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Chiral model of the nucleon and Δ isobar: The coherent-pair approximation

K. Goeke
Institut für Kernphysik, Kernforschungsanlage Jülich, D-5170 Jülich, West Germany
and Institut für Theoretische Kernphysik, Universität Bonn, D-5300 Bonn, West Germany

M. Harvey
Chalk River Nuclear Laboratories, Chalk River, Ontario, Canada

F. Grümmer
Institut für Kernphysik, Kernforschungsanlage Jülich, D-5170 Jülich, West Germany

J. N. Urbano
Centro di Fisica Teorica (INIC) and Departamento de Fisica, Universidade de Coimbra, P-3000 Coimbra, Portugal
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The linear chiral σ model with quark fields and elementary pion and σ fields is used to describe static properties of the nucleon and the Δ resonance. To this end baryon Fock states with good spin and isospin are constructed from summations of components in which three quarks in s states with the SU(2)×SU(2) quantum numbers of the nucleon and Δ are coupled to multipion states and a scalar-isoscalar coherent state of the σ fields. The multipion states are constructed to have coherent properties as well as definite spin-isospin quantum numbers. Ignoring the effects of vacuum polarization the baryon energy is made stationary, resulting in a set of four coupled nonlinear eigenvalue equations and a diagonalization procedure between frozen fields. The corresponding solitonic solutions are used to evaluate the relevant nucleon properties. A comparison is made with the results from the cloudy-bag model, the projected mean-field chiral-soliton model, and the Skyrme approach.

I. INTRODUCTION

Quantum chromodynamics (QCD) is the currently accepted fundamental theory of the strong interaction.[1] However, its long-distance, nonperturbative regime has so far defied easy solutions. Only lattice gauge calculations[2] have provided some information on this regime, but the tremendous numerical difficulties strain the capacity of available computers. An alternative approach to correlating low-energy hadronic properties is the use of effective field theories which replace the complicated interacting quark-gluon system by a simpler one involving interacting fermions (quarks) and/or appropriate boson fields.

Recently, there has been much interest in σ or soliton models.[3] These models are similar in spirit to the Friedberg-Lee model,[4] where, e.g., a nucleon is described by a solitonic solution of appropriate effective meson fields stabilized by the coupling to quark fields.[5,6] The so-called chiral-soliton models based on the Lagrangian of Gell-Mann and Lévy[7] seem to be particularly suitable,[8–12] since they have broken chiral symmetry and the partially conserved axial-vector current (PCAC) arises in a natural way through quark and boson fields.

An alternative type of soliton model, also of current interest, was proposed by Skyrme[13] and is based on the nonlinear σ model[7] without explicit quark degrees of freedom. Instead, the model has a conserved topological winding number which is interpreted as the baryon number. Compared to soliton models with explicit quarks it provides problems with the quantization since its boson field is defined over a curved function space. Therefore, a clear definition of Fock states with proper spin and isospin quantum numbers is difficult and one is forced to use classical and semiclassical arguments.[14]

In this paper we consider the Lagrangian of the chiral linear soliton model first suggested by Gell-Mann and Levy.[7] Here the fermion fields are identified with quarks rather than with nucleons as was done in the original formalism. The model describes a flavor doublet (isospin doublet) of quarks interacting with pions (pseudoscalar, isovector) and σ mesons (scalar, isoscalar). The quark-meson coupling reflects the SU(2)×SU(2) chiral symmetry of the underlying QCD theory. The self-interactions are chosen to make the theory renormalizable[10] and to allow the chiral symmetry to be realized in the hidden mode (spontaneous chiral-symmetry breaking). The σ field has a nonzero vacuum value, which gives the quarks their (dynamical) mass. The pions are considered as Goldstone bosons which are given a small mass to break chiral symmetry explicitly.

The linear chiral-soliton model has been considered recently by several authors.[8–12] Most of the Fock states that have been considered are classical and based on the hedgehog ansatz[15] for the quark states and pion field. The resulting solution corresponds to a mean-field description of the system in terms of coherent states and does not, therefore, carry a definite angular momentum and isospin. Very recently projection techniques have been suggested[16] and applied to extract physical Fock states with proper quantum numbers and to allow therefore for the evaluation of observable nucleon properties.[12,24]

The purpose of this paper is to provide a static solitonic solution of the linear chiral-soliton model using a coherent-pair trial Fock state with proper spin and isospin quantum numbers.[17] In contrast with the mean-field projection technique the coherent-pair approximation provides a systematic expansion method for the description of a boson field. It is more like a shell model based on a paired background state and provides, therefore, an alternative way to describe boson clouds, which does not go through a classical state and is, therefore, quantum mechanical from the start. In addition, it avoids assumptions like the hedgehog structure of the quark and pion fields. In this paper we will restrict ourselves to the coherent-pair approximation with one unpaired pion in order to expose the techniques required. We derive energies and other observable properties of the nucleon and the Δ resonance and compare them with the outcome of other models and with experiment. Some preliminary results have already been published in a recent paper.[18]

II. THE LAGRANGIAN

A simple renormalizable Lagrangian involving fermion fields $\hat{\psi}(x)$ and boson fields $\hat{\sigma}(x)$ and $\hat{\vec{\pi}}(x)$, which is chirally invariant, has been given by Gell-Mann and Levy.[7] If one breaks explicitly the chiral symmetry in order to give the pion a small finite mass, one obtains

$$\begin{aligned}\hat{\mathcal{L}}(x)=&\hat{\bar{\psi}}(x)(i\gamma^\mu\partial_\mu)\hat{\psi}(x)+\tfrac{1}{2}\partial^\mu\hat{\sigma}(x)\partial_\mu\hat{\sigma}(x)\\&+\tfrac{1}{2}\partial^\mu\hat{\vec{\pi}}(x)\cdot\partial_\mu\hat{\vec{\pi}}(x)\\&-g\hat{\bar{\psi}}(x)[\hat{\sigma}(x)+i\gamma_5\vec{\tau}\cdot\hat{\vec{\pi}}(x)]\hat{\psi}(x)\\&-U(\hat{\sigma}(x),\hat{\vec{\pi}}(x))\end{aligned}\tag{2.1}$$

with the self-interaction $U(\hat{\sigma},\hat{\vec{\pi}})$ given by

$$\begin{aligned}U(\hat{\sigma},\hat{\vec{\pi}})=&\frac{\lambda^2}{4}[\hat{\sigma}(x)^2+\hat{\vec{\pi}}(x)\cdot\hat{\vec{\pi}}(x)-\nu^2]^2\\&-m_\pi^2 f_\pi\hat{\sigma}(x)+U_0\ .\end{aligned}\tag{2.2}$$

The potential $U(\hat{\sigma},\hat{\vec{\pi}})$ has the shape of a "Mexican hat" in the σ-π plane slightly distorted by the linear term in σ. The parameter ν is chosen such that the classical minimum of this potential occurs for a nonzero vacuum value of the σ field and a vanishing value of the pion field:

$$\begin{aligned}\langle 0|\hat{\sigma}(\mathbf{r})|0\rangle&=\sigma_v=f_\pi\ ,\\ \langle 0|\hat{\vec{\pi}}(\mathbf{r})|0\rangle&=\pi_v=0\ ,\end{aligned}\tag{2.3}$$

with f_π being the pion decay constant, i.e., $f_\pi=0.093$ GeV. This yields, with $(\partial U/\partial\sigma)_{\sigma_v}=(\partial U/\partial\pi)_{\pi_v}=0$,

$$\nu^2=f_\pi^2-\frac{m_\pi^2}{\lambda^2}\ .\tag{2.4}$$

The model contains then two adjustable parameters: the mass of the σ field, m_σ, and the coupling constant g or, equivalently, the dynamical mass of the quarks, m_q. They are related via $(\partial^2U/\partial\sigma^2)_{\sigma_v}=m_\sigma^2$ and $(\partial^2U/\partial\pi^2)_{\pi_v}=m_\pi^2$ as

$$\lambda^2=\frac{m_\sigma^2-m_\pi^2}{2f_\pi^2}\ ,\tag{2.5}$$

$$m_q=gf_\pi\ .\tag{2.6}$$

Whereas the mass of the pion is accurately known from experiment ($m_\pi=0.138$ GeV), it is not clear which value to assume for m_σ since there are various candidates in the meson and glueball spectrum with the appropriate quantum numbers. Hence we will consider in this paper m_σ as a free parameter in the range of 0.3 GeV $\leq m_\sigma$. We will see that the results do not depend very much on the actual choice of m_σ.

