Scattering and decay theory for quaternionic quantum mechanics, and the structure of induced T nonconservation

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We develop a scattering theory and decay theory for nonrelativistic quaternionic quantum mechanics. We show that the far-zone part of scattering states lies in the complex $\mathbb{C}(1,i)$ subspace of quaternionic Hilbert space picked out by the kinetic part of the Hamiltonian; intrinsically quaternionic terms are present in the wave function but have exponential spatial decay. Hence, scattering phase shifts are necessarily complex in quaternion quantum mechanics, and the test for quaternionic effects suggested by Peres gives a null result. Integrating out the quaternionic components, the complex scattering problem can be expressed in terms of an optical potential $V_{\rm opt}$ which is Hermitian but time-reversal nonconserving. The corresponding decay problem for $\mathbb{C}(1,i)$ initial states can be expressed in terms of the same optical potential. Solving the decay problem to order $V_{\rm opt}^2$, the phenomenological form of the induced T nonconservation is seen to be "milliweak," with T nonconservation arising both from the mass and the decay matrices, and hence is compatible with the phenomenology of T nonconservation in the standard model. When the complex $\mathbb{C}(1,i)$ part of the quaternionic Hamiltonian has bound states, the optical potential develops isolated pole singularities. These lead to resonances at scattering energies equal to the bound-state binding energies, with widths proportional to the square of the quaternionic part of the potential. Inclusion of a positive-rest-mass term in the Hamiltonian shifts the location of these resonances towards lower kinetic energy, and if large enough eliminates them altogether.

I. INTRODUCTION

Our aim in this paper is to give a parallel treatment of the closely related topics of scattering theory and decay theory in nonrelativistic quaternionic quantum mechanics. In the course of doing so we will extend and correct results we obtained in an earlier discussion[1] of quaternionic time-dependent perturbation theory and quaternion-induced time-reversal (T) nonconservation;[1,2] a certain amount of the introductory material of Ref. 1 is repeated here so as to make the present treatment self-contained. Quaternionic quantum mechanics is characterized by a quaternion-valued wave function Ψ,

$$\Psi=\Psi_0+i\Psi_1+j\Psi_2+k\Psi_3 , \tag{1a}$$

with $\Psi_{0,1,2,3}$ real and with i,j,k the quaternion units satisfying

$$i^2=j^2=k^2=-1 ,\qquad ij=-ji=k . \tag{1b}$$

The inner product is

$$(\Psi,\Phi)=\int\overline{\Psi}\Phi \tag{2}$$

and the inner-product-preserving (unitary) dynamics is

$$\frac{\partial\Psi}{\partial t}=-\tilde{H}\Psi , \tag{3}$$

with $\overline{\Psi}=\Psi_0-i\Psi_1-j\Psi_2-k\Psi_3$ the conjugate of Ψ, and with $\tilde{H}=-\tilde{H}^{\dagger}$ a quaternion–anti-self-adjoint Hamiltonian. In discussing decays and time-dependent perturbation theory, we assume $\tilde{H}$ to be the sum of an unperturbed Hamiltonian $\tilde{H}_0$ and a time-independent perturbation $\tilde{V}$, and we invoke the spectral theorem[3] for quaternion–anti-self-adjoint operators to write $\tilde{H}_0$ in the form

$$\tilde{H}_0=I_0H_0 ,\qquad I_0=\sum_n |n\rangle i\langle n| ,\qquad H_0=\sum_n |n\rangle E_n\langle n| ,\quad E_n=\overline{E}_n\geq 0 , \tag{4}$$

with $|n\rangle$ a complete set of eigenstates of H_0. We take the perturbation $\tilde{V}$ to have the form

$$\tilde{V}=\tilde{V}_0+I_0\tilde{V}_1+J_0\tilde{V}_2+K_0\tilde{V}_3 ,\qquad \begin{Bmatrix} J_0 \\ K_0 \end{Bmatrix}=\sum_n |n\rangle \begin{Bmatrix} j \\ k \end{Bmatrix} \langle n| , \tag{5}$$

with $\tilde{V}_{0,\ldots,3}$ commuting with I_0,J_0,K_0. As long as H_0 is T conserving, we may assume that both the zeroth-order energy eigenkets $|n\rangle$ and the position eigenkets $|x\rangle$ are real (since the wave functions $\langle x|n\rangle$ can be chosen real) and so we have

$$\begin{Bmatrix} I_0 \\ J_0 \\ K_0 \end{Bmatrix} = \sum_n |n\rangle\langle n| \begin{Bmatrix} i \\ j \\ k \end{Bmatrix} = \begin{Bmatrix} i \\ j \\ k \end{Bmatrix} ,$$
$$\begin{Bmatrix} I_0 \\ J_0 \\ K_0 \end{Bmatrix} |x\rangle = |x\rangle \begin{Bmatrix} i \\ j \\ k \end{Bmatrix} . \quad (6)$$

Hence in what follows we can neglect the formal distinction between the operators I_0, J_0, K_0 and the quaternion units i,j,k, giving

$$\tilde{H} = iH_0 + \tilde{V} ,$$
$$\tilde{V} = \tilde{V}_0 + i\tilde{V}_1 + j\tilde{V}_2 + k\tilde{V}_3 , \quad (7)$$

with $\tilde{V}_{0,\ldots,3}$ real. Introducing complex $\mathbb{C}(1,i)$ symplectic components $\tilde{V}_\alpha, \tilde{V}_\beta$ defined by

$$\tilde{V}_a = \tilde{V}_0 + i\tilde{V}_1, \quad \tilde{V}_\beta = \tilde{V}_2 - i\tilde{V}_3 , \quad (8)$$

we can rewrite $\tilde{V}$ as

$$\tilde{V} = \tilde{V}_\alpha + j\tilde{V}_\beta . \quad (9)$$

For the matrix elements of $\tilde{V}$ we correspondingly have

$$\langle n | \tilde{V} | l \rangle = \tilde{V}_{0nl} + i\tilde{V}_{1nl} + j\tilde{V}_{2nl} + k\tilde{V}_{3nl}$$
$$= \tilde{V}_{\alpha nl} + j\tilde{V}_{\beta nl} , \quad (10)$$
$$\tilde{V}_{\alpha nl} = \tilde{V}_{0nl} + i\tilde{V}_{1nl}, \quad \tilde{V}_{\beta nl} = \tilde{V}_{2nl} - i\tilde{V}_{3nl} .$$

