Splitting of the P states in heavy quarkonium systems and the nature of the spin-dependent potential

Claudio O. Dib and Frederick J. Gilman
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

Paula J. Franzini
Theory Division, CERN, CH-1211 Geneva 23, Switzerland
(Received 14 September 1987)

We consider the behavior of the splittings between the $L=1$ states in heavy quarkonium systems. With standard assumptions, the ratio of mass splittings, $R_\chi=(M_2-M_1)/(M_1-M_0)$, is shown to grow slowly as we go from the lowest P states to their radial excitations, a behavior which is in accord with other theoretical calculations and is consistent with experiment. This behavior is the opposite of what is expected from a naive argument which we present. We show why the naive argument fails and how the phenomenological division of the potential into components which correspond to the exchange of a Lorentz scalar and vector can be made so as to obtain other behaviors.

The bound states of heavy quarks and antiquarks have proven to be a key element in our understanding of strong interactions by giving us a subset of all hadrons whose spectroscopy we can calculate quantitatively. The insight thereby gained can then also be used to understand at least semiquantitatively hadronic systems involving light quarks and, if only by the process of elimination, to highlight those states which cannot be interpreted as quark-antiquark bound states and which then must have their origin in more exotic quark and/or gluon combinations.[1] We consider here a limited but interesting aspect of the spectroscopy of heavy quark-antiquark systems, related to the character of the spin-dependent forces. We examine the behavior of the level splittings between the $L=1$ states as we go from the lowest P states to their radial excitations.

The general form for the spin-dependent potential in a system composed of a heavy quark and a heavy antiquark has been shown to be[2]

$$V_{\rm SD}(r)=\left[\frac{\mathbf{S}_1\cdot\mathbf{L}}{2m_1^2}+\frac{\mathbf{S}_2\cdot\mathbf{L}}{2m_2^2}\right]\left[\frac{-dV(r)}{r\,dr}+2\frac{dV_2(r)}{r\,dr}\right]+\frac{(\mathbf{S}_1+\mathbf{S}_2)\cdot\mathbf{L}}{m_1m_2}\frac{dV_2(r)}{r\,dr}+\frac{1}{6m_1m_2}(6\mathbf{S}_1\cdot\hat{\mathbf{r}}\mathbf{S}_2\cdot\hat{\mathbf{r}}-2\mathbf{S}_1\cdot\mathbf{S}_2)V_3(r)$$
$$+\frac{2}{3m_1m_2}\mathbf{S}_1\cdot\mathbf{S}_2V_4(r)\ , \tag{1}$$

where $V(r)$ is the usual spin-independent potential and V_2, V_3, and V_4 spin-dependent ones. These extra terms originate in expectation values of color-electric and -magnetic fields which are different than those that enter in the spin-independent potential and in principle are new objects which are not simply related to $V(r)$.

Nevertheless, it provides some physical insight to consider the nonrelativistic reduction of the four-fermion interaction arising from the exchange of vector and scalar fields between quark and antiquark. In momentum space this is represented by an interaction:

$$L_{\rm int}=\bar{s}(q^2)\bar{u}u\bar{v}v+\bar{v}(q^2)\bar{u}\gamma_\mu u\bar{v}\gamma^\mu v\ . \tag{2}$$

If we do an expansion in powers of v^2/c^2, the static limit is the spin-independent potential $v(r)+s(r)$, while the spin-dependent terms give the Breit-Fermi potential, which in configuration space for the equal-mass case is

$$V_{\rm SD}(r)=\frac{\mathbf{S}\cdot\mathbf{L}}{2m^2}\left[-\frac{dv(r)+ds(r)}{r\,dr}+4\frac{dv(r)}{r\,dr}\right]+\frac{1}{12m^2}(6\mathbf{S}\cdot\hat{\mathbf{r}}\mathbf{S}\cdot\hat{\mathbf{r}}-2\mathbf{S}\cdot\mathbf{S})\left[\frac{dv(r)}{r\,dr}-\frac{d^2v(r)}{d^2r}\right]+\frac{1}{6m^2}(2\mathbf{S}\cdot\mathbf{S}-3)\nabla^2v(r)\ .$$

