Thermodynamical properties of a class of models with non-Abelian internal symmetries at finite temperature and baryon number

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The thermodynamical properties of a particular class of gauge models with non-Abelian internal symmetries at finite temperature with exactly implemented baryon-number conservation are investigated. In this context we analyze a model for both SU(2)- and SU(3)-color symmetries in the canonical and grand canonical ensembles. For the case of the grand canonical ensemble we examine particular structure arising with a complex chemical potential. The phase structure of the SU(2)-spin model is studied in the mean-field approximation using the complex chemical potential. These results could possibly be useful for Monte Carlo computations in lattice gauge theory at finite temperature and finite baryon number.

I. INTRODUCTION

The numerous numerical investigations of lattice quantum chromodynamics (QCD) at finite temperature performed up to this time have shown that the hadronic matter exhibits a behavior similar to a phase transition leading to the deconfined quark-gluon (QG) plasma.[1] However, this nonperturbative approach to the description of the thermodynamics of the strongly interacting matter still has some shortcomings. These are mainly connected with the various analyses of the models with fermionic (quark) degrees of freedom and with nonzero baryon number. Also the influence of the systematic finite-size and -volume effects on the lattice thermodynamics is still not very well established.

In the pure gauge theory the existence of the deconfinement phase transition can be confirmed by means of the measurements of the Wilson loop.[2,3] In a special gauge L can be identified as the character of the fundamental representation of the gauge group.[3,4] The expectation value of this gauge-invariant quantity $\langle L \rangle$ determines the free energy of a single quark relative to the vacuum. Thus quarks will be confined if $\langle L \rangle$ is zero which means that the free energy of a single static quark is infinite. In contrast, in the deconfined phase $\langle L \rangle$ is always different from zero. Therefore, in the pure gauge theory the Wilson loop plays the role of the order parameter for the deconfinement phase transition. In the model with the fermions the static quarks can always be neutralized. Consequently $\langle L \rangle$ will always remain different from zero even in the low-temperature phase. Thus, in this case $\langle L \rangle$ cannot be considered to be an order parameter of the model. The formulation of the thermodynamics with some other order parameter for gauge theories containing fermions still has no final form.

The description of lattice gauge theory (LGT) in the canonical ensemble respecting the baryon-number conservation has been recently proposed and analyzed.[5,6] We must keep in mind, however, the difficulties mentioned above connected with the evaluation of the model with nonzero chemical potential as well as those related with the formulation of the order parameter in the model with fermions. In the canonical (C) formulation of the model the absolute value instead of the average value of the baryon number is assumed to be conserved.

The fundamental quantity for the evaluation of the partition function with a prescribed value of the baryon number is the grand canonical (GC) partition function with the complex chemical potential θ. Although this partition function has no direct physical meaning, nevertheless, it can give some useful information about the phase structure of the model with fermions. It has recently been indicated[5] that the Z_N symmetry which is present in a pure gauge theory implies some nontrivial structural dependence on the complex chemical potential for both the free energy and the Wilson loop. In the high-temperature continuum QCD the Wilson loop exhibits a discontinuous structure as a function of the complex chemical potential.[5] On the other hand, in the strongly coupled Wilson lattice theory both the Wilson loop and the free energy are continuous functions of θ (Ref. 5). Thus by looking at the properties of the free energy and the Wilson loop with respect to θ one could distinguish between the confining and the deconfining phases of a gauge theory with fermions.

At this point it is natural to ask the question how the above properties of the model with the complex chemical potential develop in the region between the very low and the very high temperatures. It is particularly interesting to see the structure of the model in the temperature range where the deconfinement phase transition can take place. This interesting physical region will be explained in this work through the example of the SU(2)-spin model with the complex chemical potential. The results obtained up to this time for the SU(N_c)-spin model at finite temperature and density are qualitatively in good agreement with the recent Monte Carlo simulations of the LTG (Ref. 7). Thus one can hope that the analysis of this model through this approach can also give some useful information for eventual further Monte Carlo computations. In terms of the SU(2)-spin model formulated in the C ensemble respecting baryon-number conservation we shall also give some crude estimation of the finite-volume effects. The comparison of the properties of the Wilson loop obtained in the C ensemble with its asymptotic GC value with zero chemical potential gives us the possibility of investigating how the finite-volume effects develop with the coupling constant.

In the asymptotically high-temperature region the QG plasma can be described as the ideal gas of quarks and gluons which are confined to a finite-volume cavity by a phenomenological bag pressure. However, in the Monte Carlo simulation of LGT one can observe that the thermodynamical quantities converge in the high-temperature limit to a slightly different value from their Boltzmann limit.[8,9] This fact could well be related with nonperturbative effects which modify the asymptotic Boltzmann limit due to the finite volume of the system. On the other hand, in realistic physical situations such as in collisions of heavy ions the expected production of the QG plasma should take place in a finite space-time region. Thus, if the volume of the QG plasma droplet produced in this heavy-ion experiment is of the order of one to ten times the nuclear volume, the constraints on the system which are imposed by some nonperturbative effects can indeed play an important role in the thermodynamics of this system.

As a first step in the nonperturbative modification of the ideal-gas approximation for the QG plasma one can take into account the constraints connected with the group structure of the model. These imposed conditions are contained in the requirement that the only physical states allowed in the system are those which transform under a singlet representation of the SU(N_c)-color gauge group.[10] There can also be imposed some further constraints connected with the conservation of the baryon number of the momentum.[4,6,11] These nonperturbative effects are contained in the gas model with SU(N_c)$\times$U$_B$(1) symmetry. This model has often been investigated recently in the literature,[9-12] for which considerable finite-volume effects were observed. In our actual considerations we shall use this model for the QG plasma in order to investigate the particular properties of the Wilson loop for the complex chemical potential in the high-temperature region. We shall discuss how these properties can change with the decreasing of the volume of the system. Also the properties of the Wilson loop L and the correlation function $LL^\dagger$ in the C and GC ensembles are analyzed in relation to the baryon-number conservation.

Our analysis will be explicitly performed for SU$_c$(2)- and SU$_c$(3)-color gauge groups for particular thermodynamic properties. In our character model a natural parameter is $2VT^3/\pi^2$, which we call c. We are particularly interested in the limits of large and small c for which we can compare the properties of SU$_c$(2) and SU$_c$(3) as well as the C and GC ensembles. We then extend our considerations to the SU(2)-spin model in the mean-field approximation which has a nontrivial phase-transition behavior with respect to temperature. The phase structure of all the above models with the chemical potential continued into the complex plane is also analyzed. Thus our study of these models leads us to clearer criteria related to the thermodynamical properties of QCD at finite temperature and baryon number.

Now we discuss briefly how we shall develop this paper. In the next section we shall give a quick summary of the formulation of the C and GC ensembles for gauge theories together with some general aspects of the group structure of our models. Then in Sec. III we shall go into some detail concerning the analysis of the thermodynamics for the SU$_c$(2)$\times$U$_B$(1) model in the C ensemble. Afterward in the following section we discuss the SU$_c$(2) gauge group in the GC ensemble and perform the analysis for the complex chemical potential. In Sec. V we make similar analyses for the color-SU$_c$(3) symmetry. Thereafter in Sec. VI the SU(2)-spin model is investigated for its phase structure in relation to the complex chemical potential. Finally, we make some concluding remarks on our work.

II. ENSEMBLE STRUCTURE OF GAUGE MODELS

Here we develop the formulation of the ensembles for the models with non-Abelian internal symmetry which will be explicitly investigated in the following sections.

First we shall generally discuss the evaluation of the partition function when we consider the baryon-number states. The canonical partition function Z_B in this case can be written as

$$Z_B = \mathrm{Tr}_B(e^{-B\hat{H}}) , \tag{2.1}$$

where the B under the trace indicates that the trace is restricted to only those states with a total baryon number B. Also, as usual, $\hat{H}$ is the Hamiltonian operator and β is the inverse temperature. Since the baryon number is an additive quantum number, this restriction on the trace in (2.1) can be simply removed by means of a δ function. This procedure yields

$$Z_B = \mathrm{Tr}(e^{-\beta H}\delta_{\hat{B},B}) , \tag{2.2}$$

where $\hat{B}$ is the baryon-number operator. Now the trace in (2.2) is taken over all the states. By using the integral representation of the δ function one can easily establish the following more convenient form of the canonical partition function:

$$Z_B = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{iB\phi} \tilde{Z}(\phi, V, \beta) , \tag{2.3}$$

where $\tilde{Z}(\phi, V, \beta)$ is the generating function. It may be defined as

$$\tilde{Z}(\phi, V, \beta) = \mathrm{Tr}(e^{-\beta(\hat{H}-i\phi)}) , \tag{2.4}$$

which can generally be related to the grand canonical partition function.

