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Perfect-fluid cosmologies with extra dimensions

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We give an analysis of the solutions of the n-dimensional vacuum Einstein equations with a metric in the form of a direct sum of a Friedmann-Robertson-Walker (FRW) metric and a Kasner-type Euclidean metric. The solutions are interpreted as four-dimensional perfect-fluid cosmological FRW models, using the simple ansatz proposed by Ibáñez and Verdaguer. We first obtain the general solution for flat models. These are perfect-fluid solutions that can be made compatible with contraction of all the extra dimensions. The general compatibility of the field equations is then discussed. It is found that for $n > 5$ both open and closed models admit a range of perfect-fluid solutions whose qualitative behavior is analyzed.

Exact solutions of Einstein's equations in more than four dimensions are of interest in several contexts. From a strictly technical point of view, they have been used as a tool for obtaining exact solutions in four dimensions,[1-3] without ascribing a physical meaning to the added dimensions. Kaluza-Klein-type theories, in more than four dimensions, have been used, on the other hand, as a way of unifying all gauge interactions with gravity. Here the extra dimensions play a physical role and its unobservability is usually explained by the assumption that they are restricted to a compact space with a very small length scale. The first simple cosmological model showing contraction of the extra dimensions as a consequence of cosmological evolution, a generalization of the anisotropic Kasner solution, was proposed by Chodos and Detweiler.[4] In a recent paper, Ibáñez and Verdaguer[5] consider a set of solutions of Einstein's equations in an n-dimensional vacuum. With a very simple ansatz they obtain homogeneous solutions with expanding three-dimensional isotropic spaces. Thus, their solutions contain a four-dimensional subspace isometric to a Friedmann-Robertson-Walker (FRW) cosmological model. This subspace is directly identified with the observable four-dimensional space. In this respect their ansatz differs from other treatments which include a nontrivial conformal transformation in the definition of the four-dimensional perspective. With their identification, Ibáñez and Verdaguer find that the n-dimensional vacuum solutions correspond to perfect-fluid four-dimensional FRW metrics. They also claim that the field equations are compatible only for a radiative perfect fluid while the extra dimensions contract as a result of cosmological evolution only in certain cases, depending on the number of extra dimensions and the type of model. Their analysis is, however, incomplete. In this paper we consider again the field equations given by Ibáñez and Verdaguer and show that they admit a much larger set of solutions than what is indicated in Ref. 5 (see Refs. 6–8).

We start our discussion reviewing briefly the derivation of the field equations and their interpretation as given by Ibáñez and Verdaguer.[5] Next, we consider some exact solutions that result with certain specializations of the relevant parameters. All these solutions correspond to a perfect fluid satisfying the ("γ fluid") equation of state,

$$p=(\gamma-1)\rho \ , \tag{1}$$

where p is the pressure, γ a constant, and ρ the energy density. Physical values of γ, i.e., satisfying the energy conditions, are restricted to the interval [1,2]. However, not all of the solutions satisfy this requirement.

Besides the radiative perfect-fluid solutions ($\gamma=\frac{4}{3}$) given in Ref. 5, we also obtain, for flat models, solutions where the γ fluid equation of state is satisfied for a range of values of γ including $\gamma=\frac{4}{3}$. As a consequence, for these models the relevant parameters can be chosen so that all extra dimensions contract.

Finally we show, analyzing the general compatibility of the field equations, that there are other perfect-fluid solutions. In these solutions the fluid does not satisfy (1), leading to qualitatively different behaviors of the models.

Following Ibáñez and Verdaguer[5] we consider vacuum solutions of the n-dimensional Einstein equations with a metric of the form

$$ds^2=(ds^2)_{\mathrm{FRW}}+\sum_{i=1}^{d} a_i(t)(dx_i)^2 \ , \tag{2}$$

where d is the number of extra dimensions ($d=n-4$) and $(ds^2)_{\mathrm{FRW}}$ is the line element of the FRW metrics in four dimensions:

$$(ds^2)_{\mathrm{FRW}}=-(dt)^2+R(t)^2\left[\frac{dr^2}{1-Kr^2}+(d\Omega)^2\right] , \tag{3}$$

where $(d\Omega)^2$ is the standard line element on the unit sphere, $R(t)$ is the radius of the FRW universe, and the constant K characterizes the different models ($K=0$ flat, $K=-1$ open, and $K=1$ closed).

