PHYSICAL REVIEW D VOLUME 37, NUMBER 2 15 JANUARY 1988

Brief Reports

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On the "averaged weak energy condition" and Penrose's singularity theorem

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(Received 21 August 1987)

In a previous paper, we used techniques developed by Tipler to prove that Penrose's singularity theorem will still hold if the weak energy condition is replaced by a weaker nonlocal energy condition and if the null generic condition is satisfied. The purpose of the present paper is to strengthen our original result somewhat by slightly weakening the nonlocal energy condition and by showing that the null generic condition is unnecessary. We also comment briefly on whether recent criticisms of Tipler's techniques affect the validity of our results. In addition, we take the opportunity to rewrite some of our previous analysis in what we feel is a more direct mathematical language and, in so doing, we provide the details of a proof which were not included in our earlier paper.

The singularity theorems of Hawking, Penrose, and Geroch [see Hawking and Ellis[1] (HE)] have been essential in demonstrating that singularities are a generic feature of general relativity and not simply the result of highly idealized situations, such as the exact spherically symmetric collapse of a star. A key assumption in all of these theorems is some sort of restriction on the stress-energy tensor of matter. The weakest (local) such condition that has been used in the theorems is the so-called "weak energy condition" (WEC):

$$T_{ab}U^aU^b \geq 0 \tag{1}$$

for all timelike vectors U^a. By continuity, this condition holds for all null vectors as well. The WEC is used primarily in theorems which prove null geodesic incompleteness, such as Penrose's singularity theorem.[1,2] This theorem can be used to prove the existence of a singularity in the end point of the evolution of certain massive stars.

Recently[3] we discussed several types of quantum vacuum stress-energy tensors that violate the WEC. Extending earlier results of Tipler,[4] we also showed that Penrose's theorem would still hold if the WEC were replaced by a weaker "averaged WEC" and if the "null generic condition" (HE, p. 101) holds. It has since been pointed out to us[5] that the latter condition is unnecessary, and that in fact our earlier result can be strengthened somewhat. The purpose of the present paper is to demonstrate this and, in so doing, to also rewrite some of our previous analysis in Sec. IV of Ref. 3 in what we feel is a clearer mathematical language. That analysis was framed, after the manner of Tipler,[4] in terms of integral restrictions on the function

$$F(\lambda)=\tfrac{1}{2}(R_{ab}K^aK^b+2\sigma^2)\ ,$$

where K^a is the tangent vector to a null geodesic $\gamma(\lambda)$ with λ being an affine parameter along the geodesic and σ being the shear of the geodesic congruence (see Ref. 1 for notation and details). Here discussion will be concerned only with integral restrictions on $R_{ab}K^aK^b$ directly, and thus we hope that our methods will be more transparent to the reader. Using this language, it will be easy to see why the null generic condition is superfluous.

Consider a congruence of null geodesics orthogonal to a spacelike two-surface S. Let $\gamma(\lambda)$ be a null geodesic in this congruence and let λ be an affine parameter along γ and K^a be the tangent vector to γ. The expansion θ of the congruence is defined by $\theta\equiv\nabla_aK^a$, and obeys the Raychaudhuri equation

$$\frac{d\theta}{d\lambda}=-R_{ab}K^aK^b-2\sigma^2-\frac{\theta^2}{2}\ . \tag{2}$$

Definition. A point p will be said to be *conjugate* to a spacelike two-surface S along the null geodesic $\gamma(\lambda)$ which intersects S orthogonally if $|\theta|\to\infty$ at p. The initial value of θ is ${}_2\chi_{ab}g^{ab}$, where χ_{ab} is the second null fundamental form of S (HE, pp. 101 and 102).

Theorem 1. Let $\gamma(\lambda)$ be a future-complete null geodesic which intersects a spacelike two-surface S orthogonally at the point $\gamma(0)\equiv\gamma(\lambda)\cap S$. Let $R_{ab}K^aK^b$ be finite and continuous along $\gamma(\lambda)$ for all $\lambda\in[0,+\infty)$.