We have previously explicitly discussed[4,6] how to treat gauge theories when conservation laws are brought into consideration. The above indicated procedure for the calculation of the canonical partition function can be extended to the situation when the conserved charges are related with a non-Abelian internal symmetry. At this point we need to further consider this procedure basing it upon group-theoretical methods.[10] The generating function is then given by

$$\tilde{Z}(g,V,\beta)=\mathrm{Tr}[e^{-\beta\hat{H}}U(g)] \ , \tag{2.5}$$

where $U(g)$ is unitary reducible representation of the symmetry group G for which $g\in G$. Because of the exact symmetry and the decomposition of $U(g)$ into the form $\sum_\alpha \oplus U^\alpha(g)$, we are able to write

$$\tilde{Z}(g,V,\beta)=\sum_\alpha \frac{\chi^\alpha(g)}{d_\alpha} Z_\alpha(\beta,V) \ , \tag{2.6}$$

where $Z_\alpha(\beta,V)$ is the usual canonical partition function given by $\mathrm{Tr}_\alpha[\exp(-\beta\hat{H})]$ which contains exactly that value of the quantum numbers which correspond to the α representation of the symmetry group G. $\chi^\alpha(g)$ and d_α are, respectively, the character and dimension of the α representation of G. By using the orthogonality properties of the group character,[10] we can find that

$$Z_\alpha(\beta,V)=\int dM(\varphi_1,\ldots,\varphi_r)\bar{\chi}^\alpha(\varphi_1,\ldots,\varphi_N) \times\tilde{Z}(\beta,V,\varphi_1,\ldots,\varphi_r) \ , \tag{2.7}$$

where dM is the Haar measure over G and $\varphi_1,\ldots,\varphi_r$ are the parameters of the maximal Cartan subgroup of G. We notice that for the U(1) group we can recover our previous result in (2.3). From (2.6) together with the analytical properties of the GC partition function Z we can find that

$$\tilde{Z}(\beta,V,\varphi_1,\ldots,\varphi_r) =Z(\beta,V,\mu_1=i\varphi_1/\beta,\ldots,\mu_r=i\varphi_r/\beta) \ . \tag{2.8}$$

Thus we are able to arrive at a formulation for our gauge model in terms of the canonical ensemble where the conserved quantity or charge is specified so as to relate to that representation of the group corresponding to the character $\chi^\alpha(g)$. Thereby we may consider the generating function $\tilde{Z}(\beta,V,\varphi_1,\ldots,\varphi_r)$ as the analytic continuation of the GC partition function taken to the pure imaginary values of the chemical potentials $\mu_1,\ldots,\mu_r$.

In the following sections of this paper we shall explicitly consider the group SU(N_c) for the color gauge symmetry using the specific values $N_c=2$ and $N_c=3$ for the thermodynamics at finite temperature $T=\beta^{-1}$, baryon number B, and chemical potential μ. Furthermore, we shall particularly investigate the role of the characters of the representations of the gauge group and the resulting analytical structure from the complex chemical potential in the grand canonical ensemble.

Now we can write down the color-singlet ($\alpha=0$) canonical partition function in general for fixed baryon number as

$$Z(N,T,V)=\int dM_c(\gamma_1,\ldots,\gamma_{N_c-1})dM_B(\phi)\cos N\phi\tilde{Z}(\phi,\gamma_1,\ldots,\gamma_{N_c-1},T,V) \ , \tag{2.9}$$

where $dM_c(\gamma_1,\ldots,\gamma_{N_c-1})$ is the Haar measure for SU(N_c) and $\gamma_1,\ldots,\gamma_{N_c-1}$ the group parameters. $dM_B(\phi)$ is the Haar measure for the $\mathrm{U}_B(1)$ group. It takes the simple form $d\phi/2\pi$. The factor $\cos N\phi$ arises with the quark number $N=N_cB$ as the real part of the character of the $\mathrm{U}_B(1)$ group. The general expression for the generating function $\tilde{Z}(T,V,\phi,\gamma)$ in the case when the quarks with mass m transform under the fundamental representation and the gluons under the adjoint representation of SU(N_c) can be written[4] as

$$\ln\tilde{Z}_Q(T,V,\mu,\gamma)=\frac{g_Q}{d_Q}\frac{m^2VT}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}K_2(nm/T)[e^{\mu n/T}\chi^Q(n\gamma)+e^{-\mu n/T}\bar{\chi}^Q(n\gamma)] \ , \tag{2.10a}$$

$$\ln\tilde{Z}_G(T,V,\gamma)=\frac{g_G}{d_G}\frac{VT^3}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^4}\chi^G(n\gamma) \ , \tag{2.10b}$$

where g_Q,g_G and d_Q,d_G, are, respectively, the quark and gluon degeneracy factors and dimension of the representations. The quark characters χ^Q and $\bar{\chi}^Q$ are the usual $\chi^{(1,0)}$ and $\chi^{(0,1)}$, whereas the gluon character is χ^G or $\chi^{(1,1)}$ for the adjoint representation. The symbol γ represents the parameters $\gamma_1,\ldots,\gamma_{N_c-1}$. The function $K_2(x)$ is the modified Bessel function of the second kind. In order to get $\tilde{Z}(T,V,\phi,\gamma)$ as $\tilde{Z}_Q(T,V,\phi,\gamma)\tilde{Z}_G(T,V,\gamma)$ we must replace μ/T by $i\phi$ in (2.10a) and set it into (2.9). After carrying out this calculation[6] we notice that the exact implementation of baryon-number conservation changes the symmetry to SU(N_c)×$\mathrm{U}_B(1)$ where the $\mathrm{U}_B(1)$ part relates to the conserved quantity given by the quark number N or the related baryon number B.

III. SU(2) CHARACTER MODEL IN CANONICAL ENSEMBLE

Now we can use the formulation from the last section to obtain some analytical results for a model with particular conservation laws. In this section we restrict ourselves to the $\mathrm{SU}_c(2)$-color symmetry. We have also taken into account the baryon-number conservation.[6] Thus we extend the symmetry of our model to $SU_c(2)\times U_B(1)$.

We first derive the colorless canonical partition function for the conserved baryon number B for the quark-gluon gas with the $SU_c(2)\times U_B(1)$ symmetry. Starting with Eqs. (2.7)–(2.9) we write the canonical partition function

$$Z(N,T,V)=\int_0^{2\pi}\frac{d\gamma}{\pi}\sin^2\frac{\gamma}{2}\times\int_0^{2\pi}\frac{d\phi}{2\pi}\cos N\phi\tilde{Z}(T,V,\phi,\gamma)\ , \tag{3.1}$$

where γ is the $SU_c(2)$ group parameter and the integrals correspond to the Haar measures of $SU_c(2)$ and $U_B(1)$, respectively. The expression for the generating function $\tilde{Z}(T,V,\phi,\gamma)$ connected with our $SU_c(2)\times U_B(1)$ character model can be obtained using the formalism from the last section. In the special case of Boltzmann statistics with $n=1$ it takes the form from Eqs. (2.10a) and (2.10b),

$$\ln\tilde{Z}(T,V,\phi,\gamma)=c_1\chi^G(\gamma)+c_2[e^{i\phi}\chi^Q(\gamma)+e^{-i\phi}\bar{\chi}^Q(\gamma)]\ , \tag{3.2}$$

where the constants c_1 and c_2 are, respectively, $8VT^3/\pi^2$ and $(4m^2VT/\pi^2)K_2(m/T)$. After a simple replacement of the characters of the funamental and adjoint representation for $SU_c(2)$ we find that

$$\ln\tilde{Z}(T,V,\phi,\gamma)=c_1\cos^2\left[\frac{\gamma}{2}\right]+c_2\cos(\phi)\cos\left[\frac{\gamma}{2}\right]-c_1/4\ . \tag{3.3}$$

We see that the quantity VT^3 has a certain physical importance in the model. For asymptotically large values of VT^3 the canonical partition function (3.1) may be evaluated by the steepest-descent method, whereby the leading contribution is just the ideal gas of quarks and gluons. The constraints imposed due to the conservation laws give the finite-volume correction to the QG plasma thermodynamics.

At this point we are able to evaluate the colorless canonical partition function in (3.1) from the generating function in (3.3) to find

$$Z(N,T,V)=\frac{2e^{-c_1/4}}{\pi}\int_0^{2\pi}d\gamma\,\sin^2\frac{\gamma}{2}e^{c_1\cos^2(\gamma/2)}I_N\left[c_2\cos\frac{\gamma}{2}\right]\ , \tag{3.4}$$

where the integration over the $U_B(1)$ group has been carried out explicitly. After using the symmetry properties of the modified Bessel function of the first kind $I_N(x)$ we can establish

$$Z(N,T,V)=c_0(1+e^{i\pi N})\int_0^1 dx\sqrt{1-x^2}\,e^{c_1x^2}I_N(c_2x)\ , \tag{3.5}$$

where $x=\cos(\gamma/2)$ and $c_0=(4/\pi)\exp(-c_1/4)$. Thus one readily sees that $Z(N,T,V)$ is zero when N is not an even number. Furthermore, $Z(N,T,V)$ may be evaluated in a power series in $y=c_2/2$ which is proportional to the single quark canonical partition function with coefficients containing known mathematical functions. We may write it as an expansion in even powers of y as

$$Z(N,T,V)=c_0\sum_{k=0}^{\infty}y^{2k+N}\Upsilon_k(N,c_1)\ , \tag{3.6}$$

where

$$\Upsilon_k(N,c_1)=\frac{B\left[\frac{3}{2},\frac{2k+N+1}{2}\right]{}_1F_1\left[\frac{2k+N+1}{2},\frac{2k+N+4}{2},c_1\right]}{k!(k+N)!}\ . \tag{3.7}$$

The coefficients $\Upsilon_k(N,c_1)$ contain the thermodynamics due to the gluons in the confluent hypergeometric function ${}_1F_1$, whereas the Euler beta function B together with the factorials in the denominator provide the combinatorial factors relating to the internal symmetries. The power of this expansion $(2k+N)$ maintains the colorlessness of $Z(N,T,V)$ from the number of noninteracting quark-antiquark pairs k in addition to the N quarks in the baryons.

