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Comments

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Comment on "Equivalence between the Thirring model and a derivative-coupling model"

R. Banerjee
Saha Institute of Nuclear Physics, 92, A.P.C. Road, Calcutta 700 009, India
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An operator equivalence between the Thirring model and the fermionic sector of a Dirac field interacting via derivative coupling with two scalar fields is established in the path-integral framework. Relations between the coupling parameters of the two models, as found by Gomes and da Silva, can be reproduced.

Recently, a couple of papers[1,2] have appeared which discuss the equivalence of the Thirring model with the fermionic sector of a derivative-coupling (DC) model. This equivalence is significant for analyzing different aspects of mass perturbation in the Thirring model. The fermionic Green's functions of the Thirring model and the DC model are shown[1,2] to be identical for a certain choice of the coupling parameters. In Ref. 1, moreover, it has been demonstrated that the degrees of freedom in the two models may be matched by the introduction of "spurions." Notwithstanding these successes, however, an incompleteness still persists. The reason is that a one-to-one correspondence between the two models at the operator level could not be furnished.

In this comment we explicitly show how this operator equivalence can be established in the path-integral framework. It ought to be noted that the operator constructions proposed in our approach are a logical consequence of the path-integral method. This is to be contrasted with the conventional bosonization technique to suggest an ansatz and subsequently verify it by different means. The identification of the models at the operator level immediately leads to certain relations between the coupling constants. Analogous relations given in Refs. 1 and 2 are reproduced. We also exhibit that different regularizations adopted for the DC model lead to solutions obtained by redefining the coupling parameters. In the case of the Thirring model also, it may be recalled, exactly the same thing occurs.[3] An interesting feature of this approach is that "spurions" are not necessary at any stage of the computations.

The DC model is described, in the Euclidean metric, by the Lagrangian

$$\begin{aligned}\mathcal{L}_{DC} &= -i\bar{\psi}\partial\!\!\!/\psi+\tfrac{1}{2}(\partial_\mu\phi_1)^2+\tfrac{1}{2}(\partial_\mu\phi_2)^2 \\ &\quad +g_1\bar{\psi}\gamma_5\gamma_\mu\partial_\mu\phi_1\psi+g_2\bar{\psi}\gamma_\mu\partial_\mu\phi_2\psi \\ &= -i\bar{\psi}D\!\!\!\!/\,(\phi_1,\phi_2)\psi+\tfrac{1}{2}(\partial_\mu\phi_1)^2+\tfrac{1}{2}(\partial_\mu\phi_2)^2 ,\end{aligned}$$

where the Euclidean γ matrices are defined by the algebra

$$\{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu\nu},\quad \gamma_\mu\gamma_5=i\epsilon_{\mu\nu}\gamma_\nu ,$$
$$\epsilon_{01}=-\epsilon_{10}=1,\quad \gamma_\mu^\dagger=-\gamma_\mu,\quad \gamma_5^\dagger=\gamma_5 .$$

A massive version of this model with $g_2=0$ has been considered by several authors.[4,5] The approach of Ref. 4, however, is closest in spirit to the ensuing presentation.

The generating functional of the DC model in the presence of sources is

$$Z=\int d\psi\, d\bar{\psi}\, d\phi_1 d\phi_2 \exp\left[\int d^2x\,[i\bar{\psi}\partial\!\!\!/\psi-\tfrac{1}{2}(\partial_\mu\phi_1)^2-\tfrac{1}{2}(\partial_\mu\phi_2)^2-g_1\bar{\psi}\gamma_5\gamma_\mu\partial_\mu\phi_1\psi-g_2\bar{\psi}\gamma_\mu\partial_\mu\phi_2\psi+\bar{\psi}\eta+\bar{\eta}\psi]\right] .$$

Since our attention is confined to the fermionic sector only, we have not included the bosonic source term. A change of variables in the fermion fields is now introduced to eliminate the interaction:

$$\psi(x)\rightarrow \exp(i\gamma_5 g_1\phi_1-ig_2\phi_2)\psi'(x),\quad \bar{\psi}(x)\rightarrow\bar{\psi}'(x)\exp(i\gamma_5 g_1\phi_1+ig_2\phi_2) . \tag{1}$$

