Some recoil corrections to the hydrogen hyperfine splitting

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We compute all of the recoil corrections to the ground-state hyperfine splitting in hydrogen, with the exception of the proton polarizability, that are required to achieve an accuracy of 1 ppm. Our approach includes a unified treatment of the corrections that would arise from a pointlike Dirac proton and the corrections that are due to the proton's non-QED structure. Our principal new results are a calculation of the relative order-$\alpha^2(m_e/m_p)$ contributions that arise from the proton's anomalous magnetic moment and a systematic treatment of the relative order-$\alpha(m_e/m_p)$ contributions that arise from form-factor corrections. In the former calculation we introduce some new technical improvements and are able to evaluate all of the expressions analytically. In the latter calculation, which has been the subject of previous investigations by other authors, we express the form-factor corrections in terms of two-dimensional integrals that are convenient for numerical evaluation and present numerical results for the commonly used dipole parametrization of the form factors. Because we use a parametrization of the form factors that differs slightly from the ones used in previous work, our numerical results are shifted from older ones by a small amount.

I. INTRODUCTION

The hyperfine splitting (hfs) between the spin-0 and spin-1 levels in the hydrogen ground state is one of the most accurately measured quantities in physics. The most recent experimental determinations of this quantity[1,2] give

$$\nu(\text{expt})=1420.405\,751\,766\,7(9)\ \text{MHz}\ . \tag{1.1}$$

Various theoretical contributions to the hydrogen ground-state hfs have been calculated over the years. The most important of these do not involve proton recoil or the dynamics of the proton structure; these "QED" contributions are

$$\Delta E(\text{QED})=E_F\left\{1+\tfrac{3}{2}(Z\alpha)^2+a_e+\alpha(Z\alpha)(\ln 2-\tfrac{5}{2})-\frac{8\alpha(Z\alpha)^2}{3\pi}\ln(Z\alpha)[\ln(Z\alpha)-\ln 4+\tfrac{281}{480}]+\frac{\alpha(Z\alpha)^2}{\pi}(15.38\pm 0.29)+\frac{\alpha^2(Z\alpha)}{\pi}D_1\right\}, \tag{1.2}$$

where m_e and m_p are the electron and proton masses, a_e is the electron's anomalous-moment coefficient, α is the fine-structure constant, and E_F is the Fermi[3] splitting:

$$E_F=\tfrac{8}{3}\alpha^4c^2\frac{m_r^3}{m_em_p}(1+\kappa)=\tfrac{16}{3}\alpha^2\left[hcR_\infty\frac{m_r}{m_e}\right]\frac{m_r^2}{m_em_p}(1+\kappa)\ . \tag{1.3}$$

Here κ is the proton's anomalous moment coefficient, m_r is the reduced mass $[m_r=m_em_p/(m_e+m_p)]$, and the Rydberg R_∞ is given by

$$R_\infty=m_ec\alpha^2/2h\ . \tag{1.4}$$

We have displayed the nuclear charge Z in (1.2) in order to make clear the separation between the binding effects and radiative corrections.

The expression (1.2) arises from contributions whose characteristic momenta are of order m_e or less. They do not involve the details of the proton structure, but they *are* sensitive to the low-energy properties of the proton. Thus, a factor $(1+\kappa)$ representing the proton's total magnetic moment appears in the Fermi splitting. The expressions are identical to those for muonium, except that the muon mass and anomalous moment have been replaced with the corresponding values for the proton. We note that in the discussion of muonium by Bodwin, Yennie, and Gregorio,[4,5] the anomalous-moment contribution is not incorporated into E_F. The definition used here is more usual for the hydrogen hfs.

The term $\frac{3}{2}(Z\alpha)^2$ is the so-called "Breit correction,"[6] which is the result of the relativistic corrections to the Fermi splitting that arise from the Dirac equation. The radiative correction proportional to $\alpha(Z\alpha)$ was first calculated by Kroll and Pollock[7,8] and Karplus, Klein, and Schwinger.[9] This result was obtained more simply by Sapirstein, Terray, and Yennie,[10] as part of a calculation directed toward radiative-recoil corrections. The $\ln\alpha$ parts of the radiative corrections proportional to $\alpha(Z\alpha)^2$ were originally computed by Layzer[11,12] and Zwanziger,[13,14] and were confirmed by Brodsky and Erickson,[15] who also estimated the corresponding nonloga-

rithmic term. The actual computation of the nonlogarithmic term was carried out by Sapirstein,[16] who used numerical techniques to evaluate the relevant integrals. The D_1 term represents radiative corrections involving two virtual photons, which have yet to be calculated.

We may now evaluate the known QED contributions to the hfs, as given in (1.2), using the lastest values for the fundamental constants:[17] $\alpha^{-1}=137.035\,989\,5(61)$, $m_e=0.510\,999\,06(15)$ MeV, $m_p=938.272\,31(28)$ MeV, and $\kappa=1.792\,847\,386(63)$. The Fermi splitting in frequency units is $\nu_F=1418.840\,25(14)$ MHz; the dominating uncertainty arises from the value of the fine-structure constant. With the corrections given in (1.2) taken into account, the theoretical value of the hyperfine splitting is

$$\nu(\mathrm{QED})=1420.451\,95(14)\ \mathrm{MHz}\ . \tag{1.5}$$

The difference between QED theory and experiment is then

$$\frac{\nu(\mathrm{QED})-\nu(\mathrm{expt})}{\nu_F}=32.56(10)\ \mathrm{ppm}\ . \tag{1.6}$$

In order to take full advantage of the refinements described so far, it would be necessary to calculate the recoil and dynamical corrections involving the proton's structure to about 0.1 ppm relative accuracy. Although such a goal has not yet been achieved, substantial progress has been made in working out these structure-dependent corrections. They are usually expressed as

$$\Delta E(\mathrm{structure})=E_F[\delta_p(\mathrm{rigid})+\delta_p(\mathrm{polarizability})]\ . \tag{1.7}$$

The quantity δ_p(rigid) is computed by using elastic form factors to approximate the electromagnetic interactions of the proton. The term δ_p(polarizability), the proton polarizability correction, contains all of the effects of the dynamics of the proton that are not included in δ_p(rigid).

The most important structure-dependent correction does not involve recoil; it is known as the "nonrelativistic size correction." In computing this correction, one treats the proton as a nonrecoiling particle with a fixed charge-current distribution of finite extent. The nonrelativistic size correction was first analyzed by Zemach;[18] hence, we denote it by δ_p(Zemach) and write

$$\delta_p(\mathrm{rigid})=\delta_p(\mathrm{Zemach})+\delta_p(\mathrm{recoil})\ , \tag{1.8}$$

where δ_p(recoil) contains all of the elastic recoil corrections to the hydrogen hfs. The nonrelativistic size correction is given by

$$\delta_p(\mathrm{Zemach})=-2m_r\alpha R_p\ , \tag{1.9}$$

where R_p is a mean radius associated with the proton's charge-current distribution. Accurate calculation of δ_p(Zemach) requires good knowledge of the elastic form factors from experiment. When evaluated with the commonly used dipole parametrization of the elastic form factors [see (5.20)], it gives a contribution of -38.72 ppm, which removes most of the difference between theory and experiment. The uncertainty in the Zemach correction is a rather complex issue, and we defer the discussion of it to Secs. V and VI. The existing literature contains an estimate of the uncertainty in the total structure-dependent contribution of 0.9 ppm (Ref. 19). We have been unable to determine the basis for that estimate.

Now let us turn to the effects of recoil. In muonium, the contributions arising from recoil corrections take the form

$$\delta_\mu(\mathrm{recoil})=\frac{1}{1+\kappa}\left[\frac{-3\alpha}{\pi}\frac{m_e m_\mu}{m_\mu^2-m_e^2}\ln\frac{m_\mu}{m_e}+\alpha^2\frac{m_e}{m_\mu}\left[2\ln\frac{1}{2\alpha}-6\ln 2+3\tfrac{11}{18}\right]\right]\ , \tag{1.10}$$

where we have ignored the distinction between m_r and m_e in the last term. The first term here is due to Arnowitt.[20] Of the terms containing α^2, the term proportional to $\ln\alpha$ was calculated by Lepage[21] and Bodwin and Yennie[22] and the constant term was calculated by Bodwin, Yennie, and Gregorio.[4,5] A sum over states contribution occurring in the constant term was calculated by Caswell and Lepage,[23] who also confirmed the total constant by a numerical calculation using an effective Lagrangian approach.[24] The absence in this term of contributions containing a factor $\ln(m_\mu/m_e)$ was first demonstrated by Bodwin, Yennie, and Gregorio,[25] and is a consequence of a phenomenon known as the Caswell-Lepage cancellation.[26] In addition to the terms shown in (1.10), there are radiative-recoil corrections computed by Caswell and Lepage[26] and Sapirstein, Terray, and Yennie;[10] they are of relative order $(\alpha/\pi)^2(m_e/m_\mu)$ and contain factors of $\ln(m_\mu/m_e)$. Terms containing factors of $\ln(m_\mu/m_e)$ cannot reliably be taken over for hydrogen hfs because they contain contributions from momenta much greater than m_e, and so would involve the proton's structure. On the other hand, the second term of (1.10), which contains no logarithm of the heavy-particle mass, involves momenta of order m_e and less. Thus, in the case of the hydrogen hfs, such contributions would not show a sensitivity to the proton's structure.

In hydrogen, the term δ_p(recoil) replaces the expression (1.10). The one-loop [relative order $\alpha(m_e/m_p)$] contributions to δ_p(recoil) correspond to the first term of (1.10). These contributions involve characteristic momenta that range from $\sim m_e$ to $\sim m_p$, since the scale of the proton's charge-current distribution is characterized by the mass of the ρ meson ($m_\rho\approx 750$ MeV). The one-loop recoil corrections were initially estimated by Arnowitt,[20] and Newcomb and Salpeter.[27] The first actual calculation was carried out by Iddings and Platzman,[28] who found that the combination of δ_p(Zemach) and δ_p(recoil) gave a contribution of -35 ppm. Grotch and Yennie[29] later arrived at a similar total (-34.6 ppm), but were able to carry out the calculation in a way that separated δ_p(recoil) from the Zemach correction. The small net size for δ_p(recoil) is deceptive, since individual parts are comparable in size to the Zemach contribution. These structure-dependent recoil and Zemach correc-

tions are one subject of this paper. Also included in δ_p(recoil) are relative order-$\alpha^2(m_e/m_p)$ corrections, corresponding to the second major term in (1.10). As we have already mentioned, these contributions involve momenta of order m_e or less. Hence, in analyzing contributions to the hydrogen hfs of this type, one can make use of any results that were derived originally for the muonium hfs. The recoil contributions in relative order α^2 that arise from the anomalous moment of the heavy particle have not previously been computed. Such anomalous-moment recoil contributions are important in the hydrogen hfs at the present level of precision, and they are the main subject of this paper.

Finally, we turn to the term δ_p(polarizability). Its size was first estimated by Iddings,[30] and it was analyzed further by Drell and Sullivan.[31] Some progress has been made since that initial work. For details we refer to the reader to the review of Hughes and Kuti.[32] Here we merely note that de Rafael[33] and Gnädig and Kuti[34] have shown how to use data from inelastic electron scattering with polarized beam and target to put a bound on δ_p(polarizability). In their review, Hughes and Kuti give $|\delta_p(\text{polarizability})| < 4$ ppm. On the other hand, if we evaluate the discrepancy between the present theory and experiment, omitting δ_p(polarizability) and using the value of Grotch and Yennie for the sum of δ_p(Zemach) and δ_p(recoil), then the result is $-2.0(9)$ ppm (theory minus experiment), which leaves little room for a polarizability correction.

As stated earlier, one aim of this paper is to compute the relative order-α^2 recoil corrections to the hydrogen ground-state hfs due to the proton's anomalous moment. These corrections are of the same order of magnitude as the present gap between theory and experiment (~ 1 ppm). In fact, it turns out that they actually reduce the gap. Since they arise from characteristic momenta that are of order m_e or less, the relative order-α^2 anomalous-moment recoil corrections do not probe the proton's structure, and we can compute them completely within the framework of QED.

Since the relative order-α^2 recoil contribution involves only QED effects, the issue of the proton's structure would seem to be irrelevant in computing that contribution. However, there are some ambiguities in separating the anomalous moment contribution from the structure-dependent contributions—particularly when one considers anomalous moment vertices between off-shell propagators. The precise nature of the separation would be of critical importance in matching to our work any future high-precision calculations of structure-dependent corrections. In order to indicate clearly the relationship of our computation to the structure-dependent contributions, we set up a formalism that includes the proton's structure. In the first part of this paper (Secs. II and III), which is concerned with relative order-α^2 recoil corrections, we specialize to the low-momentum region, where the form-factor dependence may be ignored. In the second part of the paper (Sec. V) we reexamine the relative order-α recoil corrections, which *do* depend on the structure functions. Our aim here is to ensure that our formalism for isolating the form-factor dependence does indeed correspond to what was done in the earlier calculations. Those earlier calculations were organized in a rather complicated manner and were specialized to particular phenomenological expressions for the form factors. Also, some of the earlier work was not based on a unified treatment of the problem, but, rather was aimed at computing corrections to previous calculations. In the present work we give a more compact and self-contained treatment in which the result is expressed in such a way that the form-factor data can be entered rather straightforwardly.

Much of the basic bound-state formalism that we use was developed in the paper of Bodwin, Yennie, and Gregorio[5] (BYG). We refer to BYG for many of the details of the formalism and also for various computational results, but we try to provide enough details so that the reader can follow the essence of the arguments independently of BYG.

The remainder of the paper is organized as follows. In Sec. II we sketch the general computational formalism and describe the perturbation kernels that we require. The role of the proton's structure is discussed in some detail there and the most important contributions involving the structure are identified. In Sec. III we organize the computation of the anomalous-moment contributions by analyzing the proton factors. Here we introduce a technical improvement, which makes use of an extension of the Gordon reduction to the off-mass-shell case. In comparison with the procedure employed by BYG to arrive at a similar expansion in inverse powers of the heavy mass, the amount of algebra is greatly reduced. We find through this analysis that much of the anomalous-moment contribution is proportional to the computation by BYG. In Sec. IV we evaluate the additional anomalous-moment contributions that are not proportional to the BYG result. Fortunately, all of the integrals that arise have already been evaluated by BYG. Section V contains a discussion of the calculation of the structure-dependent terms by means of our formalism. Numerical results are given for the dipole parametrization of the form factors. These results differ from those given in earlier work[28,29] by an amount which is comparable to the difference between experiment and known theory. A summary of our results is given in Sec. VI. Although the contributions discussed in this paper are smaller in magnitude than the uncertainty associated with unknown terms, their computation improves our implicit knowledge of the size of the proton polarizability. The Appendix contains a tabulation of certain frequently occurring integrals that were originally computed by BYG.

II. COMPUTATIONAL FORMALISM AND OVERVIEW OF THE CALCULATION

A. General considerations

In this section we adapt the basic formalism developed by BYG to the problem at hand. There are several important distinctions between the approach used here and that of BYG, which we enumerate below.

One difference is that in the present work we wish to

incorporate the effects of the proton's structure into the formal basis of our calculation. As we shall see in the course of the analysis, the anomalous-moment contributions at the order of interest in the present work arise from momenta of order m_e or less, so we would expect them to be independent of the details of the proton's structure. Nevertheless, there is some ambiguity with regard to the separation of the anomalous-moment contribution from the structure-dependent part. In order to specify that separation precisely, we start with a formalism that incorporates the proton's structure and identify within it the structure-independent terms that we actually calculate.

We shall take the point of view that the proton can be treated as a Dirac particle with various phenomenological couplings to the electron through the electromagnetic field. If the proton is on mass shell, gauge invariance and Lorentz covariance lead to the requirement that the single-photon vertex be proportional to γ_μ or $\sigma_{\mu\nu}q^\nu$. Then, we can incorporate the "on-mass-shell" structure-dependent effects by taking the single-photon vertex to be

$$e\Gamma_\mu = e\left[\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m_p}\kappa F_2(q^2)\right] . \tag{2.1}$$

F_1 and F_2 are, respectively, the Dirac and Pauli form factors describing the proton's electromagnetic interaction; they may be deduced directly from experiment. The proton's anomalous moment is given in terms of κ. Alternatively, the interaction may be expressed in terms of G_E and G_M [see (5.1)], which give the separate interactions with the charge and current distributions. Even when the proton can be "off mass shell," we can continue to use (2.1) to describe a prominent piece of the interaction, and $e\Gamma_\mu$ can appear any number of times in a graph, with proton propagators between each occurrence.

