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Skyrmions and meson-baryon reactions

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Mattis, using a Skyrmion model for the baryons, has expressed the s-channel isospin partial-wave amplitudes for the reaction $\phi+B\to\psi+B'$, where ϕ and ψ represent arbitrary, nonstrange mesons and B, B' denote either the nucleon or Δ, in terms of a set of reduced partial-wave amplitudes. Using the expression proposed by Mattis, I show that if one crosses to t-channel isospin amplitudes, the partial-wave sums may be carried out explicitly. The result is that the spin-projection amplitudes for given I_t may be written as linear combinations of unknown reduced amplitudes which depend upon the mesons, but not upon whether the pair (B,B') is (N,N), (N,Δ), or (Δ,Δ). There are, in general, fewer reduced amplitudes than spin-projection amplitudes, leading to linear relations among the latter, as well as linear relations among amplitudes involving (N,N), (N,Δ), and (Δ,Δ). From these relations I extract a considerable number of observable consequences, among them the predictions that the π^+p and π^-p elastic-scattering differential cross sections are identical at all energies and angles, that the polarization asymmetries are equal but opposite, and that there is no polarization in $\pi^-p\to\pi^0n$. While these and many other such predictions are not strictly true, the model offers a picture of two-body reactions which often coincides with much of the Regge phenomenology of the recent past. It may represent ultimately a viable link between the fundamental theory of strong interactions, QCD, and the enormous amount of data on two-body hadron reactions.

I. INTRODUCTION

In a recent Letter, Mattis[1] has obtained predictions for the production of vector mesons via the reaction $\pi N\to\rho N$ using the Skyrmion model for baryons. Subsequently I pointed out that the relations obtained by Mattis, which concerned the partial-wave amplitudes, could be extended to predictions for either the helicity amplitudes or the transversity amplitudes, corresponding to isospin zero in the t channel.[2] In particular, of the six possible transversity amplitudes, two were identically zero, while the four remaining amplitudes were determined by one unknown function. This result was obtained only after a rather laborious calculation and appeared to be a consequence of some remarkable properties of the Wigner d functions. The truth is that my result is only a special case of a much more general formula which I establish in this paper, and which holds for the generic reaction $\phi+B\to\psi+B'$, where ϕ and ψ represent nonstrange mesons of arbitrary spins S_ϕ,S_ψ, isospins I_ϕ,I_ψ, and parity, and B,B' denote either the nucleon or Δ. The essential point is that if one uses the expression given by Mattis for the s-channel isospin partial-wave amplitudes, and crosses to t-channel isospin amplitudes, the sum over J, the total angular momentum, can be carried out explicitly. The formulas necessary for this task may be found in the monograph of Yutsis, Levinson, and Vanagas.[3] The result is that the t-channel isospin amplitudes $A^{I_t}_{m_cm_dm_am_b}$ may be written as $(c,d,a,b,$ refer to ψ,B',ϕ,B, respectively)

$$A^{I_t}_{m_cm_dm_am_b}=\sum_{yLM}C^{I_tyLM}_{m_cm_dm_am_b}F^{I_t}_{yLM}\ , \tag{1}$$

where the $C^{I_tyLM}_{m_cm_dm_am_b}$ are known numerical factors which carry all the dependence on spin projections, and the $F^{I_t}_{yLM}$ are unknown functions of the dynamical variables. The reduced amplitudes $F^{I_t}_{yLM}$ depend upon the mesons ϕ and ψ, but do not depend on whether the initial or final baryons are nucleon or Δ, except that the physically realizable t-channel isospins depend on whether the pair (B,B') is (N,N), (N,Δ), or (Δ,Δ). The sum over M is purely formal insofar as the coefficient $C^{I_tyLM}_{m_cm_dm_am_b}$ is zero unless $M=m_a+m_b-m_c-m_d$. The sums over the variables L and y extend over all values permitted by the nonvanishing of certain 3-j and 6-j symbols appearing in either the $C^{I_tyLM}_{m_cm_dm_am_b}$ or $F^{I_t}_{yLM}$.

There are two distinct sorts of experimentally verifiable predictions that may be extracted from Eq. (1). One sort relates the $I_t=1$ cross sections for the reaction $\phi+B\to\psi+B'$ with (N,N) and (N,Δ) as baryons. The differential cross section for the latter is simply twice that for the former, for arbitrary mesons, and for all energies and angles. The second kind of prediction refers to a given reaction and arises from the fact that the number of reduced amplitudes is generally smaller than the number of independent spin-projection amplitudes. This may lead to relations among observable quantities such as density-matrix elements or polarization asymmetries. Indeed, the number of independent spin-projection amplitudes $A^{I_t}_{m_cm_dm_am_b}$ is given by

$$(2R+1)(2R'+1)(2S_\phi+1)(2S_\psi+1)/2\ ,$$

where R,R' are the spins of the baryons, whereas the number of unknown reduced amplitudes $F^{I_t}_{yLM}$ is

$$[(2I_t+1)(2S_\phi+1)(2S_\psi+1)+P_\phi P_\psi(-1)^{I_t+S_\phi+S_\psi}]/2 ,$$

where P denotes the parity of the corresponding meson. One sees that for the reaction $\pi N \to \rho N$ this expression allows only one reduced amplitude for t-channel isospin zero, as compared with six spin-projection amplitudes. Similarly for the reactions $\pi N \to \pi N$, $I_t=0$, $\pi N \to \pi N$, $I_t=1$, and $\pi N \to \pi\Delta$, $I_t=1$, there is only one reduced amplitude, compared to two, two, and four, respectively. Furthermore, the last two reactions are described by the same reduced amplitude. Perhaps even more striking is the prediction that if ϕ and ψ are spinless mesons of opposite parity, there is no reduced amplitude with $I_t=0$, hence no cross section for such hypothetical reactions as $\pi^0 p \to \delta^0 p$ or $\eta p \to \epsilon p$. An extremely important consequence of the expression follows from the fact that the coefficients $C^{I_t yLM}_{m_c m_d m_a m_b}$ satisfy an orthogonality relation; this implies that the differential cross section for any of the reactions under consideration is the incoherent sum of squares of absolute values of the dynamical amplitudes $F^{I_t}_{yLM}$. Consequently the reactions $\pi^+ p \to \psi^+ p$ and $\pi^- p \to \psi^- p$ are predicted to have identical differential cross sections, where ψ denotes any charged meson. Although this prediction is obviously wrong for πp elastic scattering at low energies, it is a reasonable approximation at energies of a few GeV. Karliner and Mattis[4] have pointed out in their detailed comparison of partial-wave analyses with the model that a major obstacle is the prediction that the nucleon and Δ have equal masses. At slightly higher energies this drawback should not be so serious, and I propose that an appropriate arena for general tests of the model of Mattis is actually the few-GeV region where meson exchange is the dominant production mechanism. Since the Skyrmion model is inspired by QCD, one may well investigate this tenuous link between the enormous amount of data on quasi-two-body reactions and the leading candidate for a fundamental theory of strong interactions. The key point of the work presented here is that the Skyrmion model of Mattis makes numerous predictions concerning observable quantities in various reactions, and that these predictions can be tested without performing partial-wave analyses of data. For example, using the optical theorem to relate forward-direction elastic amplitudes to total cross sections, I show that the prediction is that all spin-averaged total cross sections are pure $I_t=0$.