The transformed generating functional is then given by

$$Z=\int d\psi\, d\bar{\psi}\, d\phi_1 d\phi_2 J_{DC} \exp\left[\int d^2x\,[i\bar{\psi}\partial\!\!\!/\psi-\tfrac{1}{2}(\partial_\mu\phi_1)^2-\tfrac{1}{2}(\partial_\mu\phi_2)^2+\bar{\psi}e^{i\gamma_5 g_1\phi_1+ig_2\phi_2}\eta+\bar{\eta}e^{i\gamma_5 g_1\phi_1-ig_2\phi_2}\psi]\right] , \tag{2}$$

where J_{DC} is the Jacobian of the fermionic transformations (1). The occurrence of such a Jacobian was first observed and elaborated by Fujikawa[6] in the context of gauge field theories. In the case of theories without any γ_5 couplings the calculation of this Jacobian is quite straightforward. Subtleties arise wherever there are γ_5 couplings because the Dirac operator is no longer Hermitian. Different prescriptions[7] have been advocated and we work with two of these. Let us first follow the general treatment of Fujikawa[7] to deal with non-Hermitian operators. We define two Hermitian operators

$$D_{\psi}=\not{D}^{\dagger}(\phi_1,\phi_2)\not{D}(\phi_1,\phi_2)=\not{D}(-\phi_1,\phi_2)\not{D}(\phi_1,\phi_2)\ ,$$
$$D_{\bar\psi}=\not{D}(\phi_1,\phi_2)\not{D}^{\dagger}(\phi_1,\phi_2)=\not{D}(\phi_1,\phi_2)\not{D}(-\phi_1,\phi_2)\ , \tag{3}$$

whose eigenfunctions are the orthonormal sets φ_n,χ_n:

$$D_{\psi}\varphi_n=\lambda_n^2\varphi_n,\quad \int\varphi_n^{\dagger}(x)\varphi_m(x)d^2x=\delta_{nm}\ ,$$
$$D_{\bar\psi}\chi_n=\lambda_n^2\chi_n,\quad \int\chi_n^{\dagger}(x)\chi_m(x)d^2x=\delta_{nm}\ . \tag{4}$$

Expand $\psi(x)$ and $\bar\psi(x)$ in terms of the orthonormal sets,

$$\psi(x)=\sum a_n\varphi_n(x),\quad \bar\psi(x)=\sum b_n\chi_n^{\dagger}(x)\ ,$$

so that the functional measure is given by

$$d\psi\, d\bar\psi=\det[\varphi_n(x)]^{-1}\det[\chi_n^{\dagger}(x)]^{-1}\prod da_n db_n\ .$$

Let us now consider the expansion of the new Fermi fields $\psi',\bar\psi'$ in (1) where the transformation is initially regarded as infinitesimal. The result for the finite transformation may be subsequently evaluated.

$$\psi'(x)=\sum a_n'\varphi_n(x),\quad \bar\psi'(x)=\sum b_n'\chi_n^{\dagger}(x)\ .$$

The new functional measure is given by

$$d\psi' d\bar\psi'=\det[\varphi_n(x)]^{-1}\det[\chi_n^{\dagger}(x)]^{-1}\prod da_n' db_n'\ ,$$

where the coefficients a_n',b_n' are related to a_n,b_n by

$$a_n=\left[\delta_{nm}+\int\varphi_n^{\dagger}(i\gamma_5 g_1\delta\phi_1-ig_2\delta\phi_2)\varphi_m\right]a_m'$$
$$=\sum C_{nm}a_m\ ,$$
$$b_n=\left[\delta_{nm}+\int\chi_n^{\dagger}(i\gamma_5 g_1\delta\phi_1+ig_2\delta\phi_2)\chi_m\right]b_m'$$
$$=\sum C_{nm}'b_m'$$

obtained from (1) and exploiting the normalization condition (4). The Jacobian is then extracted from

$$d\psi\, d\bar\psi=(\det C)^{-1}(\det C')^{-1}d\psi' d\bar\psi'$$

and can be estimated by the standard procedure:[7]

$$(\det C)^{-1}(\det C')^{-1}=\lim_{M\to\infty}\exp\left\{-\mathrm{tr}\int d^2x\left[\sum\left[\varphi_m^{\dagger}(x)(i\gamma_5 g_1\delta\phi_1-ig_2\delta\phi_2)e^{-\lambda_m^2/M^2}\varphi_m(x)+\chi_m^{\dagger}(x)(i\gamma_5 g_1\delta\phi_1+ig_2\delta\phi_2)e^{-\lambda_m^2/M^2}\chi_m(x)\right]\right]\right\}$$
$$=\lim_{M\to\infty}\exp\left\{-\mathrm{tr}\int d^2x\left[\sum\left[\varphi_m^{\dagger}(x)(i\gamma_5 g_1\delta\phi_1-ig_2\delta\phi_2)e^{-D_\psi/M^2}\varphi_m(x)+\chi_m^{\dagger}(x)(i\gamma_5 g_1\delta\phi_1+ig_2\delta\phi_2)e^{-D_{\bar\psi}/M^2}\chi_m(x)\right]\right]\right\}$$
$$=\lim_{M\to\infty}\exp\left[-\mathrm{tr}\left[\int d^2x\int\frac{d^2k}{(2\pi)^2}e^{-ik\cdot x}[(i\gamma_5 g_1\delta\phi_1-ig_2\delta\phi_2)e^{-D_\psi/M^2}+(i\gamma_5 g_1\delta\phi_1+ig_2\delta\phi_2)e^{-D_{\bar\psi}/M^2}]e^{ik\cdot x}\right]\right]=1\ ,$$

