Autonomous $\lambda\phi^4$ theory in a time-dependent space-time

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After performing a special type of wave-function renormalization procedure, we obtain the autonomous $\lambda\phi^4$ Gaussian effective potential in time-dependent curved space. The bare coupling constant λ_B is allowed to be positive but infinitesimal ($\lambda_B \to 0^+$) for the occurrence of spontaneous symmetry breaking. Some properties different from those for flat space-time are also discussed.

I. INTRODUCTION

Recently, we have seen a renaissance of the triviality problem of the $\lambda\phi^4$ model in the literature.[1] By means of the Gaussian-effective-potential (GEP) method, several cases have been discussed.

(i) Introducing a large but finite moment cutoff Λ, one finds that the bare coupling constant λ_B is constrained by the condition $0<\lambda_B<8\pi/\ln(2\Lambda/\mu)$ (μ=mass parameter) for exhibiting the spontaneous-symmetry-breaking (SSB) phenomenon, and which in turn can be used in the Higgs mechanism.[2]

(ii) After taking $\Lambda\to\infty$, one gets the so-called precarious phase where the bare coupling constant is negative and infinitesimal, $\lambda_B \sim -1/\ln\Lambda$ (Ref. 3) and no SSB occurs.

(iii) By performing a wave-function renormalization trick, one can find an autonomous theory[4,5] in which the bare coupling constant is positive and infinitesimal. This theory exhibits a SSB phase.

In Refs. 6 and 7, the precarious theory has been discussed for Robertson-Walker space-time and time-dependent de Sitter space-time, respectively. In this paper we shall discuss the autonomous theory for de Sitter space-time.

II. CALCULATION

Following the notations of Ref. 7, the line element in half the de Sitter space can be expressed as

$$ds^2=dt^2-e^{2t/\alpha}\sum_{i=1}^{3}(dx^i)^2$$
$$=\left[\frac{\alpha}{\eta}\right]^2\left[d\eta^2-\sum_{i=1}^{3}(dx^i)^2\right], \tag{2.1}$$

where

$$\eta=-\alpha e^{-t/\alpha} \quad (-\infty<\eta<0) . \tag{2.2}$$

The Lagrangian of the $\lambda\phi^4$ model in curved space reads[8]

$$\mathcal{L}=\tfrac{1}{2}(-g)^{1/2}[\partial^\alpha\phi\partial_\alpha\phi-(m_B^2+\xi R)\phi^2-2\lambda_B\phi^4] , \tag{2.3}$$

where

$$g=\det(g_{\mu\nu}) .$$

From (2.1)–(2.3) the energy density can be written as

$$T_{00}=\frac{1}{2}\left[\frac{\partial\phi}{\partial\eta}\right]^2-(2\xi-\tfrac{1}{2})\sum_{i=1}^{3}\left[\frac{\partial\phi}{\partial x^i}\right]^2$$
$$+\frac{\alpha^2}{2\eta^2}\left[m_B^2+\frac{6\xi}{\alpha^2}\right]\phi^2+\frac{\lambda_B\alpha^2}{\eta^2}\phi^4+\frac{2\xi\alpha^2}{\eta^2}\phi\Box\phi$$
$$-2\xi\phi\frac{\partial^2\phi}{\partial\eta^2}-\frac{2\xi}{\eta}\phi\frac{\partial\phi}{\partial\eta} , \tag{2.4}$$

where the scalar curvature

$$R(\eta)=\frac{12}{\alpha^2}=\text{const} \tag{2.5}$$

and the D'Alembertian

$$\Box\equiv\frac{\eta^4}{\alpha^4}\left[\frac{\partial}{\partial\eta}\left[\frac{\alpha^2}{\eta^2}\frac{\partial}{\partial\eta}\right]-\sum_{i=1}^{3}\frac{\partial}{\partial x^i}\left[\frac{\alpha^2}{\eta^2}\frac{\partial}{\partial x^i}\right]\right] \tag{2.6}$$

have been used.

