Wick-Cutkosky model in the large-temperature limit

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We study the Bethe-Salpeter equation in the simple scalar-scalar model of Wick and Cutkosky in the framework of the instantaneous approximation and the large-temperature limit. Using the stereographic projection technique due to Fock and a theorem for integral equations due to Funk and Hecke, we are led to a second-order difference equation. The condition for the existence of nontrivial solutions of this equation, obtained by use of a Poincaré theorem, then yields the temperature-dependent discrete energy spectrum of the model, which is briefly discussed.

I. INTRODUCTION: WICK-CUTKOSKY MODEL AT FINITE TEMPERATURE

The initial motivation for studying the finite-temperature behavior of a class of relativistic field theories[1] was provided by the following question: In such theories, can finite temperature restore a symmetry which at zero temperature was broken either dynamically or spontaneously? Since then, the domain of inquiry of finite-temperature field theories has been considerably enlarged. Thus, one-loop radiative corrections in finite-temperature QED have been comprehensively dealt with in the paper by Donoghue *et al.*,[2] which also refers to a variety of other applications of finite-temperature theory. Within the same domain of inquiry, it seems to be of general interest to ask how the energy spectrum of bound states of a field theory changes with temperature. Indeed, similar investigations have been carried out for the ϕ^4 theory and the Gross-Neveu model[3] in one space dimension, and for hydrogenlike atoms.[4]

In this note we study the large-temperature behavior of the Bethe-Salpeter equation in the simple scalar-scalar model of Wick and Cutkosky,[5] using the stereographic projection technique due to Fock[6] and a theorem for integral equations due to Funk and Hecke.[7] We are thus led to a second-order difference equation, the condition for the existence of nontrivial solutions of which then yields the energy spectrum of the model. We note that the earlier results[5,8] of this model correspond to the case when temperature is zero.

The equation to be studied is[9]

$$\left[\left(\frac{P}{2}+p\right)^2+m^2\right]\left[\left(\frac{P}{2}-p\right)^2+m^2\right]\psi(p)=\frac{-i\lambda}{\pi^2}\int\frac{d^4p'\psi(p')}{(p-p')^2-i\epsilon}\,, \quad (1)$$

where P is the sum of the four-momenta of the particles (each of mass m) forming the bound state, p their relative four-momentum, λ the square of the coupling constant, and $\psi(p)$ the Bethe-Salpeter amplitude. In order to be able to make contact with earlier work in the limit $T=0$, also for the sake of simplicity, we work within the framework of the instantaneous approximation. Thus, specializing to the frame

$$P=(0,0,0,iE)$$

we have

$$\left[\mathbf{p}^2-\left(p_0+\frac{E}{2}\right)^2+m^2\right]\left[\mathbf{p}^2-\left(p_0-\frac{E}{2}\right)^2+m^2\right]\psi(p)=\frac{-i\lambda}{\pi^2}\int\frac{d^4p'\psi(p')}{(\mathbf{p}-\mathbf{p}')^2}\,. \quad (2)$$

Since the right-hand side (RHS) of Eq. (2) is a function of $\mathbf{p}$ alone, we may define

$$S(\mathbf{p})=\left[\mathbf{p}^2-\left(p_0+\frac{E}{2}\right)^2+m^2\right]\left[\mathbf{p}^2-\left(p_0-\frac{E}{2}\right)^2+m^2\right]\psi(p)\,, \quad (3)$$

to obtain an equation for $S(\mathbf{p})$:

$$S(\mathbf{p})=\frac{-i\lambda}{\pi^2}\int\frac{d^4p'S(\mathbf{p}')}{(\mathbf{p}-\mathbf{p}')^2\left[\mathbf{p}'^2-\left(p_0'+\frac{E}{2}\right)^2+m^2\right]\left[\mathbf{p}'^2-\left(p_0'-\frac{E}{2}\right)^2+m^2\right]}\,. \quad (4)$$

We can now introduce temperature dependence in the theory by following the recipe given in Ref. 10: viz.,

$$p_0' \to \frac{2\pi n}{-i\beta} \quad \left[\beta = \frac{1}{kT}\right], \quad \int_{-\infty}^{+\infty} dp_0' \to \left[\frac{2\pi}{-i\beta}\right] \sum_{n=-\infty}^{n=+\infty} .$$