It is quite paradoxical that the model, which is ostensibly based on baryons in the s channel, actually makes predictions concerning amplitudes which are dominated by mesonic Regge poles. Obviously my proposal to take the model of Mattis at face value for all energies and angles represents a vast extension of the original work on the Skyrmion model for baryons, which aimed at calculating low-energy or static properties of nucleons and Δ. Although I can offer no theoretical reasons to justify my proposal, the fact that Mattis's model leads directly to Eq. (1), which itself contains no mention of limits on energy or angles, lends some support. Its validity must therefore rest on how well the predictions of Eq. (1) compare with experiment. It is moderately surprising that the model restricts considerably $I_t=0$ exchanges such as Pomeron, ω, η, and f^0, while offering rather more freedom to $I_t=1$ exchanges such as π, ρ, and A_2. Even more freedom is offered to $I_t=2$ exchanges, which are known empirically to correspond to tiny cross sections such as $\pi^+ n \to \pi^- \Delta^{++}$. As shall be shown in the body of the paper, there exist no relations among the reduced amplitudes of different t-channel isospin; hence, the model provides no explanation for the weakness of exotic exchange. On the other hand, the observed fact that exotic exchanges are weak provides relevant information which could be used in subsequent refining of the model.

In Sec. II a detailed derivation of Eq. (1) is presented and explicit expressions for the coefficients are displayed. The most important special case, that of spinless mesons in the initial state, for which Eq. (1) simplifies considerably, is discussed in Sec. III. In order to describe elastic scattering, a slight modification of Eq. (1) is required. This is presented in Sec. IV, along with a number of interesting predictions concerning total cross sections obtained *via* the optical theorem. The conclusions are given in Sec. V.

II. DERIVATION OF EQ. (1)

The starting point of our derivation of Eq. (1) is the expression obtained by Mattis relating the partial-wave amplitudes for the reaction $\phi+B \to \psi+B'$ to a set of reduced partial-wave amplitudes. Mattis writes

$$A^{I_s J}_{l'S'lS} = \sum_{K\tilde{K}\tilde{K}'} \eta\eta' G_{K\tilde{K}\tilde{K}'ll'} , \tag{2}$$

where $A^{IJ}_{l'S'lS}$ denotes the partial-wave amplitude corresponding to initial and final orbital angular momenta l and l', initial and final spins S and S', total angular momentum J, and s-channel isospin I_s, and where the reduced amplitudes $G_{K\tilde{K}\tilde{K}'ll'}$ depend upon the quantum numbers, K, $\tilde{K}$, and $\tilde{K}'$ as well as l and l'. The quantity $\tilde{K}$ is the composition of the orbital angular momentum l with the isospin of the meson I_ϕ; $\tilde{K}'$ results from adding l' and I_ψ. The conserved quantity K is the common result of adding S_ϕ to $\tilde{K}$ or S_ψ to $\tilde{K}'$. The factors η and η' are defined by

$$\eta=[(2K+1)(2\tilde{K}+1)(2R+1)(2S+1)]^{1/2} \begin{Bmatrix} l & I_\phi & \tilde{K} \\ S & R & S_\phi \\ J & I_s & K \end{Bmatrix} , \tag{3a}$$

$$\eta' = [(2K+1)(2\tilde{K}'+1)(2R'+1)(2S'+1)]^{1/2} \begin{Bmatrix} l' & I_\psi & \tilde{K}' \\ S' & R' & S_\psi \\ J & I_s & K \end{Bmatrix} . \tag{3b}$$

In these expressions the 9-j symbols reflect the coupling among the various angular momenta, and one sees the crucial roles played by R and R' in that they represent both spin and isospin for the baryons. At this point all dynamics is contained in the unknown reduced amplitudes $G_{K\tilde{K}\tilde{K}'ll'}$.

The relation between the partial-wave amplitudes and the spin-projection amplitudes may be found in standard references such as Goldberger and Watson,[5] and may be written as

$$A^{I_s}_{m_c m_d m_a m_b}(\theta',\phi',\theta,\phi) = \sum_{\substack{lml'm'\\ S\mu S'\mu' JM'}} (2J+1)[(2S+1)(2S'+1)]^{1/2}(-1)^{\Omega} \begin{bmatrix} S_\phi & R & S \\ m_a & m_b & -\mu \end{bmatrix} \begin{bmatrix} S_\psi & R' & S' \\ m_c & m_d & -\mu' \end{bmatrix}$$

$$\times \begin{bmatrix} l & S & J \\ m & \mu & M' \end{bmatrix} \begin{bmatrix} l' & S' & J \\ m' & \mu' & M' \end{bmatrix} Y^{m'}_{l'}(\theta',\phi') Y^{m\,*}_{l}(\theta,\phi) A^{I_s J}_{l'S'lS} ,$$

$$\Omega = -S_\psi + R' - \mu' - S_\phi + R - \mu + l - l' + S - S' . \tag{4}$$

In this l-S formalism the choice of axes in the center-of-momentum frame is completely arbitrary. The spin projections in the various particle rest frames then refer to axes obtained from those in the center-of-momentum system by a parallel Lorentz transformation or boost along the directions of the corresponding particles. The angles (θ,ϕ) refer to the direction of the incident meson while (θ',ϕ') refer to the direction of the final meson in the c.m. frame. If the axes are chosen such that the initial meson moves along the positive y direction while the final meson momentum lies in the x-y plane and makes an angle of χ with the y axis (that is $\theta=\theta'=\phi=\pi/2$, $\phi'=\chi+\pi/2$), then the spin-projection amplitudes are simply related to the standard transversity amplitudes: namely,

$$A^{I_s}_{m_c m_d m_a m_b}(\pi/2,\pi/2+\chi,\pi/2,\pi/2) = e^{-i[\chi(m_c+m_d)+\pi(m_a+m_c)]} T^{I_s}_{m_c m_d m_a m_b}(\chi) . \tag{5}$$

If the standard helicity amplitudes of Jacob and Wick[6] are desired, the usual relation may be written in terms of Wigner rotation functions as

$$H_{\lambda_c\lambda_d\lambda_a\lambda_b}(\chi,0) = (-1)^{R+R'-\lambda_b-\lambda_d} \sum_{m_c m_d m_a m_b} D^{S_\psi}_{m_c\lambda_c} D^{R'}_{m_d\lambda_d} D^{S_\phi^*}_{m_a\lambda_a} D^{R^*}_{m_b\lambda_b} T_{m_c m_d m_a m_b}(\chi) , \tag{6}$$

where the Euler angles are $(\pi/2,\ \pi/2,\ -\pi/2)$. In this manner it is apparent that the spin-projection amplitudes may be related, by a judicious choice of the axes, to the amplitudes commonly used in phenomenology. However, nothing in what follows depends upon the choice, and one may leave it open.

In order to establish Eq. (1) one replaces $A^{I_s J}_{l'S'lS}$ in Eq. (4) by Eq. (2), performs the M' summation using the standard identity involving a Racah coefficient or 6-j symbol,[7] thereby introducing the variables L and M (note that L is not the orbital angular momentum)

$$\sum_{M'} \begin{bmatrix} l & S & J \\ m & \mu & M' \end{bmatrix} \begin{bmatrix} l' & S' & J \\ m' & \mu' & M' \end{bmatrix} = \sum_{LM} (2L+1)(-1)^{\mu-J-L-m'} \begin{Bmatrix} l' & S' & J \\ S & l & L \end{Bmatrix} \begin{bmatrix} S & S' & L \\ -\mu & \mu' & M \end{bmatrix} \begin{bmatrix} l' & l & L \\ m' & -m & -M \end{bmatrix} , \tag{7}$$

and writes the t-channel isospin amplitudes in terms of the crossing matrices given by Rebbi and Slansky:[8]

$$A^{I_t} = \sum_{I_s} (2I_s+1)(-1)^{I_s+I_t+R} \begin{Bmatrix} R' & R & I_t \\ I_\phi & I_\psi & I_s \end{Bmatrix} A^{I_s} . \tag{8}$$