Taking the trial fields in the form of

$$\phi(x)=\phi_0+\phi_\Omega(x) \tag{2.7}$$

with ϕ_0 being a constant background field and $\phi_\Omega(x)$ being a free quantum field with mass Ω in the curved space, one has[7]

$$\phi_\Omega=\int\frac{d^3k}{(2\pi)^{3/2}}[U_{\mathbf{k}}(\mathbf{x},\eta,\alpha)a(\mathbf{k})+U_{\mathbf{k}}^*(\mathbf{x},\eta,\alpha)a^\dagger(\mathbf{k})] , \tag{2.8}$$

$$U_{\mathbf{k}}(\mathbf{x},\eta,\alpha)=e^{i\mathbf{k}\cdot\mathbf{x}}U_k(\eta,\alpha) , \tag{2.9}$$

$$U_k(\eta,\alpha)=\eta^{3/2}\frac{\sqrt{\pi}}{2\alpha}\exp\left[-i\frac{\pi}{2}(\nu+\tfrac{1}{2})\right]H_\nu^{(2)}(k\eta) , \tag{2.10}$$

and

$$\nu^2=\tfrac{9}{4}-\alpha^2\Omega^2-12\xi \tag{2.11}$$

with $H_\nu^{(2)}$ being the Hankel function of second kind.

As in Ref. 7, the GEP $V_G(\phi_0)$ can be found as

$$V_G(\phi_0)\equiv\left[\frac{\eta}{\alpha}\right]^2 \min_{\Omega} {}_{\Omega}\langle 0\,|\,T_{00}\,|\,0\rangle_{\Omega}$$

$$=\min_{\Omega_1}[\,I_1(\Omega_1)+\tfrac{1}{2}(m_1^2-\Omega_1^2)I_0(\Omega_1)+\tfrac{1}{2}m_1^2\phi_0^2+\lambda_B\phi_0^4+6\lambda_B\phi_0^2I_0(\Omega_1)+3\lambda_B I_0^2(\Omega_1)]\ , \tag{2.12}$$

where

$$\Omega_1^2\equiv\Omega^2+12\xi/\alpha^2\ , \tag{2.13}$$

$$m_1^2\equiv m_B^2+6\xi/\alpha^2 \tag{2.14}$$

while $I_0(\Omega_1)$ and $I_1(\Omega_1)$ are given by

$$I_0(\Omega_1)=\frac{1}{8\pi\alpha^2}[\,-2\Lambda^2-\tfrac{1}{2}(\alpha^2\Omega_1^2-2)\ln\Lambda^2+S_0(\Omega_1)]\ , \tag{2.15}$$

$$I_1(\Omega_1)=\frac{1}{8\pi\alpha^4}[\,12\Lambda^4+\Lambda^2(-2+\alpha^2\Omega_1^2)-\tfrac{1}{8}\alpha^2\Omega_1^2(-2+\alpha^2\Omega_1^2)\ln\Lambda^2+S_1(\Omega_1)]\ , \tag{2.16}$$

$$S_0(\Omega_1)=\tfrac{1}{2}(\tfrac{1}{4}-\nu^2)[\,3-2\psi(2)-\ln 4+\psi(\tfrac{3}{2}-\nu)+\psi(\tfrac{3}{2}+\nu)]\ , \tag{2.17}$$

and

$$S_1(\Omega_1)=\tfrac{1}{4}\alpha^2\Omega_1^2S_0(\Omega_1)+\tfrac{9}{4}-\alpha^2\Omega_1^2-\tfrac{1}{16}\alpha^4\Omega_1^4\ , \tag{2.18}$$

with

$$\psi(z)\equiv\Gamma'(z)/\Gamma(z)\ .$$

It should be emphasized that Eqs. (2.12)–(2.18) are valid only for $\alpha^2\Omega_1^2>2$, which reduces to the condition $\Omega^2>0$ in flat space.