Thus, we have

$$I \equiv \int \frac{dp_0'}{\left[\mathbf{p}'^2 - \left[p_0' + \frac{E}{2}\right]^2 + m^2\right]\left[\mathbf{p}'^2 - \left[p_0' - \frac{E}{2}\right]^2 + m^2\right]}$$

$$= \frac{2\pi}{-i\beta} \sum_n \frac{1}{\left[\left[\frac{2\pi n}{\beta} - \frac{iE}{2}\right]^2 + \mathbf{p}'^2 + m^2\right]\left[\left[\frac{2\pi n}{\beta} + \frac{iE}{2}\right]^2 + \mathbf{p}'^2 + m^2\right]}$$

$$= \frac{i\pi}{E(\mathbf{p}'^2+m^2)^{1/2}\left[\mathbf{p}'^2+m^2-\frac{E^2}{4}\right]}$$

$$\times \left\{ \left[\frac{E}{4} + \tfrac{1}{2}\sqrt{\mathbf{p}'^2+m^2}\right] \coth\left[\beta\left[\tfrac{1}{2}\sqrt{\mathbf{p}'^2+m^2} - \frac{E}{4}\right]\right] - \left[-\frac{E}{4} + \tfrac{1}{2}\sqrt{\mathbf{p}'^2+m^2}\right] \coth\left[\beta\left[\tfrac{1}{2}\sqrt{\mathbf{p}'^2+m^2} + \frac{E}{4}\right]\right] \right\} . \tag{5}$$

Equation (5) holds for arbitrary β; its substitution into Eq. (4) generalizes the Wick-Cutkosky equation to arbitrary temperatures. For $\beta \to \infty$,

$$I = \frac{i\pi}{2(\mathbf{p}'^2+m^2)^{1/2}\left[\mathbf{p}'^2+m^2-\frac{E^2}{4}\right]} ,$$

which is precisely the result of integration over p_0' in Eq. (4).

It is thus explicitly seen that the $T=0$ limit of the present investigation, at this stage, corresponds exactly to the earlier investigations of the Wick-Cutkosky model in the instantaneous approximation.

II. LARGE-TEMPERATURE LIMIT OF THE MODEL AND ITS ANALYSIS THROUGH FOCK'S TECHNIQUE AND THE FUNK-HECKE THEOREM

It will be seen that the substitution of Eq. (5) into Eq. (4) leads to a very complicated equation. To explore a range of β other than the well-studied $\beta \to \infty$ limit discussed above, we now consider the opposite limit, the case when β is small. In this limit, Eq. (5) leads to

$$I = \frac{2i\pi}{\beta\left[\mathbf{p}'^2+m^2-\frac{E^2}{4}\right]^2} , \tag{6}$$

whence Eq. (4) yields

$$S(\mathbf{p}) = \frac{2\lambda}{\pi\beta} \int \frac{d^3\mathbf{p}' S(\mathbf{p}')}{(\mathbf{p}-\mathbf{p}')^2\left[\mathbf{p}'^2+m^2-\frac{E^2}{4}\right]^2} . \tag{7}$$

Now, let

$$\frac{S(\mathbf{p})}{\left[\mathbf{p}^2+m^2-\frac{E^2}{4}\right]^2} = \phi(\mathbf{p}) \tag{8}$$

and

$$\phi(\mathbf{p}) = g(p) Y_l^m(\theta,\phi) \quad (p = |\mathbf{p}|) ;$$

then, noting that

$$\frac{1}{(\mathbf{p}-\mathbf{p}')^2} = \frac{1}{2pp'(\eta - \cos\Theta)}$$

$$= \frac{4\pi}{2pp'} \sum_{K,M} Q_K(\eta) Y_{KM}(\Omega') Y_{KM}(\Omega) ,$$

where

$$\eta = \frac{p^2+p'^2}{2pp'}, \quad \cos\Theta = \frac{\mathbf{p}\cdot\mathbf{p}'}{pp'} ,$$

and $Q_k(\eta)$ is an associated Legendre function of the second kind, we can integrate over the angles in Eq. (7), to obtain

$$(p^2+m^2b^2)^2 g(p) = \frac{8\lambda}{\beta} \int \frac{dp' p'^2 g(p') Q_l(\eta)}{2pp'} , \tag{9}$$