The "off-shell-mass" dependence can enter in various ways. One way is through the dependence of the form factors on the square of the initial or final proton four-momentum. Another way is through the presence of additional interaction vertices that vanish when the proton is on mass shell. Gauge invariance, as manifested through the Ward identities, requires corresponding modifications of the proton propagator. Since all of these off-mass-shell contributions vanish as the proton goes on shell, they have the effect of "killing" a pole associated with a proton propagator. This reduces their relative importance as compared with the leading contributions, which tend to emphasize low momentum. In principle, all the structure-dependent effects can be incorporated into intrinsic multiple-photon vertices. These are specified by requiring that they contain no poles in the proton momentum at the proton mass shell. For example, an arbitrary amplitude involving two photons would consist of a part that is the iteration of $e\Gamma_\mu$ plus some additional interactions. These additional interactions would comprise the intrinsic two-photon interaction. Similarly, an amplitude involving three photons would consist of iterations of $e\Gamma_\mu$ and the intrinsic two-photon vertex plus an intrinsic three-photon vertex. In the present calculation, we study only the structure-dependent effects that can be incorporated through the "on-mass-shell" form factors of (2.1). The intrinsic two-photon interaction corresponds to the contribution δ_p(polarizability) discussed in Sec. I. We shall argue in Sec. II C that the intrinsic three-photon interaction is too small to consider at the present time. Some structure-dependent vertices are illustrated in Fig. 1.

Another difference between the present work and that of BYG is the initial choice of gauge. Most bound-state calculations to date have been performed in Coulomb gauge. There are two reasons for this. First, one usually wants to use the exact solutions to the Coulomb-Schrödinger or Coulomb-Dirac wave equation as a starting point for a perturbation expansion; an explicit Coulomb interaction in the Bethe-Salpeter kernels is a convenience for this. Second, if one starts in a covariant gauge—treating the difference between the time-time component of the covariant photon and a Coulomb photon as a perturbation—then difficulties can arise in the bound-state perturbation series. In particular, for certain choices of starting wave equation, such as the Salpeter equation, individual perturbation kernels lead to contributions that are of a spuriously low order in α. These spurious contributions ultimately cancel, but only after one sums over all numbers of loops in the perturbation kernels.[35] This particular difficulty can be expected to arise whenever the starting wave function is independent of the time component of the relative momentum. In the BYG paper, the Coulomb-gauge starting point was used in conjunction with a starting wave function in which the time component of relative momentum was fixed by a δ function to be a constant; we refer to this as an "energy-fixing δ function." Ultimately, a transformation was made to Feynman gauge. The small "gauge" correction that resulted from the transformation of the first order perturbation theory kernels was canceled by a "gauge" term from the second-order perturbation theory. This procedure worked, in part, because of the energy-fixing δ function in the starting wave function.

In the present calculation, we streamline the procedure of BYG by starting in Feynman gauge. We also

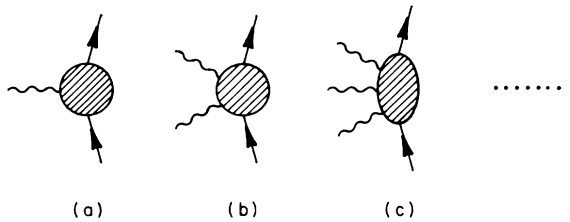

FIG. 1. Various structure-dependent photon-proton vertices. (a) represents the one-photon vertex corresponding to (2.1). (b) represents that part of the two-photon interaction which does not contain any mass-shell pole from an intermediate free proton propagator. We refer to it simply as a two-photon vertex. It includes the off-mass-shell structure of the propagator as well as intrinsic two-photon vertices. (In fact, these two effects are indistinguishable.) (c) is a similarly defined three-photon vertex.

use a starting wave equation with a relative-energy-fixing δ function. In the starting wave equation we include only the time-time component of the Feynman-gauge photon, which, in that context, becomes equal to a Coulomb interaction. The spurious contributions from the perturbation kernels mentioned above cancel, provided that one sums over all permutations of photon connections for perturbation kernels containing a given number of loops. These spurious terms are actually nonrecoil contributions, whose cancellation is discussed in detail by BYG. Because of the energy-fixing δ function, the need for a sum over all loop orders is eliminated.[36]

B. The starting wave equation and the perturbation kernels

Now let us sketch briefly the bound-state perturbation formalism that we use. A more detailed discussion of many of the concepts can be found in the BYG paper. We take the point of view that, since to first approximation the basic binding mechanism is highly nonrelativistic, we can arrive at a reasonable starting wave equation by ignoring the effects of the proton structure. That is, for purposes of obtaining a starting wave equation, we treat the proton as a pointlike Dirac particle. The structure-dependent corrections can the be added later as perturbations.

As mentioned above, we carry out the calculation in Feynman gauge. Following BYG, we find it convenient to distinguish between the temporal component of the exchanged photon, denoted by O, and the spatial components, denoted by V.

As did BYG, we split the total energy E, so that in ladder structures an amount $E'+p_0$ is routed through the electron line and an amount $E''-p_0$ is routed through the proton line (see Fig. 2), where

$$E=E'+E'' . \tag{2.2a}$$

We are free to choose the way the total energy splits between the electron and proton when fixed by the energy-fixing δ function. It is convenient to define this split by

$$E'^2-m_e^2=E''^2-m_p^2=-\gamma^2 , \tag{2.2b}$$

which leads to

$$E'=\frac{E^2-m_p^2+m_e^2}{2E} , \quad E''=\frac{E^2-m_e^2+m_p^2}{2E} . \tag{2.2c}$$

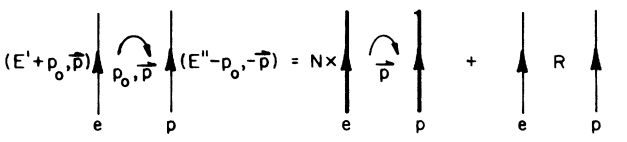

FIG. 2. The product of free electron and proton propagators between two-particle-irreducible interaction kernels. As explained in the text, the decomposition on the RHS of the graphical equation is used *only* between single-photon exchanges in which the temporal index of the photon polarization has been singled out and the proton's structure has been neglected (O' photon ladders). The first term in the decomposition corresponds to taking the energy-fixing δ function from the first term of (2.5) together with the complete numerator of the electron propagator and the large-component projector for the proton. This factor is used to build the reference equation. R represents the remainder of the O'-photon ladder. The R and crossed-ladder factors are used to build the perturbation kernels.

For the QED ground state,

$$\gamma\approx\alpha m_r , \quad E'\approx m_e-\frac{\alpha^2 m_r^2}{2m_e} , \quad E''\approx m_p-\frac{\alpha^2 m_r^2}{2m_p} . \tag{2.2d}$$

The rationalized electron and proton propagators are

$$S_F^e=\frac{E'\gamma_0+m_e+\gamma^e\cdot p}{D_e(p)} , \quad S_F^p=\frac{E'\gamma_0+m_p-\gamma^p\cdot p}{D_p(-p)} , \tag{2.3}$$

where

$$D_e(\pm p)=p^2-\gamma^2\pm 2E'p_0+i\epsilon , \quad D_p(\pm p)=p^2-\gamma^2\pm 2E''p_0+i\epsilon . \tag{2.4}$$

In order to arrive at a starting wave equation we decompose the propagator denominators as follows:

$$\frac{1}{D_e(p)D_p(-p)}=\frac{-2\pi i\delta(p_0)}{-2E(\mathbf{p}^2+\gamma^2)}-\frac{1}{2E}\left[\frac{1}{(p_0+i\epsilon)D_e(p)}+\frac{1}{(-p_0+i\epsilon)D_p(-p)}\right] . \tag{2.5}$$

In the two-particle propagator for our starting equation we take the first term of (2.5) in conjunction with the complete electron-propagator numerator and the part of the proton-propagator numerator that dominates in the nonrelativistic limit, namely,

$$\tfrac{1}{2}(E''+m_p)(1+\gamma_0^p) . \tag{2.6}$$

Thus, we obtain as our starting two-particle propagator a modified Dirac propagator:

$$N\bar{s}(p)(-2\pi i)\delta(p_0)=\frac{N\tfrac{1}{2}(1+\gamma_0^p)}{E'\gamma_0^e-m_e-\boldsymbol{\gamma}_e\cdot\mathbf{p}}(-2\pi i)\delta(p_0) , \tag{2.7a}$$

where

$$N=\frac{E''+m_p}{2E} \ . \tag{2.7b}$$

The decomposition of the exact two-particle propagator into (2.7a) plus a remainder is indicated diagrammatically in Fig. 2. The starting two-particle propagator (2.7a) is denoted by heavy fermion lines, and the difference between (2.7a) and the exact two-particle propagator is denoted by R.

In order to obtain a solvable equation that takes into account the essentially nonrelativistic physics of binding, we separate the ladder O photons into a pointlike part O', in which the form factors have been set equal to 1 and κ has been set equal to 0, and a remainder $O-O'$. (In order to pack as much physics as possible into the basic equation, some further small refinements are used by BYG, but we omit them here since they are unimportant for the present application.) We take (2.7a) in conjuction with the O' part of the photon propagator to obtain our starting wave equation

$$\Psi=\frac{N\frac{1}{2}(1+\gamma_0^p)(-2\pi i)\delta(p_0)}{E'\gamma_0^e-m_e-\boldsymbol{\gamma}_e\cdot\mathbf{p}}\gamma_0^e V_C\Psi \ , \tag{2.8}$$

where V_C is the Coulomb potential:

$$V_C=\frac{-\alpha}{r} \ . \tag{2.9}$$

The solution of this modified Coulomb-Dirac equation is discussed by BYG. Because of the factor N, reduced mass effects appear in the result to the necessary level for the present work. For many parts of the present calculation an adequate approximation for the ground-state wave function is the Coulomb-Schrödinger wave function with small components appended. The electron's small components arise naturally as a part of the solution of the modified Coulomb-Dirac equation (2.8). As explained by BYG, it is convenient to append small components to the proton leg as well, the resulting corrections being negligible. (One can regard these small components as arising from additional loops involving R corrections on the outside of the perturbation kernels.) For future reference, we record this approximate form of the ground-state wave function here:

$$\Psi(\text{approx})=\left[1+\frac{p\cdot\gamma^e}{2m_e}\right]\left[\frac{1+\gamma_0^e}{2}\right] \otimes\left[1-\frac{p\cdot\gamma^p}{2m_p}\right]\left[\frac{1+\gamma_0^p}{2}\right]\phi_{\text{nr}}\chi_{\text{spin}} \ , \tag{2.10a}$$

where

$$\phi_{\text{nr}}=\frac{(8\pi\gamma^5)^{1/2}}{(\mathbf{p}^2+\gamma^2)^2}(-2\pi i)\delta(p_0) \tag{2.10b}$$

and χ_{spin} is an electron-proton spin eigenfunction.

Next let us turn to the perturbation kernels involving exchanged photons (we do not consider radiative corrections here). They consist of those original Bethe-Salpeter kernels that were not included in the starting equation, namely, the V-photon kernels and the $O-O'$ parts of the O photon kernels, plus corrections due to the omission of the R part of the two-particle propagator in our starting equation. The discussion of these effects parallels that of Sec. III of BYG, but it is simpler here because it avoids the additional complication of transforming the kernels from the Coulomb gauge to the covariant gauge. Here we sketch it very briefly. We construct the perturbation kernels by taking all combinations of V-photon ladders, $O-O'$-photon ladders, O'-photon ladders with R's between the rungs, and crossed-photon ladders involving V and O photons. As explained by BYG, in relative order α^2, we need to consider kernels containing up to three photons. That is, with our choice of perturbation expansion, each loop in one of these perturbation kernels adds at least one power of α relative to the Fermi splitting. Following BYG we rewrite the R's in external loops as the exact two-particle propagator minus the heavy line contribution (2.7a). We use the wave equation to absorb the heavy line factors into the external wave functions. The perturbation kernels can then be reorganized so that they take the form of three-photon kernels with exact fermion and photon propagators minus similar two photon kernels. One can understand the two-photon subtractions as a mechanism for removing the double counting of

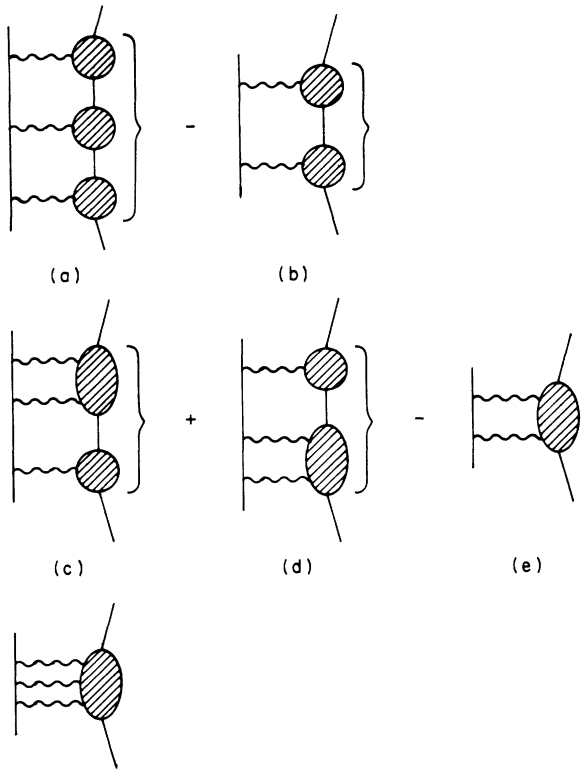

FIG. 3. The photon-exchange kernels that contribute to the order of interest (up to and including order $E_F\alpha^2 m_e/m_p$). The two-photon subtractions remove contributions that are double counted in the three-photon kernels. The curly brackets indicate that connections to the proton leg (including the blobs) are to be permuted in all ways while the electron connections are held fixed.

heavy line contributions that is implicity in the three-loop kernels. The reorganized perturbation kernels are shown in Fig. 3. This is the form for the kernels that will be used in the present calculation.

C. The role of the proton's structure

In this subsection we describe the role of the proton's structure in a general analysis of the hydrogen hfs. We estimate the orders of magnitude of the various structure-dependent kernels and identify those that are of importance at the order of current interest.

As a preliminary to that discussion, it is useful to describe a particular feature of the hfs, known as the Caswell-Lepage cancellation,[26] to which we shall refer frequently throughout this paper. If, for a given three-photon kernel, one pairs that kernel with a second kernel in which the photon connections to the electron line are reversed, then there is a cancellation of the leading contributions to the hfs for momenta larger than the electron's mass. That is, if the electron's mass is neglected and the labeling of photon momenta is held fixed, the hfs parts of the electron factors cancel completely. (Note, however, that the spin-independent terms add.) This cancellation is responsible for the absence of $\ln(m_e/m_p)$ corrections in the relative order-$\alpha^2(m_e/m_p)$ contribution to the hfs (Ref. 25).

Let us begin our analysis of the effects of the proton's structure by examining the orders of magnitude of the contributions that arise from the multiphoton vertices. Some examples of these are pictured in Figs. 3(c)–3(f). For these contributions, gauge invariance plays an important role in our analysis. Since gauge invariance holds order by order in α, the multiphoton vertices are separately gauge invariant at each level of multiplicity. (Here we imagine that external legs are on mass shell; the deviations caused by this approximation generally introduce higher powers of α and are related to sum-over-states corrections that involve perturbations of higher order in the number of kernels.) The requirement of gauge invariance means that these vertices must contain factors of the momenta of the exchanged photons. The factors of the photon momenta prevent the small-momentum region from being important in the Feynman integral. Thus, inverse powers of γ cannot arise from the integration, and one can obtain the leading contribution by neglecting wave-function momenta inside the loop; the wave-function integrations converge independently. The net result is an overall factor of $\gamma^3\alpha^n$, where n is the multiplicity of the vertex. The scale of the remaining internal intergration is set by the scale of the proton structure, which we call Λ. On dimensional grounds, then, the integral should produce a factor of order $1/\Lambda^2$ or smaller. This rules out the cases $n \geq 4$ at the present level of interest. The case $n=3$ [see Fig. 3(f)] must be examined carefully. If the integral were of order $1/\Lambda^2$, the result would be of the same order as the new contributions presented in this paper. However, the Caswell-Lepage cancellation permits us to argue that these terms are also too small to be of interest. Since, as we have already argued, the contribution comes only from momenta much greater than m_e, the Caswell-Lepage cancellation leads to at least one extra numerator power of m_e. Then, on the basis of dimensional analysis, the integral is of order m_e/Λ^3, which is too small to matter at the present level of the theory. The only multiphoton vertices that can contribute at the present level of accuracy are the two-photon vertices, which are associated with proton polarizability. Because of the Caswell-Lepage cancellation at the two-loop level, these two-photon vertices can contribute to the order of interest only at the one-loop level. In that case there is one less power of α and, furthermore, there is no Caswell-Lepage cancellation to bring an extra factor of m_e into the numerator. Such a contribution is nominally of the same order as the contributions that arise from the form-factor dependence in the kernels without multiphoton vertices, but experimental results suggest that it is actually much smaller.[32] We do not analyze these terms further in this paper.