In this notation the anti-Hermiticity condition on $\tilde{V}$ becomes

$$\tilde{V}^*_{\alpha nl} = -\tilde{V}_{\alpha ln}, \quad \tilde{V}_{\beta nl} = \tilde{V}_{\beta ln} , \quad (11)$$

where an asterisk denotes complex conjugation with respect to $\mathbb{C}(1,i)$, in other words, in the $|n\rangle$ basis [or in any other basis related to it by a complex $\mathbb{C}(1,i)$ unitary transformation] $\tilde{V}_\alpha$ is $\mathbb{C}(1,i)$ anti-Hermitian and $\tilde{V}_\beta$ is $\mathbb{C}(1,i)$ symmetric. In discussing scattering theory, we specialize Eq. (7) to the case

$$H_0 = \frac{\mathbf{p}^2}{2m} = \frac{-\nabla_x^2}{2m} ,$$
$$\tilde{V}_\alpha = i\tilde{V}_1(x), \quad \tilde{V}_\beta = \tilde{V}_\beta(x) , \quad (12)$$

thus identifying H_0 with the kinetic term in the Hamiltonian and $\tilde{V}$ with a quaternion-valued local scattering potential.

II. SCATTERING THEORY: TIME-INDEPENDENT FORMALISM

We begin by discussing quaternionic scattering theory using the time-independent formalism. To set this up, we start from the spectral representation of the full Hamiltonian $\tilde{H}$,

$$\tilde{H} = \sum_E |E\rangle E i_E \langle E| , \quad E = \bar{E} \geq 0 , \quad (13)$$

with the sum extending over all eigenstates of $\tilde{H}$, and with i_E a general unit quaternion. Thus a given eigenket $\Psi = |E\rangle$ obeys

$$\tilde{H}\Psi = \Psi i_E E , \quad (14)$$

and by Eq. (3) has the exponential time dependence

$$\Psi(t) = \Psi(0) e^{-i_E E t} . \quad (15)$$

When Eqs. (14) and (15) are multiplied by a general unit quaternion ω from the right they become

$$\tilde{H}\Psi\omega = \Psi\omega E \bar{\omega} i_E \omega ,$$
$$\Psi(t)\omega = \Psi(0)\omega e^{-\bar{\omega} i_E \omega E t} . \quad (16)$$

Since for a suitable ω we have $\bar{\omega} i_E \omega = i$, there is no loss of generality in assuming $i_E = i$ in Eqs. (14) and (15) (this corresponds to choosing a particular ray-representative for our state), giving

$$\tilde{H}\Psi = \Psi i E ,$$
$$\Psi(t) = \Psi(0) e^{-iEt} , \quad (17)$$

which is the starting point for our scattering analysis.

Before doing the general case, let us consider the one-dimensional problem of scattering by a δ-function spike. Thus we have

$$\tilde{H} = i\left[\frac{-d^2}{dx^2}\frac{1}{2m}\right] + \delta(x)(\tilde{V}_\alpha + j\tilde{V}_\beta) , \quad (18)$$

with $\tilde{V}_{\alpha,\beta}$ complex constants. Writing $\Psi = \Psi_\alpha + j\Psi_\beta$ and substituting Eq. (18) into Eq. (17) gives the pair of coupled complex equations for the symplectic components $\Psi_{\alpha,\beta}$:

$$-\frac{i}{2m}\frac{d^2}{dx^2}\Psi_\alpha(x) + \delta(x)[\tilde{V}_\alpha\Psi_\alpha(0) - \tilde{V}^*_\beta\Psi_\beta(0)] = iE\Psi_\alpha(x),$$
$$\frac{i}{2m}\frac{d^2}{dx^2}\Psi_\beta(x) + \delta(x)[\tilde{V}_\beta\Psi_\alpha(0) + \tilde{V}^*_\alpha\Psi_\beta(0)] = iE\Psi_\beta(x) . \quad (19)$$

When $|x| \neq 0$, these simplify to

$$-\frac{i}{2m}\frac{d^2}{dx^2}\Psi_\alpha(x) = iE\Psi_\alpha(x) ,$$
$$\frac{i}{2m}\frac{d^2}{dx^2}\Psi_\beta(x) = iE\Psi_\beta(x) , \quad (20)$$

which have the general solutions

$$\Psi_\alpha(x) = C_{\alpha+} e^{ipx} + C_{\alpha-} e^{-ipx} ,$$
$$\Psi_\beta(x) = C_{\beta+} e^{px} + C_{\beta-} e^{-px} , \quad (21)$$
$$p = (2mE)^{1/2} .$$

We see from Eq. (21) the very interesting fact that the intrinsically quaternionic (Ψ_β) part of the wave function has no running wave solutions. Imposing finiteness of the wave function at $x = \pm\infty$, and assuming an incoming wave e^{ipx} incident from the left, we have, for the wave function in the regions $x < 0$ and $x > 0$,

$$\left.\begin{aligned}\Psi_\alpha(x)&=e^{ipx}+Re^{-ipx},\\ \Psi_\beta(x)&=Se^{px},\end{aligned}\right\}\quad x<0\;, \tag{22}$$

$$\left.\begin{aligned}\Psi_\alpha(x)&=Te^{ipx},\\ \Psi_\beta(x)&=S'e^{-px},\end{aligned}\right\}\quad x>0\;, \tag{23}$$

with R,S,S',T complex constants. Continuity of $\Psi_{\alpha,\beta}(x)$ at $x=0$ gives the two conditions

$$\begin{aligned}1+R&=T\;,\\ S'&=S\;,\end{aligned} \tag{24}$$

while integrating Eq. (19) across $x=0$ gives the two jump conditions

$$\begin{aligned}&-\frac{i}{2m}\frac{d\Psi_\alpha}{dx}\bigg|_{0^-}^{0^+}+\tilde{V}_\alpha\Psi_\alpha(0)-\tilde{V}_\beta^*\Psi_\beta(0)=0\;,\\ &\frac{i}{2m}\frac{d\Psi_\beta}{dx}\bigg|_{0^-}^{0^+}+\tilde{V}_\beta\Psi_\alpha(0)+\tilde{V}_\alpha^*\Psi_\beta(0)=0\;,\end{aligned} \tag{25}$$

which in terms of R,S become

$$\begin{aligned}&-\frac{i}{2m}[ip(1+R)-ip(1-R)]+\tilde{V}_\alpha(1+R)-\tilde{V}_\beta^*S=0\;,\\ &\frac{i}{2m}(-pS-pS)+\tilde{V}_\beta(1+R)+\tilde{V}_\alpha^*S=0\;.\end{aligned} \tag{26}$$