The constant U_0 is chosen such that at the lowest minimum of the (tilted) Mexican hat one has $U(\sigma=\sigma_v,\vec{\pi}=0)=0$. This yields

$$U_0=\tfrac{1}{2}f_\pi^2m_\pi^2\frac{2m_\sigma^2-3m_\pi^2}{m_\sigma^2-m_\pi^2}\ .\tag{2.7}$$

The above Lagrangian is partially invariant under chiral transformations. The corresponding axial-vector current is given by

$$\hat{\vec{J}}{}_\mu^A=\hat{\bar{\psi}}\tfrac{1}{2}\gamma^\mu\gamma_5\vec{\tau}\hat{\psi}+\hat{\sigma}\partial^\mu\hat{\vec{\pi}}-\hat{\vec{\pi}}\partial^\mu\hat{\sigma}\ .\tag{2.8}$$

It is partially conserved (PCAC), yielding

$$\partial^\mu\hat{\vec{J}}{}_\mu^A=-m_\pi^2f_\pi\hat{\vec{\pi}}\ .\tag{2.9}$$

In order to obtain this result, confirmed by experiment, the vacuum value of σ had to be chosen as $\sigma_v=f_\pi$. The total Hamiltonian density is

$$\begin{aligned}\hat{\mathcal{H}}(\mathbf{r})=&\tfrac{1}{2}\{\hat{P}_\sigma(\mathbf{r})^2+[\nabla\hat{\sigma}(\mathbf{r})]^2+\hat{\vec{P}}_\pi(\mathbf{r})^2+[\nabla\hat{\vec{\pi}}(\mathbf{r})]^2\}\\&+U(\hat{\sigma},\hat{\vec{\pi}})+\hat{\psi}^\dagger(\mathbf{r})(-i\boldsymbol{\alpha}\cdot\nabla)\hat{\psi}(\mathbf{r})\\&+g\hat{\psi}^\dagger(\mathbf{r})[\beta\hat{\sigma}(\mathbf{r})+i\beta\gamma_5\vec{\tau}\cdot\hat{\vec{\pi}}(\mathbf{r})]\hat{\psi}(\mathbf{r})\ .\end{aligned}\tag{2.10}$$

In the above expressions $\hat{\psi}$, $\hat{\sigma}$, and $\hat{\vec{\pi}}$ are quantized field operators with the appropriate static expansions

$$\hat{\sigma}(\mathbf{r})=\int\frac{d^3k}{[(2\pi)^3 2\omega_\sigma(k)]^{1/2}}[c^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}+c(\mathbf{k})e^{+i\mathbf{k}\cdot\mathbf{r}}]\ ,\tag{2.11}$$

$$\begin{aligned}\hat{\pi}^t(\mathbf{r})=&\int\frac{d^3k}{[(2\pi)^3 2\omega_\pi(k)]^{1/2}}\\&\times[a_t^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}+(-)^t a_{-t}(\mathbf{k})e^{+i\mathbf{k}\cdot\mathbf{r}}]\ ,\end{aligned}\tag{2.12}$$

$$\hat{\psi}(\mathbf{r})=\sum_{njm}(\langle r\,|\,jm\,\rangle d_{njm}+\langle r\,|\,njm\,\rangle^* d_{\bar{n}jm}^\dagger)\ .\tag{2.13}$$

For the pion field the isospin label t ($\pm1,0$) indicates the three pion fields of an isovector. The corresponding conjugate momentum fields have the expansions

$$\hat{P}_\sigma(\mathbf{r})=i\int d^3k\left[\frac{\omega_\sigma(k)}{2(2\pi)^3}\right]^{1/2}[c^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}-c(\mathbf{k})e^{+i\mathbf{k}\cdot\mathbf{r}}]\ , \tag{2.14}$$

$$\hat{P}^t_\pi(\mathbf{r})=i\int d^3k\left[\frac{\omega_\sigma(k)}{2(2\pi)^3}\right]^{1/2}\times[a_t^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}-(-)^t a^\dagger_{-t}(\mathbf{k})e^{+i\mathbf{k}\cdot\mathbf{r}}]\ . \tag{2.15}$$

Here the $c(\mathbf{k})$ destroys a σ quantum with momentum $\mathbf{k}$ and frequency $\omega_\sigma(k)=(k^2+m_\sigma^2)^{1/2}$ and $a_t(\mathbf{k})$ a pion with momentum $\mathbf{k}$ and a corresponding $\omega_\pi(k)=(k^2+m_\pi^2)^{1/2}$. The $|njm\rangle$ and $|\bar{n}jm\rangle$ form a complete set of quark and antiquark spinors with angular-momentum quantum numbers and spin-isospin quantum numbers j and m, respectively. One should note that the above choices of the expansion bases and frequencies are not unique.

For the pion field it is convenient to change the basis by introducing new pion operators

$$a^\dagger_{tlm}(k)=(-i)^l\int d\Omega_k\, Y_{lm}(\Omega_k)a_t^\dagger(\mathbf{k})\ , \tag{2.16}$$

yielding

$$\hat{\pi}^t(\mathbf{r})=\left[\frac{2}{\pi}\right]^{1/2}\int dk\, k^2\left[\frac{1}{2\omega_\pi(k)}\right]^{1/2}\sum_{lm}j_l(kr)[a^\dagger_{tlm}(k)+(-)^{m+t}a_{-tl-m}(k)]Y^*_{lm}(\Omega_r) \tag{2.17}$$

and

$$\hat{P}^t_\pi(\mathbf{r})=i\left[\frac{2}{\pi}\right]^{1/2}\int dk\, k^2\left[\frac{\omega_\pi(k)}{2}\right]^{1/2}\sum_{lm}j_l(kr)[a^\dagger_{tlm}(k)-(-)^{m+t}a_{-tl-m}(k)]Y^*_{lm}\ . \tag{2.18}$$

The commutation rules are now

$$[a_{tlm}(k),a^\dagger_{t'l'm'}(k')]=\frac{1}{k^2}\delta(k-k')\delta_{tt'}\delta_{ll'}\delta_{mm'}\ . \tag{2.19}$$

III. THE FOCK STATE

The objective of this paper is to provide a static description of the nucleon and the Δ resonance by means of a variational ansatz on states that have definite angular momentum and isospin properties. A simple, in fact too simple, way is to assume the Fock state of a nucleon and delta to be

$$|NT_3J_z\rangle=[\alpha|nT_3J_z\rangle+\beta(|n\rangle\otimes b^\dagger)_{T_3J_z}+\gamma(|\delta\rangle\otimes b^\dagger)_{T_3J_z}]|0\rangle|\Sigma\rangle \tag{3.1}$$

and

$$|\Delta T_3J_z\rangle=[\alpha'|\delta T_3J_z\rangle+\beta'(|\delta\rangle\otimes b^\dagger)_{T_3J_z}+\gamma'(|n\rangle\otimes b^\dagger)_{T_3J_z}]|0\rangle|\Sigma\rangle\ . \tag{3.2}$$

Here $b^\dagger$ is a p-wave isovector pion creation operator for a state to be defined, cf. Eq. (3.10). The states $|nt_3j_z\rangle$ are SU(6) three-quark configurations coupled in the u-d sector to nucleonic quantum numbers.[19]

The two-spinor corresponding to a quark moving in a spherical 1s orbit is defined by

$$q(r)=\langle r|d^\dagger_{n=0,j=1/2}|0\rangle=\begin{bmatrix}u(r)\\ iv(r)\sigma\cdot\hat{\mathbf{r}}\end{bmatrix}\ . \tag{3.3}$$

Thus we assume right at the beginning that deformation effects are not important. The variational quantities in $q(r)$ are the upper and lower radial components $u(r)$ and $v(r)$, respectively.

The Fock state for the sigma field is assumed to be a coherent state of the form

$$|\Sigma\rangle=\frac{1}{N}\exp\left[\int d^3k\,\eta(k)c^\dagger(\mathbf{k})\right]|0\rangle \tag{3.4}$$

with the normalization coefficient given by

$$N=\exp\left[\tfrac{1}{2}\int d^3k\,\eta^*(k)\eta(k)\right]\ . \tag{3.5}$$

Clearly the coherent state has the property

$$c(\mathbf{k})|\Sigma\rangle=\eta(k)|\Sigma\rangle\ .$$

Since η depends only on $k=|\mathbf{k}|$ the coherent state $|\Sigma\rangle$ has the quantum numbers of a scalar and isoscalar state. For parity reasons we have $\eta(k)=\eta^*(-k)$ and hence

$$\langle\Sigma|\hat{P}_\sigma(\mathbf{r})|\Sigma\rangle=0 \tag{3.6}$$

and

$$\langle\Sigma|\hat{\sigma}(\mathbf{r})|\Sigma\rangle=\sigma(r)\ . \tag{3.7}$$

Because of the general properties of the coherent states we have, furthermore,

$$\langle\Sigma|{:}P^n_\sigma(\mathbf{r}){:}|\Sigma\rangle=0 \tag{3.8}$$

and

$$\langle\Sigma|{:}\sigma^n(\mathbf{r}){:}|\Sigma\rangle=\sigma^n(r)\ , \tag{3.9}$$

where : : indicates normal ordering with regard to the vacuum of the $c(\mathbf{k})$. The expectation value $\langle \Sigma | :\hat{\sigma}(\mathbf{r}): | \Sigma \rangle$ can be expressed in terms of $\eta(k)$ and equivalently in terms of $\sigma(r)$. Thus the $\sigma(r)$ will be used as a variational function, being associated to the degrees of freedom in the spherical σ Fock state $| \Sigma \rangle$.

The $|0\rangle$ in Eq. (3.1) is the boson vacuum. The b_{tm} are related to the basis operators $a_{tlm}(k)$ as

$$b_{tm} = \int dk\, k^2 \xi(k) a_{tlm}(k) \tag{3.10}$$

with $\xi(k)$ being a real function representing the variational degrees of freedom in b_{tm} with respect to a_{tlm}. Actually, later we have to replace the vacuum $|0\rangle$ and the one-pion states $b^{\dagger}_{tm}|0\rangle$ by coherent states $|P^{00}\rangle$ and $|P^{11}_{tm}\rangle$ with the corresponding quantum numbers in order to improve the Fock state. This, however, will be done in the next section. For the simple explanation of the variational properties it is sufficient to consider zero- and one-pion states. Rather than varying $\xi(k)$ explicitly, it will turn out to be convenient to vary the function $\phi(r)$ instead with

$$\phi(r) = \frac{1}{2\pi} \int dk\, k^2 \frac{\xi(k)}{[\omega(k)]^{1/2}} j_1(kr) . \tag{3.11}$$

Thus the variational degrees of freedom of the Fock state $|NT_3J_z\rangle$ or $|\Delta T_3 J_z\rangle$ are $u(r)$, $v(r)$, $\sigma(r)$, $\phi(r)$ and α, β, γ. The variations are not free, but are subjected to various normalization conditions.