If for any $\delta>0$, $\exists\lambda_1$ such that

$$\int_0^{\lambda} R_{ab}K^aK^b d\lambda \geq -\delta,\quad \forall\lambda\geq\lambda_1\ ; \tag{3}$$

and if the initial value of the expansion $\theta_0={}_2\chi_{ab}g^{ab}$ is negative, then there will be a point conjugate to S along

$\gamma(\lambda)$ for some value of $\lambda \in (0, +\infty)$. [The inequality (3), and some parts of the following proof were suggested to the author by Borde.[5]]

Proof. Our style of proof is similar to that of Tipler.[6] We want to show that $\theta \to -\infty$ at some $\lambda > 0$. Let us assume to the contrary that θ is finite everywhere in the interval $\lambda \in [0, +\infty)$. From the Raychaudhuri equation (2), we can write

$$\theta(\lambda) = -\int_0^\lambda R_{ab} K^a K^b d\lambda - \int_0^\lambda 2\sigma^2 d\lambda - \int_0^\lambda \frac{\theta^2}{2} d\lambda + \theta_0 . \quad (4)$$

Since the inequality (3) must be true for *any* $\delta > 0$, let us arbitrarily pick $\delta = -\theta_0/2$, with $\theta_0 < 0$. Then by assumption there must exist *some* λ_1, such that

$$\int_0^\lambda R_{ab} K^a K^b d\lambda \geq \frac{\theta_0}{2}, \quad \forall \lambda \geq \lambda_1 . \quad (5)$$

Using (5) and the fact that $\sigma^2 \geq 0$ and $\theta_0 < 0$, we have

$$-\int_0^\lambda R_{ab} K^a K^b d\lambda - \int_0^\lambda 2\sigma^2 d\lambda + \frac{\theta_0}{2} \leq 0, \quad \forall \lambda \geq \lambda_1 . \quad (6)$$

Adding $\theta_0/2$ to both sides of (6) and using (4) then gives

$$\theta \leq -\int_0^\lambda \frac{\theta^2}{2} d\lambda + \frac{\theta_0}{2}, \quad \forall \lambda \geq \lambda_1 . \quad (7)$$

Recalling that θ_0 is negative by assumption, we can also write

$$\theta < -\int_0^\lambda \frac{\theta^2}{2} d\lambda - c < 0, \quad \forall \lambda \geq \lambda_1 , \quad (8)$$

where $c \equiv -\theta_0/4 > 0$. Now let

$$N(\lambda) \equiv \int_0^\lambda \frac{\theta^2}{2} d\lambda + c > 0 .$$

Therefore,

$$N'(\lambda) = \frac{\theta^2}{2} > \frac{1}{2}\left[\int_0^\lambda \frac{\theta^2}{2} d\lambda + c\right]^2 = \frac{N^2}{2}, \quad \forall \lambda \geq \lambda_1 ,$$

using (8). Since $N > 0$ and nondivergent if θ is nondivergent, we can write $N'(\lambda)/N^2(\lambda) > \frac{1}{2}$.

Choose $\lambda_1 < \lambda_2$ and integrate

$$\int_{\lambda_1}^{\lambda_2} \frac{N'(\lambda)}{N^2} d\lambda > \tfrac{1}{2} \int_{\lambda_1}^{\lambda_2} d\lambda$$

to give

$$-\frac{1}{N(\lambda_2)} > \frac{\lambda_2 - \lambda_1}{2} - \frac{1}{N(\lambda_1)} , \quad (9)$$

for all $\lambda \in (\lambda_1, +\infty)$. If we choose λ_2 large enough so that the right-hand side of (9) is positive [i.e., choose $\lambda_2 > 2/N(\lambda_1) + \lambda_1$] then that implies $N < 0$. But this is a contradiction since $N(\lambda) > c > 0$, in this interval. So N must diverge in this interval. Therefore, θ must also diverge somewhere in the interval $\lambda \in (0, +\infty)$. If θ is diverging in some interval, then $-\theta^2$ diverges negatively in this interval. From the Raychaudhuri equation (2), we see that as long as $-R_{ab} K^a K^b$ is finite and continuous at each point (whatever its sign), then for large enough values of θ, $d\theta/d\lambda < 0$. So if θ diverges it must diverge to $-\infty$. [We could also have seen this from Eq. (8). Since $\theta(\lambda) < -N(\lambda)$ and if N diverges for large enough λ, then θ is always decreasing in that region and so $d\theta/d\lambda < 0$.]