where

$$b^2 = \left[1 - \frac{E^2}{4m^2}\right] . \tag{10}$$

Equation (9) is a one-dimensional equation. We adopt the strategy of transforming it into an equation that holds on the surface of a unit four-dimensional Euclidean sphere. To this end, additional integrations must be introduced. The first of these is introduced through the representation

$$Q_l(\eta)=\frac{1}{2}\int_0^{\pi}\frac{2pp'd\theta'\sin\theta' P_l(\cos\theta')}{p^2+p'^2-2pp'\cos\theta'} ;$$

a second angle is introduced by simply multiplying the RHS of Eq. (9) by $(1/2\pi)\int_0^{2\pi}d\phi$. We thus obtain

$$(1+\rho^2)^2 g(\rho) = \frac{2\lambda}{\pi\beta}\frac{1}{m^3b^3}\int\frac{d\rho' d\theta' d\varphi' \rho'^2 g(\rho')\sin\theta' P_l(\cos\theta')}{\rho^2+\rho'^2-2\rho\rho'\cos\theta'} , \tag{11}$$

where

$$p=mb\rho . \tag{12}$$

The use of Fock's transformation variables now brings about the desired mapping on a unit four-sphere:

$$\rho=\tan\frac{\psi}{2},\quad \rho'=\tan\frac{\psi'}{2} . \tag{13}$$

We obtain

$$H(\psi)=\frac{\lambda}{2\pi\beta}\frac{\cos^2\left[\frac{\psi}{2}\right]}{m^3b^3}\int\frac{d\Omega_4' H(\psi')P_l(\cos\theta')}{1-\cos\gamma} , \tag{14}$$

where

$$H(\psi)=\sec^4\left[\frac{\psi}{2}\right]g\left[\tan\frac{\psi}{2}\right] ,$$

$$d\Omega_4'=\sin^2\psi'\sin\theta' d\psi' d\theta' d\varphi , \tag{15}$$

$$\cos\gamma=\cos\psi\cos\psi'+\sin\psi\sin\psi'\cos\theta' .$$

$\cos\gamma$ here is the angle between the unit vectors with directions $(\psi,0,0)$ and (ψ',θ',φ'), respectively.

The form of Eq. (14) makes it transparent that if, through a suitable choice of $H(\psi')$, we could make $[H(\psi')P_l(\cos\theta')]$ a surface harmonic of degree n, the integration over $d\Omega_4'$ via the Funk-Hecke theorem would lead back to $[H(\psi)P_l(\cos\theta)]$. Thus, the choice

$$H(\psi)=\frac{1}{n+1}(\sin\psi)^l C_{n-l}^{l+1}(\cos\psi) \tag{16}$$

leads to

$$m^3b^3=\frac{2\lambda\pi}{\beta(n+1)}\cos^2\left[\frac{\psi}{2}\right] . \tag{17}$$

From Eqs. (12) and (13),

$$\cos^2\left[\frac{\psi}{2}\right]=\frac{m^2b^2}{p^2+m^2b^2}\simeq 1 , \tag{18}$$

if $p^2 \ll m^2b^2$. It thus follows that the functions in Eq. (16) are eigenfunctions of our problem only in the limit $p^2 \ll m^2b^2$, and in that limit the eigenvalue spectrum is given by

$$mb=\left[\frac{2\lambda\pi}{\beta(n+1)}\right]^{1/3} . \tag{19}$$

Intuitively, it might seem that the approximation in Eq. (18) is not inconsistent with the instantaneous approximation scheme[11] within which we are working; i.e., the infinite velocity of propagation of the interaction scarcely allows the participating particles to move with respect to each other. However, since the eigenfunctions are now known, one can explicitly calculate $\langle \mathbf{p}^2\rangle$. One finds, in fact, that

$$\langle \mathbf{p}^2\rangle=\frac{1}{2}m^2b^2 ,$$

showing that the approximation in Eq. (18) is inconsistent with the result obtained. In the next section we follow up Eq. (14) without using Eq. (18).