At this point Eq. (A4.8) of Ref. 3, which I reproduce here,

$$\begin{Bmatrix} j_1 & & j_2 & & j_3 & & j_1' & & j_2' & & j_3' & & j_1'' & & j_2'' & & j_3'' & \\ & k_1 & & k_2 & & k_3 & & k_1' & & k_2' & & k_3' & & k_1'' & & k_2'' & & k_3'' \end{Bmatrix}$$

$$= \sum_{x_2 x_2} (2x_1+1)(2x_2+1)(-1)^{\Omega} \begin{Bmatrix} k_1 & k_1' & k_1'' \\ x_1 & j_1 & j_2 \end{Bmatrix} \begin{Bmatrix} k_3 & k_3' & k_3'' \\ j_1'' & x_2 & j_3' \end{Bmatrix} \begin{Bmatrix} x_1 & x_2 & k_2'' \\ j_1 & k_3'' & j_3'' \\ k_1'' & j_1'' & j_2'' \end{Bmatrix} \begin{Bmatrix} x_2 & & x_1 & & k_1' & & j_2' & \\ & k_2'' & & j_2 & & j_1' & & j_3' \\ k_2' & & k_2 & & j_3 & & k_3 & \end{Bmatrix},$$

$$\Omega = j_2 + j_2'' + j_3' - j_3'' + k_1' + k_2'' + k_3 + 2j_1' - x_1 + x_2 \ , \tag{9}$$

allows one, after making the identification

$x_1 = J, \ x_2 = I_s \ ,$

$j_1 = l', \ j_2 = l, \ j_3 = I_\phi, \ j_1' = 0, \ j_2' = S, \ j_3' = R \ ,$

$j_1'' = R', \ j_2'' = S_\psi, \ j_3'' = \tilde{K}', \ k_1 = L, \ k_2 = \tilde{K}, \ k_3 = I_\phi \ ,$

$k_1' = S, \ k_2' = S_\phi, \ k_3' = I_t, \ k_1'' = S', \ k_2'' = K, \ k_3'' = I_\psi$

to carry out the sums over J and I_s in terms of an 18-j symbol with $j_1' = 0$. It should be noticed that the 12-j symbol appearing in Eq. (9) is written in the notation of Ord-Smith,[9] rather than that of Jahn and Hope.[10] Since it contains a zero entry in the middle row it is really only a 9-j symbol. (Purists may complain that an 18-j symbol with a zero entry is, at worst, only a 15-j symbol. They would be right, but this derivation is easier.) The result is the following expression for the amplitude:

$$A^{I_t}_{m_c m_d m_a m_b}(\theta',\phi',\theta,\phi)$$

$$= [(2R+1)(2R'+1)(2I_\phi+1)]^{1/2}$$

$$\times \sum_{\substack{S\mu S'\mu' K\tilde{K} \\ LMlml'm'\tilde{K}'}} (-1)^{\Omega} N(S,S',L,K,\tilde{K},\tilde{K}')$$

$$\times \begin{Bmatrix} l' & & l & & I_\phi & & 0 & & S & & R & & R' & & S_\psi & & \tilde{K}' & \\ & L & & \tilde{K} & & I_\phi & & S & & S_\phi & & I_t & & S' & & K & & I_\psi \end{Bmatrix} G_{K\tilde{K}\tilde{K}'ll'}$$

$$\times \begin{bmatrix} S & S' & L \\ -\mu & \mu' & M \end{bmatrix} \begin{bmatrix} l' & l & L \\ m' & -m & -M \end{bmatrix} \begin{bmatrix} S_\phi & R & S \\ m_a & m_b & -\mu \end{bmatrix} \begin{bmatrix} S_\psi & R' & S' \\ m_c & m_d & -\mu' \end{bmatrix} Y_{l'}^{m'}(\theta',\phi') Y_l^{m*}(\theta,\phi) \ ,$$

$$N(S,S',L,K,\tilde{K},\tilde{K}') = (2L+1)(2K+1)(2S'+1)[(2S+1)^3(2\tilde{K}+1)(2\tilde{K}'+1)]^{1/2} \ , \tag{10}$$

$$\Omega = R' - \mu' - S_\phi + R - l' - m' - S' - L + I_t + \tilde{K}' - K - I_\phi \ .$$

However, the 18-j symbol which appears in Eq. (10) possesses the following symmetry, again from Ref. 3:

$$\begin{Bmatrix} j_1 & & j_2 & & j_3 & & j_1' & & j_2' & & j_3' & & j_1'' & & j_2'' & & j_3'' & \\ & k_1 & & k_2 & & k_3 & & k_1' & & k_2' & & k_3' & & k_1'' & & k_2'' & & k_3'' \end{Bmatrix}$$

$$= \begin{Bmatrix} j_1' & & j_3 & & j_2 & & j_1 & & j_3'' & & j_2'' & & j_1'' & & j_3' & & j_2' & \\ & k_3 & & k_2 & & k_1 & & k_3'' & & k_2'' & & k_1'' & & k_3' & & k_2' & & k_1' \end{Bmatrix} \ . \tag{11}$$

If one uses this symmetry to rewrite the 18-j symbol in Eq. (10) together with Eq. (9) (now with the zero in the j_1 slot) one obtains the result

$$A^{I_t}_{m_c m_d m_a m_b}(\theta',\phi',\theta,\phi) = [(2R+1)(2R'+1)]^{1/2} \times \sum_{\substack{yS\mu S'\mu' K\tilde{K} \\ LMlml'm'\tilde{K}'}} (-1)^{\Omega'} \tilde{N}(S,S',L,K,K,K') \times \begin{Bmatrix} L & & y & & I_t & & I_\psi & \\ & S_\psi & & S_\phi & & I_\phi & & l' \\ \tilde{K}' & & K & & \tilde{K} & & l & \end{Bmatrix} \begin{Bmatrix} R' & R & I_t \\ S_\phi & y & S \end{Bmatrix} \begin{Bmatrix} R' & S' & S_\psi \\ L & y & S \end{Bmatrix} G_{K\tilde{K}\tilde{K}'ll'}$$

$$\times \begin{bmatrix} S & S' & L \\ -\mu & \mu' & M \end{bmatrix} \begin{bmatrix} l' & l & L \\ m' & -m & -M \end{bmatrix} \begin{bmatrix} S_\phi & R & S \\ m_a & m_b & -\mu \end{bmatrix} \begin{bmatrix} S_\psi & R' & S' \\ m_c & m_d & -\mu' \end{bmatrix} Y_{l'}^{m'}(\theta',\phi') Y_l^{m*}(\theta,\phi) ,$$

$$\tilde{N}(S,S',L,K,K,K') = (2L+1)(2y+1)(2S+1)(2S'+1)(2K+1)[(2\tilde{K}+1)(2\tilde{K}'+1)]^{1/2} , \quad (12)$$

$$\Omega' = R' - \mu' + R - S_\psi + l' + S' - m' + \tilde{K}' - K + I_\phi .$$

If the numerical factor $C^{I_t yLM}_{m_c m_d m_a m_b}$ is defined by

$$C^{I_t yLM}_{m_c m_d m_a m_b} = [(2y+1)(2L+1)(2R+1)(2R'+1)]^{1/2}(-1)^{R+R'} \times \sum_{S\mu S'\mu'} (2S+1)(2S'+1)(-1)^{S'-\mu'} \times \begin{Bmatrix} R' & R & I_t \\ S_\phi & y & S \end{Bmatrix} \begin{Bmatrix} R' & S' & S_\psi \\ L & y & S \end{Bmatrix} \begin{bmatrix} S & S' & L \\ -\mu & \mu' & M \end{bmatrix} \begin{bmatrix} S_\phi & R & S \\ m_a & m_b & -\mu \end{bmatrix} \begin{bmatrix} S_\psi & R' & S' \\ m_c & m_d & -\mu' \end{bmatrix} , \quad (13)$$