Solving Eqs. (26) for R,S and using Eqs. (8) and (12), we get

$$\begin{aligned}R&=-\frac{iC}{\frac{p}{m}+iC}\;,\\ T&=1+R=\frac{\frac{p}{m}}{\frac{p}{m}+iC}\;,\\ S&=-\frac{i\tilde{V}_\beta}{D}T\;,\\ D&\equiv\frac{p}{m}+\tilde{V}_1,\quad C\equiv\tilde{V}_1+(\tilde{V}_2^2+\tilde{V}_3^2)/D\;.\end{aligned} \tag{27}$$

We see that the intrinsically quaternionic part of the wave function is confined to an exponentially decaying near-zone piece, while the outgoing reflected and transmitted waves are contained in the $\mathbb{C}(1,i)$ part of the wave function, with coefficients R and T which exhaust the unitarity sum rule

$$1\,|R|^2+|T|^2\;. \tag{28}$$

These features of the one-dimensional model will be seen to carry over to the general three-dimensional problem.

Turning to the three-dimensional problem, we have

$$\tilde{H}=i\left[-\frac{\nabla_x^2}{2m}\right]+\tilde{V}_\alpha(x)+j\tilde{V}_\beta(x)\;, \tag{29}$$

which when substituted with $\Psi=\Psi_\alpha+j\Psi_\beta$ into Eq. (17) gives the coupled complex equations

$$\begin{aligned}&i\left[-\frac{\nabla_x^2}{2m}\right]\Psi_\alpha+\tilde{V}_\alpha\Psi_\alpha-\tilde{V}_\beta^*\Psi_\beta=iE\Psi_\alpha\;,\\ &-i\left[-\frac{\nabla_x^2}{2m}\right]\Psi_\beta+\tilde{V}_\beta\Psi_\alpha+\tilde{V}_\alpha^*\Psi_\beta=iE\Psi_\beta\;.\end{aligned} \tag{30}$$

We assume the potentials to be of bounded support; in the region where the potentials are zero Eq. (30) becomes

$$\begin{aligned}&(\nabla_x^2+p^2)\Psi_\alpha=0\;,\\ &(\nabla_x^2-p^2)\Psi_\beta=0\;,\\ &p=(2mE)^{1/2}\;,\end{aligned} \tag{31}$$

and we again see that the Ψ_β part has no propagating wave solutions, but instead decays exponentially at spatial infinity. Thus the asymptotic scattering states for the Hamiltonian of Eq. (29) lie entirely in the complex $\mathbb{C}(1,i)$ part of the wave function. This being the case, it is useful to completely eliminate Ψ_β from the problem by the methods of formal scattering theory. Writing $H_0=-\nabla_x^2/2m$, the second equation in Eq. (30) may be solved for Ψ_β to give

$$\Psi_\beta=-i(H_0+E+i\tilde{V}_\alpha^*)^{-1}\tilde{V}_\beta\Psi_\alpha\;, \tag{32}$$

which when substituted into the equation for Ψ_α yields

$$[H_0+V_{\rm opt}(E)]\Psi_\alpha=E\Psi_\alpha\;, \tag{33}$$

with the optical potential $V_{\rm opt}(E)$ defined by

$$V_{\rm opt}(E)=-i\tilde{V}_\alpha+\tilde{V}_\beta^*\frac{1}{H_0+E+i\tilde{V}_\alpha^*}\tilde{V}_\beta\;. \tag{34}$$

Since $\tilde{V}_\alpha=-\tilde{V}_\alpha^\dagger$ and $\tilde{V}_\beta=\tilde{V}_\beta^T$, the optical potential is Hermitian:

$$V_{\rm opt}(E)=V_{\rm opt}(E)^\dagger\;, \tag{35}$$

and hence Eq. (33) defines a standard complex quantum-mechanical scattering problem (with nonlocal potential). Letting $\Psi_{\rm p}$ be the incident plane wave,

$$\Psi_{\rm p}=\frac{1}{(2\pi)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{x}},\quad|\mathbf{p}|=(2mE)^{1/2}\;, \tag{36}$$

we can define a corresponding outgoing wave scattering solution Ψ_p^+ by the Lippmann-Schwinger equation

$$\Psi_{\rm p}^+=\Psi_{\rm p}+\frac{1}{E-H_0+i\epsilon}V_{\rm opt}(E)\Psi_p^+\;, \tag{37}$$

and the solution to the quaternionic scattering problem is then

$$\begin{aligned}&\Psi_\alpha=\Psi_p^+\;,\\ &\Psi_\beta=-i(H_0+E+i\tilde{V}_\alpha^*)^{-1}\tilde{V}_\beta\Psi_\alpha\;.\end{aligned} \tag{38}$$

Since

$$\langle x\,|(H_0+E)^{-1}|\,x'\rangle=\frac{2m}{4\pi}\frac{e^{-p|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\;, \tag{39}$$

Eq. (38) shows explicitly that $\Psi_\beta\sim e^{-p|\mathbf{x}|}$ at infinity. Introducing a complex transition matrix to the state with final momentum **q** by

$$T_{\mathbf{qp}}=(\Psi_{\mathbf{q}}, V_{\text{opt}}(E)\Psi_{p}^{+})_c \ , \tag{40}$$

with $(\ , \ )_c$ the usual complex $\mathbb{C}(1,i)$ inner product, we will see in the next section that the squared matrix elements $|T_{\mathbf{qp}}|^2$ give the transition probability per unit time via the usual golden-rule formula. Hence the usual complex quantum-mechanics scattering theory discussion[4] of the S matrix, unitarity and the optical theorem follows from Eqs. (37) and (40) without modification. We have thus demonstrated that in nonrelativistic quaternionic quantum mechanics, the S matrix is necessarily complex—there are no quaternionic phases in the far zone outgoing scattered wave. Hence the tests for quaternionic quantum effects suggested by Peres,[5] which look for nonvanishing quaternionic scattering phases, will necessarily give a null result.