With this partition function $Z(N,T,V)$ we can analyze the analytical structure of the thermodynamics of the quark-gluon system with $SU_c(2)\times U_B(1)$ symmetry possessing the exact conservation of baryon number. Let us first consider the energy density ϵ_N which can be obtained from $Z(N,T,V)$ as usual:

$$\epsilon_N=\frac{T^2}{VZ(N,T,V)}\frac{\partial Z(N,T,V)}{\partial T}\ , \tag{3.8}$$

which we compare with the classical Stefan-Boltzmann limit ϵ_{SB} as $(42/\pi^2)T^4$. After dividing out this factor we find

$$\frac{\epsilon_N}{\epsilon_{\rm SB}}=-\frac{1}{7}+\frac{4c_0}{7Z(N,T,V)}\left[\int_0^1 dx\, x^2\sqrt{1-x^2}\,e^{c_1x^2}I_N(c_2x)+\frac{1}{2}\int_0^1 dx\, x\sqrt{1-x^2}\,e^{c_1x^2}[I_{N+1}(c_2x)+I_{N-1}(c_2x)]\right] . \tag{3.9}$$

In terms of the analytical solution we can find a similar series expansion containing the same functions as in $Z(N,T,V)$ in (3.6). However, with this analytical solution one can easily deduce the asymptotic properties of ϵ_N with $N=0$ for the limit of c_1 and c_2 going to zero. With $N=0$ ϵ_N goes to zero for small values of $c=2VT^3/\pi^2$ as shown numerically in Fig. 1(a). Furthermore, in this figure we notice that the nonphysical behavior in the low-c region is due to the classical statistics which we have taken. Thus the higher terms in the series (2.10a) and (2.10b) must be added to correct for the quantum effects at small c. This region would be interesting for the investigation of other critical phenomena such as condensation which we shall not explicitly discuss here. The quantum effects are especially significant when the total baryon number increases. In this case these quantum properties appear at even larger values of c as can be seen in this figure. Finally, it is known that when the quantum effects are included in a similar SU(N_c) model[13] for nonzero N that $\epsilon_N/\epsilon_{\rm SB}$ approaches unity for small values of c.

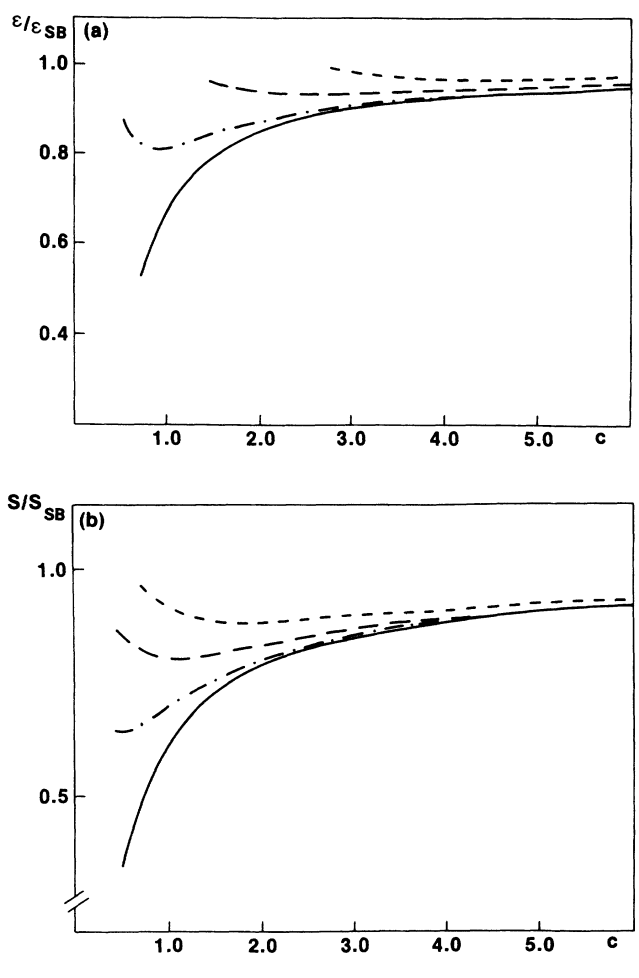

FIG. 1. Thermodynamical functions are compared to their Stefan-Boltzmann value in the SU(2)$\times$U(1) gas model as a function of the parameter $c=2VT^3/\pi^2$ for various values of the quark number N; $N=0$ (solid curve), $N=2$ (dashed-dotted curve), $N=4$ (dashed curve), and $N=6$ (dotted curve). (a) energy density $\epsilon/\epsilon_{\rm SB}$, (b) entropy density $S/S_{\rm SB}$.

The nonperturbative constraints imposed upon the system due to the conservation laws provide significant deviations from the asymptotic Boltzmann behavior of the thermodynamical functions. Not only the energy density has these properties in the limit of small volume and temperature. It is immediately clear that the pressure which is related with $\epsilon/\epsilon_{\rm SB}$ in the usual way will also exhibit the same behavior. If we are only interested in the temperature of the QG plasma around 200 MeV and when the volume is taken to be of the order of the nuclear volume, then the parameter c is of the order of unity. Thus the physically possible values of the relevant parameter is larger than unity. In the physical region we would expect deviations from the asymptotic value of thermodynamical functions up to about 30%.

The entropy shows an even more drastic change in this small-volume limit. This effect comes from the logarithmic structure present in the entropy. If we compare the entropy density s_N with its classical Stefan-Boltzmann limit of $(56/\pi^2)T^3$, we find that

$$\frac{s_N}{s_{\rm SB}}=\frac{3\epsilon_N}{4\epsilon_{\rm SB}}+\frac{\ln Z(N,T,V)}{Vs_{\rm SB}} . \tag{3.10}$$

Because of the logarithm this form is not readily amenable to a similar analytical solution. However, the numerical evaluation of s_N in Fig. 1(b) shows unusual behavior for all N in the small-volume limit. Again we notice that the deviation from the expected results increases with larger N. Furthermore, one can also see by comparing these results for the entropy with those of the energy density in Fig. 1(a) that the effects from the constraints are quantitatively different for both of these quantities. The energy density converges more rapidly to its asymptotic value. Finally we note that with increasing baryon number the quantum effects upon the energy density are more pronounced than those of the entropy.

Now let us look at the evaluation of the Wilson loop $\langle L\rangle_N$ for the canonical ensemble with fixed N. This quantity is especially interesting for gauge theories since in the pure-gluon system it is the order parameter. In the asymptotic region where the system is considered to be noninteracting, the Wilson loop is then equal to one. However, it is interesting to find out how the deviations from the asymptotic value appear when the constraints are imposed on the system. In the actual consideration these constraints arise from the requirement that the only physical states are those which transform as color singlets with respect to the $SU_c(2)$-color symmetry. There are also constraints due to the exact implementation of baryon-number conservation.[4-6,11,13] In this model we identify[4] the Wilson loop as just the character of the fundamental representation of the SU(2) group. We can set up this integral in a way similar to that for $Z(N,T,V)$ in (3.1) and carry out the $U_B(1)$ integration in the same manner. We find now for

$$\langle L \rangle_N = \frac{c_0}{2Z(N,T,V)} \int_0^{2\pi} d\gamma \, \sin^2\frac{\gamma}{2} \cos\frac{\gamma}{2} e^{c_1 \cos^2(\gamma/2)} I_N \left[c_2 \cos\frac{\gamma}{2} \right] . \tag{3.11}$$

However, the presence of the cosine in the integrand changes the entire structure of the integral causing $\langle L \rangle_N$ to vanish identically for all values of N after the integration over the $SU_c(2)$ group measure.

Nevertheless, we are able to get around this total vanishing of $\langle L \rangle_N$ by considering instead the related quantity $\langle \, | L | \, \rangle_N$. This averaged absolute value of the Wilson loop is often considered in SU(N) lattice gauge theory to remove the cancellation due to the inherent Z_N symmetry. This modified Wilson loop with a prescribed value of the baryon number can also be analytically evaluated in the form

$$\langle \, | L | \, \rangle_N = \frac{c_0}{Z(N,T,V)} \times \sum_{k=0}^{\infty} y^{2k+N-1} \Upsilon_k(N+1, c_1) . \tag{3.12}$$

The significance of these analytical results is even more clearly demonstrated in the case of the pure-gluon gas where $c_2 = 0$. This special case yields for $N = 0$ a simple form

$$\langle \, | L | \, \rangle_0 = \frac{4}{3\pi} \frac{{}_1F_1(1, \frac{5}{2}, c_1)}{{}_1F_1(\frac{1}{2}, 2, c_1)} , \tag{3.13}$$

which clearly demonstrates the correct asymptotic properties for $\langle \, | L | \, \rangle_0$ giving unity at large values of c_1. However, as c_1 goes to zero then $\langle \, | L | \, \rangle_0$ does not vanish but achieves a limiting value of $4/3\pi$. Thus we may generally conclude for all N that there is an important qualitative difference between $\langle L \rangle_N$ which is always zero and $\langle \, | L | \, \rangle_N$ which remains nonzero for all values of the parameters c_1 and c_2. The actual quantitative dependence of $\langle \, | L | \, \rangle_N$ on the temperature, volume, and baryon number has been presented in our earlier work.[6]

Another physical quantity which is normally used in LGT as a parameter for the description of the deconfinement properties is the correlation function $\langle LL^\dagger \rangle$ which in the above model can be related with the thermal average of the character χ^G of the adjoint representation as a function of c in Fig. 2 given by

$$\langle \chi^G \rangle_N = \tfrac{4}{3} \langle LL^\dagger \rangle_N - \tfrac{1}{3} . \tag{3.14}$$