Now let us minimize the energy density to find the mass parameter Ω_1. This gives

$$I_1'(\Omega_1)+\tfrac{1}{2}(m_1^2-\Omega_1^2)I_0'(\Omega_1)-\tfrac{1}{2}I_0(\Omega_1)+6\lambda_B[\phi_0^2+I_0(\Omega_1)]I_0'(\Omega_1)=0\ , \tag{2.19}$$

where a prime implies the derivative with respect to Ω_1^2.

Differentiating the GEP, Eq. (2.12) once and twice with respect to ϕ_0, we get

$$\frac{\partial V_G(\phi_0)}{\partial\phi_0}=m_1^2\phi_0+4\lambda_B\phi_0^3+12\lambda_B\phi_0 I_0(\Omega_1) \tag{2.20}$$

and

$$\frac{\partial^2 V_G(\phi_0)}{\partial\phi_0^2}=m_1^2+12\lambda_B\phi_0^2+12\lambda_B I_0(\Omega_1)-\frac{[12\lambda_B\phi_0 I_0'(\Omega_1)]^2}{I_1''(\Omega_1)+\frac{1}{2}(m_1^2-\Omega_1^2)I_0''(\Omega_1)-I_0'(\Omega_1)+6\lambda_B\{\phi_0^2I_0''(\Omega_1)+[I_0'(\Omega_1)]^2+I_0''(\Omega_1)I_0(\Omega_1)\}}\ . \tag{2.21}$$

In order to obtain more information than that in Ref. 7, we carry out a wave-function renormalization procedure similar to that of Ref. 4. Then in a renormalized theory, one has

$$\lambda_B=\frac{a}{12I_{-1}(\mu)}\ , \tag{2.22}$$

$$m_1^2+12\lambda_B I_0(\mu_0)=b/I_{-1}(\mu)+c\ , \tag{2.23}$$

and the field is rescaled by

$$\phi_0^2=I_{-1}(\mu)\Phi_0^2 \tag{2.24}$$

with

$$I_{-1}(\mu)\equiv-2I_0'(\mu)\equiv-2\frac{\partial}{\partial\mu^2}I_0(\mu)\ . \tag{2.25}$$

Here we introduce two mass parameters μ and μ_0 as well as a, b, and c being three constants. In some sense, μ is a substitute for λ_B while b is that for m_1. The constant c, on the other hand, is not independent of the reference mass parameter μ_0. Since λ_B is of order $1/I_{-1}$, a change in μ_0 would result in a finite change of c, or vice versa. However, we shall see later in (2.33) that one cannot adjust μ_0 such that $c=0$. The above prescriptions, as we shall see later also, will render the would-be divergent theory convergent by adjusting the values of a and c and removing an infinite constant into the redefinition of vacuum energy.

In order to prove that the theory resulting from (2.22)–(2.24) is really finite, one might start from Eq. (2.20) or (2.21) and integrate it with respect to Φ_0 once or twice just as has been done in Ref. 4 or 5 for flat space. But now for curved space-time the integration would be a formidable task due to the involvement of the ψ function in a complicated manner. So instead we resort to Eqs. (2.12) and (2.19) directly.

From (2.15) and (2.16), we can get

$$I_0(\Omega_1)-I_0(\mu)=-\frac{1}{16\pi}(\Omega_1^2-\mu^2)\ln\Lambda^2+\frac{1}{8\pi\alpha^2}[S_0(\Omega_1)-S_0(\mu)]\ , \tag{2.26}$$

$$I_1(\Omega_1)-I_1(\mu)=-\frac{1}{8\pi\alpha^2}(\Omega_1^2-\mu^2)\Lambda^2+\frac{1}{8\pi\alpha^4}[S_1(\Omega_1)-S_1(\mu)]-\frac{\ln\Lambda^2}{64\pi\alpha^2}[\,\Omega_1^2(-2+\alpha^2\Omega_1^2)-\mu^2(-2+\alpha^2\mu^2)]\ , \quad (2.27)$$

