient to rewrite the Lagrangian (1.1) in terms of real fields. To this end we replace $\phi \rightarrow \phi e^{i\zeta}$ and $\sigma \rightarrow \sigma e^{i\vartheta}$. The Lagrangian becomes

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{4}F_{ab}F^{ab}-\tfrac{1}{4}B_{ab}B^{ab}-\tfrac{1}{2}\nabla_a\sigma\nabla^a\sigma \\ & -\tfrac{1}{2}\nabla_a\phi\nabla^a\phi-\tfrac{1}{2}\phi^2(\nabla_a\zeta+gR_a)^2 \\ & -\tfrac{1}{2}\sigma^2(\nabla_a\vartheta+eA_a)^2-V(\phi,\sigma)\end{aligned} \tag{2.1}$$

and one easily gets

$$\begin{aligned}&\nabla_a\nabla^a\sigma-\sigma(\nabla_a\vartheta+eA_a)^2-\sigma(\lambda_\sigma\sigma^2+f\phi^2-m^2)=0\ , \\ &\nabla_a\nabla^a\phi-\phi(\nabla_a\zeta+gR_a)^2-\phi[\tfrac{1}{2}\lambda_\phi(\phi^2-\eta^2)+f\sigma^2]=0\ , \\ &\nabla^a[\sigma^2(\nabla_a\vartheta+eA_a)]=0,\quad \nabla^a[\phi^2(\nabla_a\zeta+gR_a)]=0\ , \\ &\nabla^aF_{ab}=e\sigma^2(\nabla_b\vartheta+eA_b),\quad \nabla^aB_{ab}=g\phi^2(\nabla_b\zeta+gR_b)\ .\end{aligned} \tag{2.2}$$

The spacetime around an infinite straight string can be described by the metric

$$ds^2=-e^{2A}dt^2+e^{2B}dz^2+e^{2C}d\theta^2+dr^2\ , \tag{2.3}$$

where A, B, and C are functions which depend on r only.

For a vortex with a winding number $n=1$ we take $\zeta=\theta$ and for the field R_a we use the ansatz $gR_a=[P(r)-1]\nabla_a\theta$. In order to have finite energy, the function $P(r)$ must vanish at large radius and $|\phi|=\eta$, while the regularity condition at small r demands that $P(0)=1$ and $|\phi(0)|=0$. The phase ϑ of the scalar field σ should have a trivial dependence on θ or else we would have a singularity at the origin. In fact, it is possible (at least for an infinite straight string) to choose a gauge in which $\vartheta=0$; in this gauge it is the vector potential A_a that describes the effects of any current present in the string. The electromagnetic potential can be written as $eA_a=I(r)\nabla_a z$, and for a large r the function $I(r)\propto I_T\ln(r)$, where I_T is the total current in the string. The nontrivial field equations are

$$\begin{aligned}&\phi''+(A'+B'+C')\phi'=\phi[P^2e^{-2C}+\tfrac{1}{2}\lambda_\phi(\phi^2-\eta^2)+f\sigma^2]\ , \\ &\sigma''(A'+B'+C')\sigma'=\sigma(I^2e^{-2B}+\lambda_\sigma\sigma^2-m^2+f\phi^2)\ , \\ &P''+(A'+B'-C')P'=g^2\phi^2P\ , \\ &I''+(A'-B'+C')I'=e^2\sigma^2I\ ,\end{aligned} \tag{2.4}$$

where a prime denotes the derivative with respect to r. The energy-momentum tensor is diagonal and can be written in terms of the energy density ρ and three pressures:

$$\begin{aligned}\rho=&\tfrac{1}{2}[(\sigma^2I^2+I'^2/e^2)e^{-2B}+(\phi^2P^2+P'^2/g^2)e^{-2C} \\ &+\phi'^2+\sigma'^2+2V(\phi,\sigma)]\ , \\ p_z=&\tfrac{1}{2}[(\sigma^2I^2+I'^2/e^2)e^{-2B}-(\phi^2P^2+P'^2/g^2)e^{-2C} \\ &-\phi'^2-\sigma'^2-2V(\phi,\sigma)]\ , \\ p_\theta=&\tfrac{1}{2}[-(\sigma^2I^2+I'^2/e^2)e^{-2B}+(\phi^2P^2+P'^2/g^2)e^{-2C} \\ &-\phi'^2-\sigma'^2-2V(\phi,\sigma)]\ , \\ p_r=&\tfrac{1}{2}[(-\sigma^2I^2+I'^2/e^2)e^{-2B}+(-\phi^2P^2+P'^2/g^2)e^{-2C} \\ &+\phi'^2+\sigma'^2-2V(\phi,\sigma)]\ .\end{aligned} \tag{2.5}$$

The Einstein equation takes the form

$$\begin{aligned}&A''+(A'+B'+C')A'=4\pi G(\rho+p_z+p_\theta+p_r)\ , \\ &-B''-(A'+B'+C')B'=4\pi G(\rho+p_z-p_\theta-p_r)\ , \\ &-C''-(A'+B'+C')C'=4\pi G(\rho-p_z+p_\theta-p_r)\ .\end{aligned} \tag{2.6}$$

Equations (2.4) and (2.6) form a complete set of equations needed to calculate the internal structure of the superconducting string and to find the geometry of the spacetime around such a string. However, this set of equations is too complicated to be attacked analytically and one is forced to use numerical methods. In order to separate gravitational effects we solve the problem first in flat spacetime. Next, we shall consider the influence of gravity on the quantum fields.