Here, in terms of the considered model, we can study the behavior of $\langle LL^\dagger \rangle$ assuming that the volume of the system is finite. We look explicitly at $\langle \chi^G \rangle_N$ in the canonical ensemble in order to describe the role of the baryon number in relation to the correlations. Again for the $SU_c(2) \times U_B(1)$ symmetry we are able to calculate an exact analytical expression

$$\langle \chi^G \rangle_N = \frac{2c_0}{3Z(N,T,V)} \times \sum_{k=0}^{\infty} y^{2k+N+2} \Upsilon_k(N+2, c_1) - \tfrac{1}{3} , \tag{3.15}$$

where $\langle \chi^G \rangle_N$ has been normalized to its asymptotic value 3. An investigation of this analytical expression in the limit of low temperatures (vanishing c_1 and c_2) shows that $\langle \chi^G \rangle_0$ vanishes while $\langle \chi^G \rangle_N$ for $N = 2, 4, \ldots$ is finite and increases to unity for large N. This first fact about the vanishing of $\langle \chi^G \rangle_0$ can also be deduced from its integral representation by using the orthogonality properties of the character and the structure of the invariant

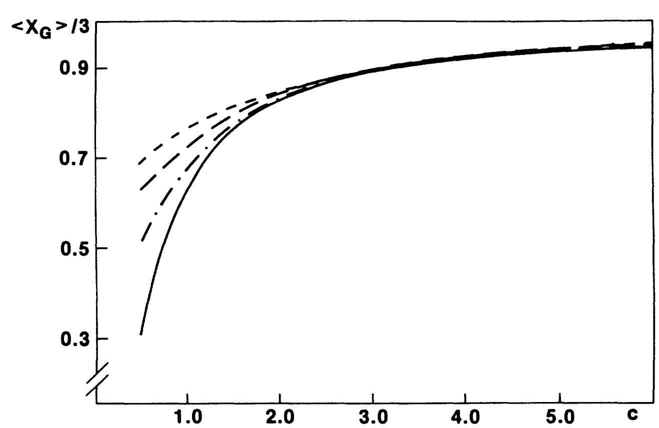

FIG. 2. Thermal average $\langle \chi^G \rangle / 3$ of the adjoint character is given by the same parameters as in Fig. 1.

measure.

At this point we consider the susceptibility which we introduce as usual as $\langle LL^{\dagger}\rangle_N - \langle L\rangle_N^2$. However, since the dynamics in this system is imposed by the constraints, one cannot expect this quality to be large. The analysis shows that for small values of c there is a finite deviation of the susceptibility from zero. However, at larger values of c this deviation is neglegible.

IV. SU(2) GAUGE MODEL IN THE GRAND CANONICAL ENSEMBLE

We consider the $\mathrm{SU}_c(2)$ character model in the GC ensemble with a finite-chemical potential μ. For the particular properties of our analysis we are especially interested in the case when μ is a pure imaginary quantity. However, we shall first develop the general formalism of the thermodynamics containing baryon-number conservation in the GC ensemble.

As a first step we find the color-singlet GC partition function.[10] For the case of $\mathrm{SU}_c(2)$-color internal-symmetry group by using (2.7) and the transformations discussed in the previous section the partition function takes the form

$$Z(T,V,\mu)=2c_0\int_0^1 dx\sqrt{1-x^2}\,e^{c_1x^2}\times\cosh[c_2(\cosh\beta\mu)x]\ . \tag{4.1}$$

The above GC partition function can be evaluated by means of the cluster decomposition.[6] For the GC ensemble we can perform similar transformations on the Wilson loop to get

$$\langle L\rangle=\frac{2c_0}{Z(T,V,\mu)}\int_0^1 dx\,x\sqrt{1-x^2}\,e^{c_1x^2}\times\sinh[c_2(\cosh\beta\mu)x]\ . \tag{4.2}$$

Furthermore, we can evaluate $\langle\,|L|\,\rangle$ in terms of the model given by $Z(T,V,\mu)$ in (4.1), which can be directly compared with $\langle L\rangle$. After taking the group structure into account we find $\langle\,|L|\,\rangle$ to be

$$\langle\,|L|\,\rangle=\frac{2c_0}{Z(T,V,\mu)}\int_0^1 dx\,x\sqrt{1-x^2}\,e^{c_1x^2}\times\cosh[c_2(\cosh\beta\mu)x]\ . \tag{4.3}$$

In the limit where c_2 vanishes the Wilson loop $\langle L\rangle$ clearly also vanishes whereas $\langle\,|L|\,\rangle$ does not. Furthermore, we can contrast these results with the C ensemble where $\langle L\rangle$ vanished identically for all values of the parameters. This fact is mainly due to the different realization of the Z_2 symmetry in these two ensembles. In the special case of the $\mathrm{SU}_c(2)$ gauge group this symmetry is contained in the replacement of L by $-L$. In the GC ensemble the action is not invariant under Z_2 symmetry. However, in the C ensemble the action is invariant under the Z_2 symmetry due to the requirement that only the even values of N are allowed.

In the special case when only the quarks are present in the system ($c_1=0$) the Wilson loop (4.2) takes on an especially simple form

$$\langle L\rangle=\frac{I_2(c_2\cosh\beta\mu)}{I_1(c_2\cosh\beta\mu)}\ . \tag{4.4}$$

For small values of the argument of the Bessel function in (4.4) we find that $\langle L\rangle$ becomes just the free energy of a single quark. In contrast with this, had we considered $\langle\,|L|\,\rangle$ instead of $\langle L\rangle$, the small-argument limit would not go over to the single-quark free energy. However, for asymptotically large values of c_2 both quantities converge to their limiting value of one.

Another special case is the ultrarelativistic limit where $c_1=c_2=4c$. In Fig. 3 we have shown the behavior of $\langle L\rangle$ in terms of c for $\mu=0$. For completeness the correlation $\langle LL^{\dagger}\rangle$ has also been computed here. As one can readily see in Fig. 3, the Wilson loop is very close to its asymptotic value even in the region of rather low values of the parameter c. However, this particular property does not appear to this extent with the other thermodynamical quantities.

Now we briefly remark on some clear differences between the GC and C ensembles with respect to the baryon-number conservation. The relation to the Z_2 symmetry is an apparent aspect which has already been discussed.[5,6] However, if we contrast directly $\langle L\rangle$ in the GC ensemble with $\mu=0$ and $\langle\,|L|\,\rangle_N$ in the C ensemble with $N=0$ or some other comparable thermodynamical quantities, then for c above the value five these quantities show approximately the same structure. Nevertheless, in the region where this parameter is less than five the differences between these two quantities can be observed up to about 10%. This comparison further confirms that $\mu=0$ does not necessarily mean zero baryon number.

Now in terms of this model we can study the thermodynamical behavior of the QG plasma with nonperturbative constraints in the high-temperature region using a

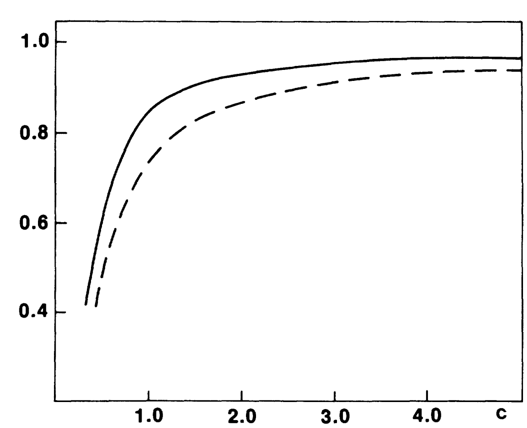

FIG. 3. Thermal average of the Wilson loop $\langle L\rangle$ (solid curve) and the correlation function $\langle LL^{\dagger}\rangle$ (dashed curve) for the SU(2)-gas model as a function of c obtained in the GC ensemble with $\mu=0$.

complex chemical potential with $\beta\mu = i\theta$. This analysis can be very useful for the understanding of the phase structure of strongly interacting matter as previously discussed.[5] In particular in terms of this model we shall show how these properties are changed when the finite-volume effects start to be important. Our partition function with $c_1 = c_2 = 4c$ takes the form

$$Z(T,V,\theta) = 2c_0 \int_0^1 dx \sqrt{1-x^2}\, e^{4cx^2} \times \cosh[4c(\cos\theta)x] \ . \qquad (4.5)$$

It is known that $Z(T,V,\theta)$ has some explicit symmetries.[5] In general there is a symmetry around π or $Z(\pi+\alpha)$ equals $Z(\pi-\alpha)$ for all parameters α. In particular for $\mathrm{SU}_c(2)$ we have $Z(\theta) = Z(\theta+\pi)$ which is related with the realization of Z_2 symmetry. In Fig. 4 we have drawn the effective potential $c' \ln Z(\theta)$ where c' is $1/4c$. For large c this quantity steadily drops until $\pi/2$ then turns sharply upward resulting in an apparent cusping form. On the other hand, at smaller values of c it has a rather smooth structure for all values of θ.

Similar indications of unusual behavior for large c at the same points are shown in our evaluation of the Wilson loop $\langle L \rangle$, which may be written as a function of θ as

$$\langle L \rangle = \frac{2c_0}{Z} \int_0^1 dx\, x\sqrt{1-x^2}\, e^{4cx^2} \times \sinh[4c(\cos\theta)x] \ . \qquad (4.6)$$

Again $\langle L \rangle$ has the same symmetry around π. Our computation of $\langle L \rangle$ in Fig. 5 clearly shows pronounced changes at $\pi/2$ whose quantitative changes in structure are greatly effected by the size of the parameter c. At large values of c the change of $\langle L \rangle$ is very abrupt for θ right around $\pi/2$ and $3\pi/2$ while for θ before and after these points $\langle L \rangle$ remains almost unchanged. This sort of behavior is commonly attributed to a phase transition although in this model we have none. On the other hand, at smaller values of c we have $\langle L \rangle$ making a very smooth transition to positive and negative values following closely the general structure of $\cos\theta$. We note that in both cases the symmetry around π of $\langle L \rangle$ is retained. However, the striking quantitative differences between these two curves do not arise from the effects of a phase transition but rather from the change in the number of degrees of freedom available in the system due to the imposed constraints.