$$I_1'(\Omega_1)=-\frac{1}{8\pi\alpha^2}\Lambda^2-\frac{\ln\Lambda^2}{32\alpha^2\pi}(-1+\alpha^2\Omega_1^2)+\frac{S_1'(\Omega_1)}{8\pi\alpha^4}\ , \quad (2.28)$$

and

$$I_0'(\Omega_1)=-\tfrac{1}{2}I_{-1}(\Omega_1)=-\frac{\ln\Lambda^2}{16\pi}+\frac{1}{8\pi\alpha^2}S_0'(\Omega_1)\ . \quad (2.29)$$

Substituting (2.26)–(2.29) into (2.19), one finds

$$\Omega_1^2=\frac{2}{2+a}\left[a\Phi_0^2+c+\frac{a}{2}\mu_0^2+\frac{1}{\alpha^2}\right]+E\ , \quad (2.30)$$

where $E\sim 1/I_{-1}$. At first sight, the E term in Ω_1^2 would provide a finite contribution to each term in the GEP $V_G(\Phi_0)$. But fortunately, after calculating, one can find that these contributions cancel each other precisely. So it is not necessary to write down the explicit form of E.

After substituting (2.26)–(2.30) into (2.12), through a tedious calculation, we are led to the surprisingly simple fact that all the divergent terms of order Λ^4 and Λ^2 (with the divergent constant erased) vanish automatically, independent of the values of a, b, c, μ_0, and μ, etc. Only two divergent terms survive. They are

$$\frac{1}{4}\left[\frac{1}{3}-\frac{a}{2+a}\right]a\Phi_0^4I_{-1}(\mu) \quad (2.31)$$

and

$$\frac{1}{2(2+a)}\left[2c+a\mu_0^2-\frac{a}{\alpha^2}\right]\Phi_0^2I_{-1}(\mu)\ . \quad (2.32)$$

So if we take the values of a and c as

$$a=1,\quad c=\frac{1}{2\alpha^2}-\frac{\mu_0^2}{2}\ , \quad (2.33)$$

all the divergences drop out and the theory becomes finite. Note that Eq. (2.33) entails the constant c to be different from zero. For if $c=0$ then $\mu_0^2=1/\alpha^2<2/\alpha^2$, which in turn implies that the theory ceases to be valid.

Finally, the GEP has the following form:

$$V_G(\Phi_0)=V_{\text{vac}}+\tfrac{1}{2}m^2\Phi_0^2+\frac{\psi_1(\Phi_0)}{32\pi\alpha^2}\left[\frac{1}{2\alpha^2}-\frac{2}{3}\Phi_0^2\right]+\frac{1}{144\pi}\left[\psi_1(\Phi_0)-\psi_2(\mu)+\frac{\alpha^2\mu^2-2}{\alpha^2}\psi_2'(\mu)-\frac{1}{2}\right]\Phi_0^4\ , \quad (2.34)$$

where

$$m^2=\tfrac{1}{3}b-\frac{S_0'(\mu)}{24\pi\alpha^4}+\frac{1}{24\pi\alpha^2}[\mu_0^2S_0'(\mu)-S_0(\mu_0)]+\frac{1}{24\pi\alpha^2}(\tfrac{9}{4}-\gamma+\ln 2) \quad (2.35)$$

with γ being the Euler constant while

$$\psi_1(\Phi_0)=\psi(\tfrac{3}{2}+(\tfrac{5}{4}-\tfrac{2}{3}\alpha^2\Phi_0^2)^{1/2})+\psi(\tfrac{3}{2}-(\tfrac{5}{4}-\tfrac{2}{3}\alpha^2\Phi_0^2)^{1/2})\ , \quad (2.36)$$

$$\psi_2(\mu)=\psi(\tfrac{3}{2}+(\tfrac{9}{4}-\alpha^2\mu^2)^{1/2})+\psi(\tfrac{3}{2}-(\tfrac{9}{4}-\alpha^2\mu^2)^{1/2})\ . \quad (2.37)$$