III. THE DIFFERENCE EQUATION AND THE EIGENVALUE SPECTRUM

In Sec. II we choose

$$H(\psi)=\frac{1}{n+1}\sin^l\psi C_{n-l}^{l+1}(\cos\psi) \equiv P_{n,l}^{(2)}(\cos\psi) . \tag{20}$$

We note that as a direct consequence of the recurrence relation satisfied by the Gegenbauer polynomials, the hyperspherical harmonics defined in Eq. (20) satisfy the following recurrence relation:

$$\cos\psi P_{n,l}^{(2)}(\cos\psi)=\frac{(n+2)(n-l+1)}{2(n+1)^2}P_{n+1,l}^{(2)}(\cos\psi) + \frac{n(n+l+1)}{2(n+1)^2}P_{n-1,l}^{(2)}(\cos\psi) . \tag{21}$$

Equations (21) and (17) then suggest that one might attempt a solution of Eq. (14) in the form

$$H(\psi)=\sum_{k=x}^{\infty} b_k P_{n+k,l}^{(2)}(\cos\psi) ,$$

where, since for any l we can only have $n+k\geq l$, the lower limit in the summation has to be $x=-n+l$. Actually, the presence of n here is quite superfluous and the expansion can simply be written as

$$H(\psi)=\sum_{k=l}^{\infty} a_k P_{k,l}^{(2)}(\cos\psi) . \tag{22}$$

We remark here that an expansion similar to the above, with the lower limit for k replaced by zero, was used by Basu and Biswas[8] in their study of the Bethe-Salpeter equation at zero temperature. A consistent analysis in such a case leads to the uninteresting situation where $a_k=0$ for all k, rather than the infinite continued fraction obtained by them.

Substitution from Eq. (22) into Eq. (14) and application of the Funk-Hecke theorem now yields

$$\sum_{k=l}^{\infty} a_k P_{k,l}^{(2)}(\cos\psi)=\gamma(1+\cos\psi)\sum_{k=l}^{\infty}\frac{a_k}{(k+1)}P_{k,l}^{(2)}(\cos\psi)\ , \tag{23}$$

where

$$\gamma=\frac{\lambda\pi}{\beta m^3 b^3}\ . \tag{24}$$

Using Eq. (21), we obtain

$$\sum_{k=l}^{\infty}\left[1-\frac{\gamma}{k+1}\right]a_k P_{k,l}^{(2)}(\cos\psi)=\gamma\sum_{k=l}^{\infty}\frac{1}{2(k+1)^3}a_k[(k-l+1)(k+2)P_{k+1,l}^{(2)}(\cos\psi)+(k+l+1)kP_{k-1,l}^{(2)}(\cos\psi)]$$
$$=\gamma\sum_{k=l+1}^{\infty}\frac{(k-l)(k+1)}{2k^3}a_{k-1}P_{k,l}^{(2)}(\cos\psi)+\gamma\sum_{k=l-1}^{\infty}\frac{(k+l+2)(k+1)}{2(k+2)^3}a_{k+1}P_{k,l}^{(2)}(\cos\psi)\ . \tag{25}$$

This equation may be rewritten as

$$\left[1-\frac{\gamma}{l+1}\right]a_l P_{l,l}^{(2)}(\cos\psi)-\gamma\frac{(2l+1)l}{2(l+1)^3}a_l P_{l-1,l}^{(2)}(\cos\psi)-\gamma\frac{(l+1)^2}{(l+2)^3}a_{l+1}P_{l,l}^{(2)}(\cos\psi)$$
$$+\sum_{k=l+1}^{\infty}P_{k,l}^{(2)}(\cos\psi)\left[\left[1-\frac{\gamma}{k+1}\right]a_k-\gamma\frac{(k-l)(k+1)}{2k^3}a_{k-1}-\gamma\frac{(k+l+2)(k+1)}{2(k+2)^3}a_{k+1}\right]=0\ . \tag{26}$$

We now equate the coefficient of each $P_{k,l}^{(2)}(\cos\psi)$ to zero. Since $P_{l-1,l}^{(2)}(\cos\psi)=0$, the terms outside the sum relate a_l to a_{l+1}. From inside the sum, we get the expression in the square brackets equal to zero for each $k\geq l+1$. This may all be set together compactly into the following difference equation for all $k\geq l$:

$$\frac{\gamma(k+l+2)(k+1)}{2(k+2)^3}a_{k+1}=\left[1-\frac{\gamma}{k+1}\right]a_k-\gamma\frac{(k-l)(k+1)}{2k^3}a_{k-1}\ , \tag{27}$$