Having dispensed with (i.e., set aside) the multiphoton contributions, we are left with the kernels pictured in Figs. 3(a) and 3(b). Since the form factor associated with a given vertex depends only on the momentum transfer at the vertex, all the algebraic manipulations involving the proton factor may be carried through without regard to the form factors. This means that for the parts of the kernels that do not involve the proton's anomalous moment we can take over the results of BYG from Secs. II, III A, III B, III C, and Appendix B. [This comment does not apply to some special features associated with the pure Coulomb interaction, as in Eqs. (2.25) and (2.26) of BYG. However, these features are unimportant for the contributions that are the subject of this paper.] In the case of the anomalous-moment terms in the perturbation kernels, the results of BYG from Appendix B are insufficient, because they are specific to the Dirac part of the proton's interaction. The additional algebra for these terms involving the anomalous moment is worked out in Sec. III.

It is useful to review at this point the types of contributions that arose in the BYG calculation and to identify for each type the important corrections that result from incorporating the proton's structure. A given perturbation kernel that is constructed from the one-photon vertices does not yield a result of unique order in α and m_e/m_p, but we can rank the kernels in importance according to the size of the largest contribution that they produce. If a given kernel yields a contribution of relative order 10^{-6} from the low-momentum region, then the proton-structure effect is much smaller and can be ignored. Kernels that are intrinsically larger or that obtain appreciable contributions from momenta of order m_p must be treated more carefully.

The most important contributions appear as expectation values of single kernels with respect to the wave functions of the unperturbed problem. There are also small contributions involving a sum over states, which have the structure of higher-order perturbations in the simplest kernels. The proton structure effects are completely negligible in these higher-order perturbation contributions. The only important term of this type, given our particular organization of the calculation, is the

second-order perturbation involving two hyperfine interactions. The result for this contribution is easily obtained from BYG by replacing the Dirac moment in each kernel with the total magnetic moment. Therefore, it is necessary only to study the modifications due to the proton's structure in the individual kernels.

The most important contributions to the hfs is the nonrecoil correction, which arises from one-photon exchange; it is treated in Sec. III D of BYG. The result of that analysis is that the leading order may be expressed as

$$E_F[1+\tfrac{3}{2}\alpha^2+3\gamma^2/(2m_e m_p)] .$$

This expression does not yet include the effect of the proton's structure. If one ignores the effect of the proton's form factors, the effect of the anomalous moment is merely to multiply this result by $1+\kappa$. Hence, it is customary to incorporate a factor $1+\kappa$ in the definition of E_F, even though in higher orders there are additional form-factor-independent terms that arise from the anomalous moment. Long ago, Zemach[18] pointed out that important form-factor effects enter through modifications of the leading-order term. These arise from a small change in the wave function due to the extended proton charge and from the extended structure of the magnetic moment of the proton; the two effects must be treated together. While the Zemach contribution has a rather transparent interpretation, it is not especially large compared to the individual contributions of other structure-dependent corrections. However, owing to a fortuitous cancellation among the more dynamical contributions, which relies on the particular numerical value of κ, the Zemach correction is approximately equal to the net structure-dependent contribution. The Zemach contribution is rederived and evaluated in Sec. V as a part of the complete treatment of the one-loop kernel.

Next we discuss the recoil corrections of relative order α. These arise from the one-loop kernels, which are treated in Sec. III E of BYG. The contribution of relative order α contains terms with the factor $\ln(m_p/m_e)$, which signals that the important integration range extends up to m_p. Thus, we cannot ignore the effects of the proton's form factors in evaluating such terms. The one-loop kernels also give rise to contributions that are of relative order α^2. It turns out that these contributions arise from the low-momentum region, so in evaluating them we can ignore the effects of the proton's form factors. However, it is often the case that a subleading contribution comes from a one-loop integral that also gives a leading contribution. Then, in order to set the form factors equal to their static values in the subleading contribution, one must first devise a procedure for separating the subleading contribution from the leading contribution at the level of the integrand. The subleading contributions appear in various places, depending on the details of the separation procedure, but the net subleading contribution is independent of the method. We give enough details of our particular method so that our results can be used as a point of departure for a future higher-precision calculation.

Finally, we discuss the recoil corrections of relative order α^2. These arise from the two-loop kernels and, as mentioned above, from the subleading contributions in the one-loop kernels. It might seem, at first sight, that the integrals that lead to these corrections could contain high-momentum contributions, and so would be sensitive to the proton's form factors. However, the Caswell-Lepage cancellation between different integrals provides an additional factor m_e/p for momenta larger than the m_e, so that such terms may be safely calculated without taking into account the proton's form factors. This cancellation is evidenced by the absence of relative order-$\alpha^2(m_e/m_\mu)\ln(m_\mu/m_e)$ contributions in the result of BYG. Many of the terms calculated by BYG can be taken over without new effort. However, the anomalous moment does introduce some new terms. Even these can be evaluated with the help of integral tables constructed by BYG. This work is carried out in Secs. III and IV.

It seems clear from the preceding discussion that all of the form-factor effects are contained in "intrinsic one loop" integrals. This terminology refers to a situation where one loop momentum is much larger than γ, so that all other loop momenta can be neglected by comparison. Integrations over those other momenta converge independently of the large-momentum loop. The detailed analysis in which we demonstrate this and extract these terms is presented in Sec. V. This is not the first careful analysis of proton-structure effects.[28,29] However, it is important to make the details of this extraction procedure precise, so that the contributions that have been calculated up to now can be unambiguously distinguished from additional contributions that could appear in some future, still more accurate calculation. Also, the earlier work was done at a time when workers did not always distinguish between F_1, F_2, and G_E, G_M. Consequently, our results differ slightly from those given in the earlier work. It would be very awkward to modify the earlier work to take into account better form-factor information. Here, we rearrange all of the structure-dependent contributions that have been calculated at present into forms, involving the proton's structure functions, that are convenient for numerical integration. In order to provide reference results to aid future work, we carry out the numerical integrations for the case of the dipole parametrization of the form factors.

III. ORGANIZATION OF THE ANALYSIS OF THE ORDER-α^2 CONTRIBUTION

As described at the end of Sec. II B, the kernels we must analyze contain either two- or three-photon exchanges. After setting aside those with multiphoton vertices, we have the kernels illustrated in Figs. 3(a) and 3(b). As explained in Sec. II B, the two-photon kernels are subtracted from the three-photon ones in order to avoid multiple counting. We need not analyze the two-photon kernels separately, since their contributions are easily identified as pieces of the three-photon contributions. Our work is based on that of BYG, but enough details are given here so that the reader should get a good overview of the methods employed.

In the BYG paper, the problem was set up initially in the Coulomb gauge, and the kernels were then converted to the Feynman gauge. Here we arrive at the kernels directly in the Feynman gauge, with on-mass-shell form factors included; these form factors are incorporated as part of the photon factor. In the BYG work, the analysis of the factor that arises from the heavy-particle propagators was made after breaking down the kernels according to the type and order of photons exchanged (O for temporal component and V for spatial component). That breakdown seemed useful and natural at the time, but the algebra involving Dirac matrices is awkward and cumbersome. Here we present a different way of handling that algebra, which is more compact and covariant. All the results of the Appendix of BYG are easily rederived from the present ones by specializing to the various photon types.

At this point, it is useful to describe our strategy briefly. We are attempting to do two rather different calculations. One is to obtain one-loop contributions of relative order $\alpha(m_e/m_p)$, which are necessarily form-factor dependent. The other is to obtain contributions of relative order $\alpha^2(m_e/m_p)$, for which form factors are irrelevant. The difficulty is that the second type of contribution is contained both in expressions where it represents the leading order, and as corrections to lower-order terms in which the form factors are important. In the process of separating the contributions of different orders, we must be careful not to make incompatible approximations which cause small terms to be misidentified. Also, we want to incorporate the work of BYG without redoing that calculation. The present covariant treatment leads to subcalculations that are quite different from those of BYG, but the final results, for a particular part of the physics, must be the same. There is no simple way to make a term by term comparison of the two calculations. Instead, we isolate a piece of the present calculation which matches precisely the total calculation of BYG, except for an overall factor, and concentrate on the calculation of the additional contributions due to the anomalous moment and the form-factor dependence.

To handle the complication that integrals containing the one-loop contributions also contain contributions of relative order $\alpha^2(m_e/m_p)$, we invent the concept of a "pure one-loop integral," which is defined so that it gives a result that is purely of relative order $\alpha(m_e/m_p)$. A pure one-loop integral is obtained from an integral involving two external loops and one internal loop by setting the external momenta equal to zero inside the internal loop and neglecting γ^2 in D_e and D_p. The external integrals then converge independently of the internal loop, and the remaining internal integration is independent of γ, so the result has a unique order in α. We emphasize that this is different, and perhaps cruder, than the decoupling procedures introduced by BYG. If we were doing a complete calculation from the beginning, it might be better to use a different procedure, but the one just suggested seems to be a useful expedient for present purposes. The pure one-loop integrals are calculated in Sec. V. Whenever a one-loop integral is encountered in our analysis of the relative order-$\alpha^2(m_e/m_p)$ contribution, the pure one-loop integral is extracted and set aside. The result is an expression which is not sensitive to the form factors and which may be evaluated to give a relative order-$\alpha^2(m_e/m_p)$ result. With this understanding, we may set the form factors equal to unity in the present and following section. At the same time we drop terms that would lead to a pure relative order-$(\alpha/\pi)(m_e/m_p)\ln(m_p/m_e)$ contribution.

The following notation is used here. A general four-momentum associated with a proton line is designated by P and an index is appended to indicate a particular value (f or i for the final or initial wave functions, respectively, or a or b for internal four-momenta). The four-momentum consists of a constant energy piece E'' and an integration variable four-momentum p:

$$P_\lambda=(E''+p_0,\mathbf{p}) . \tag{3.1}$$

For a wave-function momentum, $p_0=0$, of course. The split of the total energy of the bound state between the electron and the proton ($E=E'+E''$) is arranged according to (2.2b). As described by BYG, (2.8) is a convenient choice for the definition of the wave function. It is less essential for present purposes, but we continue to use it.

It is convenient to define a spinor factor

$$Q=\frac{\not{P}+m_p}{2m_p} , \tag{3.2a}$$

which occurs in wave functions and in rationalized proton propagators. As a factor in wave functions, this is almost a projector. For example,

$$Q_f^2=Q_f+\frac{P_f^2-m_p^2}{4m_p^2}=Q_f-\frac{\gamma^2+\mathbf{p}_f^2}{4m_p^2}\approx Q_f . \tag{3.2b}$$

The last term in the second form is clearly negligible for a characteristic wave-function momentum [which is $O(\gamma)$]. The reader may note that the expression Q_f given here is not quite the same as the factor occurring in (2.10a), but they are equivalent, to the accuracy of the present calculation.

In order to analyze the proton factor we wish to introduce the analogue of the Gordon reduction for a proton with intermediate propagators. We will use this approach for one, two, or three photon attachments to the proton line. Here we illustrate the technique with three attachements; the application to the other cases is obvious. For the purposes of this initial discussion, we ignore the proton's anomalous moment, keeping only the first term on the right-hand side (RHS) of (2.1). We will incorporate the effect of the proton's anomalous moment later. The form factor F_1 does not appear explicitly in our discussion since, for purposes of this section, it can be set equal to unity; however, it can, in principle be included in the photon factor. Using (3.2b) we may write the proton factor as

$$Q_f\gamma_\rho\frac{1}{\not{P}_a-m_p+i\epsilon}\gamma_\nu\frac{1}{\not{P}_b-m_p+i\epsilon}\gamma_\sigma Q_i , \tag{3.3}$$

which is to be taken between wave functions. Here ρ, ν, and σ are the photon polarization indices, and P_a and P_b are the four-momenta carried by the proton lines. Since we have specified the four-momenta on the electron lines, the four-momenta on the internal proton lines depend on the arrangement of photon lines.

Next it is convenient to reorganize the structure on the left-hand side (LHS) of (3.3) using

$$Q_f\gamma_\rho=\frac{P_{f\rho}+P_{a\rho}}{2m_p}+\frac{[\not P_f-\not P_a,\gamma_\rho]}{4m_p}-\gamma_\rho\frac{\not P_a-m_p}{2m_p} . \tag{3.4}$$

The terms on the RHS are, in succession, a convection vertex, a spin vertex, and (after one cancels a propagator) a seagull. For any of these terms, we write the next (uncanceled) propagator on the right as

$$\frac{1}{\not P-m_p+i\epsilon}=\frac{2m_p}{P^2-m_p^2+i\epsilon}Q . \tag{3.5}$$

Then Q is rearranged with the next vertex as in (3.4). Finally, a factor $\not P_i-m_p$ acting on the RHS wave function produces a term too small to be kept. The resulting expression for the proton factor (to be put between wave functions) is

$$\left[\frac{P_{f\rho}+P_{a\rho}+\frac{1}{2}[\not P_f-\not P_a,\gamma_\rho]}{2m_p}\right]\frac{2m_p}{P_a^2-m_p^2+i\epsilon}\left[\frac{P_{a\nu}+P_{b\nu}+\frac{1}{2}[\not P_a-\not P_b,\gamma_\nu]}{2m_p}\right]\frac{2m_p}{P_b^2-m_p^2+i\epsilon}\left[\frac{P_{b\sigma}+P_{i\sigma}+\frac{1}{2}[\not P_b-\not P_i,\gamma_\sigma]}{2m_p}\right]$$
$$-\frac{g_{\rho\nu}}{2m_p}\frac{2m_p}{P_b^2-m_p^2+i\epsilon}\left[\frac{P_{b\sigma}+P_{i\sigma}+\frac{1}{2}[\not P_b-\not P_i,\gamma_\sigma]}{2m_p}\right]-\left[\frac{P_{f\rho}+P_{a\rho}+\frac{1}{2}[\not P_f-\not P_a,\gamma_\rho]}{2m_p}\right]\frac{2m_p}{P_a^2-m_p^2+i\epsilon}\frac{g_{\nu\sigma}}{2m_p} . \tag{3.6}$$

In the seagull terms (the last two terms) we have retained only the symmetric part of the product $\gamma_\rho\gamma_\nu$. The antisymmetric part vanishes under interchange of the two photon connections (however, that would not be true for non-Abelian gauge theories). Having made a Gordon reduction, we now use the term "propagator" to refer to rationalized expressions of the form of the first factor on the RHS of (3.5), including the numerator $2m_p$.

Now we pick out the terms from (3.6) which contribute to the order of interest. A complete analysis of these terms would lead to a simplified derivation of the results of Appendix B of BYG, possibly in a more compact form. Here our aim is only to identify the terms that were calculated by BYG, so that we can compare them with the anomalous moment contributions. First let us describe the orders of magnitude of the various factors in terms of inverse powers of m_p. (An inverse power of m_p implies a factor of one of the small quantities m_e/m_p or γ/m_p.) The propagators are of order m_p^0 [see (2.4)]. In a vertex, the zero component of the convection term has a piece $2E''$ which is of order m_p^0; the remainder of the vertex is of order $1/m_p$. A seagull factor is of order $1/m_p$. The hyperfine structure must contain at least one spin factor, and therefore the leading order is $1/m_p$. Recoil corrections are of relative order $1/m_p$; terms of still higher order in $1/m_p$ are ignored in the present work [except in the discussion of the one-loop contributions, which involve $\ln(m_p/m_e)$]. While most of the terms contributing to the hfs are explicit in (3.6), there is also a recoil contribution from the three convection factors taken with the small components of both wave functions. The latter gives a contribution of the required order only for all indices 0 and only for the ladder order of photon exchanges. Thus, to the order of interest, terms with zero, one, or two explicit spin factors contribute.