First, the quark states must be normalized

$$4\pi \int_0^{\infty} dr\, r^2 [u^2(r) + v^2(r)] = 1 . \tag{3.12}$$

Second, the boson operators b_{tm} have to be normalized, which requires

$$\int_0^{\infty} dk\, k^2 \xi^2(k) = 1 . \tag{3.13}$$

Since $\xi(k)$ can be extracted from $\phi(r)$ by

$$\xi(k) = 4[\omega(k)]^{1/2} \int_0^{\infty} r^2 j_1(kr) \phi(r) dr , \tag{3.14}$$

the normalization condition reads

$$8\pi \int dr\, r^2 \phi(r) \phi_p(r) = 1 \tag{3.15}$$

with

$$\phi_p(r) = \int_0^{\infty} \omega(r,r') \phi(r') r'^2 dr' . \tag{3.16}$$

The kernel $\omega(r,r')$ is given by

$$\omega(r,r') = \frac{2}{\pi} \int_0^{K} dk\, k^2 \omega(k) j_1(kr) j_1(kr') . \tag{3.17}$$

For $K \to \infty$ this kernel is ill defined. However, when it is applied to a well-behaved pion function $\phi(r)$ the K can be assumed finite and the $\phi_p(r)$ is well defined.

Since the σ coherent state $|\Sigma\rangle$ is normalized from the beginning, the last normalized condition is

$$\alpha^2 + \beta^2 + \gamma^2 = 1 . \tag{3.18}$$

Actually, the present approach neglects from the start any contributions to the energy from the quark Dirac sea of negative-energy orbitals. A crucial quantity to indicate the importance of the vacuum polarization of the negative-energy Dirac sea continuum is $m_q R = g f_\pi R$, where R measures the size of the soliton. For $m_q R \gg 1$ (in practice for $m_q R \gtrsim 2.5$) the negative-energy continuum leads to a baryon-number density, which is more diffuse than that obtained from the valence orbit alone. The solitons studied here have an effective $m_q R = 2-3$ and, therefore, the vacuum-polarization effects are certainly there; however, they should not be so big as to invalidate the qualitative results of our model. The explicit calculations of Kahana and Ripka[10,20] indicate that for this range of $m_q R$ values the properties of the soliton are affected to about 20% by the polarization of the sea quarks. Clearly this point deserves further studies which are, however, beyond the scope of this paper.

IV. SYMMETRY-CONSERVING COHERENT STATES

If the Fock state of the system is formulated as Eq. (3.1), one allows at most for one-pion excitations of the vacuum. This cannot be sufficient since terms such as $:\vec{\pi}^4:$ would have a vanishing expectation value. Therefore the Fock state of the pions should have components involving excitations of many bosons. One possibility would be to write down explicitly one-pion, two-pion, etc., excitations coupled to the proper quantum numbers. Another, more promising way consists of constructing "coherent" states with the proper angular momentum and isospin quantum numbers.[17] We will discuss this possibility separately for $L=0$ and $L \neq 0$ partial waves. The concepts are then easily generalized to tensor product spaces such as spin and isospin spaces.

A. Coherent states of $L=0$ partial waves

Consider a multipole expansion of the form (2.19) of a general boson field

$$\hat{\chi}(\mathbf{r}) = \sum_{\alpha LM} \phi_{\alpha L}(r) Y^{*}_{LM}(\Omega_r) [c^{\dagger}_{\alpha LM} + (-)^M c_{\alpha L -M}] , \tag{4.1}$$

where α is any quantum number other than angular momentum L and its third component necessary to define the state completely. If we define $|Q^0\rangle$ to be the coherent Fock state made up from $L=0$ quanta, we have

$$|Q^0\rangle = N \exp\left[\sum_{\alpha} \tfrac{1}{2} \lambda_\alpha c^{\dagger}_{\alpha 00}\right] |0\rangle \tag{4.2}$$

with some amplitude λ_α. Since $|Q^0\rangle$ is an eigenstate of the annihilation operator $c_{\alpha 00}$,

$$c_{\alpha 00} |Q^0\rangle = \tfrac{1}{2} \lambda_\alpha |Q^0\rangle , \tag{4.3}$$

it follows directly that

$$\langle Q^0 | \hat{\chi}(\mathbf{r}) | Q^0 \rangle = Y^{*}_{00}(\Omega_r) \sum_{\alpha} \phi_{\alpha 00}(r) \lambda_\alpha = \chi(r) . \tag{4.4}$$

The function $\chi(r)$ is to be identified with the classical mean field in the same way as $\sigma(r)$ of Eq. (3.9) represented the classical σ field. In fact, the above equations show that assuming a coherent state of the form (3.6) and of the form (4.2) is exactly the same thing. Altogether it is clear now that the coherent states $|Q^0\rangle$ and $|\Sigma\rangle$ carry the quantum numbers of the vacuum since they are constructed out of scalar operators.

B. Coherent states of $L\neq 0$ partial waves

The simple property mentioned at the end of the previous paragraph obviously does not hold for coherent states built up from creation operators of $L\neq 0$ partial waves. In addition, for $L\neq 0$ partial waves an eigenvalue equation of the form (4.3) cannot be true. Hence the states we are going to construct bear certainly some coherence and show some corresponding properties; however, they are not coherent states in the strict sense.

We construct a coherent-pair state with the quantum numbers of the vacuum from $L\neq 0$ partial waves by

$$|P^0\rangle=\sum_n \frac{f_n}{(2n)!}(c_L^\dagger\cdot c_L^\dagger)^n|0\rangle \ , \tag{4.5}$$

where

$$c_L^\dagger\cdot c_L^\dagger=\sum_M(-)^M c^\dagger_{\alpha LM}c^\dagger_{\alpha L-M} \tag{4.6}$$

for a particular α. The α is arbitrary but fixed. In the applications the lowest field excitation corresponding to $\alpha=0$ will be preferred. In the following we will drop the index α. A coherent state $|P_M^L\rangle$ can be defined as

$$(-)^M c_{L-M}|P^0\rangle=a\,|P_M^L\rangle \tag{4.7}$$

with some coefficient a, yielding

$$|P_M^L\rangle=\frac{1}{a}\sum_{n=0}^{\infty}\frac{f_{n+1}}{(2n+1)!}c^\dagger_{LM}(c_L^\dagger\cdot c_L^\dagger)^n|0\rangle \ . \tag{4.8}$$

Applying c_{LM} to $|P_M^L\rangle$ again and projecting out the state with vacuum quantum numbers yields

$$\sum_M c_{LM}|P_M^L\rangle=\tilde{L}b\,|\tilde{P}^0\rangle \tag{4.9}$$

with $\tilde{L}=2L+1$ and some coefficient b. The pair of Eqs. (4.7) and (4.9) yields the equation

$$c_L^\dagger\cdot c_L^\dagger|P^0\rangle=x\,|\tilde{P}^0\rangle \tag{4.10}$$

with $x=\tilde{L}ab$. We shall show that x is a convenient coherence parameter in terms of which a and b can be determined. If we demand that $|\tilde{P}^0\rangle=|P^0\rangle$ then Eq. (4.10) is an eigenvalue equation which implies the recurrence relation

$$f_{n+1}=\frac{x(2n+1)}{\tilde{L}+2n}f_n \ , \tag{4.11}$$

from which we can deduce

$$f_n=x^n\frac{(2n-1)!!(\tilde{L}-2)!!}{(\tilde{L}-2+2n)!!}f_0 \ . \tag{4.12}$$

If we assume $|P^0\rangle$ to be normalized $\langle P^0|P^0\rangle=1$ then $f_0=f_0(x)$ is defined by the series

$$\begin{aligned} f_0^{-2}&=\sum_n\frac{x^{2n}(\tilde{L}-2)!!}{(\tilde{L}-2+2n)!!(2n)!!}\\ &=(\tilde{L}-2)!!2^{\bar{x}}\partial_y^{\bar{x}}\cosh x \end{aligned} \tag{4.13}$$

with $\bar{x}=(\tilde{L}+1)/2$, $\partial_y=\partial/\partial y=(1/2x)\partial/\partial x$ and $y=x^2$. Normalization of the state $|P_M^L\rangle$, $\langle P_M^L|P_M^L\rangle=1$, defines the parameter a in Eq. (4.7) as

$$a=x\left[\frac{2}{\tilde{L}}\frac{\partial_y^{\bar{x}}\cosh x}{\partial_y^{\bar{x}-1}\cosh x}\right]^{1/2} \ . \tag{4.14}$$

We note the following general properties of $|P^0\rangle$ and $|P_M^L\rangle$: Both are eigenstates of the scalar operator $c_L\cdot c_L$ with eigenvalue x. Furthermore, one needs the matrix elements

$$\langle P^0|d\cdot d|P^0\rangle=\langle P^0|d^\dagger\cdot d^\dagger|P^0\rangle=x \tag{4.15}$$

and similarly

$$\langle P_M^1|d\cdot d|P_M^1\rangle=\langle P_M^1|d^\dagger\cdot d^\dagger|P_M^1\rangle=x \ . \tag{4.16}$$

The mean number of quanta is given by the expectation value of the operator $\hat{N}=\sum_M c^\dagger_{LM}c_{LM}$ yielding

$$\langle P^0|\hat{N}|P^0\rangle=a^2\tilde{L},\quad \langle P_M^1|\hat{N}|P_M^1\rangle=c^2\tilde{L} \tag{4.17}$$

with

$$c^2=\frac{1}{\tilde{L}}\left[1+2x^2\frac{\partial_y^{\bar{x}+1}\cosh x}{\partial_y^{\bar{x}}\cosh x}\right] \ . \tag{4.18}$$