III. STRING IN FLAT SPACETIME

In flat spacetime the fields equations can be significantly simplified. It is convenient to introduce dimensionless variables. The natural unit of length is the radius of the string $\delta=1/\sqrt{\lambda_\phi}\eta$, so we write $x=\sqrt{\lambda_\phi}\eta r$. From now on, primes denote derivatives with respect to x. It is also natural to rescale the fields: $\sigma=Y\eta$, $\phi=X\eta$, and $I=Q\eta$. Then the field equations become

$$\begin{aligned}&X''+\frac{1}{x}X'=X\left[\frac{P^2}{x^2}+\tfrac{1}{2}(X^2-1)+\bar{f}Y^2\right]\ , \\ &Y''+\frac{1}{x}Y'=Y(Q^2+\bar{\lambda}Y^2-\bar{m}^2+\bar{f}X^2)\ , \\ &P''-\frac{1}{x}P'=\bar{g}^2X^2P,\quad Q''+\frac{1}{x}Q'=\bar{e}^2Y^2Q\ ,\end{aligned} \tag{3.1}$$

where $\bar{\lambda}=\lambda_\sigma/\lambda_\phi$, $\bar{f}=f/\lambda_\phi$, $\bar{g}^2=g^2/\lambda_\phi$, $\bar{e}^2=e^2/\lambda_\phi$, and $\bar{m}^2=m^2/(\lambda_\phi\eta^2)$. The "natural" range for these parameters is around unity, so that all constants in the potential are of the same magnitude, but they have to satisfy certain constraints[1,6] in order to preserve the gauge symmetry of electromagnetism far away from the string. The first condition $\bar{f}\geq\bar{m}^2$ follows from the demand that the field σ vanishes far away from the string and cannot be relaxed. The second condition $\bar{\lambda}\geq 2\bar{m}^4$ should be satisfied or else the solution with broken gauge symmetry of electromagnetism would be energetically favorable. This condition does not preclude the existence of solutions with $\sigma(\infty)=0$ when $\bar{\lambda}<2\bar{m}^4$, but such solutions are probably unstable.

The problem at hand differs from the standard cosmic string case by the presence of the two additional fields: the scalar field Y and electromagnetic potential Q. However, from the numerical point of view the crucial equation is the one for Y, for the reason that will be presently explained. For the moment let us concentrate on this equation.

We need a solution for the Y field that is sufficiently regular inside the string and that vanishes at infinity in order to preserve the gauge symmetry of electrodynamics. By "sufficiently regular" we mean that the linear energy density of the given field configuration must be finite. This is a weaker condition than the requirement that the field itself be regular everywhere. However, the difference is not very significant since even a mild logarithmic singularity at $x=0$ leads to an infinite linear energy density.

First we look for the solution with a vanishing electromagnetic current, so we can set $Q=0$. Assuming that the field X does not change significantly from its form for the ordinary cosmic string we can approximate X by $\tanh(x/2)$. With these simplifications the equation for Y decouples from the other equations and becomes

$$\frac{d^2Y}{dx^2}+\frac{1}{x}\frac{dY}{dx}=Y[\bar{\lambda}Y^2+\bar{f}\tanh^2(x/2)-\bar{m}^2] . \tag{3.2}$$

Let us make a few remarks about this equation. First, if Y goes to zero for large x, then equation (3.2) reduces to the modified Bessel equation, and the desired solution is $Y\propto K_0((\bar{f}-\bar{m})^{1/2}x)$. This function has to be matched with a regular solution at small x. Now, if we assume that there is a regular solution with a nonvanishing first derivative at $x=0$, then the singular term in the differential equation must be canceled by the second derivative, but this is only possible if $Y\rightarrow\ln(x)$ or if $Y\rightarrow 1/\sqrt{\bar{\lambda}}x$, which are not regular, contrary to the assumption. So at $x=0$ the first derivative must vanish. Consequently, near the origin, Y has the form $Y=Y(0)(1-ax^2+\cdots)$.

Second, despite the fact that (3.2) is an ordinary differential equation of second degree, it is not a trivial task to integrate it numerically. The first thing to check is that if one integrates equation (3.2) from a very small value of x with regular initial conditions, then invariably the solution blows up at large x, because of the presence of the growing mode I_0 which overwhelms the desirable mode K_0. Exactly for the same reason it is not safe to integrate with some initial conditions specified at large x; such a method does not give correct results even for the field ϕ. The way to proceed is to solve Eq. (3.2) as a boundary-value problem, that is, with the values of Y (or Y') specified at two points: at large x and at $x=0$. However, even this method is not good enough because any inaccuracy in the boundary condition at large x leads to a spurious singularity at $x=0$. Therefore one is forced to impose boundary conditions on $Y(0)$ *and* $Y'(0)$ and of course on $Y(x_\infty)$. This means that *three* conditions have to be imposed on the solution of the equation of *second* degree. This is only possible—if at all—for a very special choice of the constants in the equation. To find such constants one can make $\bar{m}^2$ a new variable satisfying the equation $d\bar{m}^2/dx=0$ and solve the two equations together without imposing any conditions on the $\bar{m}^2$. Then one deals with two coupled equations of second and first degree and one can impose three boundary conditions.[7] In other words, instead of searching for the correct boundary conditions which correspond to a given set of constants $\bar{f}$ and $\bar{m}^2$, we choose the boundary conditions and the constant $\bar{f}$ and we look for the correct constant $\bar{m}^2$. Of course, there is no guarantee that such a constant can be found for any set of boundary conditions. Searching the suitable range of boundary conditions we can find all $\bar{m}^2$ for which the superconducting solution exists. This boundary-value -eigenvalue method works well even when the remaining fields are unfrozen and the set of equations (3.1) is solved simultaneously. The boundary-value method should be used also for the fields X and P. However, for the electromagnetic field Q it is advantageous to use the initial-value method since we can easily obtain analytically the initial conditions for Q at $x=0$ and we expect Q to grow first quadratically and then logarithmically. This method allows one to avoid the question of the consistency of the conditions imposed at $x=0$ and at infinity. In practice we solve Eq. (3.1) separately for X and P, Y and Q and iterate until the change between subsequent solutions is very small. The relative accuracy of 10^{-4} can be reached without any problems for almost any choice of the constants and boundary conditions.