and the function (of energy and angles) $F^{I_t}_{yLM}$ is defined by

$$F^{I_t}_{yLM}(\theta',\phi',\theta,\phi) = [(2y+1)(2L+1)]^{1/2} \times \sum_{\substack{lml'm' \\ K\tilde{K}\tilde{K}'}} (2K+1)[(2\tilde{K}+1)(2\tilde{K}'+1)]^{1/2}(-1)^{\Omega''} \times \begin{Bmatrix} L & & y & & I_t & & I_\psi & \\ & S_\psi & & S_\phi & & I_\phi & & l' \\ \tilde{K}' & & K & & \tilde{K} & & l & \end{Bmatrix} \begin{bmatrix} l' & l & L \\ m' & -m & -M \end{bmatrix} G_{K\tilde{K}\tilde{K}'ll'} Y_{l'}^{m'}(\theta',\phi') Y_l^{m*}(\theta,\phi) ,$$

$$\Omega'' = -S_\psi + l' - m' + \tilde{K}' - K + I_\phi , \quad (14)$$

it is clear that Eq. (12) is precisely of the form of Eq. (1). It is readily shown from Eq. (13) that changing the sign of all spin projections simply yields an overall phase, $C^{I_t yLM}_{m_c m_d m_a m_b} = (-1)^{\Omega} C^{I_t yLM}_{-m_c -m_d -m_a -m_b}$, where $\Omega = R + R' - 1 - L + S_\phi + S_\psi$. From the explicit form of Eq. (14) one observes that the $F^{I_t}_{yLM}$ transform irreducibly under rotations of axes, with L being the index of the representation. Although the $F^{I_t}_{yLM}$ generally display no symmetry, two particular choices of the z axis lead to simplifications. If the z axis is chosen normal to the production plane, the $F^{I_t}_{yLM}$ are zero unless $P_\phi P_\psi(-1)^M$ is even, which is the usual constraint of parity conservation. If the z axis is chosen along the direction of the incident meson, with the final momenta in the x-z plane, then one may show that $F^{I_t}_{yLM} = P_\phi P_\psi (-1)^{L+M} F^{I_t}_{yL-M}$. In addition, as the scattering angle χ approaches zero, the $F^{I_t}_{yLM}$ behave as $(\sin\chi)^{|M|}$. An important feature of Eqs. (13) and (14) is that I_t is coupled to the baryon isospins R and R' in (13), whereas in (14) I_t is coupled to the meson isospins I_ϕ and I_ψ. This implies that the set of (I_t,y,L) values for which the $C^{I_t yLM}_{m_c m_d m_a m_b}$ are different from zero need not coincide with that for which the $F^{I_t}_{yLM}$ are nonzero, and conversely. Of course, for a physically allowed value of I_t both will normally be nonzero. Another very important insight can be gained from Eq. (14). The 12-j symbols appearing in Eq. (14) satisfy the orthogonality property[3]

$$\sum_{K\tilde{K}\tilde{K}'} (2K+1)(2\tilde{K}+1)(2\tilde{K}'+1) \begin{Bmatrix} L & & y & & I_t & & I_\psi & \\ & S_\psi & & S_\phi & & I_\phi & & l' \\ \tilde{K}' & & K & & \tilde{K} & & l & \end{Bmatrix} \begin{Bmatrix} L' & & y' & & I_t' & & I_\psi & \\ & S_\psi & & S_\phi & & I_\phi & & l' \\ \tilde{K}' & & K & & \tilde{K} & & l & \end{Bmatrix} = \delta_{I_t I_t'} \delta_{yy'} \delta_{LL'} / [(2I_t+1)(2y+1)(2L+1)] , \quad (15)$$

as well as a similar expression where the roles of $(K,\tilde{K},\tilde{K}')$ and (I_t,y,L) are interchanged. This allows one to introduce the quantities $\tilde{G}_{I_t yLll'}$ via

$$\sum_{K\tilde{K}\tilde{K}'}(2K+1)[(2K+1)(2K'+1)]^{1/2}(-1)^{\tilde{K}'-K}G_{K\tilde{K}\tilde{K}'ll'}\begin{Bmatrix} L & y & I_t & I_\psi \\ S_\psi & S_\phi & I_\phi & l' \\ \tilde{K}' & K & \tilde{K} & l \end{Bmatrix}=\tilde{G}_{I_t yLll'}\ , \tag{16}$$

where the inverse relation is

$$\sum_{I_t yL}(2I_t+1)(2y+1)(2L+1)\tilde{G}_{I_t yLll'}\begin{Bmatrix} L & y & I_t & I_\psi \\ S_\psi & S_\phi & I_\phi & l' \\ \tilde{K}' & K & \tilde{K} & l \end{Bmatrix}=[(2\tilde{K}+1)(2\tilde{K}'+1)]^{-1/2}(-1)^{\tilde{K}'-K}G_{K\tilde{K}\tilde{K}'ll'}\ . \tag{17}$$

Given these expressions, it is evident that the reduced partial-wave amplitudes of Mattis and my $\tilde{G}_{I_t yLll'}$ are fully equivalent, in the sense that one may pass from one set to the other by using Eqs. (16) and (17). Expressed in terms of the $\tilde{G}_{I_t yLll'}$, the $F^{I_t}_{yLM}$ may be written quite compactly as

$$F^{I_t}_{yLM}(\theta',\phi',\theta,\phi)=[(2y+1)(2L+1)]^{1/2}(-1)^{-S_\psi+I_\phi}\sum_{lml'm'}(-1)^{l'-m'}\begin{pmatrix} l' & l & L \\ m' & -m & -M \end{pmatrix}\tilde{G}_{I_t yLll'}Y^{m'}_{l'}(\theta',\phi')Y^{m\,*}_{l}(\theta,\phi)\ . \tag{18}$$

On the basis of these equations I assert that Eq. (1) incorporates the full contents of the Skyrmion model proposed by Mattis. Given the $F^{I_t}_{yLM}$ one may unravel the equations to obtain the $\tilde{G}_{I_t yLll'}$, just as one may build the $F^{I_t}_{yLM}$ from the $G_{K\tilde{K}\tilde{K}'ll'}$.

The number of allowed values of y and L for fixed I_t is determined by the usual triangular inequalities which may be read from the 6-j symbols in Eq. (13). The permissible values are those which satisfy $\{I_t y S_\phi\}$ and $\{yLS_\psi\}$. If parity is neglected, there are $2L+1$ allowed M values, and a simple computation shows that there are $(2I_t+1)(2S_\phi+1)(2S_\psi+1)$ allowed y,L,M values. The well-known constraint of parity is that transversity amplitudes vanish unless $(-1)^M=P_\phi P_\psi$, where P denotes the parity of the corresponding meson. Including parity in this way, one finds the number of independent amplitudes to be

$$[(2I_t+1)(2S_\phi+1)(2S_\psi+1)+P_\phi P_\psi(-1)^{I_t+S_\phi+S_\psi}]/2\ ,$$

as indicated in the Introduction. This number is *independent* of the baryon spins. The standard number of allowed spin-projection amplitudes is $[(2R+1)(2R'+1)(2S_\phi+1)(2S_\psi+1)/2]$ for any I_t; the model is most constraining for the smallest I_t, and least constraining for the largest.