There are, however, characteristic quaternionic effects which appear in the complex transition matrix. Using Eq. (12), Eq. (34) becomes

$$\begin{aligned}
V_{\text{opt}}(E)&=\tilde{V}_1+\tilde{V}_\beta^{*}\frac{1}{H_0+E+\tilde{V}_1}\tilde{V}_\beta \\
&=V_{\text{opt}}^{\text{even}}(E)+V_{\text{opt}}^{\text{odd}}(E) \ , \\
V_{\text{opt}}^{\text{even}}(E)&=\tilde{V}_1+\tilde{V}_2\frac{1}{H_0+E+\tilde{V}_1}\tilde{V}_2+\tilde{V}_3\frac{1}{H_0+E+\tilde{V}_1}\tilde{V}_3, \\
V_{\text{opt}}^{\text{odd}}(E)&=i\left[\tilde{V}_3\frac{1}{H_0+E+\tilde{V}_1}\cdot\tilde{V}_2-\tilde{V}_2\frac{1}{H_0+E+\tilde{V}_1}\tilde{V}_3\right] \ ,
\end{aligned} \tag{41}$$

with $V_{\text{opt}}^{\text{even}}$ and $V_{\text{opt}}^{\text{odd}}$, respectively, even and odd under complex $\mathbb{C}(1,i)$ time-reversal symmetry. Taking the $\langle x\,|\,\cdots\,|\,x'\rangle$ matrix element of $V_{\text{opt}}^{\text{odd}}(E)$, we have

$$\langle x\,|\,V_{\text{opt}}^{\text{odd}}(E)\,|\,x'\rangle=i[\tilde{V}_3(x)\tilde{V}_2(x')-\tilde{V}_2(x)\tilde{V}_3(x')]\times\left\langle x\left|\frac{1}{H_0+E+\tilde{V}_1}\right|x'\right\rangle \ , \tag{42}$$

which is nonvanishing in general, and vanishes only when $\tilde{V}_2(x)$ and $\tilde{V}_3(x)$ are linearly dependent (as is the case in the δ-function example solved explicitly above). Thus we see that the underlying quaternionic structure is reflected in the complex S matrix by the appearance of time-reversal-nonconserving effects.

III. TIME-DEPENDENT PERTURBATION THEORY

We turn now to the use of time-dependent perturbation theory to discuss the scattering and decay problems. Since we have seen in the previous section that the asymptotic wave function is complex $\mathbb{C}(1,i)$, we will be interested in the scattering of a complex $\mathbb{C}(1,i)$ state specified at time $t=-\infty$, or the decay of a complex $\mathbb{C}(1,i)$ state specified at time $t=0$. We start from the time-dependent Schrödinger equation

$$\frac{\partial\Psi}{\partial t}=-\tilde{H}\Psi \ , \tag{43}$$

and expand Ψ on a basis of zeroth-order eigenkets $|\,l\,\rangle\exp(-iE_l t)$, with time-dependent quaternionic coefficients $c_l(t)$:

$$\Psi=\sum_l |\,l\,\rangle e^{-iE_l t}c_l(t) \ . \tag{44}$$

Making a symplectic decomposition of c_l and Ψ, we have

$$\begin{aligned}
\Psi&=\Psi_\alpha+j\Psi_\beta \ , \\
c_l(t)&=c_{l\alpha}(t)+jc_{l\beta}(t) \ ,
\end{aligned} \tag{45}$$

giving, when substituted into Eq. (44),

$$\begin{aligned}
\Psi_\alpha&=\sum_l |\,l\,\rangle e^{-iE_l t}c_{l\alpha}(t) \ , \\
\Psi_\beta&=\sum_l |\,l\,\rangle e^{iE_l t}c_{l\beta}(t) \ .
\end{aligned} \tag{46}$$

Substituting into Eq. (43) gives the following complex $\mathbb{C}(1,i)$ equations for the coefficients $c_{n\alpha,\beta}(t)$:

$$\frac{d}{dt}c_{n\alpha}=-\sum_l(\tilde{V}_{\alpha nl}e^{i(E_n-E_l)t}c_{l\alpha}-\tilde{V}_{\beta nl}^{*}e^{i(E_n+E_l)t}c_{l\beta}) \ , \tag{47a}$$

$$\frac{d}{dt}c_{n\beta}=-\sum_l(\tilde{V}_{\beta nl}e^{-i(E_n+E_l)t}c_{l\alpha}+\tilde{V}_{\alpha nl}^{*}e^{i(E_l-E_n)t}c_{l\beta}) \ . \tag{47b}$$

Equation (47) forms the basis for the treatment of both the scattering and decay problems.

Turning first to the scattering problem, we seek to solve Eq. (47) with the initial conditions

$$\left.\begin{aligned}
&c_{s\alpha}\to 1 \\
&c_{n\alpha}\to 0, \quad n\neq s \\
&c_{n\beta}\to 0 \quad \text{all } n
\end{aligned}\right\} \text{ as } t\to-\infty \ . \tag{48}$$