Now let us consider the effect of the proton's anomalous moment, which corresponds to the second term in (2.1). Suppose we insert an anomalous-moment interaction in place of one of the pointlike vertices. We can again apply the rearrangement corresponding to (3.4), but this time working out from the anomalous-moment vertex in both directions. To the order of interest, the terms that occur are all proportional to terms in (3.6), but there are some differences in the way the form factors enter. The one-spin term is exactly the same as in (3.6) except that it has the factor $\kappa F_2\to\kappa$ in place of $F_1\to 1$ at the spin vertex. Thus, the combination of the one-spin term arising from the anomalous moment interaction with the one-spin term from (3.6) produces the factor $1+\kappa$, mentioned below (2.1), at the one-spin vertex. Each two-spin term occurs twice, with an $F_1\to 1$ at one vertex and an $F_2\to 1$ at the other. In both cases, the nonspin vertices have the factor $F_1\to 1$. Finally, the term from the small components is missing.

We conclude that we can account for the contributions to the order of interest that arise from one anomalous-moment interaction by multiplying (3.6) by $(1+\kappa)$ and adding to it

$$\frac{\kappa}{(4m_p)^2}\left[[\not P_f-\not P_a,\gamma_\rho]\frac{2m_p}{P_a^2-m_p^2+i\epsilon}[\not P_a-\not P_b,\gamma_\nu]\frac{2m_p}{P_b^2-m_p^2+i\epsilon}g_{\sigma 0}\right.$$
$$+[\not P_f-\not P_a,\gamma_\rho]\frac{2m_p}{P_a^2-m_p^2+i\epsilon}g_{\nu 0}\frac{2m_p}{P_b^2-m_p^2+i\epsilon}[\not P_b-\not P_i,\gamma_\sigma]$$
$$\left.+g_{\rho 0}\frac{2m_p}{P_a^2-m_p^2+i\epsilon}[\not P_a-\not P_b,\gamma_\nu]\frac{2m_p}{P_b^2-m_p^2+i\epsilon}[\not P_b-\not P_i,\gamma_\sigma]-(4m_p)^2g_{\rho 0}\frac{2m_p}{P_a^2-m_p^2+i\epsilon}g_{\nu 0}\frac{2m_p}{P_b^2-m_p^2+i\epsilon}g_{\sigma 0}\right] . \tag{3.7}$$

The expression (3.6) (now multiplied by $1+\kappa$) incorporates all the three-photon contributions previously worked out by BYG. (Recall that, in order to avoid double counting of contributions, we must subtract the two-photon-exchange graphs and make some other minor adjustments.) As just described, this means that we take over the complete one-kernel result from BYG and delete the pure one-loop contributions.

Now let us return to the correction (3.7), which contains the terms arising from one anomalous moment insertion that are not proportional to the calculation in BYG. Consider the proton factor that one obtains by summing over all ways of inserting the photon lines into the proton leg, starting with the ladder order. The hfs content of the various numerator terms is independent of the ordering of the spin factors, except that a minus sign appears if the spin factors are permuted an odd number of times. (To see this, one may use the property that if all the γ matrices are reversed in order, one obtains the same hfs content except for a sign change.) For example, for the third term of (3.7) one finds the set of propagator combinations

$$\frac{4m_p^2}{D_p(-p')D_p(-p)}+\frac{4m_p^2}{D_p(-p)D_p(p'-p-p_f)}+\frac{4m_p^2}{D_p(p'-p-p_f)D_p(p'-p_f-p_i)}-\frac{4m_p^2}{D_p(-p')D_p(p-p'-p_i)}$$
$$-\frac{4m_p^2}{D_p(p-p_f-p_i)D_p(p-p'-p_i)}-\frac{4m_p^2}{D_p(p-p_f-p_i)D_p(p'-p_f-p_i)}\ . \tag{3.8}$$

Here we have made use of the abbreviated notation for the propagator denominators of (2.4).

Aside from places where relativistic single-loop integrals might result, we may approximate D_p by

$$D_p(q)\approx 2E''q_0+i\epsilon\ . \tag{3.9}$$

For denominators that might occur in a single loop, we may neglect momenta external to that loop but otherwise keep the complete expression. The six terms then simplify (using $E''\approx m_p$) to

$$-2\pi i\delta(p_0')\left[\frac{2m_p}{D_p(-p)}-\frac{2m_p}{D_p(p)}\right]\ . \tag{3.10}$$

Repeating this procedure for the other terms in (3.7) we find that, to the order of interest, the contribution to the muon factor is

$$\frac{\kappa}{4m_p^2}\left[g_{\rho 0}\tfrac{1}{2}[\not p'-\not p,\gamma_\nu]\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma](-2\pi i)\delta(p_0')\left[\frac{2m_p}{D_p(-p)}-\frac{2m_p}{D_p(p)}\right]\right.$$
$$+g_{\sigma 0}\tfrac{1}{2}[\not p_f-\not p',\gamma_\rho]\tfrac{1}{2}[\not p'-\not p,\gamma_\nu](-2\pi i)\delta(p_0)\left[\frac{2m_p}{D_p(-p')}-\frac{2m_p}{D_p(p')}\right]$$
$$+g_{\nu 0}\tfrac{1}{2}[\not p_f-\not p',\gamma_\rho]\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma](-2\pi i)\delta(p_0'-p_0)\left[-2\pi i\delta(p_0)-\frac{2}{p_0+i\epsilon}\right]$$
$$\left.-g_{\rho 0}g_{\nu 0}g_{\sigma 0}\not p_f\not p_i(-2\pi i)^2\delta(p_0')\delta(p_0)\right]\ . \tag{3.11}$$

Here, and in all subsequent discussions, we assume that the proton factor is to be evaluated between large components. That is, there is an implicit projector $(1+\gamma_0)/2$ on either side of the proton factor. In the third term of (3.11) there is no one-loop contribution, so we have kept only the leading-order approximation for the denominators.

Finally, we consider the contribution from two anomalous-moment interactions. Given our analysis of the contribution from one anomalous-moment interaction, this is very easy to work out. We do not include some pure one-loop contributions which have no decoupling corrections; these are incorporated in (5.5). Then, a factor of Q between anomalous-moment interactions may be replaced by

$$\Lambda_+=\frac{1+\gamma_0}{2}\ . \tag{3.12}$$

The net effect of this is to insert a factor $(1+\kappa\Lambda_+)$ between the spin factors in the first three terms in (3.11). Thus,

$$\kappa\tfrac{1}{2}[\not p'-\not p,\gamma_\nu]\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma]\rightarrow\kappa\tfrac{1}{2}[\not p'-\not p,\gamma_\nu][1+\kappa\Lambda_+]\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma]$$
$$\rightarrow\kappa(1+\kappa)\tfrac{1}{2}[\not p'-\not p,\gamma_\nu]\Lambda_+\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma]+\kappa\tfrac{1}{2}[\not p'-\not p,\gamma_\nu]\Lambda_-\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma]\ , \tag{3.13}$$

etc. It proves to be convenient to treat the Λ_+ and Λ_- terms separately. We treat the last term of (3.11) together with the Λ_- terms.

In summary, we find that the contributions arising from one or two anomalous-moment insertions lead to terms that are just κ times the BYG result, plus new terms, to be calculated, that correspond to the proton factors

$$\mathcal{F}_p(\Lambda_+)=\frac{\kappa(1+\kappa)}{4m_p^2}\left[g_{\rho 0}\tfrac{1}{2}[\not p'-\not p,\gamma_\nu]\Lambda_+\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma](-2\pi i)\delta(p_0')\left[\frac{2m_p}{D_p(-p)}-\frac{2m_p}{D_p(p)}\right]\right.$$
$$+g_{\sigma 0}\tfrac{1}{2}[\not p_f-\not p',\gamma_\rho]\Lambda_+\tfrac{1}{2}[\not p'-\not p,\gamma_\nu](-2\pi i)\delta(p_0)\left[\frac{2m_p}{D_p(-p')}-\frac{2m_p}{D_p(p')}\right]$$
$$\left.+g_{\nu 0}\tfrac{1}{2}[\not p_f-\not p',\gamma_\rho]\Lambda_+\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma](-2\pi i)\delta(p_0'-p_0)\left[-2\pi i\delta(p_0)-\frac{2}{p_0+i\epsilon}\right]\right] \tag{3.14a}$$

and

$$\mathcal{F}_p(\Lambda_-)=\frac{\kappa}{4m_p^2}\left[g_{\rho 0}\tfrac{1}{2}[\not p'-\not p,\gamma_\nu]\Lambda_-\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma](-2\pi i)\delta(p_0')\left[\frac{2m_p}{D_p(-p)}-\frac{2m_p}{D_p(p)}\right]\right.$$
$$+g_{\sigma 0}\tfrac{1}{2}[\not p_f-\not p',\gamma_\rho]\Lambda_-\tfrac{1}{2}[\not p'-\not p,\gamma_\nu](-2\pi i)\delta(p_0)\left[\frac{2m_p}{D_p(-p')}-\frac{2m_p}{D_p(p')}\right]$$
$$+g_{\nu 0}\tfrac{1}{2}[\not p_f-\not p',\gamma_\rho]\Lambda_-\tfrac{1}{2}[\not p-\not p_i,\gamma_\sigma](-2\pi i)\delta(p_0'-p_0)\left[-2\pi i\delta(p_0)-\frac{2}{p_0+i\epsilon}\right]$$
$$\left.-g_{\rho 0}g_{\nu 0}g_{\sigma 0}\not p_f\not p_i(-2\pi i)^2\delta(p_0')\delta(p_0)\right]. \tag{3.14b}$$

The contributions that arise from these proton factors are calculated in Sec. IV.

IV. EVALUATION OF NEW RECOIL CONTRIBUTIONS OF RELATIVE ORDER α^2

Our aim in this section is to work out contributions of $O(\alpha^2 m_e/m_p)$ that arise from (3.14a) and (3.14b). As described in Sec. III, this effort is complicated by the fact that, in some cases, the contributions of the order of interest are embedded in one-loop integrals that also contain larger contributions, such as the first term of (1.10). There we invented the procedure, which we apply in this section, of subtracting a pure one-loop contribution that precisely removes the recoil contribution of relative order α.

We report all results as multiples of $(\alpha^2 m_e/m_p)E_F$ [where the factor $(1+\kappa)$ is included in E_F]. As much as possible we make use of results previously obtained by BYG. In addition to specific contributions to the hfs, certain frequently occurring integrals were worked out and tabulated by BYG. They are called $K_n(\cdots)$, where the argument consists of a numerator polynomial in spatial momentum and γ^2. The subscript n is used to distinguish various forms of the p_0 and p_0' dependence in the integrands; these forms are given in the Appendix. In the discussion of this section, we try to provide enough detail so that the interested reader can follow the general outlines of the argument without referring to the BYG work. Of course, to check the complete result, one must make use of the detailed calculations given by BYG. We use the phrase "to adequate accuracy" to indicate that we have taken approximations such as $E'\approx m_e$, which follow from the smallness of α, and also to indicate that we have dropped certain numerator terms whose momentum dependence is such that they cannot contribute to the order of interest. (The Appendix of BYG contains a discussion of methods for estimating orders of magnitude.)

A. Λ_+ terms

Because of the Λ_+'s, for each term in (3.14a) all of the γ matrices in both commutators must be spatial. Following the conventions of the previous paragraph, we note that all these terms have an overall factor of κ.

1. *The first two terms of (3.14a)*

The first two terms of (3.14a) give equal contributions. The subtraction kernel involving two photons cancels one of them. Thus, we need consider only the first term. The accompanying electron factor (to adequate accuracy) is

$$\frac{(2m_e+\not p')\gamma_\nu(m_e+\gamma_0 m_e+\not p)\gamma_\sigma[1+\not p_i/(2m_e)]}{D_e(p')D_e(p)}, \tag{4.1}$$

where ν and σ are spatial indices. Here the spin dependence of the initial wave function is shown explicitly. Throughout the remainder of the paper, large component projectors $[(1+\gamma_0)/2]$ are implied on either side

of the electron factor.

Let us first consider the part of (4.1) that contains no factor p_0. Comparing with BYG we see that the work of computing this contribution has already been carried out. The relevant starting point is (4.16a) of BYG, in which the "spin-spin" combination has been selected in the muon factor, and the term containing the p_0 part of $\not{p}$ has been omitted in the electron factor. The result of that lengthy calculation is contained in (4.19c) of BYG; it yields the coefficient

$$\kappa\left[\frac{3}{2}\ln\frac{1}{2\alpha}-\frac{3}{4}\right]. \tag{4.2}$$

Next consider the part of (4.1) containing the factor p_0. Rearranging it (using $E'\approx m_e$) we obtain

$$\frac{-2m_e p_0\gamma_\nu\gamma_\sigma}{D_e(p')D_e(p)}\approx\frac{[-D_e(p)+(p^2-\gamma^2)]\gamma_\nu\gamma_\sigma}{D_e(p')D_e(p)}. \tag{4.3}$$

The $D_e(p)$ term in the numerator does not contribute to the hyperfine splitting because of symmetry (consider the transformation $p_0\rightarrow -p_0$ and neglect terms of higher order in $1/m_p$). After doing the necessary angular averaging, we find that the contribution arising from the product of the first term of (3.14a) with (4.3) is given by the integral

$$\kappa E_F\frac{4\pi\alpha(8\pi\gamma)^2}{2m_p}\int\frac{d^3p'd^4p\,d^3p_i}{-(2\pi)^{10}i}\frac{(\mathbf{p}-\mathbf{p}')\cdot(\mathbf{p}-\mathbf{p}_i)(p^2-\gamma^2)}{(\mathbf{p}'^2+\gamma^2)^2[(p'-p)^2+i\epsilon]D_e(p)[(p-p_i)^2+i\epsilon](\mathbf{p}_i^2+\gamma^2)^2}\left[\frac{2m_p}{D_p(-p)}-\frac{2m_p}{D_p(p)}\right]. \tag{4.4}$$

Here, we have carried out the p_f integration. The denominator $(\mathbf{p}_i^2+\gamma^2)^2$ arises from the initial (nonrelativistic) ground-state wave function.