From the standard orthogonality properties of the 3-j and 6-j symbols it follows that

$$\sum_{\substack{m_a m_b\\ m_c m_d}} C^{I_t yLM}_{m_c m_d m_a m_b}C^{I_t' y'L'M'}_{m_c m_d m_a m_b}=\delta_{I_t I_t'}\delta_{yy'}\delta_{LL'}\delta_{MM'}(2R+1)(2R'+1)/(2I_t+1)\ , \tag{19}$$

$$\sum_{\substack{I_t y\\ LM}}(2I_t+1)C^{I_t yLM}_{m_c m_d m_a m_b}C^{I_t yLM}_{m_c' m_d' m_a' m_b'}=\delta_{m_c m_c'}\delta_{m_d m_d'}\delta_{m_a m_a'}\delta_{m_b m_b'}(2R+1)(2R'+1)\ . \tag{20}$$

From Eq. (19) it is evident that in the differential cross section, summed over the spin projections of all four particles, the different reduced amplitudes $F^{I_t}_{yLM}$ do not interfere, but rather enter only via their squared moduli. In particular, differential cross sections for the reactions $\phi^+p\to\psi^+p$ are identical to those for $\phi^-p\to\psi^-p$, for any charged mesons ϕ and ψ. I must insist upon the fact that this is an unavoidable consequence of the Skyrmion model of Mattis. Indeed, this extraordinary prediction is wrong by almost an order of magnitude for πp elastic scattering in the energy region of the $\Delta(1232)$, but for beam momenta above 1 GeV/c it is reasonably accurate, save in the backward direction. The source of the difficulty has been pointed out by Mattis and Karliner; the model supposes that the nucleon and Δ are degenerate and will fail whenever this is important.

An alternative way of expressing my result is to say that certain linear combinations of fixed I_t amplitudes are zero. From Eqs. (1) and (19) one finds easily

$$\sum_{\substack{m_a m_b\\ m_c m_d}} C^{I_t' yLM}_{m_c m_d m_a m_b}A^{I_t}_{m_c m_d m_a m_b}=\delta_{I_t I_t'}F^{I_t}_{yLM}(2R+1)(2R'+1)/(2I_t+1) \tag{21}$$

for arbitrary values of I_t, I_t', y, L, and M. If I_t is chosen equal to I_t', the reduced amplitudes are expressed in terms of the physical amplitudes; if not, the result is zero. In the latter case, I_t' need not be a physically allowed value; it is sufficient that it satisfy a triangular inequality with R and R', the baryon isospins. It should be remembered that the $F_{yLM}^{I_t}$ do not depend on whether the baryons are nucleons or Δ; hence, Eq. (21) also provides constraints among different reactions. Since it is hard to imagine anything but nucleons as target baryons only the particular case of $I_t=1$ will be useful in practice.

III. SOME EXPERIMENTAL CONSEQUENCES OF EQ. (1)

A. Relations among $I_t=1$ cross sections leading to N or Δ

If one wishes to confront experimental data with the predictions I have obtained from the Skyrmion model of Mattis, one must realistically suppose that the incident mesons have spin zero and that the target baryon is a nucleon. In fact, only charged pions are conceivable as incident mesons, although some evocation of π^0 induced reactions may be useful. Two distinct sorts of predictions may be extracted from Eq. (1). One sort relates the $I_t=1$ cross section leading to a final-state nucleon to that leading to a final state Δ, since the same reduced amplitudes occur in both reactions. Taking into account the orthogonality and normalization properties of Eq. (19), one may derive the following relation among the observable differential cross sections for the production of an arbitrary meson ψ:

$$d\sigma(\pi^+p\to\psi^0\Delta^{++})=\tfrac{3}{2}d\sigma(\pi^-p\to\psi^0 n)+\tfrac{1}{8}d\sigma(\pi^+n\to\psi^-\Delta^{++}) . \qquad (22)$$

If the meson ψ has isospin zero, the second term on the right-hand side of the expression is simply absent, as it corresponds to pure $I_t=2$ exchange. This equation is, of course, subject to the usual caveats about comparing cross sections having different kinematics caused by the $N-\Delta$ mass difference. In addition, the fact that the Δ has a large width makes absolute measurements of such cross sections a quite difficult task. Nonetheless, I consider that it brings the model of Mattis to a stage where direct comparison with experimental data on quasi-two-body reactions is feasible. Although I have used charged pions as incident mesons in this example, the equation does not depend on the spin of the incident meson, and is valid for any meson whose isospin is unity.

B. Simplifications occurring for spinless incident mesons

The second sort of predictions follows from the fact that the number of reduced amplitudes is generally smaller than the number of independent amplitudes for a given reaction of fixed I_t. Accordingly, one may be able to make definite predictions concerning observable quantities such as spin-density-matrix elements or polarization asymmetries. While this is true for arbitrary meson spins, the equations derived in the previous section simplify considerably if S_ϕ is set to zero. The 12-j symbols turn into relatively familiar 9-j symbols, and the mysterious variable y becomes equal to I_t, and may be omitted. It suffices to rewrite Eq. (13) for the special case $S_\phi=0$, omitting y and m_a; namely,

$$C_{m_c m_d m_b}^{I_t LM}=[(2L+1)(2R+1)(2R'+1)]^{1/2}\sum_{S'\mu'}(2S'+1)(-1)^{M+R+I_t-S'}\times\begin{Bmatrix}R' & S' & S_\psi\\ L & I_t & R\end{Bmatrix}\begin{bmatrix}R & S' & L\\ -m_b & \mu' & M\end{bmatrix}\begin{bmatrix}S_\psi & R' & S'\\ m_c & m_d & -\mu'\end{bmatrix} . \qquad (23)$$

The most important aspect of this equation may be read off the 6-j symbol; the quantities L, S_ψ, and I_t must satisfy the triangle inequality in order for the coefficient $C_{m_c m_d m_b}^{I_t LM}$ not to vanish. In particular, if I_t is zero, then $L=S_\psi$. This means that the number of reduced amplitudes $F_{LM}^{I_t}$ for $I_t=0$ is $(S_\psi+1)$ if the initial and final mesons have the same naturality [defined as $P(-1)^S$], and S_ψ when they have opposite naturality. Since the number of independent helicity amplitudes is $(4S_\psi+2)$, a roughly fourfold reduction in the number of amplitudes occurs. If only pion induced reactions are considered, the number of reduced amplitudes is zero if ψ has spin-parity 0^+, one if ψ is 0^- or 1^-, and two if $\psi=$ is 1^+ or 2^+. This permits one to understand in a simple way the results I derived previously for the reaction $\pi N\to\rho N$ only after a lengthy explicit calculation.[2] Parity conservation requires M to be even if the z axis is chosen normal to the plane of production, while the model imposes $L=1$ for $I_t=0$. This leaves F_{10}^0 as the unique reduced amplitude, in terms of which all helicity or transversity amplitudes may be expressed.

Perhaps the most striking result concerning $I_t=0$ is the prediction that there is no reduced amplitude if the final meson is spinless and of opposite parity to the spinless incident meson. Mechanically, this happens because parity conservation requires M to be odd in a transverse basis, while the model imposes $L=0$, an impossible situation. This means that the cross section for the unobservable reaction $\pi^0 p\to\delta^0 p$ is zero [the δ is now also called the $a_0(980)$]. Although at first glance this appears to be a rather risky prediction, it is worthwhile to push the reasoning somewhat, in terms of possible meson Regge-pole exchanges. It is well known that the δ decays into $\pi+\eta$; hence, the $\pi\eta\delta$ coupling is nonzero. Then, in order to have zero cross section the η must not couple to the nucleon. This conclusion, forced upon us by the model, is in general agreement with the phenomenology of quasi-two-body reactions, which never succeeded in showing convincing evidence for η exchange. I believe this totally unexpected prediction to be one of the most beautiful results to emerge from the model. If an experimental test is desired, the absence of $I_t=0$ amplitudes implies the following relation among the directly observable reactions:[11]

$$d\sigma(\pi^+p\to\delta^+p)+d\sigma(\pi^-p\to\delta^-p)=d\sigma(\pi^-p\to\delta^0n). \quad (24)$$

Although the most spectacular predictions for $I_t=0$ amplitudes are those mentioned above, the model also makes restrictive predictions for pion-induced reactions when the final meson is π (one amplitude instead of two), A_1 (two instead of six), and A_2 (two instead of ten). For these reactions one may derive numerous relations among observable quantities such as spin-density-matrix elements and/or polarization asymmetries. Once again, such predictions may be tested experimentally by forming the appropriate linear combinations of observable reactions.