Following the method[4] used in the complex case, we make the ansatz

$$\begin{aligned}
c_{n\alpha}(t)&=-\tilde{T}_{\alpha ns}\frac{e^{i(E_n-E_s)t+\epsilon t}}{i(E_n-E_s)+\epsilon}+\delta_{ns} \ , \\
c_{n\beta}(t)&=-\tilde{T}_{\beta ns}\frac{e^{-i(E_n+E_s)t+\epsilon t}}{-i(E_n+E_s)} \ ,
\end{aligned} \tag{49}$$

with the limit $\epsilon\to 0^+$ to be taken at the end of the calculation. From Eq. (49) we can immediately compute the transition probability per unit time into the various final states:

$$\frac{d}{dt}\,|\,c_{n\alpha}(t)\,|^2=\frac{d}{dt}\left[\frac{|\,\tilde{T}_{\alpha ns}\,|^2e^{2\epsilon t}}{(E_n-E_s)^2+\epsilon^2}\right]=\frac{2\epsilon}{(E_n-E_s)^2+\epsilon^2}\,|\,\tilde{T}_{\alpha ns}\,|^2e^{2\epsilon t}\underset{\epsilon\to 0^+}{\to}2\pi\delta(E_n-E_s)\,|\,\tilde{T}_{\alpha ns}\,|^2\ ,$$
$$\frac{d}{dt}\,|\,c_{n\beta}(t)\,|^2=\frac{d}{dt}\left[\frac{|\,\tilde{T}_{\beta ns}\,|^2e^{2\epsilon t}}{(E_n+E_s)^2}\right]=\frac{2\epsilon}{(E_n+E_s)^2}\,|\,\tilde{T}_{\beta ns}\,|^2e^{2\epsilon t}\underset{\epsilon\to 0^+}{\to}0\ . \tag{50}$$

Consistent with the results of Sec. II, where we saw that the outgoing scattering state is complex $\mathbb{C}(1,i)$, we see in Eq. (50) that the transition probability per unit time to the intrinsically quaternionic part of the final state vanishes, while that to the complex part has the standard golden-rule form with the transition matrix $\tilde{T}_{\alpha ns}$. Substituting the ansatz of Eq. (49) into Eq. (47), and setting[4] $\epsilon t=0$, the time dependence completely cancels out and we get the following coupled equations for the transition matrix elements:

$$\tilde{T}_{\alpha ns}=\tilde{V}_{\alpha ns}-\sum_l\frac{\tilde{V}_{\alpha nl}\tilde{T}_{\alpha ls}}{i(E_l-E_s)+\epsilon}-\sum_l\frac{\tilde{V}^*_{\beta nl}\tilde{T}_{\beta ls}}{i(E_l+E_s)}\ ,$$
$$\tilde{T}_{\beta ns}=\tilde{V}_{\beta ns}-\sum_l\frac{\tilde{V}_{\beta nl}\tilde{T}_{\alpha ls}}{i(E_l-E_s)+\epsilon}+\sum_l\frac{\tilde{V}^*_{\alpha nl}\tilde{T}_{\beta ls}}{i(E_l+E_s)}\ . \tag{51}$$

To solve these, we define column vectors $\tilde{T}_{\alpha,\beta s}$ and $\tilde{V}_{\alpha,\beta s}$ according to

$$\tilde{T}_{\alpha,\beta ns}\equiv\langle n\,|\,\tilde{T}_{\alpha,\beta s}\ ,$$
$$\tilde{V}_{\alpha,\beta ns}\equiv\langle n\,|\,\tilde{V}_{\alpha,\beta s}\ ; \tag{52}$$

factoring $\langle n\,|$ away from the left of Eq. (51) and using an operator form for the sum over intermediate states then gives the following column vector equations:

$$\tilde{T}_{\alpha s}=\tilde{V}_{\alpha s}-\tilde{V}_\alpha\frac{1}{i(H_0-E_s)+\epsilon}\tilde{T}_{\alpha s}-\tilde{V}^*_\beta\frac{1}{i(H_0+E_s)}\tilde{T}_{\beta s}\ ,$$
$$\tilde{T}_{\beta s}=\tilde{V}_{\beta s}-\tilde{V}_\beta\frac{1}{i(H_0-E_s)+\epsilon}\tilde{T}_{\alpha s}+\tilde{V}^*_\alpha\frac{1}{i(H_0+E_s)}\tilde{T}_{\beta s}\ . \tag{53}$$

Solving the second equation for $\tilde{T}_{\beta s}$, we get

$$\frac{1}{i(H_0+E_s)}\tilde{T}_{\beta s}=-i\frac{1}{H_0+E_s+i\tilde{V}^*_\alpha}\times\left[\tilde{V}_{\beta s}-\tilde{V}_\beta\frac{1}{i(H_0-E_s)+\epsilon}\tilde{T}_{\alpha s}\right]\ . \tag{54}$$

Substituting Eq. (54) into Eq. (53) for $\tilde{T}_{\alpha s}$, we get

$$\tilde{T}_{\alpha s}=\left[\tilde{V}_{\alpha s}+i\tilde{V}^*_\beta\frac{1}{H_0+E_s+i\tilde{V}^*_\alpha}\tilde{V}_{\beta s}\right]-\left[\tilde{V}_\alpha+i\tilde{V}^*_\beta\frac{1}{H_0+E_s+i\tilde{V}^*_\alpha}\tilde{V}_\beta\right]\frac{i}{E_s-H_0+i\epsilon}\tilde{T}_{\alpha s}\ , \tag{55}$$

or comparing with Eq. (34),

$$-i\tilde{T}_{\alpha s}=V_{\rm opt}(E_s)_s+V_{\rm opt}(E_s)\frac{1}{E_s-H_0+i\epsilon}(-i\tilde{T}_{\alpha s})\ . \tag{56}$$

Now rewriting Eq. (37) in the notation of the present section ($\mathbf{p}\to s, E\to E_s$) we have

$$\Psi_s^+=\Psi_s+\frac{1}{E_s-H_0+i\epsilon}V_{\rm opt}(E_s)\Psi_s^+\ , \tag{57}$$

and multiplying from the left by $V_{\rm opt}(E_s)$ gives

$$V_{\rm opt}(E_s)\Psi_s^+=V_{\rm opt}(E_s)\Psi_s+V_{\rm opt}(E_s)\frac{1}{E_s-H_0+i\epsilon}V_{\rm opt}(E_s)\Psi_s^+\ , \tag{58}$$

which reproduces Eq. (56) with the identification

$$T_{\alpha s}\equiv-i\tilde{T}_{\alpha s}=V_{\rm opt}(E_s)\Psi_s^+\ , \tag{59}$$

providing the promised justification of Eq. (40) of Sec. II.