The first term, involving $-[dv(r)+ds(r)]/r\,dr$, is due to the familiar Thomas precession, and it is followed by usual spin-orbit, tensor, and spin-spin interactions, each with a coefficient related to $v(r)$ or $s(r)$. We can compare this to

the more general spin-dependent potential in Eq. (1) in the equal-mass case, rewriting it a bit in the form

$$V_{\rm SD}(r)=\frac{\mathbf{S}\cdot\mathbf{L}}{2m^2}\left[\frac{-dV(r)}{r\,dr}+4\frac{dV_2(r)}{r\,dr}\right]+\frac{1}{12m^2}(6\mathbf{S}\cdot\hat{\mathbf{r}}\mathbf{S}\cdot\hat{\mathbf{r}}-2\mathbf{S}\cdot\mathbf{S})V_3(r)+\frac{1}{6m^2}(2\mathbf{S}\cdot\mathbf{S}-3)V_4(r)\ . \tag{4}$$

Comparing what is in Eq. (3) to the generic decomposition in Eq. (4) involving V_2, V_3, and V_4 we see first that the spin-independent potential V is given by the sum of the vector and scalar potentials, $v+s$. Second, the spin-dependent potentials V_2, V_3, and V_4 are all expressible in terms of derivatives of only the vector part of the potential, v. Hence, if v is related to gluon exchange and its associated $1/r$ behavior, then the potentials V_2, V_3, and V_4 are all short range in character.

This encourages the following standard division: the scalar term is long range and associated with quark confinement, while the vector term is short range and associated with one-gluon exchange. The associated physical picture[3] has confinement due to a color flux tube that connects the quark and antiquark; as they rotate around each other the flux tube rotates along with them. Consequently there are no spin-dependent forces generated from this part of the potential, aside from the Thomas term which comes in with a minus sign and is generated from the spin rotation associated with Lorentz transforming from the center-of-mass to the quark or antiquark rest frame. It is from the Coulomb-type piece that one obtains the spin-dependent terms that we are long accustomed to in atomic physics: a spin-orbit interaction (minus the piece due to Thomas precession), a tensor interaction, and a spin-spin interaction. This description has been very successful with respect to predicting and interpreting the data on heavy quarkonium systems and it is within its context that we shall work in this paper.

Let us focus on the splittings of the ^3P_J states, i.e., the quark-antiquark states with $L=1$ and $J=0$, 1, or 2. The spin-orbit and tensor terms cause this splitting, and within the picture of vector and scalar exchanges their contributions to the ^3P_J state masses are[4]

$$M(^3P_2)=\overline{M}+a-2b/5\ , \tag{5a}$$

$$M(^3P_1)=\overline{M}-a+2b\ , \tag{5b}$$

$$M(^3P_0)=\overline{M}-2a-4b\ , \tag{5c}$$

where the matrix elements a and b are defined as

$$a=\frac{1}{2m^2}\left\langle-\frac{ds}{r\,dr}+3\frac{dv}{r\,dr}\right\rangle\ , \tag{6a}$$

$$b=\frac{1}{12m^2}\left\langle\frac{dv}{r\,dr}-\frac{d^2v}{dr^2}\right\rangle\ . \tag{6b}$$

We can summarize the relative values of the matrix elements in terms of one number by forming the ratio

$$R_\chi=\frac{M(^3P_2)-M(^3P_1)}{M(^3P_1)-M(^3P_0)}=\frac{2a-\frac{12}{5}b}{a+6b}\ . \tag{7}$$

If only the Coulomb-type vector part of the potential $v(r)$ is present, $R_\chi=0.8$. As the strength of the scalar term $s(r)$ is increased, there is more cancellation between the two terms on the right-hand side of Eq. (6a) and the matrix element a decreases, as then does R_χ.