In our earlier paper,[3] we used instead of inequality (3) the condition

$$\liminf_{\lambda' \to +\infty} \int_0^{\lambda'} R_{ab} K^a K^b d\lambda \geq 0 , \quad (10)$$

the equality holding only if $R_{ab} K^a K^b \equiv 0$ at every point of γ for $\lambda \in [0, +\infty)$. Inequality (10) with the "greater than" sign holding means that there is a number $\lambda_1 > 0$ and a number $c > 0$ for which

$$\int_0^\lambda R_{ab} K^a K^b d\lambda \geq c \quad \text{for any } \lambda \in (\lambda_1, +\infty) . \quad (11)$$

Inequality (11) says that the integral on the left-hand side must be positive for large enough values of λ. One drawback of the focusing condition (10) and the analogous focusing conditions of Tipler[4,6] is that they do not cover cases in which there exists just enough positive $R_{ab} K^a K^b$ to counterbalance any negative $R_{ab} K^a K^b$, so that the above integral exactly equals zero without having $R_{ab} K^a K^b \equiv 0$ at *every* point of γ. Such cases would most likely occur only under highly contrived conditions and thus are probably not physically realistic. However, as pointed out by Borde,[5] we can also cover these cases by replacing inequality (10) by inequality (3). This requirement essentially says the same thing as our original inequalities (10) and (11), except for the slightly weaker formal restriction that $\int_0^\lambda R_{ab} K^a K^b d\lambda$ is now only required to be non-negative, but not necessarily positive. Since δ can be chosen to be arbitrarily small, the right-hand side of inequality (3) can be made arbitrarily close to zero.

We are now ready to prove the following.

Theorem 2 (modified Penrose theorem). Spacetime $(\mathcal{M}, g)$ cannot be null geodesically complete if (1) there is a closed trapped surface $\mathcal{T}$ in $\mathcal{M}$, (2) for any $\delta > 0$, $\exists \lambda_1$ such that

$$\int_0^\lambda R_{ab} K^a K^b d\lambda \geq -\delta, \quad \forall \lambda \geq \lambda_1 ,$$

along every future-complete null geodesic $\gamma(\lambda)$ orthogonal to $\mathcal{T}$ [K^a is the tangent vector to $\gamma(\lambda)$, λ is an affine parameter, and $\gamma(0) \equiv \gamma(\lambda) \cap \mathcal{T}$], and (3) there is a noncompact Cauchy surface $\mathcal{H}$ in $\mathcal{M}$.

Note. The expression $\int_0^\lambda R_{ab} K^a K^b d\lambda \geq -\delta$ and the Einstein equations $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$ imply that

$$\int_0^\lambda T_{ab} K^a K^b d\lambda \geq -\delta \quad (12)$$

for any $\delta > 0$ and $\forall \lambda \geq \lambda_1$, provided K^a is a null vector. This condition says that for large enough values of the affine parameter λ, the integral on the left-hand side of (12) must be non-negative. Therefore, if inequality (12) is satisfied, then the WEC holds on the average along a null geodesic orthogonal to $\mathcal{T}$, where the average is taken over the history of the null geodesic to the future of $\mathcal{T}$. Relation (12) is what we refer to as the "averaged weak energy condition" and it implies condition (2) above.

Proof. Suppose $\mathcal{M}$ were null geodesically complete. By the definition of a closed trapped surface, ${}_1\chi_{ab}g^{ab}$ and ${}_2\chi_{ab}g^{ab}$, the two second null fundamental forms of $\mathcal{T}$ are negative. Therefore, conditions (1) and (2) together with our theorem 1 imply that there will be a point conjugate to $\mathcal{T}$ along every complete future-directed null geodesic orthogonal to $\mathcal{T}$ for some value of $\lambda \in (0, +\infty)$. The rest of the proof is the same as in Ref. 3.