a result obtained after inserting $D_\psi,D_{\bar\psi}$ from (3) and performing the Dirac algebra. Thus, remarkably, the fermionic measure remains unaffected. Towards the end of this paper we will deal with a different regularization which produces a nontrivial Jacobian. In the present case, however, the generating functional (2) assumes the form

$$Z=\int d\psi\, d\bar\psi\, d\phi_1 d\phi_2\exp\left[\int d^2x[i\bar\psi\not\partial\psi-\tfrac{1}{2}(\partial_\mu\phi_1)^2-\tfrac{1}{2}(\partial_\mu\phi_2)^2+\bar\psi e^{i\gamma_5 g_1\phi_1+ig_2\phi_2}\eta+\bar\eta e^{i\gamma_5 g_1\phi_1-ig_2\phi_2}\psi]\right]\ .$$

Performing the integration over the fermion fields,

$$Z=\int d\phi_1 d\phi_2\exp\left[\int d^2x[-\tfrac{1}{2}(\partial_\mu\phi_1)^2-\tfrac{1}{2}(\partial_\mu\phi_2)^2-\bar\eta e^{i\gamma_5 g_1\phi_1-ig_2\phi_2}S_F e^{i\gamma_5 g_1\phi_1+ig_2\phi_2}\eta]\right]\ ,$$

where $S_F(x)$ is the free Fermi propagator

$$i\not\partial S_F(x)=\delta^{(2)}(x)\ .$$

It is now simple to compute the two-point function:

$$\langle T\psi(x_1)\bar{\psi}(x_2)\rangle = \frac{\delta^2 Z}{\delta\bar{\eta}(x_1)\delta\eta(x_2)}\bigg|_{\eta=\bar{\eta}=0}$$

$$= \int d\phi_1 d\phi_2 S_F(x_1-x_2)\exp\left[\int[-\tfrac{1}{2}(\partial_\mu\phi_1)^2-\tfrac{1}{2}(\partial_\mu\phi_2)^2]\right]$$

$$\times\exp[-i\gamma_5 g_1\phi_1(x_1)-ig_2\phi_2(x_1)+i\gamma_5 g_1\phi_1(x_2)+ig_2\phi_2(x_2)]$$

$$= \exp\{(g_1^2+g_2^2)[D_F(0)-D_F(x_1-x_2)]\}S_F(x_1-x_2)\ , \tag{5}$$

where $D_F(x)$ is the free scalar propagator,

$$\Box D_F(x)=\delta^{(2)}(x)\ .$$

From this path-integral analysis we may write the classical solution of the Fermi field by inspecting (1):

$$\psi(x)=\exp[i\gamma_5 g_1\phi_1(x)-ig_2\phi_2(x)]\psi_F(x)\ ,$$

where $\psi_F(x)$ is the free massless Fermi field and $\phi_1(x),\phi_2(x)$ are the free massless scalar fields.

To quantize this solution, we follow the usual prescription of normal ordering the exponential with respect to the scalar fields:

$$\psi(x)={:}\exp[i\gamma_5 g_1\phi_1(x)-ig_2\phi_2(x)]{:}\psi_F(x)\ . \tag{6}$$

Using the above expression for $\psi(x)$ and employing Wick's theorem, the structure for the two-point function given in (5) is reproduced which furnishes a simple check on the proposed operator fit. This completes our discussion of the DC model and we focus our attention on the Thirring model.