This brings us to the question at issue in this paper. How should the ratio R_χ behave as we go from one set of P states to another? A naive argument goes as follows. The $2P$ states are characterized by an average radius which is larger than that of the $1P$ states (and similarly for the $3P$ compared to the $2P$, etc.). Therefore these states should "live" more in the confining, Lorentz-scalar part of the potential and the value of the ratio R_χ should be smaller for the $2P$ (sometimes called χ') as compared to the $1P$ (χ) states of a given quarkonium system. This argument can be cast in a more quantitative form by noting that if $v(r)\propto 1/r$, then all the terms in Eq. (6) that involve v are proportional to $\langle 1/r^3\rangle$, while if $s(r)\propto r$ the term in Eq. (6a) involving s is proportional to $\langle 1/r\rangle$ and the whole discussion boils down to the assertion that naively $\langle 1/r\rangle/\langle 1/r^3\rangle$ should increase as we go from the $1P$ to $2P$ states.

This argument is wrong. It disagrees with evidence provided by the bottomonium system where almost all of the theoretical predictions[5-10] would have $R_\chi<R_{\chi'}$. Furthermore, the most recent experimental results[11] also tend in the same direction, with $R_\chi=0.67\pm0.06$ and $R_{\chi'}=0.69\pm0.05$.

How does the naive argument go wrong? It is a somewhat subtle point. Even though it is true that $\langle r^2\rangle$ is bigger for the χ' than the χ states, it does not necessarily follow that $\langle 1/r\rangle/\langle 1/r^3\rangle$ is bigger. The latter statement depends on the potential. With Coulomb wave functions, it does indeed grow, as $\langle 1/r\rangle/\langle 1/r^3\rangle=6r_B^2$

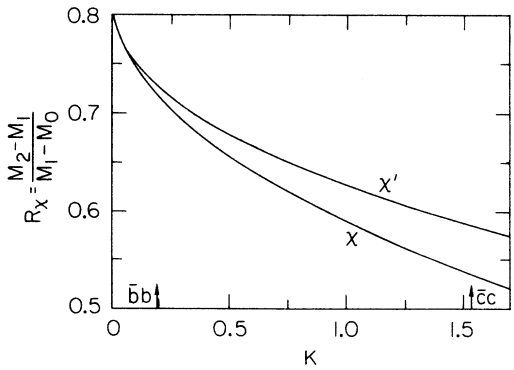

FIG. 1. The ratio R_χ for the potential of Eichten *et al.* as a function of the scaled, dimensionless variable K for the χ and χ' states, respectively. The arrows indicate the values of K corresponding to charmonium and bottomonium with $m_c=1.84$ GeV and $m_b=5.17$ GeV, respectively.

for the χ states and $9r_B^2$ for the χ' states (r_B being the Bohr radius). But with a three-dimensional harmonic-oscillator potential this same quantity is a factor of $\frac{9}{13}$ smaller for the χ' states. The wave function does spread out in space; $\langle r \rangle$ and $\langle r^2 \rangle$ increase for the χ' states, but the ratio $\langle 1/r \rangle / \langle 1/r^3 \rangle$ does not. By numerical calculation it appears that more generally $\langle 1/r \rangle / \langle 1/r^3 \rangle$ increases with radial excitation when the potential is of the form $V(r) \propto -r^\alpha$ with $\alpha < 0$, and conversely, it decreases when the potential is of the form $V(r) \propto r^\alpha$ with $\alpha > 0$.

The more interesting question now is what happens in the physical case when the potential has both "Coulomb" (one-gluon-exchange) and confining (linear) components. From the above discussion we expect that the ratio of mass splittings will tend to decrease when the wave functions are mostly sensitive to the one-gluon-exchange part of the potential and will increase when they are sensitive to the confining part. This is indeed the case, as is seen by studying the situation in various regimes with the potential of Eichten *et al.*:[12]

$$V(r) = \frac{-\beta}{r} + kr = \frac{-0.52}{r} + \frac{r}{(2.34\ \text{GeV}^{-1})^2}\ , \qquad (8)$$

with the two coefficients having been adjusted to fit the charmonium spectrum, although the model does a quite adequate job in describing bottomonium as well. The Schrödinger equation can be put in dimensionless form by using the variables $\rho = \mu\beta r = r/r_B$ and $K = k/(\beta^3\mu^2)$, where μ is the reduced mass and $r_B = 1/\beta\mu$ is the Bohr radius of the corresponding purely Coulomb potential problem.