If $I_t=1$, the values of L allowed by the triangle inequality are $S_\psi-1$, S_ψ, and $S_\psi+1$. The number of reduced amplitudes is rather closer to the number of independent helicity amplitudes, at least when the final baryon is a nucleon. For example, if ψ has spin-parity 0^-, 0^+, 1^-, 1^+, or 2^+, the corresponding numbers of reduced amplitudes are 1, 2, 5, 4, and 8, respectively. These are to be compared with 2, 2, 6, 6, and 10, respectively, when the final baryon is a nucleon and twice as many when it is a Δ. It should be remembered that the reduced amplitudes are the same in both cases. However, considering only reactions with final-state nucleons, it is rather difficult to obtain relations among observable quantities, such as spin-density-matrix elements, since the number of reduced amplitudes is roughly three-fourths the number of helicity amplitudes. The main interest of the model for $I_t=1$ amplitudes is, in my opinion, the possibility of relating observable quantities in reactions where a Δ is produced to those measured when the final baryon is a nucleon. Unfortunately, this can only be done on a reaction-by-reaction basis, as I have not found any simple formula relating the numerical coefficients $C^{1LM}_{m_cm_dm_b}$ for a final nucleon to those for a Δ.

A notable exception occurs when the final meson has spin-parity 0^- (assuming incident pions and final nucleons). The triangle inequality is satisfied only if $L=I_t$. If the quantization axis is chosen such that the incident meson moves along the z direction and the final momentum is in the xz plane, parity conservation implies that the reduced amplitude F^1_{10} is zero. For $I_t=1$, therefore, only the $M=1$ reduced amplitude survives, which implies $m_b=-m_d$. In contrast, for $I_t=0$, $L=M=0$, which implies $m_b=m_d$. Expressed directly in terms of spin-projection amplitudes $A^{I_t}_{m_dm_b}$, the prediction is that A^0_{+-} and A^1_{++} are zero, where $\pm$ denotes $\pm\frac{1}{2}$. Thus the model makes the remarkable prediction that in πN elastic scattering the $I_t=0$ amplitude conserves the nucleon spin projection, while the $I_t=1$ amplitude flips it. The optical theorem then implies that the $I_t=1$ total cross section is zero, or that total cross sections for π^+p and π^-p are equal. The spin behavior predicted by the model is similar to but not identical with the phenomenologically popular s-channel helicity conservation for $I_t=0$, and the notion that the charge-exchange amplitude was pure helicity flip. The difference is that the s-channel helicity z axis makes an angle χ with the boosted z axis in the rest frame of the final nucleon, where χ is the c.m. scattering angle. Once again, the model has produced a reasonably successful prediction in an unexpected direction. It is amusing to note that the orthogonality properties of the coefficients $C^{I_tLM}_{m_cm_dm_b}$ may be held responsible for this prediction: since the $I_t=0$ amplitude preserves the nucleon spin, the only way the cross section can be an incoherent sum of contributions from different isospin is that the $I_t=1$ amplitude be purely nucleon spin flip. Although a general result concerning nucleon target polarization effects is presented below, it is worthwhile to give here the predictions of the model for polarization in πN scattering. Since the $I_t=1$ amplitude is purely nucleon spin flip, there can be no polarization in $\pi^-p\to\pi^0n$. For $\pi^+p\to\pi^+p$, the polarization corresponds to interference between the nonflip and flip amplitudes, and the model imposes no constraint on this quantity. However, the same interference occurs in $\pi^-p\to\pi^-p$, but with opposite sign, leading to the prediction that the polarizations are equal but opposite in sign (the differential cross sections are predicted to be equal for all energies and angles). These predictions, while certainly not strictly true, do bear some resemblance to the experimental data at energies above the resonance region.

C. Comparison with quark-model predictions and exotic exchange

The $I_t=1$ amplitudes in the reactions $\pi N\to\pi\Delta$ or $\pi N\to\eta\Delta$ are also described by a single reduced amplitude (the same one which occurs when the final-state baryon is a nucleon), which has $L=1$ and $M=0$, provided that the z axis is transverse. This implies $m_d=m_b$, which means that the $\pm\frac{3}{2}$ transversity states of the Δ are not produced. This prediction is identical to that of the ρ-exchange model of Stodolsky and Sakurai,[12] as well as with the quark-model predictions of Białas and Zalewski.[13] However, the general quark-model prediction that amplitudes involving two units of spin flip at the baryon vertex vanish is not verified in the model. For example, the amplitudes in the reaction $\pi N\to\omega\Delta$, which is pure $I_t=1$, are related to five reduced amplitudes F^1_{00}, F^1_{10}, F^1_{20}, F^1_{22}, and F^1_{2-2}. The last two reduced amplitudes contribute to some amplitudes which contain two units of transversity flip at the N-Δ vertex. If the reduced amplitudes with $L=2$ were arbitrarily set equal to zero, the single-flip rule could be recovered, but the model provides no reason for doing so.

Perhaps one of the most clearly established rules of the phenomenology of quasi-two-body reactions was the absence of forward and backward direction peaks when the quantum numbers of the exchanged system were "exotic."[14] For example, the reaction $\pi^- p \to \pi^+ \Delta^-$, which is pure $I_t = 2$, has a cross section which decreases dramatically with energy above the resonance region.[15] In the backward direction, the u channel for this reaction is not exotic, but generally backward direction peaks were found to be much smaller than forward peaks. It is, therefore, surprising that the Skyrmion model of Mattis, which otherwise contains a great deal of reasonable phenomenology, imposes few constraints on $I_t = 2$ amplitudes which the standard phenomenology has shown to be very small in comparison with nonexotic amplitudes. The model makes no prediction concerning the relative importance of different t-channel isospin amplitudes. I should specify that by prediction, I mean a conclusion that may be drawn from Eq. (1) without making hypotheses concerning the reduced amplitudes. The empirical fact that $I_t = 2$ amplitudes are small can be used, by means of Eqs. (16), (17), and (18), to obtain information and to place limits on the primordial Skyrmion reduced partial-wave amplitudes $G_{K\tilde{K}\tilde{K}'ll'}$. It is clear, however, that suppression of exotic exchange must be imposed dynamically in the model proposed by Mattis.

D. Predictions of the model for polarized nucleon targets

Although the predictions of the model for polarization in πN elastic scattering have been given above, one may derive a more general prediction concerning target polarization effects, when the incident meson is spinless. By using the sum rule of Biedenharn[16] and Elliott,[17] one may derive the following relation concerning the $C^{I_t LM}_{m_c m_d m_b}$:

$$\sum_{m_c m_d} C^{I_t LM}_{m_c m_d m_b} C^{I_t' L' M'}_{m_c m_d m_b'} = (2R+1)(2R'+1)[(2L+1)(2L'+1)]^{1/2}(-1)^{M-m_b-R'-S_\psi-L-L'} \times \sum_{Tn} (2T+1) \begin{Bmatrix} I_t & I_t' & T \\ R & R & R' \end{Bmatrix} \begin{Bmatrix} I_t & I_t' & T \\ L' & L & S_\psi \end{Bmatrix} \begin{pmatrix} R & R & T \\ m_b' & -m_b & n \end{pmatrix} \begin{pmatrix} L & L' & T \\ M & -M' & -n \end{pmatrix} . \quad (25)$$