The vacuum Einstein equations in n dimensions lead to the following equations:

$$3\frac{\dot{R}^2}{R^2}+3\frac{K}{R^2}=\frac{\ddot{D}}{D}-\frac{1}{8}\left[\sum_{i=1}^{d}\frac{\dot{a}_i}{a_i}\right]^2+\frac{1}{8}\sum_{i=1}^{d}\left[\frac{\dot{a}_i}{a_i}\right]^2, \quad (4a)$$

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}+\frac{K}{R^2}=\frac{\dot{R}\dot{D}}{RD}+\frac{1}{8}\left[\sum_{i=1}^{d}\frac{\dot{a}_i}{a_i}\right]^2-\frac{1}{8}\sum_{i=1}^{d}\left[\frac{\dot{a}_i}{a_i}\right]^2, \quad (4b)$$

and

$$\ddot{a}_i+3\dot{R}\dot{a}_i/R+\dot{D}\dot{a}_i/D-\dot{a}_i^2/a_i=0, \quad (4c)$$

where $D(t)$ is a function defined by

$$D(t)^2=\prod_{i=1}^{d}a_i(t). \quad (5)$$

Equation (4c) implies that

$$R\ddot{D}=-3\dot{R}\dot{D}. \quad (6)$$

Replacing in (4c), we find

$$a_i=D^{2P_i}, \quad \sum_{i=1}^{d}p_i=1, \quad (7)$$

where the p_i are constants. Equation (6) can be integrated once to give

$$\dot{D}R^3=\beta^3 \quad (8)$$

where β is a constant.

Using (7) we may write (4a) and (4b) in a more compact and convenient form. Defining

$$\alpha=\frac{1}{2}\left[\sum_{i=1}^{d}p_i^2-1\right], \quad (9)$$

Eqs. (3a) and (3b) can be written

$$3\frac{\dot{R}^2}{R^2}+3\frac{K}{R^2}=\frac{\ddot{D}}{D}+\alpha\frac{\dot{D}^2}{D^2}, \quad (10a)$$

$$-2\frac{\ddot{R}}{R}-\frac{\dot{R}^2}{R^2}-\frac{K}{R^2}=-\frac{\dot{D}\dot{R}}{DR}+\alpha\frac{\dot{D}^2}{D^2}. \quad (10b)$$

Together with (6) [or, equivalently, (8)], (10a) and (10b) make up a system of three nonlinear ordinary differential equations for the two unknowns $R(t)$ and $D(t)$. Such an overdetermined system has in general no solution unless some conditions, that effectively reduce the number of independent equations, are satisfied. We shall come back to this question but will first consider the physical interpretation of the model. As indicated in Ref. 5, the four-dimensional Einstein equations for the metric (2), with a perfect-fluid energy-momentum tensor of the form

$$T_{\mu\nu}=(\rho+p)U_\mu U_\nu+p(g_{\mathrm{FRW}})_{\mu\nu}, \quad (11)$$

where $U=\partial/\partial t$ and $\mu,\nu=0,1,2,3$, are identical to (10a) and (10b) with the right-hand sides replaced, respectively, by ρ and p. Namely, we make the identifications

$$\rho=\frac{\ddot{D}}{D}+\alpha\frac{\dot{D}^2}{D^2}, \quad (12a)$$

$$p=-\frac{\dot{D}\dot{R}}{DR}+\alpha\frac{\dot{D}^2}{D^2}. \quad (12b)$$

The solutions can therefore be interpreted as isotropic homogeneous perfect-fluid cosmologies with extra dimensions. We remark, however, that contrary to what happens in four dimensions, the possible equations of state relating ρ and p are here restricted by the condition of compatibility of the system (6), (10a), and (10b). Thus, Ibáñez and Verdaguer[5] find that the only solutions with an equation of state of the form (1) are those with $\gamma=\frac{4}{3}$, i.e., radiative perfect fluids. However, as we show in what follows, their analysis is incomplete.

It will be convenient to eliminate R using (8). From (10a) and (10b) we get

$$\frac{1}{3}\frac{\ddot{D}^2}{\dot{D}^2}+3\frac{K}{\beta^2}(\dot{D})^{2/3}=\frac{\ddot{D}}{D}+\alpha\frac{\dot{D}^2}{D^2}, \quad (13a)$$

$$\frac{2}{3}\frac{\dddot{D}}{\dot{D}}-\frac{\ddot{D}^2}{\dot{D}^2}-\frac{K}{\beta^2}(\dot{D})^{2/3}=\frac{1}{3}\frac{\ddot{D}}{D}+\alpha\frac{\dot{D}^2}{D^2}. \quad (13b)$$