with the condition

$$a_{l-1}=0\ . \tag{28}$$

It will be convenient to set Eq. (27) in the form

$$a_{k+1}=A_k a_k-B_k a_{k-1}\ , \tag{29}$$

where

$$A_k=\left[\frac{1}{\gamma}-\frac{1}{k+1}\right]\Bigg/\frac{(k+l+2)(k+1)}{2(k+2)^3}$$

and

$$B_k=\frac{k-l}{2k^3}\Bigg/\frac{k+l+2}{2(k+2)^3}\ . \tag{30}$$

We now indicate how, in an elementary manner, one may investigate the condition for the existence of nontrivial solutions of the above difference equation. We note that Eq. (29) is subject only to the boundary condition $a_{l-1}=0$; for the exact solution given by Eq. (22), $a_k\neq 0$ for all $k\geq l$. Nevertheless, an intuitive and step-by-step method of obtaining the final result consists of subjecting Eq. (29) to one additional boundary condition, which is relaxed in a systematic manner. Thus, assume first that Eq. (29) is to be solved subject to $a_{l-1}=0$ and $a_{l+1}=0$. For a_l to be nonzero, we must have

$$A_l=0\ . \tag{31}$$

Next, let the boundary conditions be $a_{l-1}=0$ and $a_{l+2}=0$. In order that a_l and a_{l+1} be nonzero, we must now have

$$\begin{vmatrix} A_l & 1 \\ B_{l+1} & -A_{l+1}\end{vmatrix}=0 \quad\text{or}\quad A_l=\frac{B_{l+1}}{A_{l+1}}\ . \tag{32}$$

Similarly, if $a_{l-1}=0$ and $a_{l+3}=0$, for nontrivial a_l, a_{l+1}, and a_{l+2}, we must have

$$A_l=\cfrac{B_{l+1}}{A_{l+1}-\cfrac{B_{l+2}}{A_{l+2}}}\ ,$$

and in the most general case, viz., $a_k\neq 0$ for all $k\geq 1$, the following compatibility condition involving an infinite continued fraction (ICF) must be satisfied:

$$A_l-\cfrac{B_{l+1}}{A_{l+1}-\cfrac{B_{l+2}}{A_{l+2}-\cfrac{B_{l+3}}{A_{l+3}}\cdots}}=0\ . \tag{33}$$

This equation can be obtained in a rigorous manner by applying a theorem of Poincaré and by using an identity due to Thiele.[12] Such a derivation is presented in the Appendix. From the definitions of A_k and B_k, Eqs. (30), it follows that Eq. (33) expresses the binding energy as a function of the coupling constant λ, the temperature T ($\sim\beta^{-1}$), and the quantum number l, and thus represents the discrete bound-state spectrum of the present problem.

We now investigate the nature of the spectrum implied by Eq. (33). Since the equation represents a power series in $(1/\gamma)$ equated to zero, it ought to have an infinity of roots. These may be numerically determined in the usual way: choose a value of l (e.g., zero), fix the accuracy to which the roots are required (e.g., up to the sixth decimal

TABLE I. The first ten roots of Eq. (33) giving values of γ as a function of l.

$l=0$	$l=1$	$l=2$	$l=3$	$l=4$
0.837 515 6	1.616 613	2.389 677	3.161 095	3.931 852
1.834 905	2.591 663	3.355 411	4.121 706	4.889 217
2.834 225	3.576 923	4.331 960	5.092 456	5.855 853
3.833 933	4.567 011	5.314 724	6.069 724	6.828 902
4.833 775	5.559 806	6.301 423	7.051 458	7.806 595
5.833 677	6.554 287	7.290 788	8.036 396	8.787 767
6.833 612	7.549 899	8.282 055	9.023 716	9.771 624
7.833 566	8.546 310	9.274 725	10.012 87	10.757 60
8.833 531	9.543 312	10.268 48	11.003 47	11.745 27
9.833 506	10.540 76	11.263 07	11.995 23	12.734 35

place) and plot the LHS of Eq. (33) as a function of γ. We find that it sufficies to retain 20–40 terms (depending upon the value of l) in the ICF for the first ten roots for the states $l=0$ to $l=9$ to become stable. The results for these states are given in Table I. It follows from this table that the lowest values of $(1/\gamma)$ decrease monotonically with l. Furthermore, from Eq. (24) we have

$$mb=\left[\frac{\lambda\pi T}{\gamma}\right]^{1/3},$$

which implies that for a given l, as γ is increased, mb decreases; in other words, the low values of γ are responsible for the prominent states of the spectrum, while large values of γ tend to push the spectrum into the continuum.