This expression is quite useful for discussing effects of target polarization when the spin projections of the final particles are not observed. The quantity T appearing in the sum corresponds to the possible polarization of the target: $T=0$, unpolarized, $T=1$, vector polarization, etc. What is apparent from the structure of this formula is that if $I_t = I_t' = 0$, then only $T=0$ can occur. It then follows that in any pure $I_t = 0$ reaction induced by spinless mesons, there can be *no* target polarization effects when the final-particle spin projections are not observed, a prediction of remarkable generality. Expressed in terms of differential cross sections for charged-pion beams on transversely polarized target protons, it reads

$$d\sigma_\uparrow(\pi^+ p \to \psi^+ p) + d\sigma_\uparrow(\pi^- p \to \psi^- p) - d\sigma_\uparrow(\pi^- p \to \psi^0 n) = d\sigma_\downarrow(\pi^+ p \to \psi^+ p) + d\sigma_\downarrow(\pi^- p \to \psi^- p) - d\sigma_\downarrow(\pi^- p \to \psi^0 n) , \quad (26)$$

where ψ is any meson of unit isospin and the arrows indicate the target polarization. In contrast, $T=1$ may arise either through interference between $I_t = 0$ and $I_t' = 1$ reduced amplitudes, or through $I_t = I_t' = 1$. Note that in the latter case the 3-j symbol is zero if $L = L'$ and $M = M' = 0$, which explains why elastic πN scattering has no polarization in the pure $I_t = 1$ state.

IV. ELASTIC-SCATTERING AMPLITUDES AND TOTAL CROSS SECTIONS

A. Constraints of time-reversal symmetry on the reduced amplitudes

The model proposed by Mattis allows one to express the spin-projection amplitudes in terms of a generally smaller number of reduced amplitudes, as indicated in Eq. (1). If one is interested in elastic scattering, Eq. (1) is not well suited to displaying the reduction of the number of amplitudes caused by time-reversal symmetry. The reason is that the variable y does not treat the initial and final mesons equivalently, since it couples the spin of the incident meson to the t-channel isospin, while coupling the spin of the final meson to the variable L. As we have seen in the previous section, for spinless mesons this simply requires $I_t = L = y$, and leads directly to the prediction that the $I_t = 1$ amplitude is pure spin flip and hence corresponds to zero total cross section. Although the only total cross sections susceptible of measurement concern pion beams, it remains interesting to study the restrictions imposed by the model on elastic-scattering amplitudes for arbitrary mesons. In particular, vector-meson scattering amplitudes are often related to static properties of N and Δ *via* sum rules. It will be shown that rather striking regularities among total cross sections are predicted by the model.

In order to obtain a formula analogous to Eq. (1), but in which the symmetry between initial and final states is manifest, I introduce the new numerical factors $\tilde{C}^{I_t xLM}_{m_c m_d m_a m_b}$, and the reduced amplitudes $\tilde{F}^{I_t}_{xLM}$, which are related to the previously defined $C^{I_t yLM}_{m_c m_d m_a m_b}$ and

$F^{I_t}_{yLM}$ as

$$\tilde{C}^{I_t xLM}_{m_c m_d m_a m_b} = \sum_y [(2x+1)(2y+1)]^{1/2} \times \begin{Bmatrix} I_t & L & x \\ S_\psi & S_\phi & y \end{Bmatrix} C^{I_t yLM}_{m_c m_d m_a m_b} , \tag{27}$$

$$\tilde{F}^{I_t}_{xLM} = \sum_y [(2x+1)(2y+1)]^{1/2} \begin{Bmatrix} I_t & L & x \\ S_\psi & S_\phi & y \end{Bmatrix} F^{I_t}_{yLM} . \tag{28}$$

From the well-known orthogonality property of the 6-j symbols, an equation very similar to Eq. (1) may be derived:

$$A^{I_t}_{m_c m_d m_a m_b} = \sum_{xLM} \tilde{C}^{I_t xLM}_{m_c m_d m_a m_b} \tilde{F}^{I_t}_{xLM} . \tag{29}$$

Explicit formulas for $\tilde{C}^{I_t xLM}_{m_c m_d m_a m_b}$ and $\tilde{F}^{I_t}_{xLM}$ are obtained by replacing the $C^{I_t yLM}_{m_c m_d m_a m_b}$ and the $F^{I_t}_{yLM}$ in Eqs. (27) and (28) by Eqs. (13) and (14). Then, upon using the identities (A6.10) and (A6.39) of Yutsis, Levinson, and Vanagas,[3] one obtains the expressions

$$\tilde{C}^{I_t xLM}_{m_c m_d m_a m_b} = [(2x+1)(2L+1)(2R+1)(2R'+1)]^{1/2}(-1)^{R+R'} \times \sum_{S\mu S'\mu'} (2S+1)(2S'+1)(-1)^{S'-\mu'} \begin{Bmatrix} I_t & L & x \\ R' & S' & S_\psi \\ R & S & S_\phi \end{Bmatrix} \times \begin{bmatrix} S & S' & L \\ -\mu & \mu' & M \end{bmatrix} \begin{bmatrix} S_\phi & R & S \\ m_a & m_b & -\mu \end{bmatrix} \begin{bmatrix} S_\psi & R' & S' \\ m_c & m_d & -\mu' \end{bmatrix} , \tag{30}$$

$$\tilde{F}^{I_t}_{xLM}(\theta',\phi',\theta,\phi) = [(2x+1)(2L+1)]^{1/2} \times \sum_{\substack{lml'm' \\ K\tilde{K}\tilde{K}'}} (2K+1)[(2\tilde{K}+1)(2\tilde{K}'+1)]^{1/2}(-1)^{\Omega} \times \begin{Bmatrix} \tilde{K} & \tilde{K}' & x \\ l & l' & L \\ I_\phi & I_\psi & I_t \end{Bmatrix} \begin{Bmatrix} \tilde{K} & \tilde{K}' & x \\ S_\psi & S_\phi & K \end{Bmatrix} \begin{bmatrix} l' & l & L \\ m' & -m & -M \end{bmatrix} G_{K\tilde{K}\tilde{K}'ll'} Y^{m'}_{l'}(\theta',\phi') Y^{m*}_{l}(\theta,\phi) ,$$

$$\Omega = -S_\psi + l - m' + \tilde{K}' - K + I_\psi - x . \tag{31}$$

The new variable x is linked in a much more symmetric way to the initial- and final-state quantum numbers S_ϕ, $\tilde{K}$, S_ψ, and $\tilde{K}'$. Accordingly one may show, from Eqs. (30) and (31), *for elastic scattering* ($\phi=\psi$, $R=R'$, $G_{K\tilde{K}\tilde{K}'ll'} = G_{K\tilde{K}'\tilde{K}l'l}$) that

$$\tilde{C}^{I_t xLM}_{m_c m_d m_a m_b} = (-1)^{I_t+x+L+M} \tilde{C}^{I_t xL-M}_{m_a m_b m_c m_d} , \tag{32}$$

$$\tilde{F}^{I_t}_{xLM}(\pi/2,\pi/2+\chi,\pi/2,\pi/2) = (-1)^{I_t+x+L+M} e^{i\chi M} \tilde{F}^{I_t}_{xL-M}(\pi/2,\pi/2+\chi,\pi/2,\pi/2) , \tag{33}$$

where χ is the scattering angle. From Eq. (33) it follows that if $M=0$, the reduced amplitude vanishes unless I_t+L+x is even. If $I_t=0$, then $x=L$, from the 9-j symbol, and the condition is always satisfied. In contrast, if $I_t=1$, then one must have $x=L\pm1$ in order that the reduced amplitude be nonzero. If only nucleon targets are considered, the $I_t=0$ and $I_t=1$ amplitudes are the only ones which occur, and one may determine the number of independent reduced amplitudes as

$I_t=0,\ 0\le x\le 2S_\phi,\ L=x,\ 0\le M\le L,\ M$ even, yielding $(S_\phi+1)^2$ amplitudes ,

$I_t=1,\ 0\le x\le 2S_\phi,\ L=x,\ 0<M\le L,\ M$ even ,

$L=x\pm1,\ 0\le M\le L,\ M$ even, yielding $(S_\phi+1)(3S_\phi+1)$ amplitudes .