The first question to ask is what is the range of parameters in the potential for which the solutions exist. The solution should be rather sensitive to the choice of constants $\bar{m}$ and $\bar{f}$. The coupling constant $\bar{g}$ is also important since it determines the radius of the vortex. On the other hand, we do not expect the solution to be very sensitive to the choice of the constant $\bar{\lambda}$: if one rescales Y by $\sqrt{\bar{\lambda}}$ this constant disappears from the equation for Y and remains only as a factor dividing the interaction term in the equation for X. We set the inner boundary condition at $x=0$ and numerical infinity to be at $x_\infty\approx 20$; this is quite sufficient, since Y should decrease exponentially outside the string. For each value of the constant $\bar{f}$ we solve the field equations several times with a different choice of $Y(0)$. In this way we find a range of $\bar{m}^2$ acceptable for this particular value of $\bar{f}$: m^2 has to satisfy two constraints $\bar{m}^2\leq\bar{f}$ and $2\bar{m}^4\leq\bar{\lambda}$ (the second condition can be relaxed), and the total energy of the solution must be smaller than the energy of the solution with $Y=0$ ("pure" string), so that the superconducting phase is energetically favorable.

First we study the dependence of $\bar{m}^2$ and of the total energy E_T on $Y(0)$ for fixed values of $\bar{f}$, $\bar{g}$, and $\bar{\lambda}$. The results are shown in Figs. 1 and 2 for $(\bar{g}^2,\bar{f},\bar{\lambda})=(1,0.85,1)$ and $(1,0.4,1)$. We start with a very small value of $Y(0)$. In this region the energy E_Y of the scalar field Y [including the interaction term $\bar{f}(XY)^2$] is posi-

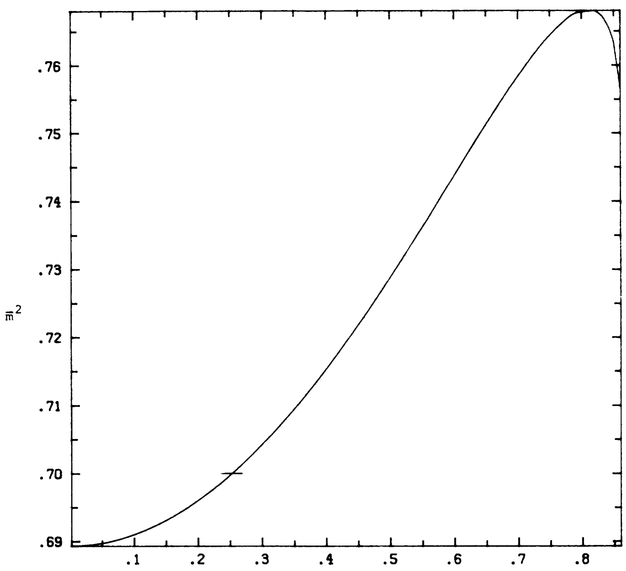

FIG. 1. The eigenvalue $\overline{m}^{\,2}$ as a function of $Y(0)$ for $(\overline{g}^{\,2},\overline{\lambda},\overline{f})=(1,1,0.85)$. The horizontal line marks the constraint $\overline{\lambda}\geq 2\overline{m}^{\,4}$.

tive, but the total energy E_T is smaller than the energy of the "pure" string. However, the difference is very small and almost any current would destroy this configuration. As $Y(0)$ increases the $\overline{m}^{\,2}$ increases also and E_Y becomes negative. The total energy E_T becomes appreciably smaller than the energy of the solution $Y=0$; this means that in this region of parameter space superconducting solutions exist. As $Y(0)$ increases further $\overline{m}^{\,2}$ grows until it reaches a maximum. It can happen that before the maximum is reached the constraint