V. THE SU(3) CHARACTER MODEL

We now proceed with a similar analysis of the character model[4] with SU(3) symmetry in order to find the thermodynamical properties corresponding to those of the previous two sections. In doing this we shall largely depend on the intuition gained from the analysis of the SU(2) model since the comparable exact analytical results for SU(3) are not immediately forthcoming.

In a recent work[6] we have presented an approach to the exact implementation of the baryon-number conservation in lattice gauge theory. This approach provides a basis for our calculation with the SU(3) character model. In this case we treat both the canonical and grand canonical ensembles equivalently and in parallel. Since there are only two mutually commuting generators of the SU(3) group, the invariant measure $dM(\gamma_1,\gamma_2)$ depends only upon the two parameters γ_1 and γ_2. The specific parametrization of the measure gives

$$dM(\gamma_1,\gamma_2) = \frac{8}{3\pi^2}\sin^2\left[\frac{\gamma_1-\gamma_2}{2}\right]\sin^2\left[\frac{2\gamma_1+\gamma_2}{2}\right] \times \sin^2\left[\frac{2\gamma_2+\gamma_1}{2}\right] d\gamma_1 d\gamma_2 \ . \qquad (5.1)$$

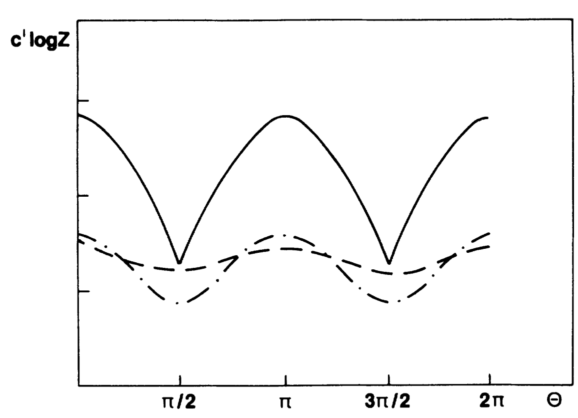

FIG. 4. The dependence of the free energy $c' \ln Z(\theta)$ for the SU(2)-gas model on the complex chemical potential $\theta = -i\mu\beta$ for various values of the parameter c; $c = 1.0$ (dashed curve), $c = 2.5$ (dashed-dotted curve), and $c = 20$ (solid curve).

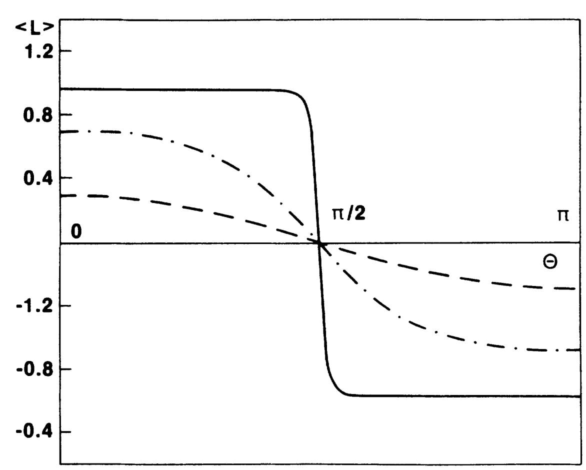

FIG. 5. Thermal average of the Wilson loop $\langle L \rangle$ as a function of θ for the SU(2)-gas model with the values of c in Fig. 4.

The other quantities essential to our SU(3) model are the real and imaginary parts of the character in the fundamental representation

$$L_R = \cos\gamma_1 + \cos\gamma_2 + \cos(\gamma_1+\gamma_2) , \quad (5.2a)$$

$$L_I = \sin\gamma_1 + \sin\gamma_2 - \sin(\gamma_1+\gamma_2) , \quad (5.2b)$$

and the character in the adjoint representation

$$\chi_G = 2[\cos(\gamma_1-\gamma_2)+\cos(2\gamma_1+\gamma_2)+\cos(2\gamma_2+\gamma_1)+1] . \quad (5.3)$$

We are now able to formulate our model[4,6] using the SU(3) group characters in the generating function (2.10a) and (2.10b) to get with $n=1$ the following:

$$\ln\tilde{Z}(\mu,V,T) = c_1\chi_G + c_2[(\cosh\beta\mu)L_R + i(\sinh\beta\mu)L_I] , \quad (5.4)$$

where $c_1 = 4VT^3/\pi^2$ and $c_2 = (2VTm^2/\pi^2)K_2(m/T)$. Thus the partition function with SU(3) symmetry takes the form

$$Z(\mu,V,T) = \int dM(\gamma_1,\gamma_2) \times \exp\{c_1\chi_G + c_2[(\cosh\beta\mu)L_R + i(\sinh\beta\mu)L_I]\} . \quad (5.5)$$

At this point we can derive the complete thermodynamics for the grand canonical ensemble. In particular we are able to evaluate any expectation value of an arbitrary quantity F:

$$\langle F \rangle = \frac{1}{Z}\int dM(\gamma_1,\gamma_2)F(\gamma_1,\gamma_2)\tilde{Z}(\mu,V,T,\gamma_1,\gamma_2) . \quad (5.6)$$

Thus we can readily evaluate $\langle L \rangle$ and $\langle LL^\dagger \rangle$ by using L_R in (5.2a) for $\langle L \rangle$ and χ_G in (5.3) for $\langle LL^\dagger \rangle$, which we show in Fig. 6 for $\mu=0$ as a function of c. For SU(3) we notice that, as was the case for SU(2) in Fig. 3, $\langle L \rangle$ is slightly different from $\langle LL^\dagger \rangle$ for small values of c. The main apparent difference between these two figures is that the asymptotic value of unity for both quantities in SU(3) is not so readily achieved as in SU(2). This situation causes $\langle LL^\dagger \rangle - \langle L \rangle^2$ to deviate from zero for SU(3) at somewhat larger values of c. These two facts may be generally interpreted as the role of the finite-volume effects in SU(3) model is more important than for SU(2).

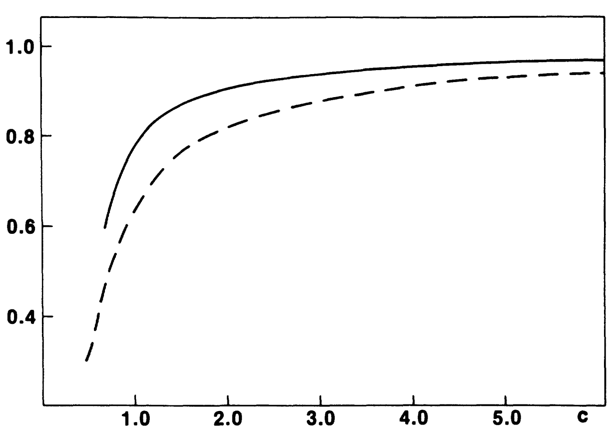

FIG. 6. Thermal average of the Wilson loop $\langle L \rangle$ (solid curve) and the correlation function $\langle LL^\dagger \rangle$ (dashed curve) for the SU(3)-gas model as a function of c obtained in the GC ensemble with $\mu=0$.

Next we go over to the canonical ensemble where we calculate the thermodynamical functions at fixed baryon number. We are able to go from $Z(\mu,V,T)$ in (5.5) to $Z(N,V,T)$ in the same way as in Sec. III for SU(2) by following our previous approach.[6] Now using $y=(L_R^2+L_I^2)^{1/2}$ we find after integration over the $U_B(1)$ group the result

$$Z(N,V,T) = \int dM(\gamma_1,\gamma_2)e^{c_1\chi_G}I_N(c_2y)T_N(L_R/y) , \quad (5.7)$$

where the $I_N(x)$ Bessel functions appear as in SU(2) but we have here in addition the Chebyshev polynomials $T_N(x)$. Furthermore, we should notice here that the structure apparent in (5.5) and (5.7) arises from the presence of a finite-imaginary part of the quark representation which can also be found in other models dealing with non-Abelian symmetry and baryon-number conservation. In particular in the lattice formulation of QCD the oscillating structure of the action in both the ensembles resemble that which is found in these two equations.[6,14] From $Z(N,V,T)$ is is a straightforward calculation using again Eq. (3.7) to find the energy density ϵ_N with respect to the SU(3) classical Stefan-Boltzmann limit ϵ_{SB} of $84T^4/\pi^2$. This calculation yields

$$\epsilon_N/\epsilon_{SB} = \frac{1}{7Z(N,V,T)}\int dM(\gamma_1,\gamma_2)e^{c_1\chi_G}T_N(L_R/y) \times [I_{N+1}(c_2y)+I_{N-1}(c_2y)] . \quad (5.8)$$

The results of the numerical computation of this quantity are shown in Fig. 7 for $N=0,3,6$. The case of zero baryon number looks rather similar to that of SU(2) except that, as it would be expected, it approaches its asymptotic limit more slowly. However, compared with SU(2) the quantum effects are considerably less drastic at the lower values of N. This means that the effects of the constraints are more evident for SU(3). Nevertheless, for larger N the quantum effects dominate the energy density for the lower values of c. Thus, in spite of the stronger effects of colorlessness and baryon-number conservation in SU(3) this model suffers badly from the quantum nature of the small volume and low-temperature region.