In order to evaluate (4.4), it is necessary to consider various numerator combinations individually. The terms in the numerator of (4.4) involving $\mathbf{p}'$ and/or $\mathbf{p}_i$ are relatively straightforward to work out. For the term $-\mathbf{p}\cdot(\mathbf{p}'+\mathbf{p}_i)$ we use the symmetry of the integrand under $p_i\leftrightarrow p'$ to obtain $-2\mathbf{p}\cdot\mathbf{p}'$. Since the resulting integral has no one-loop contribution [i.e., $O(\alpha(m_e/m_p)\ln(m_p/m_e))$], we may use (3.9) to simplify the last factor in (4.4), which becomes

$$-2\pi i\delta(p_0)-\frac{2}{p_0+i\epsilon}.$$

(Incidentally, this expression manifests the Caswell-Lepage cancellation: in the high-momentum region for the electron, the remaining factors of the integrand become symmetric in p_0, and the two terms cancel.) Next the p_i integration can be "decoupled" by making use of the substitution $(p-p_i)^2+i\epsilon\rightarrow p^2-\gamma^2+i\epsilon$. It is a particular feature of this substitution that the corrections to it are of higher order in α. [This may be seen most easily by noting that the corrections involve a three-loop integral. The δ-function piece of the last factor of (4.4) yields zero by cancellation, and the remainder can be seen by power counting to be of higher order.] The integration over p_i yields a factor $(2\pi)^3(8\pi\gamma)^{-1}$. Then the p_0^2 part of $p^2-\gamma^2$ leads to the integral $4K_7(\mathbf{p}\cdot\mathbf{p}')$. The remaining part leads to the integral $2K_1(\mathbf{p}\cdot\mathbf{p}'(\mathbf{p}^2+\gamma^2))$, so that the total contribution is

$$\kappa[4K_7(\mathbf{p}\cdot\mathbf{p}')+2K_1(\mathbf{p}\cdot\mathbf{p}'(\mathbf{p}^2+\gamma^2))]=\kappa\left[-\ln\frac{1}{2\alpha}+1\right]. \tag{4.5}$$

For the $\mathbf{p}'\cdot\mathbf{p}_i$ term, we must take the nonrelativistic approximation to obtain a contribution of the order of interest. That is, we replace the last factor in (4.4) by $-2\pi i\delta(p_0)$. The resulting expression is proportional to the three-loop integral I_1 defined by BYG and gives the contribution

$$\kappa\frac{\gamma}{4\pi^6}I_1=\kappa\tfrac{1}{4}. \tag{4.6}$$

Finally, the $\mathbf{p}^2$ term of (4.4) is the most complicated since it is dominated by a one-loop contribution which contains terms that are larger [relative order $\alpha(m_e/m_p)$] than those we are trying to extract. We follow the procedure described at the beginning of this section, ignoring the proton's structure and subtracting the relative order-$\alpha(m_e/m_p)$ terms. We describe the main features of the calculation here briefly. First we decouple the wave functions to obtain a one-loop integral by using the substitution described above for both photon propagators. Only terms containing at most one decoupling correction are of the order of importance. The terms with exactly one decoupling correction give

$$2\kappa K_1((\mathbf{p}'^2-2\mathbf{p}\cdot\mathbf{p}'-\gamma^2)\mathbf{p}^2)=-\kappa\tfrac{3}{4}. \tag{4.7}$$

The one-loop contribution (with no decoupling correction) is contained in

$$\kappa E_F\frac{2\pi\alpha}{m_p}\int\frac{d^4p}{-(2\pi)^4i}\frac{\mathbf{p}^2}{(p^2-\gamma^2)D_e(p)}\times\left[\frac{2m_p}{D_p(-p)}-\frac{2m_p}{D_p(p)}\right]. \tag{4.8}$$

This is evaluated to sufficient accuracy and the pure relative order-α recoil correction is removed, leaving the residual piece of the order of interest. It turns out to be

$$-\kappa\tfrac{3}{4}. \tag{4.9}$$

The total contribution from the first two terms of (3.14a) is

$$C((3.14a)-\mathrm{I,II})=\kappa\left[\frac{1}{2}\ln\frac{1}{2\alpha}-1\right]. \tag{4.10}$$

2. *The third term of (3.14a)*

Now we turn to the third term from (3.14a). As was the case with the first term, a large part of the work of calculating this term, which is rather lengthy, has already been carried out by BYG. The electron factor which accompanies the third term of (3.14a) is

$$\frac{[1+\not p_f/(2m_e)]\gamma_\rho(m_e+\gamma_0 m_e+\not p')\gamma_0(m_e+\gamma_0 m_e+\not p)\gamma_\sigma[1+\not p_i/(2m_e)]}{D_e(p')D_e(p)} . \tag{4.11}$$

Setting aside the $p_0' p_0$ piece of the electron factor temporarily, we find that the resulting integral is very similar to the spin-spin piece of the term *VOV* of BYG. The p_0, p_0' dependence of the third term of (3.14a) is slightly different than in the muon factor of *VOV*. The net effect of this difference is to replace K_1 by K_3, which changes nonlogarithmic terms, but does not alter the coefficient of $\ln[1/(2\alpha)]$. The four-loop integral is identical to the one in BYG since it arises from the $\delta(p_0)\delta(p_0')$ part of the proton factor. The net result is

$$\kappa\left[\frac{5}{4}\ln\frac{1}{2\alpha}-\frac{1}{16}-\ln 2\right] . \tag{4.12}$$

There remains the additional contribution from the term in the electron factor that is proportional to $p_0 p_0'$. For it, we may neglect the numerator dependence on $\mathbf{p}_f$ and $\mathbf{p}_i$ and decouple the wave-function integrations as before, with negligible decoupling corrections. The resulting contribution is

$$\kappa[2K_9(\mathbf{p}'\cdot\mathbf{p})] = -\tfrac{1}{16}\kappa . \tag{4.13}$$

The total contribution (aside from the one-loop part) from the third term of (3.14a) is then

$$C((3.14a)-\text{III}) = \kappa\left[\frac{5}{4}\ln\frac{1}{2\alpha}-\ln 2-\frac{1}{8}\right] . \tag{4.14}$$

B. Λ_- terms

Between proton large component projectors, each commutator in (3.14b) must contain one spatial γ and one γ_0. This leads to three different types of contributions.

1. No indices spatial

First consider the contribution where all the photon indices are 0. Since there is no possibility of one-loop contributions, the proton factor simplifies to

$$\frac{\kappa g_{\rho 0} g_{\nu 0} g_{\sigma 0}}{4m_p^2(1+\kappa)}\left[-\not p_f \not p'(-2\pi i)\delta(p_0)\left[-2\pi i\delta(p_0')-\frac{2}{p_0'+i\epsilon}\right]-\not p\not p_i(-2\pi i)\delta(p_0')\left[-2\pi i\delta(p_0)-\frac{2}{p_0+i\epsilon}\right]\right.$$
$$\left.-\not p'\not p\left[(-2\pi i)^2\delta(p_0')\delta(p_0)+\frac{4\pi i\delta(p_0)}{p_0'+i\epsilon}+\frac{4\pi i\delta(p_0')}{p_0+i\epsilon}+\frac{-4\pi i\delta(p_0-p_0')}{p_0+i\epsilon}\right]\right] . \tag{4.15}$$

Terms which do not contribute to the order of interest have been discarded. When the two-photon subtraction is made, the first two terms are canceled. With any term, the electron factor is (to sufficient accuracy)

$$\frac{-(\not p_f \not p'+\not p'\not p+\not p\not p_i)}{D_e(p')D_e(p)} . \tag{4.16}$$

On performing the angular integrations, one finds that the first and last terms of the electron factor do not contribute to the hfs when paired with the last term of (4.15). (This is easily seen, for example, by noting that when the initial wave-function integration is carried out, the last term of the electron factor is transformed according to $\not p\not p_i \to \not p\,\gamma\cdot\mathbf{p} \doteq -\mathbf{p}^2$, which has no spin dependence.) The result is precisely *OOO* contribution of BYG, to be multiplied by κ, which gives the coefficient

$$C((3.14b)-\text{no spatial}) = \frac{\kappa}{1+\kappa}\left[\frac{1}{4}\ln\frac{1}{2\alpha}-\frac{3}{8}\right] . \tag{4.17}$$

2. One index spatial

The contribution from (3.14b) with one photon index spatial is

$$\frac{\kappa}{4m_p^2(1+\kappa)}\left[g_{\rho 0}(-2\pi i)\delta(p_0')p_0[g_{\nu 0}(\not p'-\not p)\gamma_\sigma+g_{\sigma 0}(\not p-\not p_i)\gamma_\nu]\left[\frac{2m_p}{D_p(-p)}-\frac{2m_p}{D_p(p)}\right]\right.$$
$$+g_{\sigma 0}(-2\pi i)\delta(p_0)p_0'[g_{\rho 0}(\not p_f-\not p')\gamma_\nu+g_{\nu 0}(\not p'-\not p)\gamma_\rho]\left[\frac{2m_p}{D_p(-p')}-\frac{2m_p}{D_p(p')}\right]$$
$$\left.+g_{\nu 0}4\pi i\delta(p_0'-p_0)[g_{\rho 0}(\not p_f-\not p')\gamma_\sigma+g_{\sigma 0}(\not p-\not p_i)\gamma_\rho]\right] . \tag{4.18}$$

The two-photon subtraction kernel eliminates the two terms proportional to $\delta(p_0)$.

Of the two terms containing a factor $\delta(p_0')$ we consider the first term and double the result in order to incorporate the contribution of the second term. In the electron factor, the relevant terms are

$$\frac{2m_e(\not{p}-\not{p}_i)\gamma_\sigma+p_0(\not{p}'-\not{p}_i)\gamma_\sigma}{D_e(p')D_e(p)} . \tag{4.19}$$

After angular averaging, the first term from the electron factor, combined with the first $\delta(p_0')$ term from the proton factor, yields the numerator form $(\mathbf{p}'-\mathbf{p})\cdot(\mathbf{p}-\mathbf{p}_i)$. This may be rearranged, using symmetry, to obtain an equivalent expression $2\mathbf{p}\cdot\mathbf{p}'-\mathbf{p}^2$, where terms of higher order have been dropped. The evaluation of the integral for the $2\mathbf{p}'\cdot\mathbf{p}$ term is straightforward and gives $-8K_{13}(\mathbf{p}'\cdot\mathbf{p})$. The $-\mathbf{p}^2$ term causes a problem because it leads to contributions that include a one-loop piece. We can rewrite the factor of p_0 by using

$$p_0=\frac{D_e(p)-(p^2-\gamma^2)}{2m_e} . \tag{4.20}$$

The $D_e(p)$ term leads to an integrand that is odd in p, so it does not contribute. For the $-(p^2-\gamma^2)$ term, there is a decoupled one-loop integral that has a residue of interest, which together with the decoupling correction gives -3. The total contribution from the $\delta(p_0')$ term of the proton factor and the first term of the electron factor is

$$\frac{\kappa}{1+\kappa}\left[-2\ln\frac{1}{2\alpha}-1\right] . \tag{4.21}$$

The first $\delta(p_0')$ term from the proton factor with the second term of the electron factor yields a contribution that is straightforward to compute. For it we use $\mathbf{p}'^2-\mathbf{p}'\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{p}_i-\mathbf{p}'\cdot\mathbf{p}_i\doteq\mathbf{p}'^2$ to obtain the integral $-4K_7(\mathbf{p}'^2)$, yielding

$$\frac{\kappa}{1+\kappa}[-4K_7(\mathbf{p}'^2)]=\frac{\kappa}{1+\kappa}\frac{1}{2} . \tag{4.22}$$

Next consider the $\delta(p_0'-p_0)$ terms of (4.18). We may calculate either term and double the result. Because of the structure of the integral, we may neglect the external momenta in the numerator and decouple the wave functions with negligible corrections, as previously described. To sufficient accuracy, the electron factor accompanying the first term is

$$\frac{2m_e\not{p}\gamma_\sigma+p_0(\not{p}'+\not{p})\gamma_\sigma}{D_e(p')D_e(p)} . \tag{4.23}$$

The first term of the electron factor yields the integral $4K_{11}(\mathbf{p}'\cdot\mathbf{p})$ and the second yields the integral $4K_9(\mathbf{p}'^2+\mathbf{p}'\cdot\mathbf{p})$, for a total contribution of

$$\frac{\kappa}{1+\kappa}[4K_{11}(\mathbf{p}'\cdot\mathbf{p})+4K_9(\mathbf{p}'^2+\mathbf{p}'\cdot\mathbf{p})]=\frac{\kappa}{1+\kappa}\left[4\ln 2-\frac{5}{2}\right] . \tag{4.24}$$

The total contribution arising from (4.18) is

$$C((3.14b)-\text{one spatial})=\frac{\kappa}{1+\kappa}\left[-2\ln\frac{1}{2\alpha}+4\ln 2-3\right] . \tag{4.25}$$

3. *Two indices spatial*

In the case where two photon indices are spatial, (3.14b) reduces to

$$\frac{\kappa}{4m_p^2(1+\kappa)}\left[g_{\rho 0}\gamma_\nu\gamma_\sigma(-2\pi i)\delta(p_0')p_0^2\left[\frac{2m_p}{D_p(-p)}-\frac{2m_p}{D_p(p)}\right]\right.$$
$$\left.+g_{\sigma 0}\gamma_\rho\gamma_\nu(-2\pi i)\delta(p_0)p_0'^2\left[\frac{2m_p}{D_p(-p')}-\frac{2m_p}{D_p(p')}\right]+g_{\nu 0}\gamma_\rho\gamma_\sigma 4\pi i\delta(p_0'-p_0)p_0\right] . \tag{4.26}$$

Again, the third term contains no one-loop contributions of relative order α, so we have taken the usual approximations for the denominators. Either of the first two main terms is canceled by the two-photon subtraction. Let us discard the second one.

The electron factor associated with the first term of (4.26) is

$$\frac{\not p'\gamma_\nu \not p \gamma_\sigma + \gamma_\nu \not p \gamma_\sigma \not p_i - 2m_e p_0 \gamma_\nu \gamma_\sigma}{D_e(p)} . \tag{4.27}$$

The first two terms of the electron factor yield the same contribution, so we calculate the first one and double the result. We can decouple the $\mathbf{p}_i$ and $\mathbf{p}_f$ integrations with negligible corrections as before to obtain $4K_8(\mathbf{p}'\cdot\mathbf{p})$, which yields

$$\frac{\kappa}{1+\kappa}[4K_8(\mathbf{p}'\cdot\mathbf{p})] = -\frac{\kappa}{1+\kappa}\frac{1}{8} . \tag{4.28}$$

For the last term of the electron factor we use (4.20) to rewrite the factor p_0. As before, the first term on the RHS of (4.20) gives an integrand that is odd in p_0, so its contribution vanishes. For the second term on the RHS of (4.20) we follow our standard decoupling procedure and then subtract the pure relative order-α contribution from the resulting integral. The remainder does not contain a piece of the order of interest. The decoupling correction yields $12K_8(\mathbf{p}^2 - 2\mathbf{p}\cdot\mathbf{p}' - \gamma^2)$ and the contribution

$$\frac{\kappa}{1+\kappa}[12K_8(\mathbf{p}^2 - 2\mathbf{p}\cdot\mathbf{p}' - \gamma^2)] = -\frac{\kappa}{1+\kappa}\frac{3}{4} . \tag{4.29}$$

The last term of the proton factor of (4.26) goes with the electron factor

$$\frac{\gamma^\rho \not p' \not p \gamma^\sigma}{D_e(p')D_e(p)} . \tag{4.30}$$

It yields a spin factor

$$\gamma_\nu\gamma_\sigma \otimes \gamma^\nu \not p' \not p \gamma^\sigma \doteq \tfrac{2}{3}\boldsymbol{\sigma}_e\cdot\boldsymbol{\sigma}_p(-3)(p_0^2 - \mathbf{p}'\cdot\mathbf{p}) .$$

After decoupling the $\mathbf{p}_i$ and $\mathbf{p}_f$ integrations with negligible corrections, we obtain $6K_{10} - 6K_9(\mathbf{p}'\cdot\mathbf{p})$ and the contribution

$$\frac{\kappa}{1+\kappa}[6K_{10} - 6K_9(\mathbf{p}'\cdot\mathbf{p})] = \frac{\kappa}{1+\kappa}\frac{3}{8} . \tag{4.31}$$

The total contribution from (4.26) is

$$C((3.14\text{b}) - \text{two spatial}) = -\frac{\kappa}{1+\kappa}\frac{1}{2} . \tag{4.32}$$

V. DETAILED ANALYSIS OF THE STRUCTURE-DEPENDENT CONTRIBUTIONS

In this section we compute the structure-dependent contributions to the hydrogen hfs that do not involve multiphoton vertices, keeping terms of the order of current interest. In the preceding sections we set aside the pure one-loop integrals, except for their subleading parts, and concentrated on the terms that are insensitive to the proton's structure. Here we wish to recover the pure one-loop contributions and to compute the associated form-factor corrections. As we shall demonstrate below, the pure one-loop integrals contain all of the structure dependence that is relevant at the order of current interest.

Experimental results are usually expressed in terms of the charge and magnetic moment distributions $G_E(q^2)$ and $G_M(q^2)$. These latter quantities refer more closely to the physical concepts of charge and current distributions, which are given as the *three-dimensional* Fourier transforms of these form factors. The precise definitions of G_E and G_M in terms of F_1 and F_2 are

$$G_M = F_1 + \kappa F_2, \quad G_E = F_1 + \frac{q^2}{4m_p^2}\kappa F_2 . \tag{5.1a}$$

The inverse of this is

$$F_1 = \frac{G_E - [q^2/(4m_p^2)]G_M}{1 - [q^2/(4m_p^2)]} ,$$
$$\kappa F_2 = \frac{G_M - G_E}{1 - [q^2/(4m_p^2)]} . \tag{5.1b}$$

The $F_i(q^2)$ are analytic functions of q^2 except for branch cuts along the real axis for $q^2 > 4m_\pi^2$, with $F_i(0) = 1$. Hence,

$$F(q^2) - 1 \sim q^2/\Lambda^2 \tag{5.2}$$

for q^2 small and spacelike, where Λ is the scale of the proton's structure, which is of the order of m_ρ.