Turning next to the decay problem,[1] we consider the time evolution of a state which is at $t=0$ the $\mathbb{C}(1,i)$ superposition of a set of degenerate eigenkets $|\,s_a\rangle$ of the unperturbed Hamiltonian H_0. Thus we want to solve Eq. (47) for $t\geq 0$ subject to the $t=0$ initial conditions

$$c_{s_a\alpha}(0)=K_a\in\mathbb{C}(1,i)\ ,$$
$$c_{n\alpha}(0)=0,\quad n\neq\{s_a\}\ ,$$
$$c_{n\beta}=0\quad\text{all } n\ . \tag{60}$$

Equivalently, we can add a δ function to the equation for $dc_{s_a\alpha}/dt$, changing Eq. (47a) to

$$\frac{d}{dt}c_{n\alpha}=-\sum_l(\tilde{V}_{\alpha nl}e^{i(E_n-E_l)t}c_{l\alpha}-\tilde{V}^*_{\beta nl}e^{i(E_n+E_l)t}c_{l\beta})+\delta(t)\sum_a\delta_{ns_a}K_a\ , \tag{61}$$

and solve the problem in the domain $-\infty<t<\infty$ subject to the boundary conditions

$$c_{n\alpha}(t)=c_{n\beta}(t)=0,\quad t<0\ . \tag{62}$$

Introducing Fourier transforms with respect to t by

$$c_{n\alpha}(t)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}dE\,e^{i(E_n-E)t}c_{n\alpha}(E)\ ,$$
$$c_{n\beta}(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dE\,e^{-i(E_n+E)t}c_{n\beta}(E)\ ,$$
$$i\delta(t)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}dE\,e^{i(E_n-E)t}\ , \tag{63}$$

Eqs. (47b) and (61) become

$$(E+i\epsilon-E_n)c_{n\alpha}(E)=-i\sum_l[\tilde V_{\alpha nl}c_{l\alpha}(E)+\tilde V^*_{\beta nl}c_{l\beta}(E)]+\sum_a \delta_{ns_a}K_a\ , \quad (64)$$

$$(E+i\epsilon+E_n)c_{n\beta}(E)=i\sum_l[\tilde V_{\beta nl}c_{l\alpha}(E)-\tilde V^*_{\alpha nl}c_{l\beta}(E)]\ ,$$

with the boundary conditions of Eq. (62) implemented by requiring $c_{n\alpha,\beta}(E)$ to be analytic in the upper half of the E complex plane. Rewriting the equation for $c_{n\beta}(E)$ as

$$\sum_l (H_0+E+i\epsilon+i\tilde V^*_\alpha)_{nl}c_{l\beta}(E)=i\sum_l \tilde V_{\beta nl}c_{l\alpha}(E)\ , \quad (65)$$

we can solve for the $c_{n\beta}$'s in terms of the $c_{n\alpha}$'s, giving

$$c_{n\beta}(E)=i\sum_{l,k}[(H_0+E+i\epsilon+i\tilde V^*_\alpha)^{-1}]_{nk}\tilde V_{\beta kl}c_{l\alpha}(E)\ , \quad (66)$$

which when substituted back into the equation for $c_{n\alpha}$ yields

$$(E+i\epsilon-E_n)c_{n\alpha}(E)=\sum_l V_{\rm opt}(E+i\epsilon)_{nl}c_{l\alpha}(E)+\sum_a \delta_{ns_a}K_a\ , \quad (67)$$

with $V_{\rm opt}$ the optical potential defined in Eq. (34). From this point on the analysis is the same as the usual discussion of the decay problem[6] in complex quantum mechanics; to accuracy of order $V^2_{\rm opt}$ the coefficients $c_{s_a\alpha}$ of the initially occupied states are governed by

$$\sum_b\left[(E+i\epsilon-E_s)\delta_{ab}-M_{ab}(E)+\frac{i}{2}\Gamma_{ab}(E)\right]c_{s_b\alpha}(E)=K_{a\alpha},$$

$$M_{ab}(E)=V_{\rm opt}(E+i\epsilon)_{s_as_b}+\sum_{l\neq\{s_c\}}V_{\rm opt}(E+i\epsilon)_{s_al}\times\frac{P}{E-E_l}V_{\rm opt}(E+i\epsilon)_{ls_b}\ , \quad (68)$$

$$\Gamma_{ab}(E)=2\pi\sum_{l\neq\{s_c\}}V_{\rm opt}(E+i\epsilon)_{s_al}\times\delta(E-E_l)V_{\rm opt}(E+i\epsilon)_{ls_b}\ ,$$

which are readily solvable if we make the Weisskopf-Wigner approximation

$$M_{ab}(E)\approx M_{ab}(E_s)\ ,\qquad \Gamma_{ab}(E)\approx\Gamma_{ab}(E_s)\ , \quad (69)$$

in which the mass and decay matrices become real constants. Because $V_{\rm opt}$ is time-reversal nonconserving, *both* the mass and the decay matrix exhibit T-nonconserving effects at leading order $\tilde V^2_\beta$ and $\tilde V_\alpha\tilde V^2_\beta$, respectively. Because our earlier calculation of Ref. 1 worked only to order $\tilde V^2$, the T-nonconserving contribution to the decay matrix was omitted, and we erroneously reached the conclusion that quarternionic physics implies a "superweak" form for T or CP nonconservation. In fact, the form of the T nonconservation predicted by Eqs. (34) and (68) is phenomenologically "milliweak," with T nonconservation of the same order ($\tilde V^2_\beta$) appearing in both the mass matrix and in decay amplitudes.