The x plays the role of a coherence parameter. According to Eqs. (4.5), (4.8), and (4.12) one sees that for $x\to 0$ the one-boson limit is obtained:

$$|P^0\rangle\underset{x\to 0}{\to}|0\rangle,\quad |P_M^L\rangle\underset{x\to 0}{\to}c^\dagger_{LM}|0\rangle \ . \tag{4.19}$$

It is easy to incorporate the coherent states $|P^0\rangle$ and $|P^1\rangle$, as they are defined in this section, into the ansatz for the Fock state of Eqs. (3.1) and (3.2). One only has to generalize the above formulas for the tensor product of two spaces, spin and isospin. Thus one uses the p-wave pions of Eq. (3.10):

$$(b^\dagger{:}b^\dagger)=\sum_{m,t=-1}^{+1}(-)^{t+m}b^\dagger_{tm}b^\dagger_{-t-m} \tag{4.20}$$

and constructs the states $|P^{00}\rangle$ and $|P^{11}_{tm}\rangle$ from the $b^\dagger_{tm}$. One also has to replace the factors $\tilde{L}=2L+1$ by $\tilde{L}=(2L+1)(2T+1)=9$. This yields then explicitly

$$a=\frac{1}{3}\left[\frac{(105+45x^2+x^4)\sinh x-(105+10x^2)\cosh x}{-(15+6x^2)\sinh x+(15+x^2)\cosh x}\right]^{1/2} \tag{4.21}$$

and

$$c=\frac{1}{3}\left[1+\frac{-(945+420x^2+15x^4)\sinh x+(945+105x^2+x^4)\cosh x}{(157+45x^2+x^4)\sinh x-(105+10x^2)\cosh x}\right]^{1/2} . \tag{4.22}$$

The final trial Fock state for the nucleon is then assumed to be

$$|NT_3J_z\rangle=[\alpha\,|nT_3J_z\rangle\,|P^{00}\rangle+\beta(\,|n\rangle\otimes|P^{11}\rangle)_{T_3J_z}+\gamma(\,|\delta\rangle\otimes|P^{11}\rangle)_{T_3J_z}]\,|\Sigma\rangle \tag{4.23}$$

and analogously for $|\Delta T_3J_z\rangle$. Actually for $x\rightarrow 0$ this ansatz goes into the one-pion approximation as it is written in Eq. (3.1).

V. THE VARIATIONAL PRINCIPLE

The objective of this section is to seek the minimum of the energy of the nucleon

$$E_N=\langle NT_3J_z\,|\int d^3r{:}\hat{\mathcal{H}}(\mathbf{r}){:}\,|NT_3J_z\rangle \tag{5.1}$$

by variation with respect to $u(r)$, $v(r)$, $\sigma(r)$, $\phi(r)$ and α, β, γ subjected to the normalizations. The variation looks for stationary states corresponding to space localized solutions, which are more bound than three free quarks, i.e, whose energy is $E_N<3gf_\pi$. These are the solitonic solutions.

By a lengthy but straightforward calculation the total energy of the system can be evaluated as

$$E_N=4\pi\int dr\,r^2E_N(r) \tag{5.2}$$

with

$$\begin{aligned}E_N(r)={}&\frac{1}{2}\left[\frac{d\sigma}{dr}\right]^2+\frac{\lambda^2}{4}[\sigma^2(r)-\nu^2]^2-m_\pi^2f_\pi\sigma(r)+U_0+3\left[u\left[\frac{dv}{dr}+\frac{2}{r}v\right]-v\frac{du}{dr}+g\sigma(r)[u^2(r)-v^2(r)]\right]\\&+[9\alpha^2(ab+a^2)+9(\beta^2+\gamma^2)(ab+c^2)]\left[\left[\frac{d\phi}{dr}\right]^2+\frac{2}{r^2}\phi^2\right]-[9\alpha^2(ab-a^2)+9(\beta^2+\gamma^2)(ab-c^2)]\phi_p^2(r)\\&+\frac{\lambda^2}{4}\{\tfrac{16}{3}x^2+72x[\alpha^2a^2+(\beta^2+\gamma^2)c^2]\}\phi^4(r)+\frac{4}{\sqrt{3}}\alpha\beta(a+b)g\phi(r)u(r)v(r)\langle n\|(d^\dagger\tilde{d})^{11}\|n\rangle\\&+\frac{4}{\sqrt{3}}\alpha\gamma(a+b)g\phi(r)u(r)v(r)\langle\delta\|(d^\dagger\tilde{d})^{11}\|n\rangle+[9\alpha^2(ab+a^2)+9(\beta^2+\gamma^2)(ab+c^2)]\lambda^2(\sigma^2(r)-\nu^2)\phi^2(r)\end{aligned} \tag{5.3}$$

with

$$(d^\dagger\tilde{d})^{11}_{mt}=-\sum_{\substack{m_1m_2\\t_1t_2}}(\tfrac{1}{2}m_1\tfrac{1}{2}m_2|\tfrac{1}{2}\tfrac{1}{2}1m)(\tfrac{1}{2}t_1\tfrac{1}{2}t_2|\tfrac{1}{2}\tfrac{1}{2}1t)\times(-)^{1/2+m_2}(-)^{1/2+t_2}d^\dagger_{m_1t_1}\tilde{d}_{-m_2-t_2}\ , \tag{5.4}$$

where the $d^\dagger_{mt}$ creates a quark in the lowest s state with spin and isospin projection m and t, respectively, and $\tilde{d}_{-m-t}=(-)^{m+t}d_{mt}$. Using explicit quark wave functions[19] one obtains eventually

$$\begin{aligned}\langle n\|(d^\dagger\tilde{d})^{11}\|n\rangle&=-5\ ,\\\langle n\|(d^\dagger\tilde{d})^{11}\|\delta\rangle&=-4\sqrt{2}\ ,\\\langle\delta\|(d^\dagger\tilde{d})^{11}\|\delta\rangle&=-10\ .\end{aligned} \tag{5.5}$$

For fixed α, β, and γ the functional variations are expressed by

$$\delta\left[\int dr\,r^2\{E_N(r)-3\epsilon[u^2(r)+v^2(r)]-\kappa\phi(r)\phi_p(r)\}\right]=0\ , \tag{5.6}$$

where ϵ and κ are Lagrange multipliers that enforce the normalization conditions (3.12) and (3.15).

The variation of most of the terms is rather straightforward. Only the variation of the term $\phi_p^2(r)$ requires some comments. An easy calculation yields

$$\delta\int dr\,r^2\phi_p^2(r)=2\int dr\,r^2\phi_Q(r)\delta\phi(r) \tag{5.7}$$

with

$$\begin{aligned}\phi_Q(r)&=\int dr'r'^2\omega(r,r')\phi_p(r')\\&=\frac{2}{\pi}\int dk\,k^2(k^2+m_\pi^2)j_1(kr)\tilde{\phi}(k)\ ,\end{aligned} \tag{5.8}$$

where

$$\tilde{\phi}(k)=\int dr\, r^2 j_1(kr)\phi(r) .$$

Because of the structure of $\phi_Q(r)$ one can show that

$$\phi_Q(r)=\left[m_\pi^2-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\frac{2}{r^2}\right]\phi(r) . \tag{5.9}$$

Using this feature the variational equations for the nucleon can be evaluated straightforwardly:

$$\begin{aligned}
\frac{du}{dr}&=-(g\sigma+\epsilon)v-\tfrac{2}{3}g\alpha\delta(a+b)\phi u ,\\
\frac{dv}{dr}&=-\frac{2}{r}v-(g\sigma-\epsilon)u+\tfrac{2}{3}g\alpha\delta(a+b)\phi v ,\\
\frac{d^2\sigma}{dr^2}&=-\frac{2}{r}\frac{d\sigma}{dr}+\lambda^2(\sigma^2-\nu^2)\sigma+2\lambda^2(x+N_\pi)\phi^2\sigma\\
&\quad+3g(u^2-v^2)-m_\pi^2 f_\pi ,\\
\frac{d^2\phi}{dr^2}&=-\frac{2}{r}\frac{d\phi}{dr}+\frac{2}{r^2}\phi+\tfrac{1}{2}\left[1-\frac{x}{N_\pi}\right]m_\pi^2\phi\\
&\quad+\tfrac{1}{2}\left[1+\frac{x}{N_\pi}\right]\lambda^2(\sigma^2-\nu^2)\phi\\
&\quad+\frac{2x}{N_\pi}(\tfrac{2}{3}x+N_\pi)\lambda^2\phi^3-\frac{\alpha\delta(a+b)}{N_\pi}guv-\kappa\phi_p ,
\end{aligned} \tag{5.10}$$

with

$$\delta=\frac{5}{\sqrt{3}}(\beta+\tfrac{4}{5}\sqrt{2}\gamma) \tag{5.11}$$

and the average pion number

$$N_\pi=9\alpha^2a^2+9(\beta^2+\gamma^2)c^2 . \tag{5.12}$$

These are four nonlinear coupled ordinary differential equations with eigenvalues ϵ and κ. They consist of a Dirac equation where σ and ϕ appear as potentials and two Klein-Gordon equations with uv and u^2-v^2 as source terms.