Figure 1 shows the ratio of mass splittings

$$R_\chi = \frac{M(^3P_2) - M(^3P_1)}{M(^3P_1) - M(^3P_0)}$$

for the $1P$ and $2P$ levels of the potential of Eichten *et al.* as a function of the scaled variable K. The critical assumption, even if it is the most straightforward one, has been made that the $-\beta/r$ piece of the potential is a Lorentz four-vector and the kr piece a Lorentz scalar *for all values of r.* As is to be expected from our previous discussion, $R_\chi < R_{\chi'}$ when K is large and the confining part of the potential plays a dominant role even for the lowest bound lying states. Indeed, it is only for very small values of K that we get an inversion of this behavior and $R_\chi > R_{\chi'}$. Inspection of Fig. 2 indicates that this happens for $K < \frac{1}{34}$.

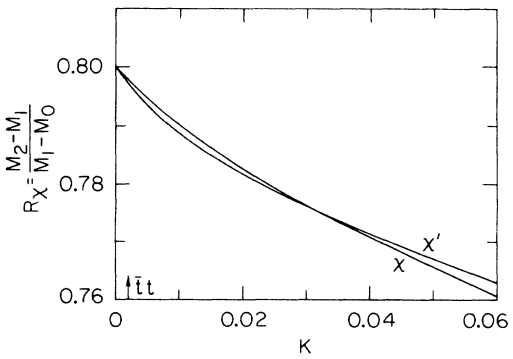

FIG. 2. The ratio R_χ for the potential of Eichten *et al.* for small values of the scaled, dimensionless variable K for the χ and χ' states, respectively. The arrow indicates the value of K corresponding to toponium with $m_t = 50$ GeV.

TABLE I. R_χ for charmonium and bottomonium computed using the potential of Eichten *et al.*

State	R_χ [experiment (Ref. 11)]	R_χ (theory)
χ_c	0.48±0.01	0.538
χ_c'		0.588
χ_b	0.67±0.06	0.717
χ_b'	0.69±0.05	0.727

The charmonium and bottomonium systems correspond to values of K indicated by the arrows in Fig. 1 (corresponding to $m_c = 1.84$ GeV and $m_b = 5.17$ GeV) and are well within the region where $R_\chi < R_{\chi'}$. The values of R_χ that correspond to these two situations are shown in Table I.

The agreement with experiment is quite good for both charmonium and bottomonium considering that nothing about spin-dependent effects was used as an input in the choice of parameters. For charmonium, however, the absolute magnitude of the χ_c splittings is about a factor of 2 smaller than experiment.

For quark masses above ~ 13 GeV with the potential of Eichten *et al.*, which corresponds to $K = \frac{1}{34}$, the situation will change and R_χ will be smaller for the χ' instead of the χ states. But even with very high mass quarks, for sufficiently high radial excitations which "live" primarily in the confining part of the potential we expect the situation to revert back again to larger values of R_χ as we go up in principal quantum number. This is seen in Table II, where we have again used the potential of Eichten *et al.* with the same assumptions except we consider the case of toponium with $m_t = 50$ GeV. The corresponding value of K is indicated by the arrow in Fig. 2.

The values of R_χ do indeed decrease until we get to the $5P$ state, and then increase again, but we need to look at four or five significant figures to see the effect. The situation is completely dominated by the Coulomb piece of the potential and all values of R_χ are very close

TABLE II. R_χ for toponium computed with the potential of Eichten *et al.* and $m_t = 50$ GeV.

State	R_χ
$1P$	0.797 63
$2P$	0.796 89
$3P$	0.796 56
$4P$	0.796 45
$5P$	0.796 42
$6P$	0.796 44
$7P$	0.796 47
$8P$	0.796 51
$9P$	0.796 56

TABLE III. Various theoretical predictions for R_χ.

Model	$R_{\chi_c}=0.48\pm0.02$	$R_{\chi_b}=0.67\pm0.06$	$R_{\chi_b'}=0.69\pm0.05$
Beavis (Ref. 5)	0.97	0.96	1.0
Buchmüller (Ref. 6)	0.61	0.76	0.75
McClary and Byers (Ref. 7)	0.35	0.45	0.48
Moxhay and Rosner (Ref. 8)	0.42	0.42	0.42
Gupta *et al.* (Ref. 9)	0.50	0.64	0.67
Olsson and Suchyta (Ref. 10)	0.55	0.78	0.79

to 0.8. Moreover, the absolute magnitudes of the mass splittings are on the order of tens of MeV, making them hard to measure to the needed accuracy.