Any proton form factor can be rearranged into four terms:

$$F[(p_1 - p_2)^2] = 1 + [F(p_1^2) - 1] + [F(p_2^2) - 1]$$
$$+ [F((p_1 - p_2)^2) - F(p_1^2) - F(p_2^2) + 1] . \tag{5.3}$$

The first term of (5.3) corresponds to neglecting the effects of form factors. The remaining three terms we call "form-factor corrections." The last term vanishes if either p_1 or p_2 is very small; i.e., it necessarily involves two adjacent loops of very high momentum ($\gtrsim \Lambda$). The second (third) term vanishes if p_1 (p_2) is small. In some cases (e.g., a contribution of second order in the anomalous moment within a single loop), this rearrangement cannot be made because the separate integrals would diverge. Then one would study a similar form where only the momentum external to the loop would be reorganized in the manner of (5.3).

As a prelude to the actual computation of the effect of the form-factor corrections, let us identify the contributions to the hfs for which they might be of importance. Consider contributions which arise from Figs. 3(a) and 3(b). We wish to retain only those contributions that are required in order to account, to sufficient accuracy, for the deviations of the form factors from their static values. To this end, it is useful to recall the general properties of various types of terms resulting from the algebraic rearrangements given by BYG and in Sec. III.

The first step in those rearrangements was to drop terms of too high an order in α and/or (m_e/m_p). Obviously, there is no way that the momentum dependence of the form factors can make these terms important, so we continue to drop them. Next let us consider the structure of the remaining contributions. In general, the three-photon kernels lead to four-loop integrals: two internal loops plus two wave-function integrations. If we are to obtain a contribution to the orders of interest (i.e., relative orders 1, α, or α^2), then the loop integrations must yield at least two factors of $1/\gamma$. These factors of $1/\gamma$ can arise from individual loop integrations that converge independently or from coupled loop integrations. In either case, nonrelativistic loop momenta (of order γ) are involved and the insertion of a form-factor correction into such a loop results in the replacement of a factor $1/\gamma$ by a factor of order $1/\Lambda$. Except in conjunction with the leading-order term, where all the loops give factors $1/\gamma$, such corrections are negligible. Even in the case of the leading-order term, it can be seen that the form-factor corrections can involve at most one loop adjacent to the V photon. This result follows from power counting and from the form of the structure functions at small momentum, as given in (5.2). Of course, important contributions can also arise from regions of integration in which one or two loop momenta are relativistic with respect to the electron mass. Because of the two-photon subtraction kernel, such relativistic loops must be internal loops. In the case of a single relativistic loop, the insertion of a form-factor correction produces a contribution of the order of interest. If two relativistic loops occur, then, because of the Caswell-Lepage cancellation, the characteristic momenta must be of order m_e or less. In the absence of form factors, such a two-loop integration gives a contribution of relative order $\alpha^2(m_e/m_p)$. The insertion of a form-factor correction into one of the relativistic loops leads to a suppression factor m_e/Λ, and the resulting contribution is negligible.

We conclude that, to the order of present interest, one need consider form-factor corrections only in the case of the one-loop contributions to the hfs. All contributions through one loop are represented diagramatically in Fig. 4. They may be derived either by making suitable approximations on the terms arising from Figs. 3(a) and 3(b), or by an *ab initio* analysis. The latter approach is probably more transparent. Starting from Fig. 4 we find that a further small simplification is necessary in order to obtain a convenient form. Our objective is to omit the form-factor dependence occurring in the wave functions, while retaining the relevant physics. By the arguments of the preceding paragraph, this may be done immediately with the part of the kernel producing terms of relative order $\alpha(m_e/m_p)$ since a form-factor correction in the wave-function loop produces a further factor of $\alpha(m_e/\Lambda)$. Terms of relative order 1 arise only from a ladder kernel in which one photon is an O photon without an anomalous-moment interaction and the other photon is a V photon. A form-factor correction in the wave-function loop that is adjacent to the O photon produces a correction of relative order $\alpha^2(m_e^2/\Lambda^2)$ because, as may be seen by power counting, two inverse powers of γ are eliminated. The same is true for the contribution of the last term of (5.3) taken in conjunction with the one-photon subtraction. Finally, form-factor corrections in the wave-function loop next to a V photon cancel between the one-loop kernel and the one-photon subtraction kernel.

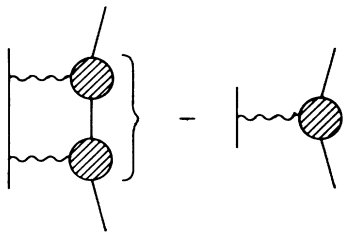

FIG. 4. The photon-exchange kernels that contribute to the proton-structure corrections in order $E_F\alpha m_e/m_p$, excluding the two-photon-proton vertex, which is not treated in this paper.

Thus, ignoring terms of relative order 10^{-8} and smaller, one may use the following prescription to compute the structure-dependent corrections to the hfs: treat the form factor dependence inside one-loop kernels, ignoring dependence on the wave-function momentum, and subtract E_F from the result. If p is the loop momentum and p_i and p_f are the wave-function momenta, then

$$F_a((p-p_i)^2)F_b((p-p_f)^2)\rightarrow F_a(p^2)F_b(p^2)\ , \qquad (5.4)$$

where the form factors F_a and F_b may be F_1 or F_2, as appropriate.

The proton factor can be worked out using the methods of Sec. II. We keep only terms that can contribute in leading and one-loop order and find the result

$$\mathcal{J}_p \doteq \left[\frac{1}{D_p(-p)}+\frac{1}{D_p(p)}\right]\left[[g_{\nu 0}(-\not{p}+\not{p}_i)\gamma_\sigma+g_{\sigma 0}\gamma_\nu(-\not{p}+\not{p}_f)]\left[F_1^2+\kappa F_1F_2+\kappa^2\frac{p^2}{4m_p^2}F_2^2\right]+\frac{\gamma_\nu\gamma_\sigma p_0 p^2}{4m_p^2}\kappa^2F_2^2\right]$$
$$+\left[\frac{1}{D_p(-p)}-\frac{1}{D_p(p)}\right]\left[\frac{\gamma_\nu\gamma_\sigma}{2m_p}p^2(F_1^2+2\kappa F_1F_2)-\frac{\gamma_\nu\not{p}\Lambda+\not{p}\gamma_\sigma}{2m_p}\kappa^2F_2^2\right], \qquad (5.5)$$

where the arguments of all F_i are p^2. Here we have dropped terms that do not give a contribution to the hfs and certain terms which are too small for our present purposes. For an hfs contribution, a term must have two or four γ matrices with spatial indices.

To relate this to the preceding sections we identify the terms of (5.5) which correspond either to terms of BYG with

the $1+\kappa$ factor included or to those terms treated in Sec. IV. In the first line of (5.5) the combination of $F_1^2+\kappa F_1 F_2$, with $F_i \to 1$, corresponds to the main part of *VO* of BYG. The part of the first term of the second line that contains the same combination corresponds to the main piece of *VV* of BYG. Hence, except for form-factor dependence, these contributions were worked out completely by BYG, including decoupling corrections. The remaining $\kappa F_1 F_2$ part of the first term of the second line is partly in (3.14a) and partly in (3.14b), and the corresponding decoupling corrections are described in Sec. IV. The κ^2 terms in the first line are nominally higher-order recoil contributions and have no correspondence to terms of BYG or of Sec. IV. They have no decoupling corrections of the order of interest. The last term of the second line occurs in (3.14a) and its decoupling correction is also described in Sec. IV. We have to treat the leading-order contribution in (5.5) accurately enough to obtain the Zemach correction. Other terms can be approximated by their pure one-loop parts.

Various options exist for rearranging this expression into a suitable form for analytic/numerical evaluation of the corrections due to form factors. In the following, we freely drop terms which do not contribute to the form-factor corrections to the order of interest. We also try to arrange expressions so as to be able to make use of results already found by BYG. As a first step we rewrite

$$\gamma_\nu \not{p} \Lambda_+ \not{p} \gamma_\sigma \doteq p^2 \gamma_\nu \gamma_\sigma + p_0 \gamma_\nu \not{p} \gamma_\sigma \ . \tag{5.6}$$

Then, using the identity

$$p_0 \left[\frac{1}{D_p(-p)} - \frac{1}{D_p(p)} \right] = \frac{-2}{2m_p} + \frac{p^2}{2m_p} \left[\frac{1}{D_p(-p)} + \frac{1}{D_p(p)} \right] \tag{5.7}$$

for the terms containing the p_0 factor, we may rearrange (5.5) to obtain

$$\begin{aligned} \mathcal{J}_{p_\phi} \doteq & \left[\frac{1}{D_p(-p)} + \frac{1}{D_p(p)} \right] \{ [g_{\nu 0}(-\not{p}+\not{p}_i)\gamma_\sigma + g_{\sigma 0}\gamma_\nu(-\not{p}+\not{p}_f)] F_1 G_M \} \\ & + \left[\frac{1}{D_p(-p)} - \frac{1}{D_p(p)} \right] \left[\frac{\gamma_\nu \gamma_\sigma}{2m_p} p^2 G_M^2 \right] - \gamma_\nu \not{p} \gamma_\sigma \frac{2\kappa^2}{4m_p^2} F_2^2 \ . \end{aligned} \tag{5.8}$$

It is easy to work out the electron factor to sufficient accuracy for our present purposes. The result is

$$\mathcal{J}_e \doteq \frac{1}{D_e(p)} [g_{\nu 0}(\not{p}-\not{p}_i)\gamma_\sigma + g_{\sigma 0}\gamma_\nu(\not{p}-\not{p}_f) - p_0 \gamma_\nu \gamma_\sigma] \ . \tag{5.9}$$

Then, contracting the electron factor into the proton factor and extracting the hfs contribution, we obtain

$$\begin{aligned} \mathcal{J}_e \times \mathcal{J}_p \doteq \tfrac{2}{3} \langle \boldsymbol{\sigma}_e \cdot \boldsymbol{\sigma}_p \rangle \Bigg[& -[(\mathbf{p}-\mathbf{p}_i)^2 + (\mathbf{p}-\mathbf{p}_f)^2] \left[\frac{1}{D_p(-p)} + \frac{1}{D_p(p)} \right] \frac{1}{D_e(p)} F_1(p^2) G_M(p^2) \\ & + \frac{3}{2m_p} p^2 p_0 \left[\frac{1}{D_p(-p)} - \frac{1}{D_p(p)} \right] \frac{1}{D_e(p)} [G_M(p^2)]^2 + \frac{1}{2m_p^2}(2p^2 + p_0^2) \frac{1}{D_e(p)} \kappa^2 [F_2(p^2)]^2 \Bigg] \ . \end{aligned} \tag{5.10}$$

This expression contains three main terms. Notice that the last term does not have a D_p denominator. If the form factors are replaced by their static values, then this term leads to a logarithmic divergence; it also yields a logarithmic dependence on the electron mass. All other terms are convergent even without the intervention of a form factor. For practical calculations, it seems convenient to arrange the work so that terms containing a factor $\ln(m_p/m_e)$ can be calculated analytically; the remainder, which is then much less sensitive to low-momentum behavior, may be evaluated numerically. To this end, we make the rearrangement in the last term of (5.10):

$$\frac{1}{D_e(p^2)} = \left[\frac{1}{D_e(p^2)} - \frac{1}{D_p(p^2)} \right] + \frac{1}{D_p(p^2)} \ . \tag{5.11}$$

The first piece of this expression leads to a $\ln(m_p/m_e)$ contribution in the absence of form-factor dependence, and the second piece leads to a contribution known as "correction No. 1."

The first main term of (5.10) contains, among other things, the Zemach correction. Since it is conventional to write the Zemach correction in terms of the form-factor combination $G_E G_M$, we arrange the form factor for this term as follows:

$$F_1 G_M = G_E G_M - \frac{p^2}{4m_p^2} \kappa F_2 G_M \ . \tag{5.12}$$

For the contribution to the hfs that corresponds to the first term in (5.12), we set up the integral in a convenient form by substituting an identity from Appendix B of BYG:

$$\left[\frac{1}{D_p(-p)}+\frac{1}{D_p(p)}\right]\frac{1}{D_e(p)}\doteq\frac{-2\pi i\delta(p_0)}{2(m_e+m_p)(-\mathbf{p}^2-\gamma^2)}+\frac{1}{m_p^2-m_e^2}\left[\frac{m_e}{D_e(p)}-\frac{m_p}{D_p(p)}\right]\frac{1}{p_0+i\epsilon}\ . \tag{5.13}$$

Here we have made use of the symmetry properties of the integrand under $p_0\rightarrow -p_0$. For the contribution to the hfs that corresponds to the second term of (5.12) we substitute the identity

$$\frac{p^2}{D_e(p)}\left[\frac{1}{D_p(-p)}+\frac{1}{D_p(p)}\right]\doteq\frac{2}{m_p^2-m_e^2}\left[\frac{-m_e^2}{D_e(p)}-\frac{-m_p^2}{D_p(p)}\right]\ . \tag{5.14}$$

In treating the second main term of (5.10) we make use of yet another identity from Appendix B of BYG:

$$\left[\frac{1}{D_p(-p)}-\frac{1}{D_p(p)}\right]\frac{p_0}{D_e(p)}=\frac{m_p}{m_p^2-m_e^2}\left[\frac{-1}{D_e(p)}+\frac{1}{D_p(p)}\right]\ . \tag{5.15}$$

Taking into account all of these arrangements of (5.10) we now write the contribution of the one-loop kernel to the hfs:

$$\begin{aligned}\Delta E(\text{one loop})\doteq E_F\frac{16\pi\alpha}{1+\kappa}\frac{m_em_p}{m_p^2-m_e^2}\int\frac{d^4p}{-(2\pi)^4i}\frac{1}{(p^2-\gamma^2+i\epsilon)^2}\\ \times\Bigg[(m_p-m_e)(-2\pi i)\delta(p_0)G_E(p^2)G_M(p^2)\\ -\frac{2\mathbf{p}^2}{p_0+i\epsilon}\left[\frac{m_e}{D_e(p)}-\frac{m_p}{D_p(p)}\right]G_E(p^2)G_M(p^2)\\ +\frac{\mathbf{p}^2}{m_p^2}\left[-\frac{m_e^2}{D_e(p)}+\frac{m_p^2}{D_p(p)}\right]\kappa F_2(p^2)G_M(p^2)\\ +\tfrac{3}{2}p^2\left[-\frac{1}{D_e(p)}+\frac{1}{D_p(p)}\right][G_M(p^2)]^2\\ +\tfrac{1}{2}(2p^2+p_0^2)\left[\frac{1}{D_e(p)}-\frac{1}{D_p(p)}\right]\kappa^2[F_2(p^2)]^2\\ +\tfrac{1}{2}(2p^2+p_0^2)\frac{1}{D_p(p)}\kappa^2[F_2(p^2)]^2\Bigg]\ .\end{aligned} \tag{5.16}$$

(For uniformity, we have taken the liberty of replacing some of the denominators m_p^2 with $m_p^2-m_e^2$.) Here we have decoupled the wave-function integrations in the following manner:

$$\frac{(\mathbf{p}-\mathbf{p}_i)^2}{(p-p_i)^2+i\epsilon}\rightarrow\frac{\mathbf{p}^2}{p^2-\gamma^2+i\epsilon}\ , \tag{5.17a}$$

$$\frac{1}{(p-p_i)^2}\rightarrow\frac{1}{p^2-\gamma^2+i\epsilon}\ , \tag{5.17b}$$

with similar expressions for $p_i\rightarrow p_f$. The corrections to these decoupling formulas can contribute in relative order-$\alpha^2(m_e/m_p)$. Indeed, in our previous discussion of the relative order-$\alpha^2(m_e/m_p)$ contributions, we retained such decoupling corrections, although the details of the decoupling procedure did not correspond exactly to the method used here. In this section, we are interested in computing the effects that arise from the difference between the form factors and their static values. However, the decoupling corrections contribute to the order of interest only when the momenta in the integrals are small; that is, they are insensitive to the proton's structure. Thus, in the terms corresponding to the form-factor corrections, the decoupling corrections lead to contributions that are suppressed by a factor m_e/Λ relative to the order of interest. Hence, we drop the decoupling corrections in this section.