IV. BOUND-STATE EFFECTS

The optical potential defined in Eqs. (34) and (41) contains the inverse of the operator $H_0+\tilde V_1+E$; hence for each complex quantum-mechanics bound state ψ_b satisfying

$$(H_0+\tilde V_1)\psi_b=-E_b\psi_b,\quad E_b>0 \quad (70)$$

there will be an isolated singularity in the optical potential of the form

$$\langle x\,|\,V_{\rm opt}(E)\,|\,x'\rangle=\text{nonsingular at } E=E_b+\frac{\tilde V^*_\beta(x)\psi_b(x)\psi^*_b(x')\tilde V_\beta(x')}{E-E_b}\ . \quad (71)$$

The singular term is explicitly Hermitian and so does not invalidate the arguments of the preceding sections, but we wish to explore here its detailed consequences for the scattering problem. We can anticipate some of the results by looking back at the δ-function spike problem discussed in Sec. II. When $\tilde V_1$ of Eqs. (18)–(27) is negative, the complex quantum mechanics problem defined by $H_0+\tilde V_1\delta(x)$ always has a single bound state. Writing

$$\tilde V_1\equiv-\frac{p_b}{m}\ , \quad (72)$$

we indeed see that the quantity C defined in Eq. (27), which in the δ-function spike problem is the analog of the optical potential, becomes

$$C=\tilde V_1+m(\tilde V^2_2+\tilde V^2_3)/(p-p_b) \quad (73)$$

and has a single pole at $p=p_b$. In the neighborhood of p_b, the coefficients R, T, and S are

$$R \approx \frac{-i\Delta}{p-p_b+i\Delta} ,$$
$$T \approx \frac{p-p_b}{p-p_b+i\Delta} ,$$
$$S \approx \frac{-im\tilde{V}_\beta}{p-p_b+i\Delta} , \tag{74}$$
$$\Delta \equiv \frac{m^2}{p_b} | \tilde{V}_\beta |^2 ,$$

so the scattering coefficients remain bounded (as required by unitarity) but exhibit resonant behavior at $p \approx p_b$, with a resonance width Δ proportional to $| \tilde{V}_\beta |^2$. We will see that these features are common to the behavior exhibited by the partial waves of the three-dimensional case as well.

To study the three-dimensional problem, we assume that the potential components $\tilde{V}_{1,2,3}$ are spherically symmetric, permitting a partial-wave analysis. We assume that the bound state ψ_b has angular dependence $Y_{lm}(\theta,\phi)$, so that we can write

$$\psi_b = \frac{u_b(r)}{r} Y_{lm}(\theta,\phi) . \tag{75}$$

We then see that the singular term in Eq. (71) contributes only to the l,m partial wave of Ψ_α. Writing

$$\Psi_\alpha \propto \frac{u_l(r)}{r} Y_{lm}(\theta,\phi) + \text{other modes} , \tag{76}$$

and substituting Eqs. (71), (75), and (76) into Eq. (33) for Ψ_α, we find that the contribution of the singular term to the radial Schrödinger equation for u_l is

$$\frac{1}{2m}\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-p^2\right]u_l(r) = C\tilde{V}^*_\beta(r)u_b(r) , \tag{77a}$$

$$C \equiv -\frac{1}{E-E_b}\int_0^\infty dr' u_b^*(r')\tilde{V}_\beta(r')u_l(r') , \tag{77b}$$

where we have written $E=p^2/2m$ and have omitted all nonsingular potential terms. To solve Eq. (77) we introduce the Green's function[7] $G(r,r')$,

$$G(r,r') = -prr'j_l(pr_<)n_l(pr_>) , \tag{78}$$

which satisfies

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-p^2\right]G(r,r') = \delta(r-r') . \tag{79}$$

Including a homogeneous solution corresponding to the incident plane wave, we can then integrate Eq. (77a) to give

$$u_l(r) = prj_l(pr) + \int_0^\infty dr' G(r,r')2mC\tilde{V}^*_\beta(r')u_b(r') . \tag{80}$$

Substituting Eq. (80) back into Eq. (77b), we get a linear equation determining C which can be solved to give

$$C = \frac{K_1}{E-E_b-K_2} ,$$
$$K_1 = -\int_0^\infty dr' u_b^*(r')\tilde{V}_\beta(r')pr'j_l(pr') , \tag{81}$$
$$K_2 = K_2^*$$
$$= -2m\int_0^\infty dr'dr'' u_b^*(r')\tilde{V}_\beta(r')G(r',r'') \times u_b(r'')\tilde{V}^*_\beta(r'') .$$

Taking the large-r asymptotic limit of Eq. (80), we find by the usual method[7] that the scattering phase shift δ_l is given by

$$\tan\delta_l = \int_0^\infty dr' r' j_l(pr')2mC\tilde{V}^*_\beta(r')u_b(r')$$
$$= -\frac{2m}{p}CK_1^* , \tag{82}$$

or, substituting Eq. (81) for C,

$$\tan\delta_l = \frac{\Gamma}{2(E_r-E)} ,$$
$$E_r = E_b + K_2 , \tag{83}$$
$$\Gamma = \frac{4m}{p} | K_1 |^2 .$$

Hence as a consequence of the bound state, the l,m scattering partial wave exhibits a resonance at energy E_b+K_2, with the resonance width Γ quadratic in the quaternionic part $\tilde{V}_\beta$ of the potential.

The fact that the resonance energy is at E_b+K_2, rather than at E_b, reflects the fact that the quaternionic dynamics shifts the bound-state energy by a term which in leading order is quadratic in $\tilde{V}_\beta$. To see this, we note that in the limit of vanishing $\tilde{V}_\beta$, the bound-state solution of Eq. (70) appears as a solution $\Psi_\alpha=0$, $\Psi_\beta=\psi_b$ of Eq. (30), reflecting our convention of choosing the quaternion ray representative so as to make E always positive. Hence the $\tilde{V}_\beta$ corrections to the bound-state energy are obtained by eliminating Ψ_α from Eq. (30) according to

$$\Psi_\alpha = \frac{P}{H_0+\tilde{V}_1-E}(-i)\tilde{V}^*_\beta\Psi_\beta , \tag{84}$$

giving as the effective equation for Ψ_β

$$\left[H_0+\tilde{V}_1+\tilde{V}_\beta\frac{P}{H_0+\tilde{V}_1-E}\tilde{V}^*_\beta+E\right]\Psi_\beta=0 . \tag{85}$$

The use of a principal-value boundary condition in Eqs. (84) and (85) is dictated by the fact that E is real, and hence the inversion in Eq. (84) must be carried out so as to produce an Hermitian effective potential in Eq. (85). Treating the $\tilde{V}_\beta$ term in Eq. (85) as a small perturbation (and making the leading-order approximation of dropping $\tilde{V}_1$ in the denominator) we get, for the corrected bound-state energy,

$$E=E_b-\int d^3x\, d^3x'\psi_b^*(x)\left\langle x\left|\tilde{V}_\beta\frac{P}{H_0-E_b}\tilde{V}_\beta^*\right|x'\right\rangle\psi_b(x')$$

$$=E_b+K_2\ , \tag{86}$$

reproducing the result inferred from the scattering analysis.