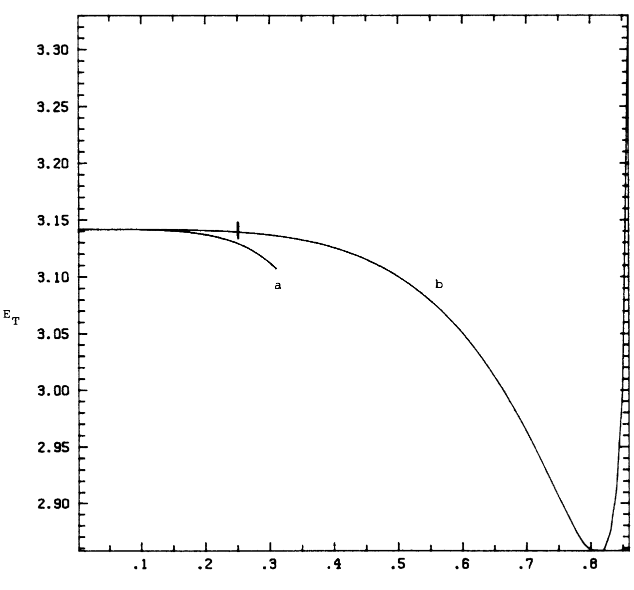

FIG. 2. The total energy E_T as a function of $Y(0)$ for $(\overline{g}^{\,2},\overline{\lambda})=(1,1)$ and (a) $\overline{f}=0.4$ and (b) $\overline{f}=0.85$. In the case (a) the curve ends because of the constraint. $\overline{f}\geq\overline{m}^{\,2}$. The vertical line marks the constraint $\overline{\lambda}\geq 2\overline{m}^{\,4}$.

$\overline{m}^{\,2}\leq\overline{f}$ is violated, in which case the line $\overline{m}^{\,2}=\overline{f}$ on the plane $(\overline{m}^{\,2},\overline{f})$ is the boundary of the region in parameter space in which superconducting solutions exist. This is the case for $\overline{f}=0.4$. For $\overline{f}=0.85$ the maximum of $\overline{m}^{\,2}$ can be reached. The total energy E_T has a minimum exactly when $\overline{m}^{\,2}$ has a maximum. When $Y(0)$ increases further $\overline{m}^{\,2}$ starts to decrease. The energy E_Y is still negative and its magnitude grows quickly, but this effect is offset by even faster growth of the energy of the string, so the total energy increases. This means that for many values of the parameter $\overline{m}^{\,2}$ there are two solutions with different values of $Y(0)$. One should note that before the maximum is reached the condition $\overline{\lambda}\geq 2\overline{m}^{\,4}$ is violated so all solutions in this region are unstable against the transition to the phase with the gauge symmetry of electrodynamics broken far away from the string. Since such a transition involves the change of the field in the whole space, it is plausible that such solutions are at least metastable. However, since the total energy of the solutions with a larger value of $Y(0)$ is larger than the total energy of the solutions with a smaller $Y(0)$, these solutions are unstable against the decay into the state with smaller $Y(0)$. The energy difference, of the order 10^{32} erg/m, is released when an unstable solution decays. We have failed to find solutions for even larger $Y(0)$. This can mean either that our code cannot handle this problem or that solutions with large initial value of Y do not exist. The behavior of $\overline{m}^{\,2}$ as a function of $Y(0)$ is very similar for other values of the constants $\overline{f}$, $\overline{g}$, and $\overline{\lambda}$. For example, the plots of the difference $\overline{m}^{\,2}(Y)-\overline{m}^{\,2}(0.002)$ for $\overline{f}=1.6$ and $\overline{f}=0.85$ almost exactly coincide if other constants are kept fixed.

Now we vary the constants $\overline{f}$, $\overline{g}$, and $\overline{\lambda}$ in order to find the region in parameter space corresponding to the superconducting phase. Needless to say, this search is very tedious even if one uses a Cray as we did. In the Figs. 3–5 we show the cross sections $(\overline{m}^{\,2},\overline{f})$ of the whole parameter space for three values of $\overline{g}$ and two values of $\overline{\lambda}$. The upper edge is determined by the requirement that the energy of the Y field $E_Y\leq 0$. The lower edge is determined either by the constraint $\overline{m}^{\,2}\leq\overline{f}$ or by the maximum possible $\overline{m}^{\,2}$. The vertical lines mark the constraint $\overline{\lambda}\geq 2\overline{m}^{\,4}$. The upper edge can be approximated analytically by the formula $\overline{f}=\alpha\overline{m}^{\,2\beta}$, with the parameters α and β given in Table I. The position of the upper edge agrees reasonably well with the results of the Hill, Hodges, and Turner. The rapid growth of the energy of the string after $\overline{m}^{\,2}$ reaches a maximum is caused by the strong interaction with the scalar field Y. This is so because when Y becomes large the potential for the X field is significantly changed at the core of the string.

Next we consider a string with a current. The principal question is what is the maximal value of the current allowed for a given set of parameters in the superconducting region? *A priori* one expects that as the current grows the energy of the string increases. When the total energy of the string with $Y\neq 0$ and a large current is larger than the energy of the "pure" string with $Y=0$ then superconductivity is no longer energetically favorable. The real story is, however, a little bit different. In

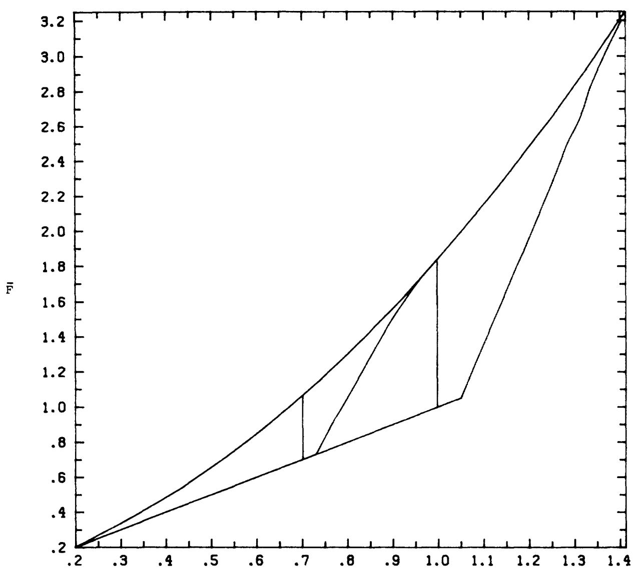

FIG. 3. The cross section $(\bar{m}^2, \bar{f})$ of parameter space for $\bar{g}^2 = 0.1$ and $\bar{\lambda} = 1$ and 2. Superconducting solutions exist for parameters inside the large triangle for $\bar{\lambda} = 2$ and small one for $\bar{\lambda} = 1$. The vertical lines mark the constraint $\bar{\lambda} \geq 2\bar{m}^4$.