For large r the σ and ϕ fields behave as

$$\sigma(r)=f_\pi+\text{const}\times\frac{e^{-m_\sigma r}}{m_\sigma r} , \tag{5.13}$$

$$\phi(r)=\text{const}\times e^{-m_\pi r}\left[\frac{1}{m_\pi r}+\frac{1}{m_\pi^2 r^2}\right] \tag{5.14}$$

and

$$u(r)=\left[\frac{gf_\pi+\epsilon}{gf_\pi-\epsilon}\right]^{1/2}\left[\frac{\pi}{2\beta r}\right]^{1/2}K_{1/2}(\beta r) , \tag{5.15}$$

$$v(r)=\left[\frac{\pi}{2\beta r}\right]^{1/2}K_{3/2}(\beta r) , \tag{5.16}$$

with $\beta=(g^2f_\pi^2-\epsilon^2)^{1/2}$ and $K_{1/2},K_{3/2}$ being the modified Bessel functions of the third kind.[21]

These equations together with the conditions at $r=0$ yield the boundary conditions for the solution of the equations. At $r=0$ they are

$$v(r)=0,\quad \frac{d\sigma}{dr}=0,\quad \phi(r)=0 , \tag{5.17}$$

and for large values of r they are

$$[r(gf_\pi-\epsilon)^{1/2}+1/(gf_\pi-\epsilon)^{1/2}]u(r) -(gf_\pi+\epsilon)^{1/2}rv(r)=0 , \tag{5.18}$$

$$(2+2m_\pi r+m_\pi^2 r^2)\phi+r(1+m_\pi r)\frac{d\phi}{dr}=0 , \tag{5.19}$$

$$r\frac{d\sigma}{dr}+(\sigma-f_\pi)(1+m_\sigma r)=0 . \tag{5.20}$$

For fixed values of α, β, γ the above differential equations with the corresponding boundary conditions are solved by using the program package COLSYS in the way described in Refs. 6 and 22. The equations have a solution for any value of ϵ and κ. Strictly speaking, we are confronted with a set of nonlinear integro-differential equations because the $\phi_p(r)$ is related by the integral (3.16) to $\phi(r)$. To do this explicitly in the internal iterations of COLSYS would be extremely time consuming. Fortunately, the system allows for an iterative solution: For a given ϵ and κ we start with a reasonably guessed $\phi_p(r)$ and solve the equations keeping it fixed. From the resulting $\phi(r)$ we evaluate a new $\phi_p(r)$ according to Eq. (3.16) and solve the differential equations again. In the actual calculations one has to introduce a relaxation factor, i.e., one works with

$$\tilde{\phi}^{\text{new}}(r)=\rho\phi^{\text{new}}(r)+(1-\rho)\phi^{\text{old}}(r) ,$$

where ρ has to be taken between 0.3 and 0.5 to yield convergence after about maximally 50 iterations. By this a complete solution of Eqs. (5.10) for a given ϵ and κ is obtained. The one with the proper normalizations (3.12) and (3.15) is found then by a systematic variation of ϵ and κ. The corresponding fields $u(r)$, $v(r)$, $\sigma(r)$, and $\phi(r)$ are now "frozen" and used in order to determine the α, β, γ by a diagonalization of the energy matrix:

$$\begin{bmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & H_{\alpha\gamma}\\ H_{\beta\alpha} & H_{\beta\beta} & H_{\beta\gamma}\\ H_{\gamma\alpha} & H_{\gamma\beta} & H_{\gamma\gamma}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=E\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} . \tag{5.21}$$

The matrix elements can be obtained from Eq. (5.3) and are given, e.g., as $H_{\alpha\beta}=4\pi\int dr\, r^2 E_{\alpha\beta}(r)$ with

$$E_{\alpha\alpha}(r)=E_0(r)+9(ab+a^2)\left[\left(\frac{d\phi}{dr}\right)^2+\frac{2}{r^2}\phi^2(r)\right]$$

$$-9(ab-a^2)\phi_p^2(r)+38\lambda^2a^2\phi^4(r)$$

$$+9(ab+a^2)\lambda^2[\sigma^2(r)-v^2]\phi^2(r)\ ,$$

$$E_{\beta\beta}(r)=E_0(r)+9(ab+c^2)\left[\left(\frac{d\phi}{dr}\right)^2+\frac{2}{r^2}\phi^2(r)\right]$$

$$-9(ab-c^2)\phi_p^2(r)+18\lambda^2c^2\phi^4(r)$$

$$+9(ab+c^2)\lambda^2[\sigma^2(r)-v^2]\phi^2(r)\ ,$$

$$E_{\alpha\beta}(r)=\frac{4}{\sqrt{3}}(a+b)g\phi(r)u(r)v(r)\langle n\|(a^\dagger\tilde{a})^{11}\|n\rangle\ , \quad (5.22)$$

$$E_{\alpha\gamma}(r)=\frac{4}{\sqrt{3}}(a+b)g\phi(r)u(r)v(r)\langle \delta\|(a^\dagger\tilde{a})^{11}\|n\rangle\ ,$$

$$E_{\gamma\gamma}(r)=E_{\beta\beta}(r),\quad E_{\beta\gamma}(r)=0\ ,$$

and

$$E_0(r)=\frac{1}{2}\left(\frac{d\sigma}{dr}\right)^2+\frac{\lambda^2}{4}[\sigma^2(r)-v^2]^2-m_\pi^2f_\pi\sigma(r)$$

$$+U_0+\tfrac{4}{3}\lambda^2x^2\phi^4(r)+3\epsilon\ . \quad (5.23)$$

Since the fields u, v, σ, and ϕ depend on the initial choice of α, β, γ we have to solve the problem iteratively. In practice, five $\alpha\beta\gamma$ iterations are sufficient, in a few cases about ten.

The equations in this section describe the proton and neutron. Since the formalism is based on Racah algebra, there are only minimal changes in the equations in order to describe the Δ_{33} resonances in terms of the Fock-state equation (3.2). One simply has to replace

$$\langle n\|(a^\dagger\tilde{a})^{11}\|n\rangle=-5\rightarrow-5\ ,$$

$$\langle \delta\|(a^\dagger\tilde{a})^{11}\|n\rangle=-4\sqrt{2}\rightarrow-2\sqrt{2}\ .$$

If one does this replacement simultaneously in the equations of motion and in the expression for the total energy, one does, in the terminology of nuclear physics, a "projection before the variation." If one solves the equations of motion for a nucleon (delta) and calculates with these fields the energy of a delta (nucleon) by replacing the doubly reduced matrix elements, then one has a "projection after the variation."

VI. NUCLEON PROPERTIES

The Fock states calculated in the previous sections can be used to evaluate various baryon properties. Electromagnetic properties, such as charge radii and magnetic moments, are calculated by taking the appropriate moments of matrix elements of the electromagnetic current

$$\hat{J}^\mu_{\rm em}(x)=\hat{\bar\psi}\gamma^\mu q_f\hat\psi-i(\hat\pi_{+1}\partial^\mu\hat\pi_{-1}-\hat\pi_{-1}\partial^\mu\hat\pi_{+1}) \quad (6.1)$$

with

$$q_f=\frac{2}{3}\frac{1+\tau_3}{2}+\left(-\frac{1}{3}\right)\frac{1-\tau_3}{2}\ . \quad (6.2)$$

For $\mu=0$ this yields the charge distribution of the proton

$$\frac{\rho_p(r)}{4\pi e}=\alpha^2(u^2+v^2)+\beta^2[\tfrac{1}{3}(u^2+v^2)+\tfrac{4}{3}\phi\phi_p]$$

$$+\gamma^2[\tfrac{4}{3}(u^2+v^2)-\tfrac{2}{3}\phi\phi_p] \quad (6.3)$$

and, for the neutron,

$$\frac{\rho_n(r)}{4\pi e}=\beta^2[\tfrac{2}{3}(u^2+v^2)-\tfrac{4}{3}\phi\phi_p]$$

$$+\gamma^2[-\tfrac{1}{3}(u^2+v^2)+\tfrac{2}{3}\phi\phi_p]\ . \quad (6.4)$$

In a similar way the operator for the magnetic moment is given by

$$\hat\mu(\mathbf{r})=\tfrac{1}{2}[\mathbf{r}\times\hat{\mathbf{J}}_{\rm em}(\mathbf{r})]\ . \quad (6.5)$$

The matrix element of its z component between the Fock states $|N,J_z=\frac{1}{2},T_3=\frac{1}{2}\rangle$ and $|N,J_z=\frac{1}{2},T_3=-\frac{1}{2}\rangle$ yields the magnetic moment of the proton and the neutron. Explicitly their distribution reads

$$\frac{\mu_p(r)}{4\pi e}=\frac{ruv}{81}(54\alpha^2+8\beta^2+40\gamma^2+32\sqrt{2}\beta\gamma)$$

$$+\frac{2c^2+1}{11}(4\beta^2+\gamma^2)\phi^2\ , \quad (6.6)$$

$$\frac{\mu_n(r)}{4\pi e}=\frac{ruv}{81}(-36\alpha^2-2\beta^2-20\gamma^2-32\sqrt{2}\beta\gamma)$$

$$-\frac{2c^2+1}{11}(4\beta^2+\gamma^2)\phi^2\ . \quad (6.7)$$

The axial-vector coupling constant, measured in neutron β decay, is a matrix element of the space part of the isovector–axial-vector current (2.9). Explicitly one is interested in g_A/g_V, where g_V is the corresponding matrix element of the isovector-vector current

$$\hat{J}^V_\mu=\hat{\bar\psi}\tfrac{1}{2}\gamma^\mu\vec\tau\hat\psi+\hat{\vec\pi}\times\partial^\mu\hat{\vec\pi}\ . \quad (6.8)$$