The experimental implications of the resonance phenomenon discussed above will require careful study. In principle, since the resonance width is governed by $|\tilde{V}_2|^2$ and $|\tilde{V}_3|^2$, the width could have an order of magnitude different from that of the T-nonconserving effect of the preceding section, which is governed in magnitude by $|\tilde{V}_2||\tilde{V}_3|$.

It should also be noted that in calculating the resonance locations from the bound-state energies, the energy zero point is a significant parameter. We have assumed that the potential $\tilde{V}_1$ vanishes at spatial infinity, but the complex quantum-mechanics bound state and scattering theory governed by $H_0+\tilde{V}_1$ are unchanged when $\tilde{V}_1$ approaches a constant μ at infinity, apart from a shift $E\to E+\mu$ of all scattering and bound-state energies. One can verify that provided $\mu>0$ (i.e., has the sign of a rest mass), the intrinsically quaternionic modes remain nonpropagating and the general framework set up in Sec. II remains valid. Since E_b is defined in Eq. (70) as the *negative* of the bound-state energy, the corresponding shift in E_b is $E_b\to E_b-\mu$, and hence the singular term in Eq. (71) is shifted according to

$$\langle x\,|\,V_{\rm opt}(E)\,|\,x'\rangle=\text{nonsingular}+\frac{\tilde{V}_\beta^*(x)\psi_b(x)\psi_b^*(x')\tilde{V}_\beta(x')}{E-E_b+2\mu}\ ,$$
$$E=p^2/2m\ . \tag{71'}$$

This shift propagates throughout the subsequent analysis, and in particular the resonance of Eq. (83) now appears at a lowered kinetic energy

$$E_r=E_b+K_2-2\mu\ , \tag{83'}$$

and is absent altogether if $2\mu\gg E_b$.

The reason why μ is a relevant physical parameter and cannot be eliminated from the physics by rephasing the quaternionic wave function was already noted in Ref. 1: Consider the time-dependent Schrödinger equation which corresponds to a shifted $\tilde{V}_1$,

$$\frac{\partial\Psi}{\partial t}=-(\tilde{H}+i\mu)\Psi\ , \tag{87}$$

and let us attempt to eliminate μ by defining a rephased wave function

$$\Psi'(t)=e^{i\mu t}\Psi(t)\ . \tag{88}$$

Substituting Eq. (88) into Eq. (87), we find that Ψ' obeys

$$\frac{\partial\Psi'}{\partial t}=-\tilde{H}'\Psi'\ ,$$
$$\tilde{H}'=e^{i\mu t}\tilde{H}e^{-i\mu t}$$
$$=i(H_0+\tilde{V}_1)+j\tilde{V}_\beta e^{2i\mu t}\ ; \tag{89}$$

hence, if $\tilde{H}$ is a time-independent operator, the quaternionic part of $\tilde{H}'$ has an harmonic time dependence. Corresponding to the fact that $\tilde{H}'$ is time dependent, if we assume Ψ to be a stationary state

$$\Psi=(\Psi_\alpha+j\Psi_\beta)e^{-iEt}\ , \tag{90}$$

then Ψ' is given by

$$\Psi'=e^{i\mu t}\Psi$$
$$=\Psi_\alpha e^{-i(E-\mu)t}+j\Psi_\beta e^{-i(E+\mu)t}\ , \tag{91}$$

and is not a stationary state unless Ψ_α or Ψ_β vanishes.

V. DISCUSSION

We see that scattering in quaternionic quantum mechanics takes a very interesting form, with the asymptotic state structure complex, and with the quaternionic effects disguised as an effective T nonconservation and as novel scattering resonances induced by bound states. Our results here lend strong support to the conjecture we have made[1,2] that the asymptotic state space of a quaternionic quantum field theory resides within a complex subspace of quaternionic Hilbert space. Complexity of the asymptotic state space automatically eliminates the two principal objections which have been raised against the possible relevance of quaternionic quantum mechanics to elementary-particle physics. The first of these objections is that quaternionic quantum mechanics has no multilinear tensor product,[8] and hence cannot accommodate multiparticle states. The second objection is that the quaternionic representations of the Poincaré group are just the usual complex representations constructed with respect to any complex subspace of the quaternions.[9] However, in quantum field theory, both the Fock-space construction and the Poincaré classification of states are relevant only to asymptotic "free particle" states and, if these states are necessarily complex for quaternionic quantum theory, then the usual complex constructs (and not nonexistent quaternionic generalizations) are all that are needed. Quaternionic quantum mechanics then becomes a new way, with an unexplored and potentially very rich structure, of generating an effective complex theory acting on the asymptotic state space. The fact that the qualitative prediction, that this effective theory is T nonconserving, is in accord with the known symmetry properties of elementary-particle physics, encourages exploration of the idea that a layer of quaternionic dynamics may underlie the effective complex dynamics of the standard model. If the scale of quaternionic T nonconserving effects is indeed set by the magnitude of the observed CP violation in the K-meson system, then[2] assuming $|\tilde{V}_2|\sim|\tilde{V}_3|$ the ratio $V_{\rm opt}^{\rm odd}/V_{\rm opt}^{\rm even}$ for typical nuclear processes is of order 10^{-16}, and will be extremely hard to detect (becoming visible in the K system only because there it is amplified by a factor $M_K/\Delta M_K \sim 10^{14}$). Similarly, by this estimate the bound-state-induced resonances discussed in Sec. III will be exceedingly narrow, and hence very hard to detect.

At a strict formal level, our results show that there exists a new class of problems to be studied in potential theory, with possibly interesting consequences for such well-developed topics as the inverse scattering problem and Levinson's theorem.

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