IV. CONCLUDING REMARKS

It is interesting to recall the case of the hydrogen atom or the Wick-Cutkosky model at zero temperature, for which the energy levels are degenerate with respect to one of the two quantum numbers which characterize the eigenfunctions [O(4) degeneracy]. In contrast, in the present case, both the energy eigenfunctions and the levels depend only on one quantum number l. However, for each l, we now have an infinite number of levels—as if there were also a hidden quantum number characterizing the system. The latter feature seems to be a consequence of the instantaneous approximation, rather than due to any temperature considerations.[13]

We conclude by pointing out that, since $0\le b\le 1$ and the lowest state for the system under investigation corresponds to $\gamma\simeq 0.838$ ($l=0$), the critical temperature in the model is given by

$$T_c\simeq\frac{0.838m^3}{\lambda\pi},$$

beyond which the bound system will definitely not survive.

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APPENDIX

The difference equation given in the text may be written as

$$a_{k+2}-A_{k+1}a_{k+1}+B_{k+1}a_k=0\,, \quad \text{(A1)}$$

where

$$k>l-1 \text{ and } a_{l-1}=0\,.$$

Let us define b_k by

$$a_k=\Gamma(k+2)b_k\,, \quad \text{(A2)}$$

in terms of which the difference equation takes the form

$$b_{k+2}+p_kb_{k+1}+q_kb_k=0\,, \quad \text{(A3)}$$

where

$$p_k=-\frac{\Gamma(k+3)}{\Gamma(k+4)}A_{k+1},\quad q_k=\frac{\Gamma(k+2)}{\Gamma(k+4)}B_{k+1}\,. \quad \text{(A4)}$$

Note that

$$p\equiv\lim_{k\to\infty}p_k=\frac{-2}{\gamma},\quad q=\lim_{k\to\infty}q_k=0\,. \quad \text{(A5)}$$

The characteristic equation corresponding to (A3) is

$$t^2+pt+q=0\,, \quad \text{(A6)}$$

which, in view of (A5), reduces to

$$t\left[t-\frac{2}{\gamma}\right]=0\,. \quad \text{(A7)}$$

The two distinct roots of this equation are

$$t_1=\frac{2}{\gamma},\quad t_2=0\,. \quad \text{(A8)}$$

Now let (A3) have a set of fundamental solutions $b_k^{(1)}$ and $b_k^{(2)}$. In terms of these solutions, we obtain, from (A3),

$$p_k=\frac{b_{k+2}^{(1)}b_k^{(2)}-b_{k+2}^{(2)}b_k^{(1)}}{b_k^{(1)}b_{k+1}^{(2)}-b_k^{(2)}b_{k+1}^{(1)}}\,, \quad \text{(A9)}$$

$$q_k=\frac{b_{k+1}^{(1)}b_{k+2}^{(2)}-b_{k+1}^{(2)}b_{k+2}^{(1)}}{b_k^{(1)}b_{k+1}^{(2)}-b_k^{(2)}b_{k+1}^{(1)}}\,. \quad \text{(A10)}$$

To proceed further, we use an identity due to Thiele (see Milne-Thomson[12]). We define

TABLE I. (*Continued*).

$l=5$	$l=6$	$l=7$	$l=8$	$l=9$
4.702 282	5.472 529	6.242 662	7.012 718	7.782 722
5.657 405	6.426 009	7.194 886	7.963 952	8.733 154
6.620 975	7.387 209	8.154 200	8.921 729	9.689 656
7.590 704	8.354 262	9.119 055	9.884 749	10.651 12
8.565 078	9.325 872	10.088 34	10.852 04	11.616 70
9.543 050	10.301 11	11.061 21	11.822 87	12.585 73
10.523 87	11.279 28	12.037 06	12.796 65	13.557 69
11.507 00	12.259 86	13.015 38	13.772 94	14.532 16
12.492 02	13.242 46	13.995 78	14.751 38	15.508 79
13.478 59	14.226 75	14.977 99	15.731 66	16.487 32

$$z_s=\frac{x_s-x_{s+1}}{x_s-x_{s+2}}\Big/\frac{x_{s+1}-x_{s+3}}{x_{s+2}-x_{s+3}},\quad s=1,2,\ldots,n-3\ , \tag{A11}$$

$$V_s=\frac{x_n-x_s}{x_n-x_{s+1}}\Big/\frac{x_s-x_{s+2}}{x_{s+1}-x_{s+2}},\quad V_{n-2}=1\ . \tag{A12}$$