The dynamics of this model is governed in the Euclidean metric, by the Lagrangian

$$\mathcal{L}_T=-i\bar{\psi}\partial\!\!\!/\psi+g(\bar{\psi}\gamma_\mu\psi)^2\ .$$

The Euclidean generating functional in the presence of sources is

$$Z=\int d\psi\, d\bar{\psi}\exp\left[\int d^2x[i\bar{\psi}\partial\!\!\!/\psi-g(\bar{\psi}\gamma_\mu\psi)^2+\bar{\psi}\eta+\bar{\eta}\psi]\right]$$

In solving this model we shall henceforth closely follow Ref. 4. A multiplicative factor

$$\int d\sigma_\mu\exp\left[g\int d^2x(\sigma_\mu-\bar{\psi}\gamma_\mu\psi)^2\right]$$

is inserted to eliminate the quartic coupling term. The modified Z is

$$Z=\int d\psi\, d\bar{\psi}\, d\sigma_\mu\exp\left[\int d^2x(i\bar{\psi}\partial\!\!\!/\psi+g\sigma_\mu^2-2g\bar{\psi}\sigma\!\!\!/\psi+\bar{\psi}\eta+\bar{\eta}\psi)\right]\ .$$

Exactly as happens in the DC model, the vector field may be decoupled from the spinor field by the following change of variables:

$$\psi(x)\to\exp[-2ig\beta(x)+2ig\gamma_5\alpha(x)]\psi'(x)\ ,$$
$$\bar{\psi}(x)\to\bar{\psi}'(x)\exp[2ig\beta(x)+2ig\gamma_5\alpha(x)]\ , \tag{7}$$
$$\sigma_\mu(x)\to\partial_\mu\beta(x)+i\epsilon_{\mu\nu}\partial_\nu\alpha(x)\ .$$

In terms of these new objects, Z may be expressed as

$$Z=\int d\psi\, d\bar{\psi}\, d\alpha\, d\beta\, J_T\exp\left[\int d^2x[i\bar{\psi}\partial\!\!\!/\psi+g(\partial_\mu\beta+i\epsilon_{\mu\nu}\partial_\nu\alpha)^2+\bar{\psi}e^{2ig(\beta+\gamma_5\alpha)}\eta+\bar{\eta}e^{2ig(-\beta+\gamma_5\alpha)}\psi]\right]\ .$$

The calculation of the Jacobian J_T of the fermionic transformation (7) has been extensively dealt with in an earlier work.[4] We insert the final result corresponding to Schwinger's normalization. Thus,

$$Z=\int d\psi\, d\bar{\psi}\, d\alpha\, d\beta\exp\left\{\int d^2x\left[i\bar{\psi}\partial\!\!\!/\psi+g(\partial_\mu\beta)^2-g\left[1-\frac{2g}{\pi}\right](\partial_\mu\alpha)^2+\bar{\psi}e^{2ig(\beta+\gamma_5\alpha)}\eta+\bar{\eta}e^{2ig(-\beta+\gamma_5\alpha)}\psi\right]\right\}$$
$$=\int d\alpha\, d\beta\exp\left\{\int d^2x\left[g(\partial_\mu\beta)^2-g\left[1-\frac{2g}{\pi}\right](\partial_\mu\alpha)^2-\bar{\eta}e^{2ig(-\beta+\gamma_5\alpha)}S_F e^{2ig(\beta+\gamma_5\alpha)}\eta\right]\right\}\ . \tag{8}$$

The two-point function is then computed as usual:

$$\langle T\psi(x_1)\bar{\psi}(x_2)\rangle=\frac{\delta^2 Z}{\delta\bar{\eta}(x_1)\delta\eta(x_2)}\bigg|_{\eta=\bar{\eta}=0}$$
$$=\exp\left[\frac{4g^2}{\pi-2g}[D_F(0)-D_F(x_1-x_2)]\right]S_F(x_1-x_2) \tag{9}$$

which reproduces Schwinger's form.[3]

It follows from the preceding analysis that the operator solution of the Thirring field may be directly written from (7) and (8):

$$\psi(x)=:\exp[-2ig\beta(x)+2ig\gamma_5\alpha(x)]:\psi_F(x)\ , \tag{10}$$

where $\alpha(x)$ and $\beta(x)$ are massless scalar fields satisfying the modified commutation relations

$$\begin{aligned}[\alpha(x_1),\alpha(x_2)]&=\frac{1}{2g\left[1-\dfrac{2g}{\pi}\right]}D(x_1-x_2)\ ,\\ [\beta(x_1),\beta(x_2)]&=-\frac{1}{2g}D(x_1-x_2)\ .\end{aligned} \tag{11}$$

$\alpha(x)$ and $\beta(x)$ are defined here in indefinite metric but translation-invariant form which is opposed to Klaiber's[3] positive-definite but translation-noninvariant form. The method of extracting the positive-definite physical subspace has been elaborated in an earlier paper.[4] Again, as was done for the DC model, we may utilize (10) and (11) to reproduce the two-point function (9) by exploiting Wick's theorem. This serves as a consistency check.