Fig. 6 we show the total energy of the string as a function of a current for $(\bar{f}, \bar{m}^2, \bar{\lambda}, \bar{g}^2) = (0.65, 0.6, 1, 1)$. One can see that the maximal current

$$0.499 \times (E_{\mathrm{GUT}} / \sqrt{\lambda_\phi} 10^{15}\ \mathrm{GeV}) 10^{21}\ \mathrm{A}$$

corresponds to the energy roughly halfway between the minimum of the total energy E_T $(Y \neq 0,\ Q = 0)$ and

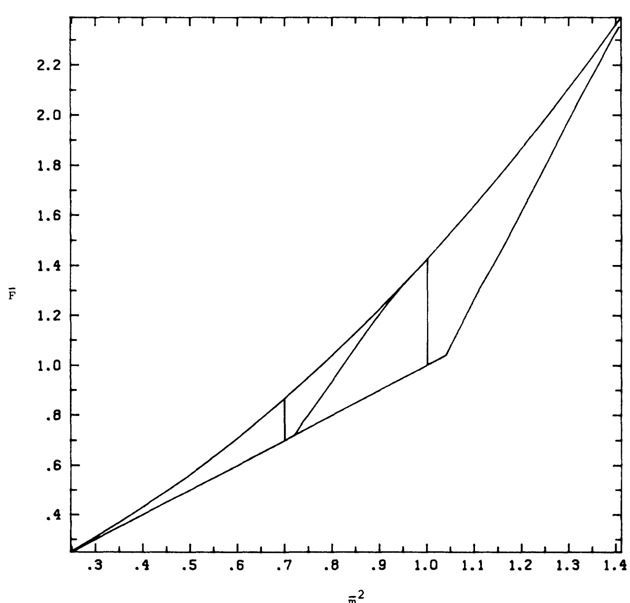

FIG. 4. The cross section $(\bar{m}^2, \bar{f})$ of parameter space for $\bar{g}^2 = 1$ and $\bar{\lambda} = 1$ and 2. Superconducting solutions exist for parameters inside the large triangle for $\bar{\lambda} = 2$ and small one for $\bar{\lambda} = 1$. The vertical lines mark the constraint $\bar{\lambda} \geq 2\bar{m}^4$.

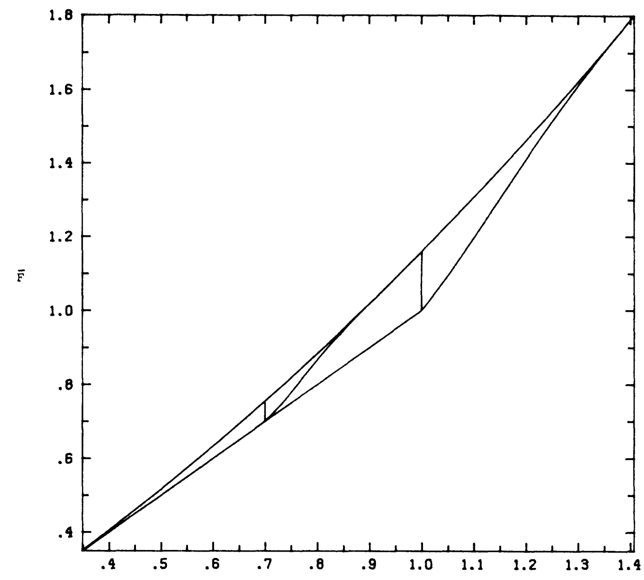

FIG. 5. The cross section $(\bar{m}^2, \bar{f})$ of parameter space for $\bar{g}^2 = 10$ and $\bar{\lambda} = 1$ and 2. Superconducting solutions exist for parameters inside the large triangle for $\bar{\lambda} = 2$ and small one for $\bar{\lambda} = 1$. The vertical lines mark the constraint $\bar{\lambda} \geq 2\bar{m}^4$.

E_T $(Y = 0)$ (we used $\lambda_\phi = 0.1$). This can be partially explained in the following way. The asymptotic form of Q as $x \to 0$ is $Q = Q_0 I_0(\bar{e} Y_0 x)$, so the derivative $Q' = Q_0 \bar{e} Y_0 I_1(\bar{e} Y_0 x)$, where I_0 and I_1 are modified Bessel functions. To increase the current in the string one monotonically increases Q_0. However, as Q_0 increases the initial value of Y must be lowered in order to keep $\bar{m}^2$ fixed. If the change is small the product $Q_0 Y_0$ increases and the current grows. When Q_0 is larger than a certain critical value Q_C its further growth induces such a change in Y_0 that the value of $Q_0 Y_0$ decreases and the derivative of Q at $x \approx 0$ becomes smaller than before. Consequently, the current goes down. The important point is that as Y_0 becomes small the energy $|E_Y|$ decreases very rapidly and therefore the "binding energy" $E_T(Y=0) - E_T(Y) \to 0$. Table II contains values of the maximal current for combinations of all parameters corresponding roughly to the centers of "good" regions and for points on the boundary $\bar{\lambda} = 2\bar{m}^4$ in Figs. 3–5. The only problem with this argument is that the magnetic energy of the infinite straight string with a current is

TABLE I. Parameters for the analytic fits to the upper edges of the regions on the plane $(\bar{m}^2, \bar{f})$ inside which the superconducting solutions exist.