The actual numbers of g_A and g_V are taken from the z component in space and the third (or zeroth) component in isospace. Since the vector part yields just $\frac{1}{2}$, one obtains altogether

$$\frac{g_A}{g_V}=2\langle NJ_z=\tfrac{1}{2}T_3=\tfrac{1}{2}\,|:\int(\tfrac{1}{2}\hat{\bar\psi}\gamma^5\tau_0\hat\psi+\hat\sigma\partial^z\hat\pi^0-\hat\pi^0\partial^z\hat\sigma)d^3r:|\,N,J_z=\tfrac{1}{2},T_3=\tfrac{1}{2}\rangle\ , \tag{6.9}$$

yielding

$$\frac{g_A}{g_V}=4\pi\int dr\,r^2\left[\left[\frac{5}{2}\alpha^2+\frac{5}{27}\beta^2+\frac{25}{27}\gamma^2+\frac{32\sqrt{2}}{27}\beta\gamma\right](u^2-\tfrac{1}{3}v^2)+\frac{8}{3\sqrt{3}}\alpha\beta(a+b)\frac{d\sigma}{dr}\phi\right]\ . \tag{6.10}$$

The πNN coupling constant can be evaluated in two different ways, as a matrix element of the pion field, $\hat{\vec\pi}(\mathbf{r})$, or of the pion source current

$$\hat{J}_\pi=(\partial^\mu\partial_\mu+m_\pi^2)\vec\pi=-g\hat{\bar\psi}i\gamma_5\vec\tau\hat\psi-\lambda^2(\hat\sigma^2+\hat{\vec\pi}^2-f_\pi^2)\hat{\vec\pi}\ , \tag{6.11}$$

yielding

$$\frac{g_{\pi NN}}{2M_N}=\langle N,J_z=\tfrac{1}{2},T_3=-\tfrac{1}{2}\,|:\int d^3r\,z\hat{J}_\pi^0(\mathbf{r}):|\,N,J_z=\tfrac{1}{2},T_3=-\tfrac{1}{2}\rangle \tag{6.12}$$

or

$$\frac{g_{\pi NN}}{2M_N}=m_\pi^2\langle N,J_z=\tfrac{1}{2},T_3=-\tfrac{1}{2}\,|:\int d^3r\,z\hat\pi^0(\mathbf{r}):|\,N,J_z=\tfrac{1}{2},T_3=-\tfrac{1}{2}\rangle\ . \tag{6.13}$$

Explicitly these expressions read as

$$\frac{g_{\pi NN}}{2M_N}=4\pi\int dr\,r^3\left[\frac{2}{3\sqrt{3}}\alpha\beta(a+b)\lambda^2[f_\pi^2-\sigma^2(r)]\phi^2(r)-x\lambda^2\frac{4\alpha\beta}{3\sqrt{3}}\left[2a+\frac{b^2+c^2}{b}\right]\phi^3(r)\right.$$
$$\left.+gu(r)v(r)(90\alpha^2+10\beta^2+50\gamma^2+64\sqrt{2}\beta\gamma)/81\right] \tag{6.14}$$

from Eq. (6.12), and as

$$\frac{g_{\pi NN}}{2M_N}=m_\pi^2 4\pi\frac{2}{3\sqrt{3}}\alpha\beta(a+b)\int dr\,r^3\phi(r) \tag{6.15}$$

from Eq. (6.13).

VII. NUMERICAL RESULTS

The present model contains two free parameters: the coupling constant g (or the asymptotic quark mass $m_q=gf_\pi$) and the σ mass m_σ. The g is varied between $4.0\le g\le 6.5$ and the m_σ in the range 0.3 GeV $\le m_\sigma\le 2.1$ GeV. In this range there are reasonable values for the energies and the observables and in all cases solitonic solutions of the equations of motion exist.

A. The soliton

The successive iterations between COLSYS and the diagonalizing of the Hamiltonian between frozen fields gives us a solution of the model for a chosen value of the coherence parameter x. The total energy should show a minimum with respect to x which defines the final solution. This is indeed the case as the curve in Fig. 1

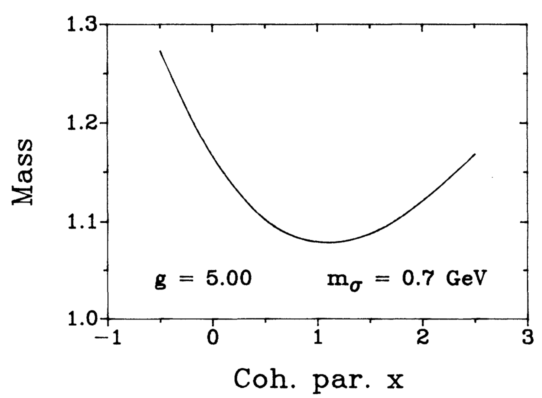

FIG. 1. The mass of the solitonic nucleon solution vs the coherence parameter x. The evaluation is performed for $g=5.00$ and $m_\sigma=0.7$ GeV. The unit of the mass is GeV.

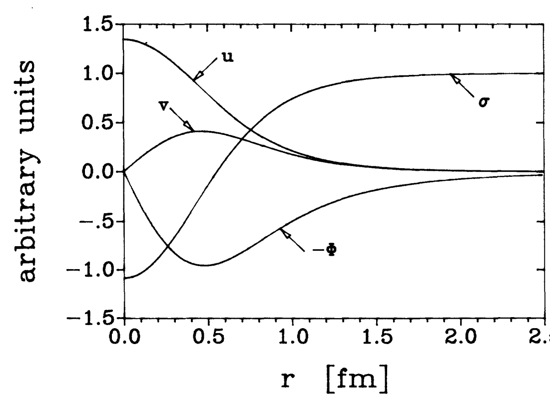

FIG. 2. The quark fields $u(r)$, $v(r)$, the σ field, $\sigma(r)$, and the pion field, $\phi(r)$, are plotted vs the radial coordinate r. The calculations are performed for $g=5.0$, $m_\sigma=0.7$.

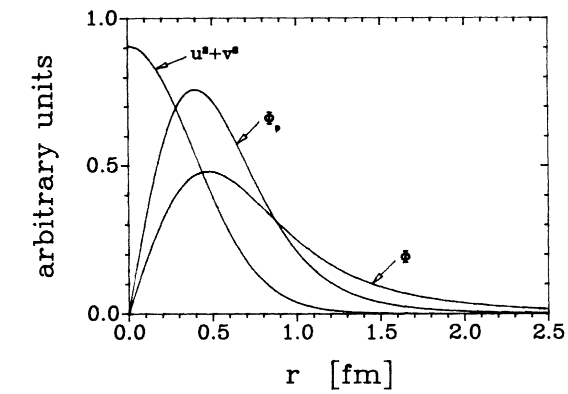

FIG. 3. The pion field, $\phi(r)$, and the corresponding momentum field, $\phi_p(r)$, are plotted together with the quark density, $u^2(r)+v^2(r)$, vs the radial coordinate r. The calculations are performed for $g=5.0$ and $m_\sigma=0.7$.

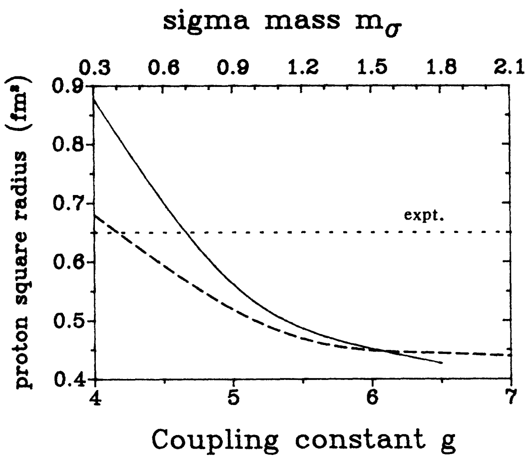

FIG. 5. The proton square charge radius is plotted vs the σ mass ($g=5.37$, solid line) and vs the coupling constant ($m_\sigma=0.7$ GeV, dashed line). The experimental value is given by a dotted line.

shows. For a typical soliton the corresponding fields are presented in Fig. 2. For $r\geq 2$ fm all fields basically show their vacuum values and the total soliton has an extension of about $r=0.6$–0.7 fm. The wall of the σ field is not very steep in contrast with many results of the Friedberg-Lee model.[6] The pion fields $\phi(r)$ and $\phi_p(r)$ are displayed together with the quark density in Fig. 3. They are nonzero everywhere with negligible values for $r\geq 2$ fm. The $\phi_p(r)$ is altogether steeper than $\phi(r)$, which requires some care in the numerical evaluation.

B. Trends

There are some simple trends of the energies and expectation values with regard to the variation of g and m_σ (see Fig. 4). For fixed m_σ the mass of the system decreases with increasing g since the coupling between quarks and mesons becomes stronger. Actually it can reach even negative values for larger g indicating a clear limitation of this model. The inclusion of sea quarks would prevent this instability. The radius of the soliton shrinks with increasing g since the pion pressure grows; this is shown in Fig. 5. A similar effect takes place with the magnetic moments and the axial-vector constant of Figs. 6 and 7. Actually, with increasing g the relative contribution of the mesons to the various expectation values grows as seen from the average pion number in Fig. 8. However, this meson part is relatively small and overpowered by the decreasing quark contribution such that the overall effect follows the trend of the quarks. The increased coupling of the quarks to the mesons results also in a decreasing eigenvalue ϵ of the quarks, see Fig. 9, although the square radius of the quark distribution is reduced.