Now we return to the grand canonical ensemble for the case where the chemical potential is complex. Analogously with SU(2) we will look at the case where $\beta\mu = i\theta$ is pure imaginary. In this case the generating function in (5.4) becomes the real quantity $\ln\tilde{Z}(\theta,T,V)$ where the

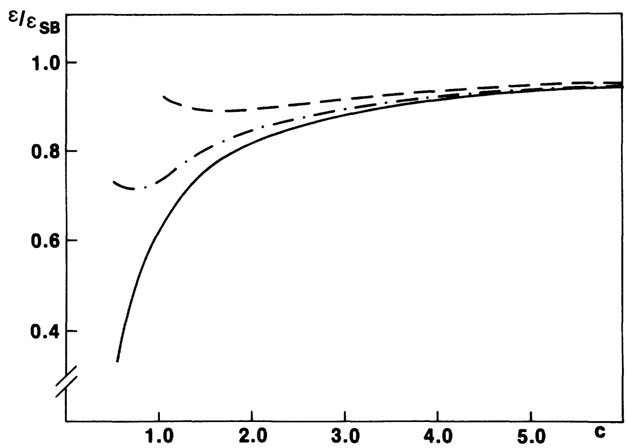

FIG. 7. The energy density is compared to its Stefan-Boltzmann value ϵ_{SB} in the SU(3)$\times$U(1)-gas model as a function of the parameter c for various values of the quark number N; $N=0$ (solid curve), $N=3$ (dashed-dotted curve), and $N=6$ (dashed curve).

hyperbolic trigonometric functions related to L_R and L_I become, respectively, $\cos\theta$ and $\sin\theta$. Because of this replacement the $\ln\tilde{Z}(\theta,T,V,\bar{\gamma})$ is free from the oscillating contributions arising from the presence of an imaginary part. Thus we can rewrite the partition function with an imaginary chemical potential as

$$Z(\theta)=\int dM(\gamma_1,\gamma_2) \times\exp\{c_1\chi_G+c_2[(\cos\theta)L_R+(\sin\theta)L_I]\} \ . \tag{5.9}$$

In Fig. 8 we look at the effective potential $(1/2c)\ln Z(\theta)$ for $0\leq\theta\leq 2\pi$, which is proportional to the negative of the free energy divided by T^4. We notice that there is a cusping structure of the effective potential at $\pi/3$, π, and $5\pi/3$ and maxima at 0, $2\pi/3$, $4\pi/3$, and 2π. These effects illustrate the basic symmetries of $Z(\theta)$ for complex chemical potential[5] where in each sector of size $2\pi/3$ the effective potential appears to be the same.

Finally we turn to the Wilson loop $\langle L\rangle$ with a complex chemical potential. For large values of the parameter c, $\langle L\rangle$ ranges between the limiting values of 1.0 and -0.5 as illustrated in Fig. 9. We notice particularly that changes between these values take place rather abruptly at $\pi/3$ and $5\pi/3$ and no place else in the fundamental interval between 0 and 2π. However, for smaller values of c the structure of $\langle L\rangle$ changes quite slowly between positive and negative values showing no sudden jump around the points $\pi/3$ and $5\pi/3$. When we compare the analytical structure of $\langle L\rangle$ and $(1/2c)\ln Z$ as a function of θ (Figs. 8 and 9), we find that the change at π for the effective potential does not appear in $\langle L\rangle$. This fact relates to the symmetry of $\langle L\rangle$ around π, for which $\langle L(\pi+\alpha)\rangle$ must be the same as $\langle L(\pi-\alpha)\rangle$. Thus a corresponding abrupt change in $\langle L\rangle$ at π is not possible. Furthermore, if we compare our results here for SU(3) with those earlier discussed for SU(2), we find that SU(2) does not have this difference between $\langle L\rangle$ and $(1/2c)\ln Z(\theta)$. Therefore the structure of $\langle L\rangle$ in SU(2) is a direct reflection of the symmetry in the free energy.

VI. SU(2) SPIN MODEL

In the last few sections we have discussed the properties of the quark-gluon plasma in terms of the model which contained the exact conservation of the global color and the baryon number. The above model can be considered as a good approximation for the description of the thermodynamical of the quark-gluon plasma only in

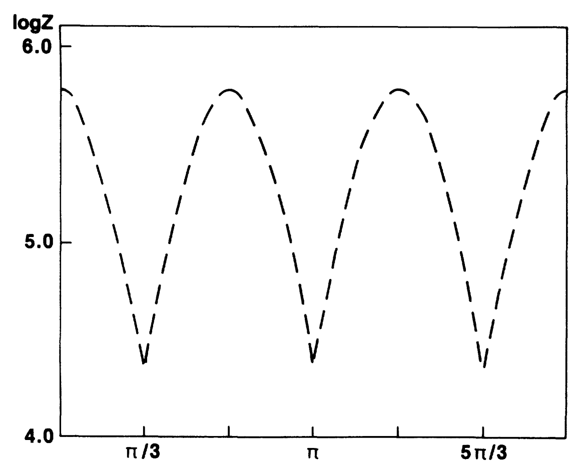

FIG. 8. The dependence of the free energy $\ln Z(\theta)/2c$ for the SU(3)-gas model on the complex chemical potential $\theta=-i\mu\beta$ for $c=15$.

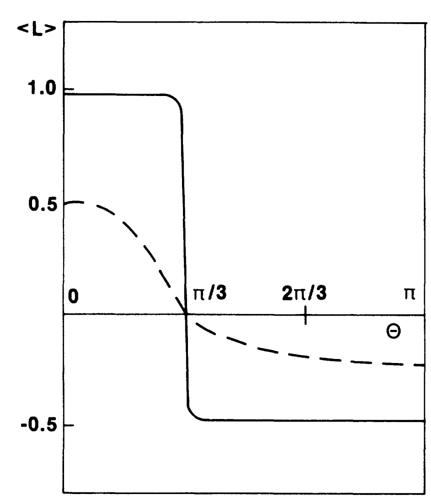

FIG. 9. Thermal average of the Wilson loop $\langle L\rangle$ as a function of θ for the SU(3)-gas model with the given values of c; $c=2.5$ (dashed curve), and $c=15$ (solid curve).

the region where the QCD coupling constant is assumed to be very small. This is the region where the deviation from the ideal-gas behavior is not very significant. It would be possible to develop this model further by taking higher perturbative corrections to the thermodynamical potential. However, even in this case one could not study the behavior of the system in the region where one expects a phase transition or where quarks and gluons are premanently confined. In order to investigate the properties of the quark-gluon plasma in the above physically interesting region, we shall consider the effective three-dimensional SU(N_c)-spin model.[7] This model is of particular interest since it describes the strong-coupling and large-quark-mass limit of lattice QCD at finite temperature and baryon-number density. The thermodynamical properties of the SU(N_c)-spin model in relation to the phase structure are already well established. As it turns out from the previous analyses, this model exhibits the deconfinement phase transition and is qualitatively in agreement with the Monte Carlo evaluations of both SU(2) and SU(3) gauge theories.[7]

In all our further considerations in this section we shall use the SU(N_c)-spin model in order to describe the behavior of the Wilson loop as well as the free energy in relation to the complex chemical potential. Furthermore, we shall discuss the possible qualitative differences between the canonical and grand canonical treatments of the baryon-number conservation.

The thermodynamics of the SU(N_c)-spin model is given by the following grand partition function:[7]

$$Z(\mu,N_\beta,N_\sigma)=\int \prod_x dL_x e^{-S}\ , \tag{6.1}$$

where N_β and N_σ represent the number of Euclidean lattice sites in the temporal (thermal) and spatial direction which replace the thermodynamical variables $T=N_\beta a$ and $V=N_\sigma^3\times a^3$ in the continuum. The action S is given by

$$\begin{aligned}S=&-\beta'\sum_{x,l}(\mathrm{Tr}L_x\mathrm{Tr}L_{x+l}^\dagger+\text{c.c.})\\&-h\sum_x(e^{\mu N_\beta a}\mathrm{Tr}L_x+\bar{e}^{\mu N_\beta a}\mathrm{Tr}L_x^\dagger)\ ,\end{aligned} \tag{6.2}$$

where L_x indicates the Wilson line variable

$$L_x=\sum_{x_4=0}^{N_\beta-1} U_{(x,x_4),x_4}\ . \tag{6.3}$$

a is the lattice spacing and $U_x\in\mathrm{SU}(N_c)$. The parameters h and β' are related to the hopping parameter κ and the coupling constant g^2 in the usual way:

$$h=-4N_f(2\kappa)^{N_\beta}\ , \tag{6.4a}$$

$$\beta'\simeq\left[\frac{2N_c}{g^2}\right]^{N_\beta}\ , \tag{6.4b}$$

where N_f is the number of flavors and N_c the number of colors.

The partition function (6.1) describes the thermodynamics of this model with baryon-number conservation formulated in the grand canonical ensemble. As we have already indicated, there is an alternative way in which we can ensure the baryon-number conservation in the system. This alternative amounts to the assumption that the absolute value instead of the average value of the baryon number is conserved. In this case the canonical partition function respecting the baryon-number conservation can be found[6] using the prescription given in Eq. (2.7). From the Eqs. (2.5) and (2.6) the generating function can be obtained as

$$\tilde{Z}(\theta,\beta,V)=\int\prod_x dM_x e^{-S(\theta)}\ , \tag{6.5}$$

where the action is

$$\begin{aligned}S(\theta)=&-\beta'\sum_{x,l}(\chi_x^{(1,0)}\chi_{x+l}^{(0,1)}+\text{c.c.})\\&-h\sum_x(\cos\theta\,\mathrm{Re}\chi_x^{(1,0)}+\sin\theta\,\mathrm{Im}\chi_x^{(1,0)})\ .\end{aligned} \tag{6.6}$$

$\chi^{(1,0)}$ indicates the character of the fundamental representation of the SU(N_c) group and dM_x is the invariant Haar measure on this group. The U(1) group parameter plays the role of the complex chemical potential. In order to derive (6.5) we have chosen a gauge where $U_{(x,x_4)}$ is fixed in the time direction so that it is time independent. In this case, as one can see from (6.6) the action $S(\theta)$ is a function of the group characters only and not on the entire group. Also the Wilson loop $\mathrm{Tr}L_x$ is equivalent to the character of the fundamental representation.