From these expressions one finds one reduced elastic amplitude for elastic scattering of spinless mesons in both $I_t=0$ and $I_t=1$, as was shown in the previous section. For elastic scattering of spin-1 mesons there are four reduced amplitudes with $I_t=0$ and eight with $I_t=1$, compared to 12 independent helicity amplitudes for each I_t. Once again a considerable reduction in the number of amplitudes is predicted by the model.

B. Forward direction reduced amplitudes and the optical theorem

The imaginary parts of the elastic-scattering amplitudes, evaluated in the forward direction, are related to the total cross section through the optical theorem. In order to examine the constraints among cross sections imposed by Eq. (29), it is convenient to choose the z axis along the incident beam direction. When this is done, conservation of angular momentum implies that only those amplitudes for which $M=0$ are nonzero in the forward direction. Furthermore, the reduced amplitudes with L odd are zero in the forward direction, and it is sufficient to consider only those reduced amplitudes $\tilde{F}^{I_t}_{xLM}$ with L even and $M=0$. The counting of such amplitudes is straightforward:

$I_t=0$, $0\le x\le 2S_\phi$, $L=x$, L even, yielding $S_\phi+1$ forward reduced amplitudes ,

$I_t=1$, $0\le x\le 2S_\phi$, $L=x\pm 1$, L even, yielding $2S_\phi$ forward reduced amplitudes .

The general number of independent nonvanishing forward amplitudes is $3S_\phi+1$ for either isospin, and once again a considerable reduction takes place.

C. The spin-averaged forward amplitude

From the definition of the coefficients $\tilde{C}^{I_t xLM}_{m_c m_d m_a m_b}$ it is easy to show that, if $I_t=x=L=M=0$,

$$\tilde{C}^{0000}_{m_c m_d m_a m_b}=\left[\frac{2R+1}{2S_\phi+1}\right]^{1/2}\delta_{S_\psi S_\phi}\delta_{RR'}\delta_{m_c m_a}\delta_{m_d m_b}\ , \tag{34}$$

which means that the corresponding reduced amplitude $\tilde{F}^{0}_{000}$ appears only in amplitudes which preserve the spin projections. As a consequence of the orthogonality properties of the $\tilde{C}^{I_t xLM}_{m_c m_d m_a m_b}$, which are identical to those of the $C^{I_t yLM}_{m_c m_d m_a m_b}$, one may prove that, for *elastic scattering*,

$$\sum_{m_a m_b} A^{I_t}_{m_a m_b m_a m_b}=[(2R+1)^3(2S_\phi+1)]^{1/2}\delta_{I_t 0}\tilde{F}^{0}_{000}\ , \tag{35}$$

which means that spin averaging the forward amplitudes is equivalent to projecting out the $I_t=x=L=0$ reduced amplitude. If one uses the isospin crossing relations,[8] one finds for the spin-averaged elastic amplitude

$$[(2R+1)(2S_\phi+1)]^{-1}\sum_{m_a m_b} A^{I_s}_{m_a m_b m_a m_b} =(2S_\phi+1)^{-1/2}\tilde{F}^{0}_{000}\ , \tag{36}$$

independent of the s-channel isospin. This implies that the spin-averaged total cross section for meson-nucleon or meson-Δ scattering is independent of the charge states of either the mesons or the baryons. This is another firm prediction of the model, which is extremely general in scope.

D. Consistency with forward dispersion relations and sum rules

In the model of Mattis, all forward-direction spin-averaged amplitudes with $I_t\neq 0$ are zero. If one considers the forward-direction dispersion relation for the difference of the π^+p and π^-p elastic amplitudes, given, for example, in Ref. 5, equality of the total cross sections means that the forward $I_t=1$ amplitude is determined entirely by the one-particle intermediate state in π^-p, the neutron. One may then ask how the model can accommodate a zero forward amplitude, instead of the one-neutron contribution. The answer is that the model predicts a Δ degenerate with the nucleon, and which also contributes through one-particle intermediate states in $\pi^\pm p$. In π^+p the Δ^{++} is an allowed intermediate state, while the Δ^0 occurs in π^-p. The latter contributes only one-third of the former, with opposed sign. Therefore the necessary condition that the one-particle intermediate states cancel is that the neutron contribution be just two-thirds of that of the Δ^{++}. In this way the model imposes a relation between the renormalized πNN and $\pi N\Delta$ coupling constants. Similarly, sum rules which depend on the integrals of $I_t=1$ total cross sections, such as that of Adler[18] and Weisberger,[19] and that of Cabibbo and Radicati,[20] are predicted to have no continuum contribution, according to the model. On the other hand, the contribution of the Δ one-particle intermediate state must be taken into account along with the neutron in evaluating the commutators. In this way relations among the static properties of the nucleon and Δ may be derived. It would be of interest to compare the results of such sum rules with the direct computation in the Skyrmion model of such static properties, such as that of Adkins, Nappi, and Witten.[21]

V. CONCLUSIONS

In this paper I have shown that in a world in which the formula derived by Mattis for the partial-wave amplitudes in meson-baryon two-body reactions were exact, then Eq. (1) would follow, and all the consequences which I have examined in Secs. III and IV would be observed experimentally. The formula derived by Mattis is based on the Skyrmion model for the baryons, which is itself supposed to represent the large-N limit of QCD (Ref. 21). It is therefore of interest to see whether the model provides a reasonable approximation to the real world. The fact that the nucleon and Δ are not degenerate indicates that one must not expect total agreement between the model and reality. Once this is admitted, however, the general trend toward reasonable agreement with some of the more sweeping predictions of the model cannot be denied. Here I have assumed that the model is fully general, and does not apply only at low energies. The fact that the model relates different amplitudes at fixed t-channel isospin, and then imposes that amplitudes of different I_t do not interfere in the differential cross sections for any reaction is a totally unexpected consequence of Mattis's expression for the partial-wave amplitudes. On the other hand, the model shows no tendency to suppress exotic $I_t=2$ exchanges, which are known empirically to be tiny. Indeed, the model always allows maximum freedom to the highest possible I_t amplitudes, while severely limiting $I_t=0$ amplitudes. It would be of great interest to find a natural mechanism in the Skyrmion model which would lead to the suppression of exotic amplitudes. Care must be taken not to achieve too much suppression, since these exchanges often correspond to allowed u-channel exchanges, and thus to backward rather than forward peaks in the differential cross sections. To this end, Eq. (17), which relates the $G_{K\tilde{K}\tilde{K}'ll'}$ of Mattis to my $\tilde{G}_{l_y L l l'}$, may permit one to achieve any desired amount of suppression, while skirting the essential problem of how to justify it in terms of the original calculation of Mattis. In my opinion the remarkable predictive powers of the model in the $I_t=0$ and $I_t=1$ sectors justify the search for a simple extension of Mattis's method which would effect the suppression of exotic exchanges. If such an extension were found one would possess a predictive model, based on a fundamental theory, which provides a reasonable approximation to the phenomenology of two-body reactions. Mattis[1] has pointed out that the $G_{K\tilde{K}\tilde{K}'ll'}$ are calculable quantities, at least in principle. I find that the approach of Mattis is remarkably predictive even when the $G_{K\tilde{K}\tilde{K}'ll'}$ are free, which leaves one the hope that two-body hadronic reactions may someday be computed from first principles.

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