The identity is then

$$V_1=1-\frac{z_1}{V_2}=1-\cfrac{z_1}{1-\cfrac{z_2}{V_3}}=1-\cfrac{z_1}{1-\cfrac{z_2}{1-\cfrac{z_3}{V_4}}}$$

$$\cdots$$

$$=1-\cfrac{z_1}{1-\cfrac{z_2}{1-\cfrac{z_3}{1-\ddots\cfrac{z_{n-4}}{1-z_{n-3}}}}}\ . \tag{A13}$$

Setting

$$x_s=\frac{b^{(1)}_{k+s-2}}{b^{(2)}_{k+s-2}}\ , \tag{A14}$$

we get

$$z_s=\frac{q_{k+s-1}}{p_{k+s-2}p_{k+s-1}} \tag{A15}$$

and

$$V_1-1=\frac{b^{(1)}_{k+n}b^{(2)}_{k+1}-b^{(2)}_{k+n}b^{(1)}_{k+1}}{b^{(1)}_{k+n}b^{(2)}_{k}-b^{(2)}_{k+n}b^{(1)}_{k}}\ . \tag{A16}$$

Substituting into the identity (A13), we get

$$\frac{b^{(1)}_{k+n}b^{(2)}_{k+1}-b^{(2)}_{k+n}b^{(1)}_{k+1}}{b^{(1)}_{k+n}b^{(2)}_{k}-b^{(2)}_{k+n}b^{(1)}_{k}}=\cfrac{-q_k}{p_k-\cfrac{q_{k+1}}{p_{k+1}-\ddots-\cfrac{q_{k+n-2}}{p_{k+n-2}}}}\ . \tag{A17}$$

Let us now choose our fundamental solutions so that

$$\lim_{n\to\infty}\frac{b^{(1)}_{k+n+1}}{b^{(1)}_{k+n}}=t_1=\frac{2}{\gamma}\ , \tag{A18}$$

$$\lim_{n\to\infty}\frac{b^{(2)}_{k+n+1}}{b^{(2)}_{k+n}}=t_2=0\ , \tag{A19}$$

which is possible from Poincaré's theorem (Milne-Thomson[12]). It then follows that

$$\lim_{n\to\infty}\left|\frac{b^{(2)}_{k+n+1}}{b^{(1)}_{k+n+1}}\Big/\frac{b^{(2)}_{k+n}}{b^{(1)}_{k+n}}\right|=0\ . \tag{A20}$$

whence

$$\lim_{n\to\infty}\frac{b^{(2)}_{k+n}}{b^{(1)}_{k+n}}=0\ . \tag{A21}$$

Equation (A17) then yields

$$\frac{b^{(2)}_{k+1}}{b^{(2)}_{k}}=\cfrac{-q_k}{p_k-\cfrac{q_{k+1}}{p_{k+1}-\cfrac{q_{k+2}}{p_{k+2}-\ddots}}}\ . \tag{A22}$$

We now put $k=l$, so that the LHS becomes simply the ratio $b_{l+1}^{(2)}/b_l^{(2)}$. Setting $k=l-1$ in (A3) and using the boundary condition

$$b_{l-1}=0\ ,$$

which follows from (A2) and Eq. (28) in the text, we obtain

$$\frac{b_{l+1}^{(2)}}{b_l^{(2)}}=\frac{\Gamma(l+2)}{\Gamma(l+3)}A_l\ . \tag{A23}$$

From Eqs. (A22) and (A4), we finally get Eq. (33) given in the text.

It may be noted that the difference equation of the type considered here will, in general, have a second solution. This has been discussed by Milne-Thomson. Because of the result $t_2=0$, such a solution is nonexistent in the present situation. The result given in the text is thus the only solution of Eq. (27).

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