From the generating function $\tilde{Z}(\theta,\beta,V)$ in (6.5) one can find the canonical partition function after carrying out the integration over the U(1) group. The final expression for it takes the form[6]

$$Z(N,N_\beta,N_\sigma,h)=\int\prod_x dM_x\exp\left[\beta'\sum_{x,l}\chi_x^{(1,0)}\chi_{x+l}^{(0,1)}+\text{c.c.}\right]I_N(hy)T_N\left[\sum_x \mathrm{Re}\chi^{(1,0)}/y\right]\ , \tag{6.7}$$

where $y=[(\mathrm{Re}\chi^{(1,0)})^2+(\mathrm{Im}\chi^{(1,0)})^2]^{1/2}$, $I_N(x)$ is the modified Bessel function, and $T_N(x)$ is the Chebyschev polynomial. Thus with the canonical partition function (6.7) together with the generating function (6.5) one can study the properties of this model in both the canonical and the ground canonical ensembles with a complex chemical potential.

In our previous consideration[6] we have mainly concentrated on the differences between the canonical and grand canonical partition functions in the thermodynamical limit where the baryon number B and the volume N_σ^3 are large but the baryon-number density B/N_σ^3 is kept fixed. Furthermore, some aspects of the realization of the Z_N symmetry in these two ensembles have been discussed up to this time.[5,6]

Let us first discuss the properties of the above model assuming a purely imaginary chemical potential. Actually we shall consider only the case of the SU(2)-spin model. For the SU(2) group the generating function has the following special form:

$$\tilde{Z}(\theta,N_\beta,N_\sigma,h)=\frac{2}{\pi}\int_{-1}^{1}\prod_x (1-L_x^2)^{1/2}dL_x\exp\left[\beta'\sum_{x,l}L_xL_{x+l}+h\sum_x\cos\theta\,\mathrm{Re}L_x\right], \tag{6.8}$$

where now $L_x=\chi_x^{(1,0)}$. In order to obtain (6.8) we have used the known expressions for the Haar measure and fundamental character for the SU(2) group.

The above SU(2)-spin model with a complex chemical potential can be analyzed by using the mean-field (MF) method. One could proceed as for the Heisenberg model where the interactions with the nearest neighbors are approximated by their mean values. Then the evaluation of the partition function from the generating function (6.8) is reduced to a single-site problem. However, in order to make use of the canonical ensemble, which will be considered later, we shall apply another somewhat simpler MF method for the evaluation of the free energy. This method is summarized by the assumption that the distribution of the SU(2) variables L_x, which are analogous to the spins, is uniform and uncorrelated from site to site.[15] In this case the MF free energy of the SU(2) model can be evaluated using the steepest-descent method. We obtain the following equation:

$$-\tilde{F}_{\mathrm{MF}}/N_\sigma^3\simeq 3\beta' M^2+\tfrac{1}{2}\ln(1-M^2)+h(\cos\theta)M\ , \tag{6.9}$$

where M is the mean-field value of the Wilson loop. It can be found as a solution of $\partial\tilde{F}_{\mathrm{MF}}/\partial M=0$.

This very simple model contains all the features which can be expected from the SU(2)-gauge theory. For the complex chemical potential with θ taken at $\pi/2$ (modπ) the generating function (6.8) is equivalent to the pure SU(2)-spin model which exhibits a deconfinement phase transition of the second order. On the other hand, at $\theta=0$ (modπ) the free energy (6.9) describes the thermodynamics of the SU(2)-spin system with an external magnetic field. This external magnetic field h implies that the system is always in the Z_N broken phase. Thus the Wilson loop for any value of β' is nonzero. However, unless the external field is very large, at some value of β' there can be observed an abrupt change in the energy density and other thermodynamical quantities.[7,15]

As we have already seen for the two prescribed values of the complex chemical potential $\theta=\pi/2$ (modπ) and $\theta=0$ (modπ) the model exhibits qualitatively different behavior with respect to the realization of Z_2 symmetry. Thus one could expect that the SU(2) as well as in general the SU(N_c)-spin model with a complex chemical potential exhibits some interesting structure.

Now we consider the behavior of the Wilson loop in a manner done previously[5] in the asymptotic region with $g^2\to\infty$ or $\beta'\to 0$. From Eq. (6.9) one can easily establish that for not too large value of the quark parameter h, the Wilson loop behaves as $h\cos\theta$. Thus it is a smooth function of the complex chemical potential. On the other hand, as we have seen in the previous section and as it had been earlier indicated,[5] in the limit of large temperature the Wilson loop exhibits a discontinuous structure in θ. In terms of the SU(2) model given by the free energy (6.9) it is also possible to study the intermediate region between very large and small β'.

The dependence of the Wilson loop on the complex chemical potential for different values of β' is shown in Fig. 10. As one can see for large values of β' (above 0.3) with θ in the range $0\le\theta<\pi/2$, the Wilson loop depends very weakly upon θ. However, for θ at $\pi/2$ (modπ) there is a discontinuous change to the negative Z_2-symmetric value. With decreasing β' the dependence of the Wilson loop on θ in the region $0\le\theta<\pi/2$ is more sensitive. Also the discontinuity at $\pi/2$ decreases. When the parameter β' is lower than some critical value β'_c, then the discontinuous structure at $\pi/2$ disappears and the behavior of the Wilson loop resembles $\cos\theta$. This critical parameter β'_c can be easily established to correspond to the value where the phase transition takes place in the pure SU(2)-spin model with no external field contributions. In order to further investigate the phase structure of the SU(2)-spin model, we have also shown in Fig. 11 the properties of the free energy for some fixed value of the complex chemical potential and coupling constant. At the values of $\theta=\pi/2$ and $3\pi/2$ and for large β' above β'_c the effective potential has two stable absolute minima. Thus it exhibits a structure typical for a system with a

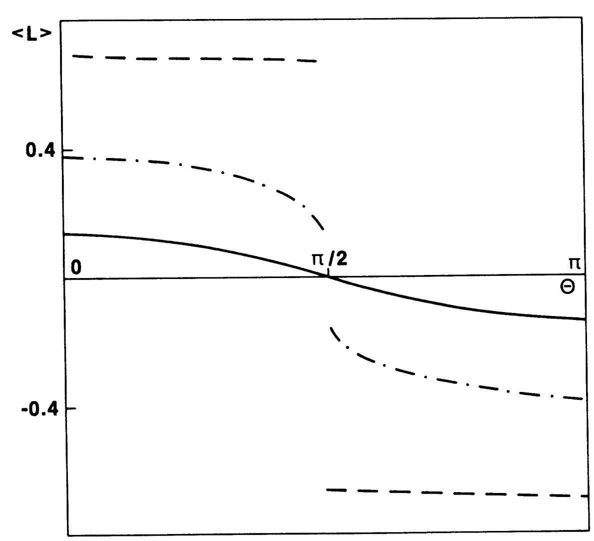

FIG. 10. Mean-field value of the Wilson loop as a function of the complex chemical potential θ for different values of the coupling constant β' in the SU(2)-spin model; $\beta'=0.1$ (solid curve), $\beta'=0.17$ (dashed-dotted curve), and $\beta'=0.3$ (dashed curve).

phase transition. At $\theta=\pi/2$ and $3\pi/2$ for low values of β' below β'_c there is only one stable minimum at $L=0$. Thus there is no phase transition. For values of θ not at $\pi/2$ and $3\pi/2$ there is only one stable minimum in the effective potential which indicates that the system stays in a single phase.

The phase structure which can be seen in Figs. 10 and 11 for the above model with the complex chemical potential θ indicates that even in the theory with the fermions, there can be observed some remnants of the Z_2 symmetry. In the deconfined phase one can always observe the discontinuous structure in the Wilson loop as a function of θ. In contrast, in the confined phase the Wilson loop is a smooth function of θ. Thus from the above assertions one can conclude that by means of analysis of the behavior of the Wilson loop on the complex chemical potential one can distinguish whether a system stays in the confined or the deconfined phase.

It would be very interesting to study the above properties of this model by means of the more realistic Monte Carlo simulation of the SU(2) or SU(3) gauge models. However, since, as we have in the above simple SU(2)-spin model, in the region of β' where one would expect a drastic change in the degrees of freedom in the system, the discontinuity of the Wilson loop with respect to θ is rather small. Thus by taking into account a typical error bar for a Monte Carlo computation one could well doubt that if one were to look at the dependence of the Wilson loop on the complex chemical potential, one would be able to distinguish whether the system would stay in the confined or the deconfined phase. These difficulties will be particularly pronounced in the region where the deconfinement transition can take place.

At this point we would like to relate the results of this section with the ones obtained previously in terms of the

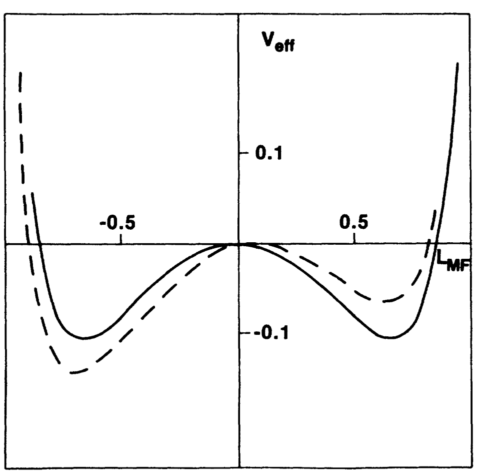

FIG. 11. The form of the effective potential $V_{\rm eff}$ in SU(2)-spin model as a function of the mean-field Wilson loop $L_{\rm MF}$ with $\beta'=0.3$ for different values of the complex chemical potential θ; $\theta=\pi/2$ (solid curve) and $\theta=\pi$ (dashed curve).