III. DISCUSSION

We see from (2.34) and (2.35) that μ_0^2 appears only in m^2. Since b is arbitrary, so is m^2. Therefore, the theory is μ_0 independent as expected. Notice that two independent parameters m^2 and μ in quantum theory now replace the original bare parameters m_B^2 and λ_B at the classical level. Furthermore, it is worthwhile to mention that the function $\psi_1(\Phi_0)$ [or $\psi_2(\mu)$] is real even if $\nu=(\frac{5}{4}-\frac{2}{3}\alpha^2\Phi_0^2)^{1/2}$ [or $\nu=(\frac{9}{4}-\alpha^2\mu^2)^{1/2}$] is imaginary. Actually, one has

$$\psi(x+iy)+\psi(x-iy)=2\,\text{Re}\psi(x+iy) \quad (3.1)$$

with real x and y.

Taking $\alpha^2\Phi_0^2\gg 1$ while keeping α^2 finite, one finds

$$\psi_1(\Phi_0)\sim\ln\Phi_0^2\ . \quad (3.2)$$

Thus it is easy to prove that this theory does exhibit SSB property for either negative or positive (but not too large) m^2, a property better than that in the flat-space case. This is because not only the coefficient of the Φ_0^4 term but also that of the Φ_0^2 term now all have a

$$\psi_1(\Phi_0)\xrightarrow[\alpha^2\Phi_0^2\gg 1]{}\ln\Phi_0^2$$

dependence whereas in the flat-space case only the Φ_0^4 term has this dependence.

Comparing the above calculation with that of Ref. 7, one discovers that all the infinitesimal terms $1/I_{-1}(\mu)$ in the precarious theory now become important in the autonomous theory and vice versa. What does it mean and which one of these two theories is the correct or better? We do not know yet.

Finally, let us go back to flat space from the previous results. In view of Eq. (2.1), when $\alpha\to\infty$, the theory should return to the flat-space case. It is easy to accomplish this if one notices that

$$\psi_1(\Phi_0)-\psi_2(\mu)\xrightarrow[\alpha\to\infty]{}\ln\left[\frac{2\Phi_0^2}{3\mu^2}\right] \quad (3.3)$$

and

$$\psi_2'(\mu)\xrightarrow[\alpha\to\infty]{}1/\mu^2\ . \quad (3.4)$$

Hence the GEP in flat space follows immediately:

$$V_G(\Phi_0)=V_{\rm vac}+\tfrac{1}{2}m^2\Phi_0^2+\frac{1}{144\pi}\left[\ln\frac{2\Phi_0^2}{3\mu^2}+\frac{1}{2}\right]\Phi_0^4\ . \tag{3.5}$$

As μ^2 and m^2 are arbitrary, Φ_0 may be rescaled at one's disposal, say,

$$\Phi_0^2=\frac{1}{\sqrt{\pi}}\Phi^2 \tag{3.6}$$

and correspondingly

$$\mu^2=\frac{2V^2}{3\sqrt{\pi}}\exp\left[1+\frac{36\pi^2M^2}{V^2}\right], \tag{3.7}$$

$$m^2=\sqrt{\pi}M^2\ , \tag{3.8}$$

then

$$V_G(\Phi)=V_{\rm vac}+\tfrac{1}{2}M\Phi^2\left[1-\frac{1}{2}\frac{\Phi^2}{V^2}\right]+\frac{\Phi^4}{144\pi^2}\left[\ln\frac{\Phi^2}{V^2}-\frac{1}{2}\right]$$

which coincides precisely with that in Ref. 4 with V being the broken vacuum value of Φ in flat space.

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