It is now simple to establish a one-to-one correspondence between the operator solution (6) of the DC model with (10) of the Thirring model by making the identifications

$$g_1\phi_1=2g\alpha,\quad g_2\phi_2=2g\beta\ .$$

The coupling constants are, of course, not independent. Recalling that whereas ϕ_1 and ϕ_2 are the usual free scalar fields while α and β are the modified ones [as defined by (11)], it follows that

$$\begin{aligned}D(x_1-x_2)&=[\phi_1(x_1),\phi_1(x_2)]\\ &=\frac{4g^2}{g_1^2}[\alpha(x_1),\alpha(x_2)]\\ &=\frac{2g}{g_1^2\left[1-\dfrac{2g}{\pi}\right]}D(x_1-x_2)\ ,\end{aligned}$$

$$\begin{aligned}D(x_1-x_2)=[\phi_1(x_1),\phi_1(x_2)]&=\frac{4g^2}{g_2^2}[\beta(x_1),\beta(x_2)]\\ &=-\frac{2g}{g_2^2}D(x_1-x_2)\ ,\end{aligned}$$

which leads to the relations

$$\begin{aligned}\left[1+\frac{g_1^2}{\pi}\right]\left[1+\frac{g_2^2}{\pi}\right]&=1\ ,\\ g_1^2g_2^2=-\frac{4g^2}{1-\dfrac{2g}{\pi}}&=-k^2\ ,\end{aligned} \tag{12}$$

where k is Klaiber's[3] coupling corresponding to Schwinger's normalization. Equation (12) reproduces the choice of parameters discussed in Refs. 1 and 2 but which, as opposed to our presentation, have been obtained by comparing the Green's functions of the respective models.

As promised earlier we will end this Comment by working with an alternative prescription for computing the Jacobian in (2). The Dirac operator $\not{D}$ in the DC model, it may be recalled, is non-Hermitian. If we make an analytic continuation $\phi_1\rightarrow i\phi_1$, however, it becomes Hermitian. We may then employ this analytically continued $\not{D}^2$ as the regulator. The final result will be obtained after continuing back to ϕ_1. This approach, it may be mentioned, leads to the consistent anomaly in a chiral gauge theory.[7] The previous prescription (i.e., using $\not{D}^\dagger\not{D}$ and $\not{D}\not{D}^\dagger$ as regulators), on the other hand, corresponds to a gauge-covariant (Schwinger-type) regularization and consequently yields the covariant anomaly.[7] In the present example the Jacobian, evaluated in the standard way,[4] turns out to be

$$J_{\rm DC}=\exp\left[\frac{1}{2\pi}\int d^2x[g_1^2(\partial_\mu\phi_1)^2-g_2^2(\partial_\mu\phi_2)^2]\right]$$

and the corresponding Z is

$$Z=\int d\phi_1 d\phi_2\exp\left\{\int d^2x\left[-\frac{1}{2}\left[1-\frac{g_1^2}{\pi}\right](\partial_\mu\phi_1)^2-\frac{1}{2}\left[1+\frac{g_2^2}{\pi}\right](\partial_\mu\phi_2)^2\right]\right\}\ ,$$

where the source term has not been written. Thus the massless scalar fields ϕ_1 and ϕ_2 occurring in the operator fit (6) are now defined via the modified commutators

$$\begin{aligned}[\phi_1(x_1),\phi_1(x_2)]&=\frac{1}{1-\dfrac{g_1^2}{\pi}}D(x_1-x_2)\ ,\\ [\phi_2(x_1),\phi_2(x_2)]&=\frac{1}{1+\dfrac{g_2^2}{\pi}}D(x_1-x_2)\ .\end{aligned} \tag{13}$$

The two operator solutions of the DC model may be matched by adopting the identifications

$$g_1\phi_1\rightarrow(g_1\phi_1)_S,\quad g_2\phi_2\rightarrow(g_2\phi_2)_S\ ,$$

where the expression with the subscript S corresponds to the previous Schwinger form. Using the commutation relations (13) it follows that the new solutions may be obtained from the Schwinger-type solutions by the replacements

$$(g_1^2)_S\rightarrow\frac{g_1^2}{1-\dfrac{g_1^2}{\pi}},\quad (g_2^2)_S\rightarrow\frac{g_2^2}{1+\dfrac{g_2^2}{\pi}}\ .$$

In view of its equivalence with the Thirring model, this feature of the DC model is quite expected. The different solutions of the Thirring model are known to be reproduced by redefining the coupling parameter.[3]

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