$\bar{g}^2$	$\bar{\lambda}$	α	β
0.1	1	1.81	1.44
0.1	2	1.87	1.51
1	1	1.39	1.27
1	2	1.45	1.33
10	1	1.14	1.14
10	2	1.17	1.18

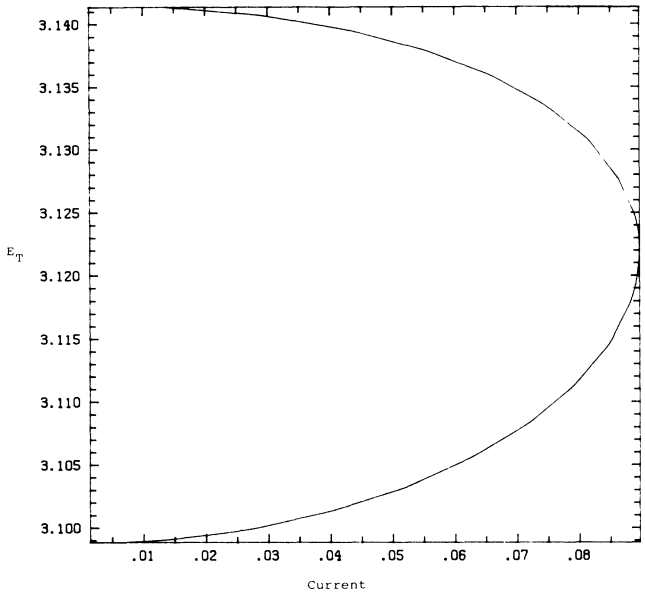

FIG. 6. The dependence of total energy E_T on the current in the string for $(\bar{g}^2,\bar{\lambda},\bar{f},\bar{m}^2)=(1,1,0.65,0.6)$. The unit of current is $2.85\times(E_{\mathrm{GUT}}/\sqrt{\lambda_\phi}10^{15}\ \mathrm{GeV})10^{21}$ A.

logarithmically divergent. To correct this one should add $0.8I_T^2\times\ln(X_{\max}/20)$ to the total energy computed numerically, where I_T is the total current and $X_{\max}$ is a cutoff which should be of the order of the radius of a string loop. Since the maximum current is $I_T\approx 0.1$, this addition is small and does not change the results.

The interesting question is what happens if one tries to generate an even larger current by, for example, dragging the string through an external magnetic field. The complete answer to this question requires the study of the time-dependent solutions, but it seems clear what should happen. The work done on the string in this way does not go into increasing the current, but is used to destroy the field Y. Consequently, the current decreases and the transition to the ordinary phase should proceed in a smooth fashion, that is, without any tunneling. Of course, when the current is dissipated the string should return to the superconducting phase rather abruptly.

TABLE II. The maximal current for different combinations of the constants in the potential. The unit of current is $(E_{\mathrm{GUT}}\sqrt{\lambda_\phi}10^{15}\ \mathrm{GeV})10^{21}$ A.

$\bar{g}^2$	$\bar{\lambda}$	$\bar{f}$	$\bar{m}^2$	Current
0.1	1.0	0.70	0.6	0.501
0.1	1.0	0.90	0.7	0.469
0.1	2.0	1.0	0.8	0.333
0.1	2.0	1.40	1.0	0.455
1.0	1.0	0.65	0.6	0.255
1.0	1.0	0.75	0.7	0.837
1.0	2.0	0.90	0.8	0.204
1.0	2.0	1.20	1.0	0.330
10	1.0	0.65	0.62	0.076
10	1.0	0.70	0.68	0.339
10	2.0	0.90	0.85	0.112
10	2.0	1.05	1.0	0.444

IV. STRING IN CURVED SPACETIME

The principal questions we are trying to answer in this section are the following: What is the metric of the spacetime and how are the solutions of the field equations affected by the curvature of the spacetime? In order to answer these questions we have to solve the Einstein equation and the field equations with the curvature taken into account. It is convenient to rewrite the Einstein equation and the field equations in the following form:

$$(KA')'=4\pi\eta^2\left[-2KV+\frac{(P')^2e^{2(A+B)}}{\bar{g}^2K}+\frac{(Q')^2K}{\bar{e}^2e^{2B}}\right],$$

$$(KB')'=4\pi\eta^2\left[-2KV+\frac{(P')^2e^{2(A+B)}}{\bar{g}^2K}-\frac{(Q')^2K}{\bar{e}^2e^{2B}}-2Y^2Q^2Ke^{-2B}\right],\tag{4.1}$$

$$K''=4\pi\eta^2\left[-6KV+\frac{(P')^2e^{2(A+B)}}{\bar{g}^2K}+\frac{(Q')^2K}{\bar{e}^2e^{2B}}-2Y^2Q^2Ke^{-2B}-\frac{2X^2P^2e^{2(A+B)}}{K}\right];$$

$$X''+\frac{K'}{K}X'=X\left[\tfrac{1}{2}(X^2-1)+\frac{P^2e^{2(A+B)}}{K^2}+\bar{f}Y^2\right],$$