A number of calculations of the splittings of the P states have been done, each with its own prescription for the Lorentz structure of the potential. As can be seen from Table III, they have varying success in accounting for the data, although several are in quite good accord with experiment.

This brings us to the more general question of the division of the potential into pieces with different Lorentz transformation properties. When a calculation of the potential is made from first principles, for example, with lattice-gauge-theory techniques, then this question will have a definitive answer. Until that time we are left in the somewhat unsatisfactory state of having to make a prescription that is in accord with theory and experiment as it is known at any given time. When we did this for the potential of Eichten *et al.* above, it was "natural" to make the Coulomb piece arise from exchange of a Lorentz four-vector and the confining linear piece correspond to exchange of a Lorentz scalar. But is this so natural at all values of r? It surely does not follow from any known physics that the $1/r$ behavior from one-gluon exchange at short distance should be continued to all r and, moreover, its Lorentz character maintained in what is, after all, a phenomenological potential.

In order to see how sensitive the splittings of the P states are to our assumptions and to see if, in particular, we could have come to opposite conclusions about the behavior of R_χ with a plausible input, we have done the following computation.

We start from the Richardson potential[13]

$$V(r)=\frac{8\pi}{33-2n_f}\Lambda\left[\Lambda r-\frac{f(\Lambda r)}{\Lambda r}\right], \tag{9}$$

with

$$f(t)=1-4\int_1^\infty \frac{dq}{q}\frac{e^{-qt}}{\ln^2(q^2-1)+\pi^2}. \tag{10}$$

In momentum space it can be written in a more transparent form as one term:

$$\tilde{V}(q^2)=-\frac{4}{3}\frac{12\pi}{33-2n_f}\frac{1}{q^2\ln(1+q^2/\Lambda^2)}, \tag{11}$$

where we take $\Lambda=0.398$ GeV and n_f is the number of light quarks at the relevant momentum scale for renormalization (taken equal to three). There is no obvious way to separate things into vector and scalar pieces, even though the Richardson potential was designed to have the correct behavior at both short and long distances. For our purposes we make the arbitrary division that

$$v(r)=V_{\text{Richardson}}(r)(e^{r^2/a^2}) \tag{12a}$$

and

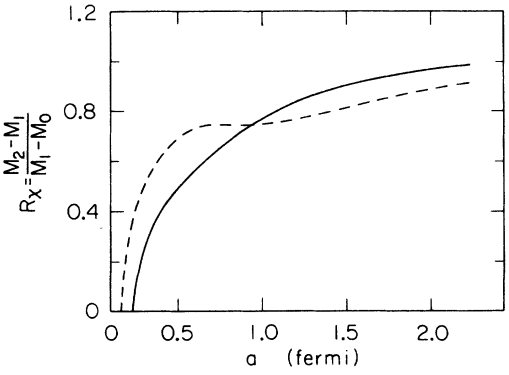

FIG. 3. The ratio R_χ for the Richardson potential with the division into scalar and vector parts as in Eq. (12) as a function of the distance a for the χ (solid line) and χ' (dashed line) states of charmonium, respectively.

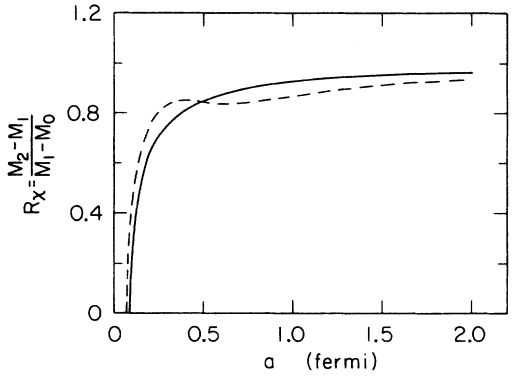

FIG. 4. The ratio R_χ for the Richardson potential with the division into scalar and vector parts as in Eq. (12) as a function of the distance a for the χ (solid line) and χ' (dashed line) states of bottomonium, respectively.

$$s(r)=V_{\text{Richardson}}(r)(1-e^{r^2/a^2}) , \tag{12b}$$

where a is a distance scale which provides the smooth division between a vector character at short distances and scalar character at large distances. By construction,

$$V_{\text{Richardson}}=v(r)+s(r) .$$

In Figs. 3 and 4 we have plotted R_χ for the charmonium and bottomonium systems, respectively, as a function of the distance scale a using Eqs. (5), (6), and (12) and masses $m_c=1.49$ GeV and $m_b=4.88$ GeV appropriate to the Richardson potential.