There are also some simple trends of the above quanti-

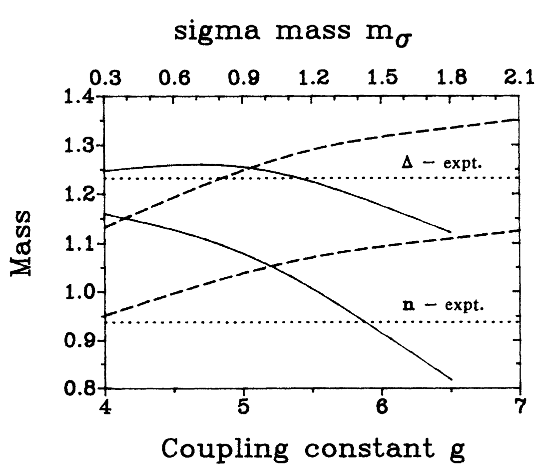

FIG. 4. The masses of the nucleon and delta for $m_\sigma=0.7$ GeV are plotted vs the coupling constant g (solid lines) and for $g=5.37$ vs the σ mass m_σ (dashed lines). The corresponding experimental numbers are indicated by dotted lines.

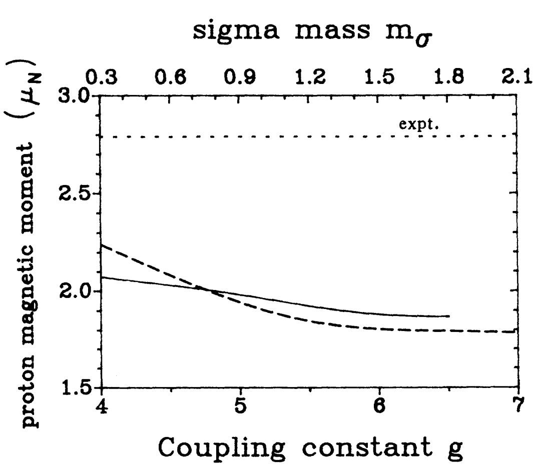

FIG. 6. The magnetic moment of the proton is plotted vs the coupling constant g ($m_\sigma=0.7$ GeV, solid line) and vs the σ mass m_σ ($g=5.0$, dashed line). The magnetic moments are given in units of the nuclear magneton.

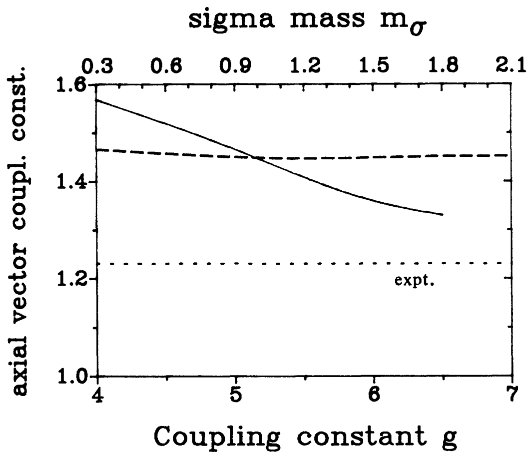

FIG. 7. The axial-vector coupling constant g_A/g_V is plotted vs the coupling constant g ($m_\sigma = 0.7$ GeV, solid line) and vs the sigma mass m_σ ($g = 5.0$, dashed line).

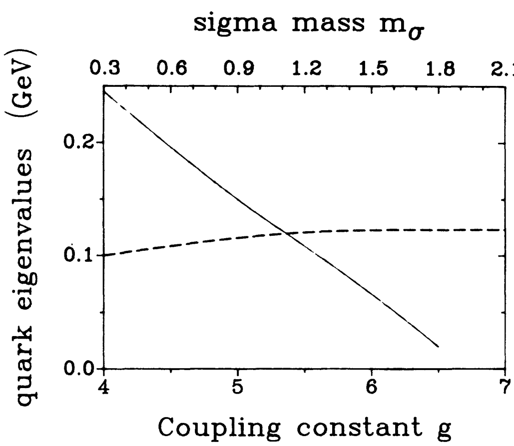

FIG. 9. The eigenvalue ϵ of the quarks is plotted vs the coupling constant g ($m_\sigma = 0.7$ GeV, solid line) and the sigma mass m_σ ($g = 5.0$, dashed line).

ties for varying σ mass but fixed g, as can be seen from the dashed curves in Figs. 4–9. Basically, all quantities level off for increasing m_σ. The reason is very clear. For $m_\sigma \to \infty$ the self-interaction becomes so repulsive that the equilibrium of the system lies on the chiral circle $\sigma^2 + \vec{\pi}^2 - f_\pi^2 = 0$.

C. Physical results

The g and m_σ have been varied in order to obtain rough agreement of the theoretical expectation values of observables with the experimental data. Because of the neglect of the sea quarks and of center-of-mass corrections the results are not expected to be accurate to more than 20–30%; thus, the fitting of g and m_σ to the data was not done very carefully. For a few typical results we present the numbers in Table I. They show an overall reasonable agreement with the experimental data. The distributions of the magnetic moments and of the charge are given as examples in Figs. 10 and 11. The delta-nucleon splitting appears somewhat too small; however, it is not clear whether this is an advantage or a disadvantage of the model. There are certainly contributions of one-gluon-exchange terms to this splitting, which are neglected presently. The axial-vector coupling constant is in good agreement with experiment. This is actually an important property, in particular, for a chiral model as the present one. The magnetic moments are too small; however, their ratio is very good. The square radii of the neutrons are also too small. Both these facts indicate that the pion component of the total Fock state is probably not large enough, a feature which will be discussed later in more detail. It should be mentioned that the above results are obtained using the experimental values for $f_\pi = 0.093$ GeV and $m_\pi = 0.138$ GeV. There is some discrepancy in the pion-nucleon coupling constant $g_{\pi NN}$. If one uses the Goldberger-Treiman relation and evaluates the $g_{\pi NN}$ as

$$g_{\pi NN}^{(\mathrm{GT})} = \frac{M_N g_A/g_V}{f_\pi} ,$$

then one obtains rather good agreement with experiment. On the other hand, the evaluation of Eq. (6.12) yields values which are by 40% too large, and of Eq. (6.13) which are by 60% too small. As Birse pointed out, the difference between Eq. (6.12) and Eq. (6.13) and the violation of the Goldberger-Treiman relation corresponds to a virial theorem of the kind discussed in Ref. 23. These virial theorems are time derivatives of expectation values of various operators. Differences should vanish for an exact eigenstate of the Hamiltonian. The extent to which an approximation satisfies these virial theorems provides a test of the approximations used. The above discrepancy, which occurs also to some extent in other models,[8,12] shows that the present calculations need to be improved, probably by adding components with more impaired pions. In the framework of the pro-

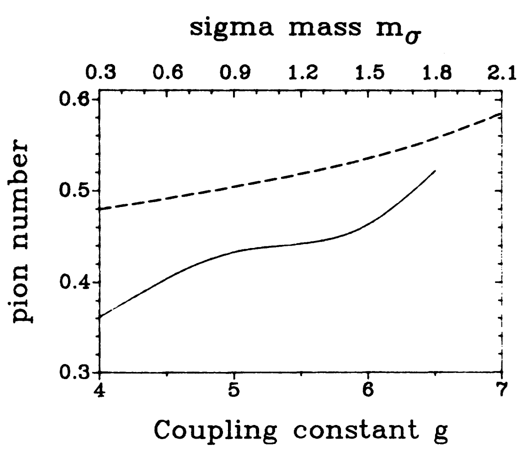

FIG. 8. The average pion number N_π is plotted vs the sigma mass ($g = 5.37$, dashed line) and vs the coupling constant g ($m_\sigma = 0.7$, solid line).

TABLE I. Results of the chiral model for the nucleon. For three sets of the coupling constant g and the σ mass m_σ the results of the present coherent-pair-approximation (CPA) model are compared with experimental numbers. Given are the nucleon mass E_N, the delta mass E_Δ, the ratio of the axial-vector coupling constant g_A, to the vector coupling constant g_V, the pion-nucleon coupling constant resulting from g_A and the Goldberger-Treiman relation, $g_{\pi NN}^{(GT)}$, the $g_{\pi NN}$ from Eq. (6.14), the magnetic moment of the proton and the neutron, μ_p and μ_n, respectively, their ratio (μ_p/μ_n), the square radius of the proton $\langle r^2\rangle_p$, and the square radius the neutron $\langle r^2\rangle_n$. Please note that the calculations are performed with $f_\pi=0.093$ GeV and $m_\pi=0.138$ GeV. The numbers are compared with the outcome of projection calculations of Fiolhais *et al.* (Ref. 24).

	CPA $g=6.00$ $m_\sigma=0.7$ GeV	CPA $g=6.11$ $m_\sigma=1.2$ GeV	CPA $g=5.37$ $m_\sigma=1.2$ GeV	Projected hedgehog $g=5.37$ $m_\sigma=1.2$ GeV	Experiment
E_N (GeV)	0.915	0.938	1.071	0.871	0.938
E_Δ (GeV)	1.176	1.215	1.291	1.023	1.232
$E_\Delta-E_N$ (GeV)	0.261	0.276	0.219	0.152	0.294
g_A/g_V	1.36	1.45	1.45	1.78	1.23
$g_{\pi NN}^{(GT)}$	13.9	14.6	16.69	17.95	12.4
$g_{\pi NN}$	19.2	22.5	22.7	16.94	13.6
μ_p (μ_N)	1.88	1.85	1.85	2.57	2.79
μ_n (μ_N)	−1.27	−1.25	−1.25	−2.23	−1.91
$\lvert\mu_p/\mu_n\rvert$	1.49	1.48	1.48	1.15	1.46
$\langle r^2\rangle_p$ (fm^2)	0.45	0.42	0.47	0.53	0.65
$\langle r^2\rangle_n$ (fm^2)	−0.01	−0.01	−0.01	−0.08	−0.12

jection theory it has been shown recently[24] that with a generalization of the hedgehog ansatz the Goldberger-Treiman relation and the pion virial theorem are satisfied.