Equation (5.16) contains six main terms. The first term, which is proportional to $\delta(p_0)$, contains twice the Fermi splitting plus the Zemach correction. The second term corresponds to one photon index spatial and one photon index temporal. It was worked out analytically by BYG for the case of constant form factors. Following BYG, we call this term "*VO*." The third term of (5.16) contains a correction to *VO* due to the difference between F_1 and G_E, which we call "correction No. 2." In the older work[29] such terms were simply dropped

(i.e., the difference between F_1 and G_E was overlooked). They appeared to be a part of the corrections of higher order in the recoil, which were very difficult to incorporate consistently at that time. Actually, as will be seen, correction No. 2 is not unimportant at the present level of comparison between theory and experiment. The fourth term of (5.16) corresponds to both indices spatial for the exchanged photons. It was also worked out analytically by BYG for the case of constant form factors. Following BYG, we call it "*VV*." The fifth term of (5.16) arises from two photon indices spatial or one spatial and one temporal. We denote it by "the κ^2 correction." A correction to this term, which contains no m_e dependence, is given in the sixth term of (5.16), which is called "correction No. 1."

Zemach correction. We now describe the computation of the Zemach correction in the present formalism. We focus on the first main term of (5.16). If the form factors are replaced by their static values, the result is twice the leading contribution E_F to the hyperfine splitting. One of these contributions of E_F is canceled by the contribution from one photon subtraction kernel; we set aside the other, since it does not depend on the proton's structure. The remaining contribution to the hfs is

$$\Delta E(\text{Zemach}) = E_F \frac{2\alpha m_r}{\pi^2} \int d^3p \frac{1}{(\mathbf{p}^2+\gamma^2)^2} \times \left[\frac{G_E(-\mathbf{p}^2) G_M(-\mathbf{p}^2)}{1+\kappa} - 1 \right] . \tag{5.18}$$

The γ^2 dependence is actually negligible, but we retain it temporarily as a limiting procedure. Then we may write the expression in terms of the three-dimensional Fourier transforms of the form factors to obtain

$$\begin{aligned} \Delta E(\text{Zemach}) &= E_F \delta_p(\text{Zemach}) \\ &= 2\alpha m_r E_F \lim_{\gamma \to 0} \int d^3r_1 d^3r_2 d^3r_3 \delta(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3) \frac{e^{-\gamma r_1}}{\gamma} [\rho_E(r_2)\rho_M(r_3) - \delta(\mathbf{r}_2)\delta(\mathbf{r}_3)] \\ &= E_F(-2\alpha m_r) \int d^3r_2 d^3r_3 \, | \, \mathbf{r}_2+\mathbf{r}_3 \, | \, \rho_E(r_2)\rho_M(r_3) = E_F(-2\alpha m_r R_p) \, . \end{aligned} \tag{5.19}$$

This is the usual result. Here ρ_E and ρ_M are normalized to unity upon spatial integration, and R_p is the mean radius of their convolution. As an illustration, and a point of reference for future work, (5.19) may be evaluated using a common parametrization of these form factors, experimentally valid in the spacelike region ($q^2 < 0$), given by

$$G_E = \frac{G_M}{1+\kappa} \left[\frac{\Lambda^2}{\Lambda^2 - q^2 - i\epsilon} \right]^2 . \tag{5.20}$$

The relationship between G_E and G_M is called "form-factor scaling," and this particular parametrization of the experimental data is known as the "dipole form." Fits to the experimental data[37,38] give the value $\Lambda = (0.898 \pm 0.013) m_p$. We have introduced the $-i\epsilon$ in the denominator of (5.20) to provide the correct analyticity in the timelike region; note, however, that the functions F_1 and F_2 have spurious poles at $q^2 = 4m_p^2$. As we mentioned previously, the exact form factors have branch cuts for $4m_{\pi^2} < q^2$. We use this approximate form only for purposes of illustration; it was used by Grotch and Yennie,[29] among others, so we may make comparisons with that earlier work. Iddings and Platzman[28] used the form on the RHS of (5.20) to represent F_1 and F_2. In practice, we use the analytic properties of the form factors to convert the necessary integrals to ones which depend only on knowledge of the form factors in the spacelike (i.e., $q^2 < 0$) region. This gives the spatial distributions

$$\rho_E(r) = \rho_M(r) = \frac{\Lambda^3}{8\pi} e^{-\Lambda r} .$$

With this choice, one finds $R_p = 35/(8\Lambda)$, giving a correction $\delta_p(\text{Zemach}) = (-38.72 \pm 0.56)$ ppm for $\Lambda = (0.898 \pm 0.013) m_p$. The error that we quote here corresponds only to the uncertainty in the determination of Λ and does not reflect any systematic errors that might arise from the inadequacy of the one-parameter dipole fit. The variations of the contributions from different terms in (5.16) with respect to changes in Λ are highly correlated. In order to display these correlations, here, and in the remainder of this section, we adopt the convention that the upper sign in the error term corresponds to a positive change in Λ.

Now we turn to a description of the various remaining contributions from (5.16). It will turn out that their net contribution is much smaller than that of the Zemach correction; but, individually, they are quite comparable. The small net contribution seems to be due purely to a fortuitous compensation. It is not our aim to make a new precision calculation here. Rather, we try to present the method in a convenient form for application to experimentally determined form factors. In each case, the discussion follows a fixed pattern: the complete four-momentum integral for the contribution is presented first. Then, assuming the usual analyticity of the form factors, the p_0 contour is rotated and a form that is more appropriate to numerical evaluation is presented. Next, if the integral can be evaluated analytically for constant form factors, the result is given. Finally, the contribution including the effect of the dipole form factors (5.20) is given.

VO contribution. The second term of (5.16) gives

$$\Delta E(VO)=E_F\frac{-32\pi\alpha m_e m_p}{m_p^2-m_e^2}\int\frac{d^4p}{-(2\pi)^4 i}\frac{\mathbf{p}^2}{(p_0+i\epsilon)(p^2+i\epsilon)^2}\left[\frac{m_e}{D_e(p)}-\frac{m_p}{D_p(p)}\right]\frac{G_E(p^2)G_M(p^2)}{1+\kappa}\ . \tag{5.21}$$

Now we rotate the p_0 contour to the imaginary axis and call the new (real) variable ξ. The only pole that requires special attention is the one from the denominator $p_0+i\epsilon$, which becomes $i(\xi+i\epsilon)$. The other integrand factors are modified as follows:

$$\frac{1}{p^2+i\epsilon}\to-\frac{1}{\mathbf{p}^2+\xi^2} \tag{5.22a}$$

and

$$\frac{1}{D_e(p)}\to\frac{-\mathbf{p}^2-\xi^2-i2m_e\xi}{(\mathbf{p}^2+\xi^2)^2+4m_e^2\xi^2}\ , \tag{5.22b}$$

with a similar expression for D_p. With a factor of ξ in the numerator, the pole at $\xi=0$ is canceled. With no such numerator factor of ξ, the integrand is even in ξ except for the pole, and we may substitute $(\xi+i\epsilon)^{-1}\to-i\pi\delta(\xi)$. Finally, it turns out that the effect of the δ function can be simulated by setting $\xi=0$ everywhere except in the electron (or proton) denominator, for which one makes the replacement

$$\frac{\pi\delta(\xi)}{(\mathbf{p}^2+\xi)^2+4m^2\xi^2}\to\frac{2m}{[(\mathbf{p}^2)^2+4m^2\xi^2]\mathbf{p}^2}\ . \tag{5.23}$$

The advantage of this manipulation is that it improves the accuracy of the numerical integrations by producing point-wise cancellations in certain regions which would otherwise be singular. The result of all this is that (5.21) becomes

$$\begin{aligned}\Delta E(VO)=E_F\frac{-8\alpha m_e m_p}{\pi^2(m_p^2-m_e^2)}\int_0^\infty d\xi\,dp\,p^4\\ \times\left[\left[\frac{2m_e^2}{(p^2+\xi^2)^2+4m_e^2\xi^2}-\frac{2m_p^2}{(p^2+\xi^2)^2+4m_p^2\xi^2}\right]\frac{G_E(-p^2-\xi^2)G_M(-p^2-\xi^2)}{(p^2+\xi^2)^2(1+\kappa)}\right.\\ \left.-\left[\frac{2m_e^2}{(p^2)^2+4m_e^2\xi^2}-\frac{2m_p^2}{(p^2)^2+4m_p^2\xi^2}\right]\frac{G_E(-p^2)G_M(-p^2)}{p^4(1+\kappa)}\right]\ .\end{aligned} \tag{5.24}$$

If the form factors are replaced with their static values, then the integral is easily evaluated analytically, with the result

$$\Delta E(VO\ \text{no structure})=E_F(-6)\frac{\alpha}{\pi}\frac{m_e m_p}{m_p^2-m_e^2}\ln\frac{m_p}{m_e}\ . \tag{5.25}$$

The size of this correction is -57.04 ppm. The change due to the form factor is most easily computed by making the replacement

$$\frac{G_E G_M}{1+\kappa}\to\frac{G_E G_M}{1+\kappa}-1\ . \tag{5.26}$$

The resulting integral is no longer sensitive to m_e, making its evaluation quite straightforward using, for example, Chebyshev integration or an adaptive Monte Carlo technique such as VEGAS (Ref. 39). Using (5.20) we find that the correction is reduced in size to (-43.98 ∓ 0.09) ppm. The uncertainty in this correction is relatively small in comparison with the similar uncertainty in the Zemach correction. The reason is that the dependence of the *VO* correction on Λ is mainly logarithmic. That is, the approximate effect of the form factor is to replace m_p by Λ in the argument of the logarithm in (5.25).

VV contribution. The fourth term of (5.16) gives the contribution

$$\Delta E(VV)=E_F\frac{24\pi\alpha m_e m_p}{m_p^2-m_e^2}\int\frac{d^4p}{-(2\pi)^4 i}\frac{1}{p^2+i\epsilon}\left[-\frac{1}{D_e(p)}+\frac{1}{D_p(p)}\right]\frac{[G_M(p^2)]^2}{1+\kappa}\ . \tag{5.27}$$

As before, we rotate the p_0 contour to the imaginary axis:

$$\Delta E(VV)=E_F\frac{6\alpha m_e m_p}{\pi^2(m_p^2-m_e^2)}\int_0^\infty d\xi\,dp\,p^2\left[\frac{1}{(p^2+\xi^2)^2+4m_e^2\xi^2}-\frac{1}{(p^2+\xi^2)^2+4m_p^2\xi^2}\right]\frac{[G_M(-p^2-\xi^2)]^2}{1+\kappa}\ . \tag{5.28}$$

This integral is also easily evaluated analytically with constant form factors; the result is

$$\Delta E(VV \text{ no structure})=E_F[3(1+\kappa)]\frac{\alpha}{\pi}\frac{m_e m_p}{m_p^2-m_e^2}\ln\frac{m_p}{m_e} . \tag{5.29}$$

This reference integral gives a correction of 79.66 ppm. The change due to the form factor is computed by subtracting $(1+\kappa)$ from the last factor of (5.28). Again, the resulting integral is easily evaluated by numerical techniques. Using (5.20), we find that this correction becomes (70.05 ± 0.10) ppm.

κ^2 *contribution.* The fifth term of (5.16) gives

$$\Delta E(\kappa^2)=E_F\frac{8\pi\alpha m_e m_p}{m_p^2-m_e^2}\int\frac{d^4p}{-(2\pi)^4 i}\frac{2p^2+p_0^2}{(p^2+i\epsilon)^2}\left[\frac{1}{D_e(p)}-\frac{1}{D_p(p)}\right]\frac{\kappa^2}{1+\kappa}[F_2(p^2)]^2 . \tag{5.30}$$

After we rotate the p_0 contour, the integral becomes

$$\Delta E(\kappa^2)=E_F\frac{-2\alpha m_e m_p}{\pi^2(m_p^2-m_e^2)}\int_0^\infty d\xi\, dp\, p^2\frac{3\xi^2+2p^2}{p^2+\xi^2}$$
$$\times\left[\frac{1}{(p^2+\xi^2)^2+4m_e^2\xi^2}-\frac{1}{(p^2+\xi^2)^2+4m_p^2\xi^2}\right]\frac{\kappa^2}{1+\kappa}[F_2(-\xi^2-p^2)]^2 . \tag{5.31}$$

To define a reference integral which is easily evaluated analytically, we set the form factor equal to its static value. The result is

$$\Delta E(\kappa^2 \text{ no structure})=E_F\left[-\frac{9}{4}\frac{\kappa^2}{1+\kappa}\right]\frac{\alpha}{\pi}\frac{m_e m_p}{m_p^2-m_e^2}\ln\frac{m_p}{m_e} . \tag{5.32}$$

The size of this correction is -24.62 ppm. The change due to the form factor is taken into account by subtracting 1 from the last factor of (5.31), and the resulting integral is easily evaluated by numerical techniques. Using (5.20), we find that the contribution is (-21.36 ∓ 0.03) ppm.

Correction No. 1. The last term of (5.16) gives a contribution

$$\Delta E(\text{No. } 1)=E_F\frac{8\pi\alpha m_e m_p}{m_p^2-m_e^2}\int\frac{d^4p}{-(2\pi)^4 i}\frac{2p^2+p_0^2}{(p^2+i\epsilon)^2}\frac{1}{D_p(p)}\frac{\kappa^2}{1+\kappa}[F_2(p^2)]^2 . \tag{5.33}$$

Since the momentum scale is governed by m_p and the form factors, the integral is well behaved numerically. Rotating the p_0 contour, we obtain

$$\Delta E(\text{No. } 1)=E_F\frac{-2\alpha m_e m_p}{\pi^2(m_p^2-m_e^2)}\int_0^\infty d\xi\, dp\, p^2\frac{3\xi^2+2p^2}{p^2+\xi^2}\frac{1}{(p^2+\xi^2)^2+4m_p^2\xi^2}\frac{\kappa^2}{1+\kappa}[F_2(-\xi^2-\mathbf{p}^2)]^2 . \tag{5.34}$$

For the illustrative example (5.20), the contribution is (-1.08 ∓ 0.01) ppm.

Correction No. 2. For the third main term of (5.16), it is easy to see that the contribution involving D_e is smaller than the contributions already studied by a factor of m_e^2/m_p^2, so we ignore it. Thus we are left with

$$\Delta E(\text{No. } 2)=E_F\frac{16\pi\alpha m_e m_p}{m_p^2-m_e^2}\int\frac{d^4p}{-(2\pi)^4 i}\frac{\mathbf{p}^2}{(p^2+i\epsilon)^2}\frac{1}{D_p(p)}\frac{\kappa}{1+\kappa}F_2(p^2)G_M(p^2) , \tag{5.35}$$

which we can rewrite as

$$\Delta E(\text{No. } 2)=E_F\frac{4\alpha m_e m_p}{\pi^2(m_p^2-m_e^2)}\int_0^\infty d\xi\, dp\, p^2\frac{p^2}{p^2+\xi^2}\frac{1}{(p^2+\xi^2)^2+4m_p^2\xi^2}\frac{\kappa}{1+\kappa}F_2(-p^2-\xi^2)G_M(-p^2-\xi^2) . \tag{5.36}$$

For the illustrative example (5.20), the contribution is (1.59 ± 0.02) ppm.