These two figures make it clear that it is possible to invert the usual ordering in which $R_\chi<R_{\chi'}$ by choosing the distance parameter a large enough, i.e., making the potential vector in character out to fairly large distances. The price is in getting values of both R_χ and $R_{\chi'}$ which are not far from 0.8, the value associated with a Coulomb potential, which is in disagreement with the data for both charmonium and bottomonium. A value of a around 0.5 F is needed to fit charmonium and about half of that for bottomonium, with the absolute magnitudes of the mass splittings coming out in good agreement as well.

We do not propose this as a serious method to calculate the splittings of the P states in quarkonium systems. Aside from being arbitrary, it does not give quantitative agreement using a single value of a with the known data. However, it does show us that other assumptions about the Lorentz character of the heavy-quark potential are possible and that one can obtain different behavior for R_χ depending on the assumptions one makes. Turning it around, the behavior of R_χ as we go from the lowest P states to their radial excitations is sensitive to the Lorentz character of the exchanges between heavy quarks as it depends on their distance and can be used as a tool to understand this more detailed and more quantitative feature of the potential.

ACKNOWLEDGMENT

This work was supported by the Department of Energy, contract DE-AC03-76SF00515.

[1]J. L. Rosner, in *Hadron '87,* proceedings of the Second International Conference on Hadron Spectroscopy, Tsukuba, Japan, 1987, edited by Y. Oyanagi, K. Takamatsu, and T. Tsoro (KEK, Tsukuba, Japan, 1987), p. 395, gives an overview of our present understanding of hadron spectroscopy.

[2]E. Eichten and F. Feinberg, Phys. Rev. D **23**, 2724 (1981); D. Gromes, Z. Phys. C **26**, 401 (1984). See E. Eichten, in *Dynamics and Spectroscopy at High Energy,* SLAC Summer Institute on Particle Physics, Stanford, California, 1983, edited by P. M. McDonough (SLAC Report No. 267, Stanford, 1983), p. 497.

[3]W. Buchmüller, Phys. Lett. **112B**, 479 (1982). See also J. L. Rosner, J. Phys. (Paris) **46**, C2, Suppl. 2 (1985); R. D. Pisarski and J. D. Stack, Nucl. Phys. **B286**, 657 (1987); M. G. Olsson and C. J. Suchyta, Phys. Rev. D **36**, 1459 (1987); **35**, 1738 (1987).

[4]J. L. Rosner, in *Proceedings of the 1985 Symposium on Lepton and Photon Interactions at High Energies,* Kyoto, Japan, 1985, edited by M. Konuma and K. Takahashi (Research Institute for Fundamental Physics, Kyoto University, Kyoto, 1986), p. 448. See also H. J. Schnitzer, Phys. Lett. **134B**, 253 (1984); Phys. Rev. Lett. **35**, 1540 (1975), for the original use of the ratio R_χ.

[5]D. Beavis *et al.*, Phys. Rev. D **20**, 2345 (1981).

[6]Buchmüller (Ref. 3).

[7]R. McClary and N. Byers, Phys. Rev. D **28**, 1632 (1983).

[8]P. Moxhay and J. L. Rosner, Phys. Rev. D **28**, 1132 (1983).

[9]R. Gupta *et al.*, Phys. Rev. D **34**, 201 (1986).

[10]Olsson and Suchyta (Ref. 3).

[11]J. Lee-Franzini, invited talk at the 1987 SLAC Summer Institute, August, 1987 (unpublished).

[12]E. Eichten *et al.*, Phys. Rev. D **17**, 3090 (1978); **21**, 203 (1980).

[13]J. L. Richardson, Phys. Lett. **82B**, 272 (1979).