Clearly, all radiative fluid solutions correspond to $\alpha=0$. Assuming $\alpha=0$, from (13a) and (13b), we obtain

$$\frac{\dddot{D}}{\dot{D}}-\frac{4}{3}\frac{\ddot{D}^2}{\dot{D}^2}-\frac{\ddot{D}}{D}=0, \quad (14)$$

where the general solution is

$$D=\frac{D_0(2\kappa t+A)}{|\kappa t^2+At+B|^{1/2}}, \quad (15)$$

where A, B, and D_0 are (not all independent) constants and $\kappa=0,\pm1$. From (12a) we have

$$\rho=\frac{3(A^2-4\kappa B)}{4(\kappa t^2+At+B)^2}. \quad (16)$$

Therefore, the energy density will be positive provided $A^2>4\kappa B$. This, in turn, means that for solutions with $\rho>0$, the denominator in (15) must vanish for some value of t. It is easily verified that in this case, for $K\neq0$, we may choose the constants such that,[5] for $K=-1$,

$$D=\frac{A+2t}{|At+t^2|^{1/2}}, \quad A>0, \quad (17)$$

for $K=+1$,

$$D=\frac{A-2t}{|At-t^2|^{1/2}}, \quad A>0, \quad (18)$$

while for $K=0$, a solution is

$$D=t^{-1/2}. \quad (19)$$

These are the solutions given by Ibáñez and Verdaguer. We remark, however, that we also obtain n-dimensional vacuum solutions when $A^2<4\kappa B$. In this case only $K=-1$ is admissible, but ρ is negative. The case $A^2=4\kappa B$ corresponds to the trivial solution $D=$const.

We consider now the general solution for $K=0$. If we choose a trial solution of the form

$$D = C_1(t - t_0)^{C_2} , \tag{20}$$

with t_0, C_1, and C_2 real constants, then (13a) and (13b) are compatible if

$$C_2 = \frac{1+\epsilon(9+12\alpha)^{1/2}}{(4+6\alpha)} , \tag{21}$$

with $\epsilon = \pm 1$. From (12a) and (12b) we have

$$\frac{p}{\rho} = \frac{1+C_2}{1-C_2} . \tag{22}$$

Namely, we obtain perfect-fluid solutions where ρ and p satisfy a "γ law" equation of state with $\gamma = 2/(1-C_2)$. The dominant energy condition $|p/\rho| \le 1$ is satisfied only if $C_2 \le 0$ and this requires $\epsilon = -1$. The resulting cosmological models contain a "big-bang"-type singularity for $t = t_0$ and expand forever for $t > t_0$. We thus find that for flat spacetimes ($K = 0$), the field equations are compatible with the equation of state (1) and the dominant energy condition if

$$\gamma = \frac{12\alpha + 8}{3(2\alpha+1) + (12\alpha+9)^{1/2}} . \tag{23}$$

This solution is not mentioned in Ref. 5. It introduces, however, an important modification in their conclusions and is therefore worth further discussion.

The reality of the p_i together with (6) imply a lower limit for α, equal to[9] $1/(2d) - \frac{1}{2}$. This, on account of (23), sets a lower limit on γ that is equal to $\frac{4}{3}$ for $d = 1$ and decreases with the number of extra dimensions but is always larger than one.[10,11] As illustrative examples, for $d = 2$ we have $\gamma > 1.26\ldots$ and for $d = 10$ we have $\gamma > 1.18\ldots$. Notice also that we recover the result $\gamma = \frac{4}{3}$ for $\alpha = 0$. The fact that α can take negative values for $K = 0$ and $d > 1$ implies, however, that for flat models, in opposition with the results given in Ref. 5, it is possible to choose all the p_i positive and thereby obtain perfect-fluid solutions with compactification in all the extra dimensions as a result of cosmological evolution.

The solutions with $\epsilon = +1$ do not satisfy, in general, the condition that the energy be positive. However, for $\alpha = 0$, we obtain $C_2 = 1$, i.e.,

$$D = C_1(t - t_0) . \tag{24}$$

This solution corresponds to Minkowski spacetime in four dimensions, but, since the p_i are restricted only by (7) and (9), with a nontrivial Kasner-type structure in the extra dimensions.