In our previous paper,[3] we used the additional assumption of the null generic condition to cover us in the "loophole" case [see inequality (10) of this paper] when $R_{ab}K^aK^b \equiv 0$. This condition guarantees that a null geodesic will experience some effective curvature at least at one point in its history. Our purpose was to ensure that two neighboring geodesics which started out parallel to one another would eventually experience some focusing. The trapped surface condition (1) guarantees that any null geodesic orthogonal to $\mathcal{T}$ will experience nonzero effective curvature at the point $\gamma(0)$. This condition, as shown by Proposition 4.4.6 of HE, is sufficient for the existence of conjugate points when $R_{ab}K^aK^b \equiv 0$. Therefore, in the original proof of the modified Penrose theorem, the assumption of the null generic condition, although not very restrictive, is superfluous.

At this juncture we take the opportunity to also point out two minor corrections to Ref. 3. On p. 3530 the definition of the local energy flow vector S^a is missing a minus sign; it should read $S^a = -T^{ab}U_b$. The second initial condition in Eqs. (21) and (23) is missing a factor of $\frac{1}{2}$; it should read

$$\left.\frac{dx}{d\lambda}\right|_{\lambda=0} = \tfrac{1}{2}\chi^a_a \ .$$

Comment. Chicone and Ehrlich[7] (CE) have recently argued that one must be careful in assuming that conjugate solutions of Tipler's equation [Eq. (21) in our original paper[3]],

$$\frac{d^2x}{d\lambda^2} + F(\lambda)x = 0 \ , \qquad (13)$$

which is essentially the Raychaudhuri equation rewritten via a change of variables [see Tipler[4,6] and CE (Ref. 7) for details], necessarily imply the existence of conjugate points in actual congruences of geodesics. They provide a counterexample involving a timelike geodesic $\gamma(\lambda)$ in a three-dimensional Minkowski spacetime along which formally $|\theta(\lambda)| \to \infty$ (i.e., $x=0$) at two points $\lambda = 0$ and $\lambda = 1$. Obviously this geodesic cannot contain any true conjugate points since the Riemann curvature tensor vanishes in flat spacetime. This seeming contradiction arises from the fact that CE deliberately chose a Jacobi tensor $A_{\alpha\beta}$ (defined on pp. 96 and 97 of HE) associated with $\gamma(\lambda)$ that does not satisfy certain specific initial conditions. They emphasize that such initial conditions on $A_{\alpha\beta}$ (again, see CE, p. 16 and HE, p. 100 for details) must be satisfied to ensure that there actually exist physical congruences of geodesics with conjugate points corresponding to the conjugate solutions of (13). No such congruences exist in their example. In our proof, we are guaranteed the existence of the relevant geodesic congruences by our assumption in theorem 2 of a closed trapped surface (CTS). A CTS is defined as a closed spacelike two-surface $\mathcal{T}$ such that the two families of null geodesics orthogonal to $\mathcal{T}$ are converging at $\mathcal{T}$. Thus the very definition of a CTS singles out congruences of null geodesics with the required initial conditions [HE, p. 102, Eq. (4.47), and p. 262].

It should be mentioned that our focusing condition (3) does not cover cases in which the sign of $R_{ab}K^aK^b$ oscillates periodically; one example would be $R_{ab}K^aK^b \sim \sin k\lambda$. These cases cannot be readily ruled out. In a more general analysis, Borde[8] has recently shown that one can obtain conjugate points in congruences of geodesics under even weaker integral restrictions than we or Tipler have assumed. His results show that it is sufficient that $\int R_{ab}K^aK^b d\lambda$ visit a neighborhood of zero frequently and that the parameter interval that it spends in this neighborhood on each visit not approach zero. Borde's focusing conditions cover some situations in which there might be regularly repeating violations of the energy conditions, as in the example given above.

The author is deeply grateful to Arvind Borde for kindly suggesting most of the improvements reported in this paper, and for many helpful discussions.

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