$\mathrm{SU}_c(2)$ and $\mathrm{SU}_c(3)$ character models with a complex chemical potential. These character models because of their own construction do not exhibit any phase transition. Quite independently of the values of the parameters, these models can only describe the properties of the QG plasma in the deconfined phase. However, as we have seen in Fig. 4 the structure of the effective potential and the Wilson loop with respect to the complex chemical potential θ were substantially different when the parameter c was changed from large to small values. In the region where c was small (around unity) the effective potential and the Wilson loop were found to be smooth functions of θ. This, however, in the $\mathrm{SU}(N_c)$ character model does not mean that the system is really in the confined phase. Thus as we have seen in the character model the continuous structure of the Wilson loop with respect to θ does not necessarily mean that the system has changed phase. The continuous structure of the Wilson loop with respect to θ can also be obtained in the deconfined phase when the finite-volume effects are significant.

In the MF analysis of the SU(2)-spin model with either real or complex chemical potential the results show that the thermodynamical quantities are independent of the volume. The MF approximation which we have applied is valid for asymptotically large values of N_σ. Thus in the GC treatment of the SU(2)-spin model together with the MF approximation the results are free from any corrections due to the finite-volume effects. However, in terms of the C formulation of the baryon-number conservation of the SU(2)-spin model in the MF approximation one can obtain some ideas concerning the dependence of some thermodynamical quantities on N_σ^3 in the different regions of the coupling constant β'.

In the C formulation of the baryon-number conservation we can derive the free energy of the system based on (6.7). After application of the MF approximation as we have previously used in order to derive the free energy (6.9), we arrive at the form

$$-F/n_\sigma^3=3\beta' L_N^2+N_\sigma^{-3}\ln I_N(N_\sigma^3 h L_N)+\tfrac{1}{2}\ln(1-L_N^2)\ . \tag{6.10}$$

Using (6.10) the Wilson loop L_N with the prescribed value of the quark number N can be found as the solution of the equation

$$6\beta' L_N-\frac{L_N}{1-L_N^2}+h\left[\frac{\partial I_N(x)}{\partial x}\right]\Big/ I_N(x)=0\ , \tag{6.11}$$

at the value $x=N_\sigma^3 h L_N$. In the limit of large N and N_σ with fixed N/N_σ^3 the thermodynamical properties of the model described by the free energy (6.10) are equivalent to the results obtained in the GC ensemble with finite-baryon number density.[6] On the other hand, in the limit of $N_\sigma\to\infty$ with fixed N the free energy (6.10) has the same analytical structure as in the GC ensemble with vanishing chemical potential ($\mu=0$).

Since both F and L_N in the C ensemble given by (6.10) and (6.11) depend explicitly upon N_σ^3 and β', one can directly examine the functional dependence of L_N on

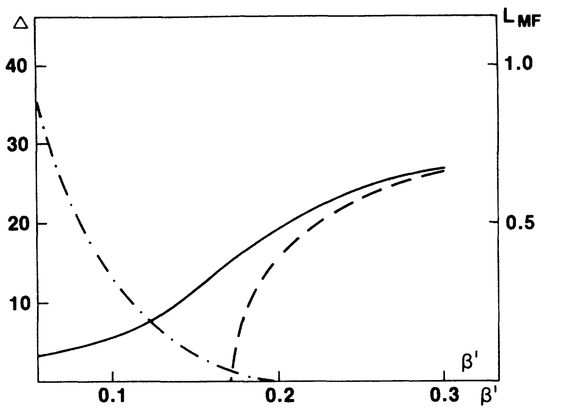

FIG. 12. The dependence of the mean-field Wilson loop L_{MF} on β' obtained for the SU(2)-spin model in the following cases: L_{MF} for the pure-gluon system (dashed curve); L_{MF} for the SU(2)-spin model with the quark parameter $h=0.06$ (solid curve); the relative error $\Delta=(L_{\mu=0}-L_{N=0})/L_{\mu=0}$ in percent between the mean-field value of the Wilson loop obtained in the GC ensemble with $\mu=0$ and that in the C ensemble with $N=0$ (dashed-dotted curve).

these parameters. With a fixed value of N_σ^3 we compare the result for L_N with $N=0$ with its asymptotic value for $N_\sigma \to \infty$ given by $L_{\mu=0}$. Thus we can see how the effect of N_σ develops with the coupling constant β'. In order to illustrate how big this difference can be, we have shown in Fig. 12 the dependence of the Wilson loop $L_{\mu=0}$ on β' as well as the relative error between $L_{\mu=0}$ and $L_{N=0}$. The behavior of $L_{\mu=0}$ in the pure SU(2) spin model ($h=0$) is also indicated in Fig. 12. As one can see even for the relatively large value of $N_\sigma^3=8^3$, the difference between $L_{N=0}$ and $L_{\mu=0}$ is not negligible. This is especially true in the region where the pure SU(2)-spin model is in the Z_2 symmetric confined phase. For β' larger than $\beta_c'=\frac{1}{6}$ the effects due to finite values of N_σ start to become unimportant.

VII. SUMMARY AND CONCLUSIONS

In this paper we have examined a class of models which can give some insight into the structure and properties of strongly interacting matter at finite temperature and density.

In the high-temperature region we have used the gas models with the $SU(N_c)\times U(1)$ internal-symmetry group as an approximation to the QG plasma thermodynamics. The constraints imposed on these models by color and baryon-number conservation imply a nonperturbative finite-volume correction to the ideal-gas approximation of the QG plasma. The natural parameter c for the above models arises as a product of temperature and volume in the form $2VT^3/\pi^2$.

When the formation of the QG plasma droplet takes place in heavy-ion collisions, the physically interesting values of c are approximately between 1 and 10. On the other hand, in LGT, $VT^3=(N_\sigma/N_\beta)^3$ remains constant for a given lattice. For a typical lattice of size $8^3\times 4$ the parameter c is of the order of one. Thus the region where c is larger than one is interesting for both phenomenological and LGT analyses of the high-temperature behavior of the QG plasma.

In terms of the above gas models we have examined the dependence of the thermal average of the Wilson loop $\langle L \rangle$ on c. In the region of small c at about one there can be observed a deviation of $\langle L \rangle$ of the order of 10% from its asymptotic value. With increasing values of c the asymptotic behavior is achieved rather quickly. For c about ten the deviation is a couple of percent. These observations are particularly true for our model with SU(2)-color symmetry. In the case of the SU(3)-color group the finite-volume effect is found to be larger than for SU(2). In this same context we have also noted the qualitative differences between $\langle L \rangle$ and $\langle |L| \rangle$ in the above models.

We have also investigated the thermodynamical quantities such as the energy density and entropy. The quantitative structure of these functions is quite similar to $\langle L \rangle$ at large c. However, at small c the effects of the classical statistics are much more extreme for these quantities than for $\langle L \rangle$ and $\langle LL^\dagger \rangle$. Particularly important is that the finite-size effects are also quite different in these two cases. At the value of $c \simeq 1$ and with the baryon number $B=0$ a deviation of the energy density and entropy from their asymptotic values can be even of the order of 30%. For SU(2) we saw that the entropy was slower at achieving its asymptotic value than was the energy density. Furthermore, if we compare the energy density in SU(2) with that in SU(3) we see the general tendency that the SU(3) asymptotic value is more slowly attained because of the apparent greater role of the constraints in SU(3).

These results for the finite-volume effects could be useful for the Monte Carlo computations in LGT. One would also expect there, that the finite-volume effects should be different for the various thermodynamical quantities. Furthermore, one expects for LGT an increase of the finite-volume effect in going from SU(2) to SU(3) gauge groups.

The complex chemical potential θ has been introduced to study the phase structure of our models including the quarks. We have seen significant changes in the structure of the effective potential and the Wilson loop as a function of θ. We also noted that this shape was dependent upon the parameter c. The fact that at large c the change at specific points was very sharp in the character model was not a proper indicator of a phase change. In the character model the dynamics is insufficient for a phase transition. However, we could note again the importance of finite-size effects.

Finally we considered the SU(2)-spin model in a mean-field approximation as an example of a system with a finite-temperature phase transition. We observed jumps in $\langle L \rangle$ as a function of θ at the same points $\pi/2$ and $3\pi/2$ as for the SU(2) character model. However, these jumps disappear as the transition point β_c' for the pure-gluon system is approached. In the region β' where the pure SU(2)-spin model confines, the free energy and the Wilson loop exhibit a continuous behavior in θ. Thus looking at the functional dependence of the Wilson loop on θ one can distinguish whether the system is in the confined or deconfined phase even when the quarks are present. However, as our character model has indicated, also in the high-temperature region where the QG plasma is in the deconfined phase, one can find the continuous behavior of $\langle L \rangle$ on θ when the finite-volume effects are important. Also we have noted the differences between the C and GC treatments of this model. It could be possible to make an extension of this treatment to $SU_c(3)$ symmetry using the same analysis.

In our considerations of the gas models the dynamics was determined by the nonperturbative constraints coming from the conservation laws. However, there do exist other nonperturbative effects which have not been discussed here. Important natural effects come also from the fact that the momentum spectrum is discrete at finite volume. All of the nonpertrubative effects should be contained in the careful analysis of the polarization tensor in finite-temperature QCD. However, we still believe that the analysis presented here can lead to further insight toward an accurate description of the properties of strongly interacting matter.

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