The constant C_2 can be related to the Hubble constant H_0 and the age of the Universe τ_0. From (20),

$$C_2 = 1 - 3H_0\tau_0 .$$

As yet, our analysis has sidestepped the fundamental question of the general compatibility of the field equations by resorting to particular solutions. Summarizing, we have explicit, exact solutions with $\alpha = 0$, K unrestricted[5] or with $K = 0$ and a range of values for α. The question is then if these are the only possible solutions. In other words, if the system (13a), (13b) is compatible for the indicated combinations of values of K and α. The problem can be analyzed as follows. Equation (13a) is of the general form

$$h(D, \dot{D}, \ddot{D}) = 0 . \tag{25}$$

From the general theory of nonlinear ordinary differential equations, it has regular solutions for any initial choice of D and $\dot{D}$, except in the neighborhood of the set defined by

$$\frac{\partial h}{\partial \ddot{D}} = 0 . \tag{26}$$

This implies

$$\ddot{D} = \frac{3}{2}\frac{\dot{D}^2}{D} . \tag{27}$$

Equation (27) determines the singular solutions of (13a). It is now easy to check, by direct differentiation of (13a), that the regular solutions of (13a) are also solutions (13b). We remark that these conditions are independent of the particular values of α and K. Also, the singular solutions of (13a) are not, in general, solutions of (13b). Therefore, Eqs. (13a) and (13b) should be considered as independent, although compatible.

These results imply that besides those already mentioned, there are other perfect-fluid solutions compatible with the ansatz (2) and leading, possibly, to contraction of the extra dimensions as a result of cosmological evolution. Even though for general values of α and K, Eq. (13a) can be solved only numerically, we may still extract useful information by analyzing the behavior of its solutions near critical points where D (or $\dot{D}$) approaches zero or infinity or in the limit $t \to \infty$.

We have already given the general solution for $K = 0$ in closed form. We notice now that for all values of K, near a singularity where $D \to \infty$, it is consistent with (13a) and (13b) to assume for D the form (21), because the leading term in the expansion of D is (as expected) independent of K. This implies that the three types of models (open, flat, and closed), admit solutions with a "big-bang"-type singularity where $R \to 0$ for $t \to t_0$.

The behavior of the solutions after the initial singularity can be analyzed qualitatively using (13a). For open models ($K = -1$), all solutions expand indefinitely. For large t we have the asymptotic expansion

$$D \simeq C_0 - \frac{\beta^2}{2} t^{-2} . \tag{28}$$

Therefore

$$R \simeq t \tag{29}$$

and the open models approach the perfect radiative fluid FRW regime irrespective of the value of α.

For all closed models ($K = 1$), D goes to zero in a finite time. The nature of this second singularity depends on the value of α. For $\alpha > 0$, the pressure grows until it equals the energy density. This happens when $\ddot{D} = 0$, where R attains its maximum value. After this time the models recollapse but the ratio p/ρ keeps growing and the dominant energy conditions are violated. For $\alpha<0$, D and $\dot{D}$ vanish at the second singularity and R is unbounded. Therefore these models do not recollapse. It can be seen that the energy conditions are also violated in this case. We remark that this behavior of the closed models is in agreement with the general results for the recollapse of FRW models given by Barrow, Galloway, and Tipler.[12] In particular, for $\alpha<0$, we have $\ddot{R}<0$ near the "big-bang" singularity and $\ddot{R}>0$ near the second singularity with the inflection point $\ddot{R}=0$ at the value of t where the ratio p/ρ goes through the "critical" value $-\frac{1}{3}$.

We thus have that for closed models, the equation of state for the matter content eventually violates the dominant energy condition, except in the particular case $\alpha=0$. For open models, near the singularity, we have $p/\rho<\frac{1}{3}$ for $\alpha<0$ and $p/\rho>\frac{1}{3}$ for $\alpha>0$. However, this ratio approaches $\frac{1}{3}$ in all cases when $t\rightarrow\infty$.

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[6] To avoid any misinterpretation regarding the purpose of our paper, it should be clear that it is restricted to the analysis of the solutions of the field equations as given in Ref. 5. Thus, we do not make any attempt here to discuss the validity of their ansatz. However, for completeness, we must point out that although their interpretation of the four-dimensional metric, leading to the effective equation of state given by Eq. (12), is in agreement with other recent articles (see, e.g., Ref. 7), a different approach to the effective four-dimensional equation of state is given in Ref. 8, where the inclusion of a Weyl factor leads to a Zeldovich-type fluid ($p=\rho$) instead of the radiative fluid of Refs. 5 and 7.

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[9] An intriguing example of a solution of (6), (10a), and (10b) not satisfying this restriction on α is $D=C\exp(mt)$, with C and m real constants. This corresponds to a four-dimensional vacuum solution with a cosmological constant $\Lambda=m^2/3$, but requires $\alpha=-\frac{2}{3}$ which can only be satisfied with complex p_i.

[10] Actually, for $d=1$ we have $\alpha=0$ and only $\gamma=\frac{4}{3}$ is allowed (Ref. 11). We also notice that the maximum physically admissible value of γ (equal to 2), cannot be reached either, because it would require one or more of the p_i to go to infinity.

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