The net one-loop contribution, as defined here, turns out to be (-33.50 ± 0.55) ppm. As mentioned earlier, the uncertainty arises entirely from the uncertainty in Λ and it does not take into account systematic deviations from the simple dipole fit. The bulk of the uncertainty comes from the Zemach term alone, since the recoil uncertainties are individually quite small and tend to cancel among themselves. The one-loop contribution that we obtain differs from the values of -35 ppm given by Iddings and Platzman[28] and of -34.6 given by Grotch and Yennie.[29] Let us now try to account for these differences by examining the assumptions that were made about the form factors in the earlier treatments. First, we note that in the earlier work a slightly different value of Λ was used: namely, $\Lambda=0.91m_p$. Replacing this value with $\Lambda=0.898m_p$ would shift the results of the earlier work by -0.50 ppm. Iddings and Platzman present their analysis in terms of F_1 and F_2 and take the

dipole form on the RHS of (5.20) to represent both of them. In the language of our analysis, this would correspond to reexpressing G_M in (5.10) in terms of F_1 and F_2 and using the expression on the RHS of (5.20) for F_1 and F_2. As a consequence, Iddings and Platzman have no correction No. 2; also, their calculations of the κ^2 contribution and correction No. 1 do not contain the kinematic denominator factor $[1-p^2/(4m_p^2)]^2$. Including correction No. 2 in their result gives a change of $+1.61$ ppm. The kinematic factor in the κ^2 contribution produces a change of $+0.13$ ppm, and the kinematic factor in correction No. 1 produces a change of $+0.07$ ppm. Since Iddings and Platzman quote their result to the nearest ppm, these changes would bring their result into agreement with ours. It is difficult to establish an exact correspondence between the calculation of Grotch and Yennie and the present one. As we remarked earlier, Grotch and Yennie ignored correction No. 2. It appears that the effect of their approach was to ignore the distinction between the F's and the G's in other respects as well. Thus, our best guess is that the Grotch and Yennie calculation should be corrected in the same manner as described for the Iddings and Platzman calculation. Their result corrected in this manner would be -33.3 ppm, in satisfactory agreement with the result of the present calculation.

VI. SUMMARY AND DISCUSSION OF RESULTS

In this section we summarize the principal results of this paper, compare the theoretical and experimental results for the hydrogen hfs, and discuss briefly the current status of the theory.

We have carried out a systematic computation, to the order of current interest, of the recoil corrections to the hydrogen hfs that arise from the non-QED nature of the proton—namely, the anomalous moment corrections and structure function corrections (excluding the polarization correction). The calculation of the new anomalous moment recoil corrections, which appear in relative order $\alpha^2(m_e/m_p)$, involves loop momenta of order m_e or less and, thus, is independent of structure function variations. We were able to obtain analytic expressions for these contributions. The structure function dependence is contained in the relative order-$\alpha(m_e/m_p)$ contributions. These have been computed previously. However, we felt it necessary to redo the calculation in order to ensure that no terms had been omitted in matching it onto the structure-independent part. All these contributions are part of δ_p(recoil), which is introduced in (1.8).

In Secs. III and IV we computed the order-$E_F\alpha^2(m_e/m_p)$ corrections to the proton hfs, with the aim of obtaining the new contributions that arise from the proton's anomalous magnetic moment. As we mentioned in Sec. III, a part of this anomalous-moment contribution is given by κ times the one-kernel part of the result of BYG. Hence, there is a contribution to the hydrogen hfs that is given by $(1+\kappa)$ times the one-kernel part of the result of BYG. This contribution yields

$$\delta_p(\text{BYG one kernel})=\alpha^2\frac{m_e}{m_p}\left[2\ln\frac{1}{2\alpha}-6\ln 2+1\tfrac{5}{8}\right]$$
$$\approx 0.17 \text{ ppm} . \quad (6.1)$$

We also mentioned in Sec. III that, owing to the particular way that our starting equation is defined, the contribution that is of second order in the perturbation kernels involves just two hyperfine interactions, each with a factor of the total magnetic moment. Accordingly, we may take over the result of BYG by multiplying it by a factor of $(1+\kappa)^2$, giving

$$\delta_p(\text{BYG second order})=\alpha^2\frac{m_e}{m_p}(1+\kappa)(1\tfrac{71}{72})$$
$$\approx 0.16 \text{ ppm} . \quad (6.2)$$

The total Λ_+ contribution from (4.10) and (4.14) is given by

$$\delta_p(\Lambda_+)=\alpha^2\frac{m_e}{m_p}\kappa\left[\frac{7}{4}\ln\frac{1}{2\alpha}-\ln 2-1\tfrac{1}{8}\right]$$
$$\approx 0.29 \text{ ppm} . \quad (6.3)$$

The total Λ_- contribution from (4.17), (4.25), and (4.32) is given by

$$\delta_p(\Lambda_-)=\alpha^2\frac{m_e}{m_p}\left[\frac{\kappa}{1+\kappa}\right]\left[-\frac{7}{4}\ln\frac{1}{2\alpha}+4\ln 2-3\tfrac{7}{8}\right]$$
$$\approx -0.16 \text{ ppm} . \quad (6.4)$$

Adding (6.1)–(6.4) we find for the total contribution in relative order $\alpha^2(m_e/m_p)$:

$$\delta_p(\alpha^2(m_e/m_p))\approx 0.46 \text{ ppm} . \quad (6.5)$$

In Sec. V we computed the order-$E_F\alpha(m_e/m_p)$ contributions that depend on the proton's structure. We have succeeded in reducing these contributions to two-dimensional integrals that are suitable for numerical integration. The integral expressions are presented in Eqs. (5.24), (5.28), (5.31), (5.34), and (5.36), and they represent a convenient starting point for incorporating form-factor data into the computation of the hydrogen hfs. In order to make a comparison with previous calculations of the structure dependence and also to establish a point of reference for future work, we have evaluated the integrals numerically, using the dipole parametrization of the form factors given in (5.20). Our result is

$$\delta_p(\alpha(m_e/m_p)\text{ structure})\approx(5.22\mp 0.01)\text{ ppm} . \quad (6.6)$$

As we discussed in Sec. V, our value for the structure-dependent contribution differs slightly from those given previously[28,29] because it is based on a somewhat different parametrization of the form factors. In particu-

lar, our result (including the Zemach correction) represents an increase of 0.9 ppm compared with the result of Grotch and Yennie.[29]

Our principal new results, (6.5) and (6.6), may be combined to give the total recoil correction to the hfs, excluding the proton polarizability, through relative order $\alpha^2(m_e/m_p)$:

$$\delta_p(\text{recoil}) \approx 5.68 \text{ ppm} , \tag{6.7}$$

where $\delta_p(\text{recoil})$ is defined in (1.8). This is a net increase of 1.4 ppm compared with the result of Grotch and Yennie. When our result (6.7) is combined with the Zemach correction of (-38.72 ± 0.56) ppm, the difference between theory and experiment becomes

$$\frac{\nu(\text{theory}) - \nu(\text{expt})}{\nu_F} = (-0.48 \pm 0.56 \pm \text{unknown}) \text{ ppm} . \tag{6.8}$$

The error of 0.56 ppm contains a small contribution from the uncertainty in α, but it arises mainly from the uncertainty in the parameter Λ in the dipole fit to the proton's elastic form factors. The primary effect of this uncertainty in Λ is in the Zemach correction, where it enters through the mean radius of convolution of the proton's electric and magnetic form factors R_p. As was seen in the analysis of Sec. V, the corresponding uncertainties in $\delta_p(\text{recoil})$ are much smaller, both because the Λ dependence is essentially logarithmic and because there is a partial cancellation between the various terms.

The part of the error labeled "unknown" in (6.8) represents all remaining uncertainties in ν(theory). The most important sources of these uncertainties are the radiative corrections to the structure-dependent contributions, the systematic errors in the parametrization of the proton's form factors, and $\delta_p(\text{recoil})$. The radiative corrections to the structure-dependent contributions, which we discuss below, could potentially contribute at the level of 1 ppm. However, they could probably be calculated to a precision of 0.01 ppm or better. More closely related to the analysis of this paper are the systematic errors in the parametrization of the proton's form factors, which are evidenced by statistically significant deviations of the scaling assumption and dipole parametrization of (5.20) from the elastic-scattering data. We have made use of (5.20) in our analysis because it represents a convenient way to summarize the gross behavior of the form factors over the region of q^2 that seems most significant for the structure-dependent corrections. Also, we wished to make comparisons between our results and the results of previous analyses that made use of the dipole form. It would be desirable to reduce this part of the systematic uncertainty in the theory by carrying out the numerical integrations of Sec. V with a more precise parametrization of the proton's form factors. Based on our experience with the Λ dependence of the corrections, it seems likely that a refinement in the treatment of the form factors would have a larger effect on the Zemach correction than on $\delta_p(\text{recoil})$. It is difficult to estimate how much the use of more precise expressions for the form factors might shift the central value of ν(theory) or how large the remaining statistical uncertainty in that determination might be. As we remarked in Sec. I, a value of 0.9 ppm has appeared in the literature as an estimate of the complete uncertainty in the $\delta_p(\text{Zemach}) + \delta_p(\text{recoil})$. We have been unable to determine the basis for that estimate, but we have traced the value 0.9 ppm to a review by Brodsky and Drell.[19] The present work has shifted the theory by more than that amount (by ≈ 1.5 ppm). Our subjective impression is that a statistical uncertainty of 1 ppm seems to be a reasonable estimate of what might be achieved through a more precise treatment of existing form-factor data, together with an evaluation of radiative corrections that are significant at that level. It would not be surprising if such an analysis were to lead to a shift of as much as 1 ppm in the central value of ν(theory). However, it is unlikely that the analysis would reveal any incompatibility with the estimate

$$|\delta_p(\text{polarizability})| < 4 \text{ ppm} ,$$

which is based on data from deep-inelastic scattering with a polarized beam and target.[32,34] In fact, by incorporating the refinements in the computation of ν(theory) that we have already mentioned, one could use the hydrogen hfs to determine $\delta_p(\text{polarizability})$ with a precision of roughly 1 ppm.

Finally, let us discuss in somewhat more detail the uncalculated radiative corrections. In the case of radiative corrections that do not involve the proton's structure, the most important contributions have already been taken into account; we mention some additional refinements in a moment. The structure-dependent radiative corrections arise from radiative corrections to exchanged-photon lines or to the electron line, taken in conjunction with structure effects. As Caswell and Lepage[26] pointed out, the most important corrections of this type arise from the insertion of a vacuum-polarization correction into an exchanged-photon line. In the region where the photon four-momentum p is very large compared to the electron mass, the vacuum polarization introduces a factor

$$\frac{\alpha}{3\pi} \ln \frac{-p^2}{m_e^2} . \tag{6.9}$$

One can estimate the size of this vacuum-polarization correction when it is taken in conjunction with the Zemach contribution by replacing $-p^2$ with Λ^2 in (6.9) and doubling the result to allow for the two ways of inserting the vacuum polarization. This yields a correction factor

$$\frac{4\alpha}{3\pi} \ln \frac{\Lambda}{m_e} \approx 0.023 , \tag{6.10}$$

which would lead to a correction of about -1 ppm. The total result for all the radiative corrections to the Zemach contribution would probably be somewhat smaller, since the corrections on the electron line tend to contribute with the opposite sign. In the case of radia-

tive corrections to δ_p(recoil), the p integration is approximately logarithmic over the range m_e to Λ, so (6.9) yields a factor

$$\frac{2\alpha}{3\pi}\ln\frac{\Lambda}{m_e}\approx 0.011 \tag{6.11}$$

in conjunction with any of the recoil contributions of Sec. V. In (6.11) we have taken into account a factor of 2 for the two ways of inserting the vacuum polarization and a factor of $\frac{1}{2}$ that arises from the integration. An estimate of the size of the largest radiative-recoil correction may be obtained by multiplying the results of Sec. V by the factor (6.11). The largest individual correction is then about 1 ppm. The individual terms are likely to cancel as in Sec. V, giving a contribution at the level of 0.03 ppm. The radiative corrections that do not contain the leading logarithm of (6.9), including corrections on the electron line, are potentially more problematic, since they would likely affect the various recoil corrections in different ways. However, their size is intrinsically smaller than that of the leading-logarithmic correction. Hence, the corresponding radiative-recoil corrections would probably contribute less than 0.1 ppm.

In addition to the radiative-Zemach and radiative-recoil corrections, there are uncalculated pure-QED corrections of relative order α^3/π; they are represented by the D_1 term of (1.2). They arise from various sources: a higher-order vacuum-polarization correction without recoil effects, two-photon corrections to the electron line which cannot be absorbed into the electron's anomalous magnetic moment, light-by-light scattering diagrams, etc. These corrections should not be sensitive to the proton's structure. Their nominal size is 0.12 ppm, so, unless the individual contributions conspire to give a large total, these radiative corrections should also be masked by the uncertainties in the proton's size.

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APPENDIX: TABULATION OF CERTAIN INTEGRALS

Certain dimensionless integrals $K_n(P)$ that occur frequently in the BYG paper are tabulated there. They take the general form

$$K_n(P)=(4\pi)^2\int\frac{d^4p'd^4p}{-(2\pi)^8}\frac{G_n(p'_0,p_0,m_e)P(\mathbf{p}',\mathbf{p},\gamma)}{(p'^2-\gamma^2+i\epsilon)D_e(p')[(p'-p)^2+i\epsilon]D_e(p)(p^2-\gamma^2+i\epsilon)}\,.$$

TABLE I. Tabulation of the integrals (a) K_1 and K_3 and (b) K_7, K_8, K_9, K_{10}, K_{11}, and K_{13}. The underscored headings indicate the polynomial P that appears in the numerator of the integrand, as explained in the Appendix.

(a)

	$\mathbf{p}^2\mathbf{p}'^2$	$\mathbf{p}^2\mathbf{p}\cdot\mathbf{p}'$	$\mathbf{p}'^2\mathbf{p}\cdot\mathbf{p}'$	$(\mathbf{p}'\cdot\mathbf{p})^2$
K_1	$-\ln\frac{m_e}{2\gamma}+\frac{11}{16}$	$-\frac{1}{2}\ln\frac{m_e}{2\gamma}+\frac{5}{8}$	$-\frac{1}{2}\ln\frac{m_e}{2\gamma}+\frac{5}{16}$	$-\frac{1}{2}\ln\frac{m_e}{2\gamma}+\frac{7}{16}$
K_3	$-\ln\frac{m_e}{2\gamma}+\frac{7}{8}$	$-\frac{1}{2}\ln\frac{m_e}{2\gamma}-2\ln 2+\frac{9}{8}$	$-\frac{1}{2}\ln\frac{m_e}{2\gamma}-2\ln 2+\frac{9}{8}$	$-\frac{1}{2}\ln\frac{m_e}{2\gamma}-2\ln 2+\frac{9}{8}$

	$\mathbf{p}^2\gamma^2$	$\mathbf{p}'^2\gamma^2$	$\mathbf{p}\cdot\mathbf{p}'\gamma^2$	γ^4
K_1, K_3	$-\frac{3}{16}$	$-\frac{3}{16}$	$-\frac{1}{16}$	$-\frac{1}{16}$

(b)

	$\mathbf{p}'^2$	$\mathbf{p}\cdot\mathbf{p}'$	$\mathbf{p}^2$	γ^2
K_7	$-\frac{1}{8}$	$-\frac{1}{32}$		0
K_8		$-\frac{1}{32}$	$-\frac{1}{8}$	0
K_9	$-\frac{3}{32}$	$-\frac{1}{32}$	$-\frac{3}{32}$	0
K_{11}	$\ln 2$	$\ln 2-\frac{1}{2}$	$\ln 2$	0
K_{13}	$\frac{1}{2}\ln\frac{m_e}{2\gamma}-\frac{1}{8}$	$\frac{1}{4}\ln\frac{m_e}{2\gamma}-\frac{1}{4}$		$\frac{1}{8}$

$K_{10}=\frac{1}{32}$

Here the G_n, which are given below, are certain generalized functions of the specified arguments. The result K_n is actually a functional of the polynomial P. These polynomials are indicated in the main text as they occur. The first and last factors of the denominator of the integrand result from the method of decoupling the wave functions described in Sec. III. Fifteen categories of K_n are worked out analytically by BYG; that calculation makes use, wherever possible, of the approximation $\alpha \ll 1$. The integrals required in the present work correspond to the subset of forms

$$G_1 = -2\pi i\delta(p_0')\left[-2\pi i\delta(p_0) - \frac{2}{p_0+i\epsilon}\right],$$

$$G_3 = -2\pi i\delta(p_0-p_0')\left[-2\pi i\delta(p_0) - \frac{2}{p_0+i\epsilon}\right],$$

$$G_7 = -2\pi i\delta(p_0')p_0 \ , \quad G_8 = -2\pi i\delta(p_0)p_0' \ ,$$

$$G_9 = -2\pi i\delta(p_0-p_0')p_0 \ ,$$

$$G_{10} = -2\pi i\delta(p_0-p_0')p_0^3 \ ,$$

$$G_{11} = -2\pi i\delta(p_0-p_0')2m_e \ ,$$

$$G_{13} = -2\pi i\delta(p_0')2m_e \ .$$

For the convenience of the reader, we tabulate the results $K_n(P)$ for these integrands in Table I.

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