$$Y''+\frac{K'}{K}Y'=Y(\bar{\lambda}Y^2+\bar{f}X^2-\bar{m}^2+Q^2e^{-2B}),$$

$$P''+\left[2A'+2B'-\frac{K'}{K}\right]P'=\bar{g}^2X^2P,\tag{4.2}$$

$$Q''+\left[\frac{K'}{K}-2B'\right]Q'=\bar{e}^2Y^2Q,$$

where $K=e^{A+B+C}\sqrt{\lambda_\phi}\eta$. These equations can be solved by iteration. We start with a flat-spacetime solution for the quantum fields and calculate the metric keeping the fields fixed. The Einstein equations can be easily solved with the initial conditions specified at $x=0$. We take $A(0)=B(0)=0$; this choice is dictated by the normalizations of the Killing vectors $\partial/\partial z$ and $\partial/\partial\varphi$ to unity on the axis. Conditions $K(0)=0$ and $K'(0)=0$ follow from the requirement that the metric be smooth[8] at $x=0$. Finally, the initial values of the derivatives $A'(0)=B'(0)=0$ can be easily obtained using Einstein equations and the known asymptotic form of all fields as $x\to 0$. Having found the metric we calculate the quantum fields again and iterate: in practice two or three iterations give the solutions with a relative accuracy better than 10^{-4}.

The results are not surprising. In Fig. 7 we plot the metric functions A and B for

$$(\eta,\bar{g}^2,\bar{\lambda},\bar{f},\bar{m}^2)=(10^{-2},0.25,1,0.75,0.65)$$

and three values of the current. For a small current the metric is the same as for the usual cosmic string[9] and

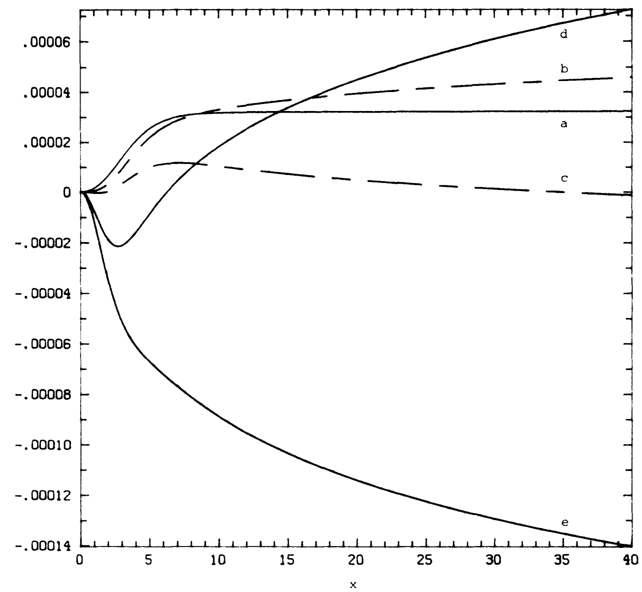

FIG. 7. The metric functions A and B for $(\eta,\bar{g}^2,\bar{\lambda},\bar{f},\bar{m}^2)=(10^{-2},0.25,1,0.75,0.65)$. (a) $e^A-1=e^B-1$ for the string with a very small current, (b) e^A-1, (c) e^B-1 for the string with a current 0.05, (d) e^A-1, and (e) e^B-1 for the string with a current 0.175. The unit of current is $2.85\times(E_{\rm GUT}/\sqrt{\lambda_\phi}10^{15}\ {\rm GeV})10^{21}$ A.

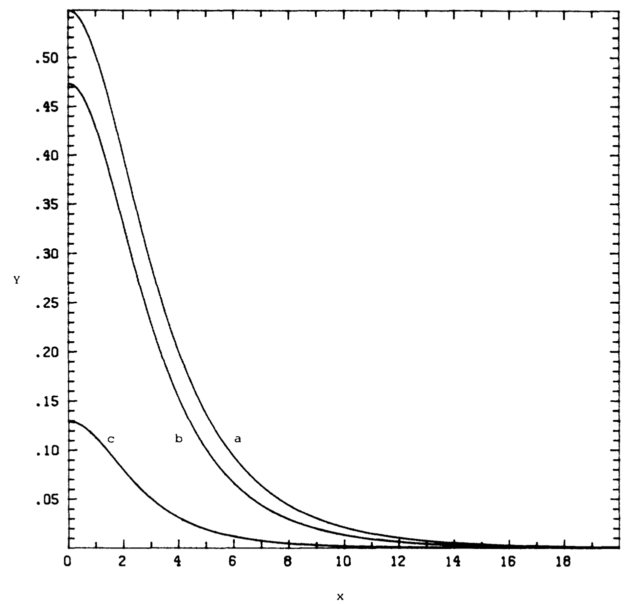

FIG. 8. The Y field for three different combinations of Y_0 and Q_0 corresponding to (a) small current, large binding energy, (b) maximal current, and (c) small current, small binding energy. $(\eta,\bar{g}^2,\bar{\lambda},\bar{f},\bar{m}^2)=(10^{-3},0.25,1,0.75,0.65)$.