The relative contribution of the mesons and the quarks to the above expectation values can be obtained from Table II. Generally the mesons contribute rather little to the final values. Because of strong coupling and the nonlinearity of the problem this does not mean that they can be ignored from the start; however, their final direct contribution is small. The only exception is the axial-vector constant for which the value comes 30% from the mesons and 70% from the quarks.

Altogether in the present model one has about $N_\pi=0.5$ pions per nucleon. This agrees with the estimates of Thomas[25] who has used the results of deep-inelastic neutrino and antineutrino scattering to obtain an upper bound of 0.5 pions per nucleon, although the uncertainties are rather large.

VIII. COMPARISON NUCLEON AND DELTA

For simplicity in Table I the delta isobar was not calculated self-consistently, but the same fields $u(r)$, $v(r)$, etc., of the nucleon were used in both angular-momentum and isospin coupling schemes. However, for $g=5.0$ and $m_\sigma=0.7$ GeV we performed an accurate comparison whose qualitative features are supported by test calculations with various parameter sets. The results are presented in Table III. In summary it turns out that the nucleon-delta splitting gets a bit reduced by the self-consistent treatment of the delta and that the quark

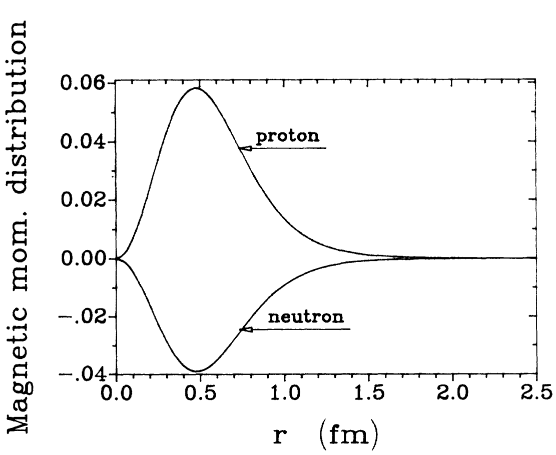

FIG. 10. The charge distributions for the proton and neutron are plotted vs the radial coordinate r.

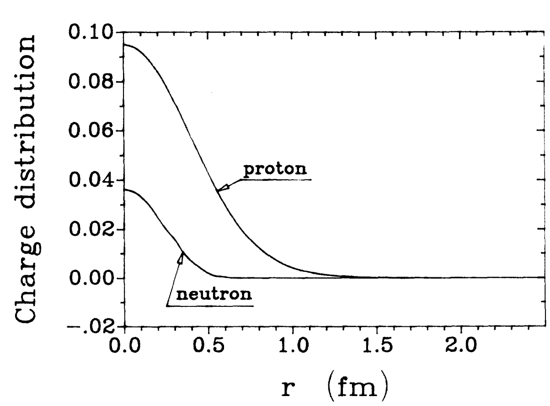

FIG. 11. The distributions of the magnetic moment for the proton and neutron are plotted vs the radial coordinate r.

TABLE II. Contributions of quarks and mesons. For the solution with $g=6.00$ and $m_\sigma=0.7$ GeV the contributions of the quarks and of the mesons to the observable quantities are given. Considered are the axial-vector coupling constant, the vector part of the magnetic moment, the quadratic radius of the charge distribution of the proton and of the neutron. The CPA results are compared with the outcome of projection calculations of Fiolhais *et al.* (Ref. 24).

	CPA $g=6.00$ $m_\sigma=0.7$ GeV		Projected hedgehog $g=5.37$ $m_\sigma=1.2$ GeV	
	Quarks	Mesons	Quarks	Mesons
g_A/g_V	0.943	0.416	0.98	0.80
μ_p (μ_N)	1.415	0.158	1.34	1.23
$\langle r^2\rangle_p$ (fm^2)	0.430	0.021	0.36	0.17
$\langle r^2\rangle_n$ (fm^2)	0.017	−0.021	0.09	−0.17

distribution of the delta shows a larger extension than that of the nucleon. In the present model the latter is an effect of about 15% for the squared quark radius $\langle r^2\rangle_q$. Presently the extraction of form factors of the nucleon and delta from experiment are not accurate enough to see if this result is or is not correct.

IX. COMPARISON WITH OTHER MODELS

It is interesting to compare the nucleon Fock state in the present approach with those used in other models. Here we consider four models which also include pionic degrees of freedom: the cloudy-bag model,[26] the mean-field model of Birse and Banerjee,[8] the projected chiral-soliton model,[12,24] and the Skyrme model.[14]

In the cloudy-bag model[26] the hadrons are described in terms of massless quarks confined inside a bag. The quarks are coupled to the pion field at the surface of the bag in order to maintain axial-vector-current conservation. The pion field is described by a nonlinear σ model whose linearized version is actually treated perturbatively provided the bag radii are larger than 0.8 fm. Actually there are striking similarities in the results between the cloudy-bag model and the present one. Both are in good agreement with experiment and both give quark wave functions with rms radii of 0.7 fm. If pions are treated to first order one obtains an average number of 0.5 pions per nucleon for bag radii of 0.8 fm. This is comparable to our results, although our coherent pion state contains multipion components.

TABLE III. Comparison of nucleon and delta. Listed are the mass, the mass splitting, the squared quark radius, the squared charge radius, the mixing coefficients α, β, γ and the quark eigenvalue. The Δ^+ (nsc) indicates the non-self-consistent calculation of the delta and the Δ^+ (sc) the self-consistent one, where the minimization of the energy is performed for the delta rather than for the nucleon.

	Proton	Δ^+ (nsc)	Δ^+ (sc)
M (GeV)	1.08	1.26	1.22
$M_\Delta - M_N$ (GeV)		0.18	0.14
$\langle r^2\rangle_q$	0.57	0.57	0.66
$\langle r^2\rangle_c$	0.58	0.59	0.67
α	0.82	0.82	0.84
β	0.38	0.43	0.47
γ	0.43	0.38	0.26
ϵ (GeV)	0.14	0.14	0.21

Birse and Banerjee[8] solved the linear chiral σ model in the mean-field approximation using the hedgehog ansatz for the quarks and for the pion field. After the variation they performed an approximate projection on angular momentum and isospin ignoring in this procedure the contribution of the pions. Birse[12] and Golli and Rosina[12] have evaluated this model further, performing proper projections even before the variation in the hedgehog approximation. Fiolhais *et al.*[24] generalized the hedgehog and performed spin and isospin projections as well. Since these authors used exactly the same Lagrangian used in this paper, one can directly compare their numbers[24] with the outcome of the present approach (see Table I). For $g=5.37$ and $m_\sigma=1.2$ GeV the hedgehog mean-field energy is $E_{\rm intr}=1.119$ GeV, projection after the variation yields for the nucleon $E_N^{\rm PAV}=0.924$ and the projection before variation $E_N^{\rm PBV}=0.871$ GeV. The present approach only allows, in this terminology, projection before the variation and yields $E_N=1.071$ GeV, which is about 200 MeV less bound than the corresponding $E_N^{\rm PBV}$. The reason for this discrepancy probably lies in a different treatment of the pion cloud, since in our model we have considered only one unpaired pion. As has been shown[27] recently there are noticeable contributions from states having two, three, and higher numbers of unpaired pions. Since those contributions are ignored in the present approximation, the neutron squared radii are very much smaller than in the projection formalism. The ratio of the magnetic moments is better in our model, on the other hand, since the pions contribute only negligibly (see Table II) which causes the absolute value of μ_p to be too small.

Another approach, which has received much attention recently, is the Skyrme model.[14] In contrast with the approaches discussed so far the Skyrme model does not include explicit quark degrees of freedom but only effective fields having the quantum numbers of a pion and a σ meson. The stable solitons of the model with topological winding-number 1 are identified with baryons. Those Skyrmions are subjected to some sort of semiclassical quantization procedure to extract observable quantities. If one compares the results of Adkins *et al.*[14] with those of Jackson *et al.*[28] one realizes that the Skyrmion models suffer under the dilemma either to reproduce g_A and f_π and to fail in the nucleon and delta masses, or to reproduce the masses and to fail in g_A and f_π. The nontopological models as, e.g., the present one seem not to have this problem although a clear conclusion cannot be drawn since the comparison is not based on identical fits.

X. SUMMARY AND CONCLUSIONS

The linear chiral-soliton model, involving quark fields and elementary pion and σ fields, has been solved in order to obtain a description of static nucleon and delta properties. To this end Fock states with good spin and isospin properties were constructed whose fermion part consists of three quarks in a spherical s orbit with $SU(2)\times SU(2)$ nucleon and delta wave functions. The boson part of the Fock state consists of a spherically symmetric scalar coherent state for the σ field and coherent pair states of pions with definite spin and isospin quantum numbers. Using these Fock states, ignoring sea-quark effects and center-of-mass corrections, the stationary state of the system was found variationally; the solution involved solving four nonlinear coupled differential equations with two Lagrange multipliers and an associated diagonalization procedure.

The resulting nucleon-delta splitting is about half the observed value suggesting the need for residual, spin-dependent gluonic interactions. The calculated nucleon properties appear in reasonable agreement with experiment. This is true in particular for the axial-vector coupling constant, to whose value the mesons contribute by more than 30%. The other properties of the nucleon are to 10% affected by the mesons. This seems to be too small as the neutron charge radius indicates. Altogether, the approach suggests about half a pion per nucleon, a number being in agreement with first-order perturbation theory of the cloudy-bag Lagrangian. However, the problems in satisfying the virial theorems and the Goldberger-Treiman relation are an indication that a better description of the pion cloud is required before a comparison with experiment can be considered successful. In the present approach this entails the inclusion of a larger number of unpaired pions. The techniques spelled out in this paper are directly applicable for this extension.

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