$A=B$, as shown analytically in Ref. 8. The geometry of the spacetime can be approximated by a conical metric with a small deflecting angle so, far away from the string, the spacetime becomes nearly flat. However, the spacetime is different if the current is large. First of all, $A\neq B$ anymore, and even more important, far away from the string the spacetime approaches the Kasner solution, in agreement with the solution given by L. Witten[10] for an infinitely thin wire with a current. To check whether the solution is Kasner or Minkowski at a very large distance from the string one has to look at the asymptotic behavior of KA' and KB'. For large currents these products approach a constant and are some four orders of magnitude larger than for ordinary strings. In the latter case their values are still decreasing when $x=40$. The Kasner solution is only acceptable close to the string where the string is approximately straight. Real strings are not straight nor infinite, and they do not reside in vacuum. Therefore, far away from the string our approximation does not have any physical meaning. The function K deviates very slightly from its flat spacetime behavior $K=x$. In all cases we have studied, K increases monotonically with $K'\to 1$ as $x\to 0$ and $K'\approx 0.98$ for large x.

The solutions are very similar for smaller values of η, but the deviations from flatness are even smaller. To see this one can calculate the deflecting angle $\Delta\phi$. As shown in Ref. 8,

$$\Delta\phi=\lim_{r\to\infty}2\pi\left[1-\frac{d(e^{-(A+B)}K)}{dr}\right]. \tag{4.3}$$

The values of $\Delta\phi$ are given in Table III for three different values of η and for three different combinations of Y_0 and Q_0 corresponding to (a) small current, large binding energy, (b) maximal current, and (c) small current, small binding energy. As for the ordinary

TABLE III. The deflecting angle and the total energy of the string for different values of η and different currents. The unit of current is $(E_{\rm GUT}/\sqrt{\lambda_\phi}10^{15}\ {\rm GeV})\ 10^{21}$ A. Deflecting angle is in radians.

η	Y_0	Q_0	Current	E_T	$\Delta\phi$
10^{-4}	0.5471	0.001	0.608×10^{-3}	4.0988	1.0499×10^{-5}
10^{-4}	0.4732	0.130	0.499	4.1595	1.0043×10^{-5}
10^{-4}	0.1290	0.260	0.052	4.2113	0.9225×10^{-6}
10^{-3}	0.5471	0.001	0.608×10^{-3}	4.0988	1.0500×10^{-3}
10^{-3}	0.4732	0.130	0.499	4.1596	1.0043×10^{-3}
10^{-3}	0.1290	0.260	0.052	4.2113	0.9225×10^{-4}
10^{-2}	0.5474	0.001	0.610×10^{-3}	4.1003	1.0514×10^{-1}
10^{-2}	0.4738	0.130	0.501	4.1611	1.0048×10^{-1}
10^{-2}	0.1290	0.260	0.055	4.2128	0.9224×10^{-2}

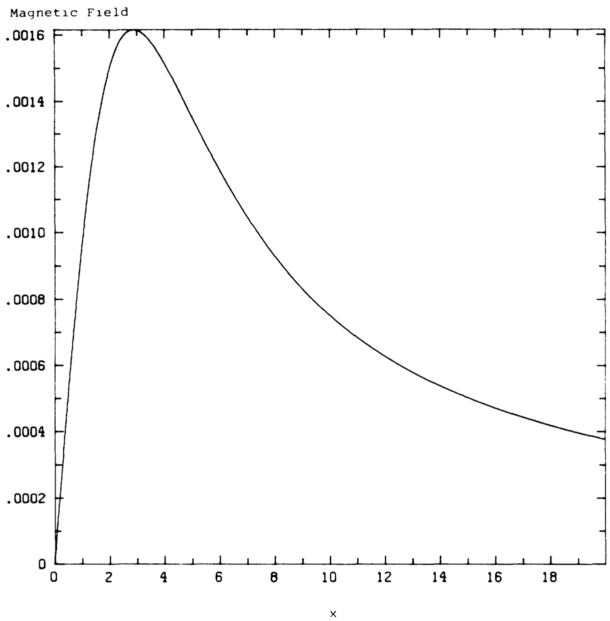

FIG. 9. The magnetic field corresponding to the maximal current for $(\eta, \bar{g}^2, \bar{\lambda}, \bar{f}, \bar{m}^2) = (10^{-3}, 0.25, 1, 0.75, 0.65)$.

cosmic string the deflecting angle scales as η^2. It is interesting to note that the deflecting angle is ten times smaller when the binding energy vanishes than in the other two cases.

Since the spacetime is nearly flat, we do not expect any significant changes in the fields Y and Q as compared with the solutions obtained for flat spacetime. Indeed, the changes are so small that it is not possible to show them in figures. The field Y is shown in Fig. 8 for three different values of the Q_0: a very small one, one that corresponds to the maximal current, and one such that the "binding energy" is nearly zero. In Fig. 9 the magnetic field is plotted for the maximal current.

V. CONCLUSIONS

In this paper we describe the numerical solutions of the field equations for a superconducting bosonic string. Direct numerical integration reveals several interesting features of the solutions. First of all, the superconducting phase exists only for a very special choice of the parameters in the potential. The "good" region is, in fact, similar to the one found with the help of the parametrized variational method.[6] Second, for many values of the parameters there are unstable solutions. The decay of such a configuration should release energy of the order 10^{32}erg/m. Third, for the parameters in the range studied here, the maximal current does not occur when superconductivity breaks down. On the contrary, when the binding energy vanishes the current vanishes also. This probably means that the transition between the superconducting phase and the ordinary one is smooth. Finally, the spacetime around a superconducting string is asymptotically Kasner and the deflection angle is not sensitive to the value of the current but decreases when the string is about to switch from the superconducting phase to the ordinary one.

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