Quantization of a constituent-string model in four dimensions

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(Received 18 November 1986)

In superstring theories, anomalies and mathematical divergences (though they cancel) and the necessity to quantize in ten dimensions are attributable to the strings being extended physical objects, i.e., strings consisting of an infinite number of "beads." Therefore, it is interesting to consider a "string" generated by a single "bead" which tends toward oscillation with infinite velocity between two spatial points in some Lorentz frame. This picture has been realized classically in earlier work on the constituent model. The Lagrangian is given in terms of a set of generalized harmonic-oscillator normal coordinates and displays a gauge symmetry due to parametric invariance. The model is quantized by going to the Hamiltonian formalism (using constraint theory) and assuming the usual quantum conditions for the normal coordinates and their conjugate momenta. In analogy to the relativistic free string model, a set of orthonormal invariant supplementary conditions are applied. Constituents are defined in terms of the normal coordinates and form two-body composite particles. For any composite (observable) particle of real mass, it is shown that in its rest frame the time oscillations of its constituents vanish and its internal orbital angular momentum is a linear function of the $(\text{mass})^2$. Only composites of equal mass can couple in this model, thereby eliminating couplings between physical and unphysical masses. The invariant scattering amplitude for two-particle (meson) scattering is calculated and is a crossing-symmetric function of a linear trajectory function $\alpha(z)=\alpha' z-\alpha_0$.

I. INTRODUCTION

It is customary in physics to treat a string mathematically as a system of N beads in a row on a line in space-time: N is allowed to approach infinity at the same time as the distance between the beads shrinks to zero. In a similar manner, fields may be thought of as strings. For example, $\phi(x,t)$ can be regarded as an (infinitely long) string, as x and t vary continuously. In both field and superstring theories, mathematical divergences and other anomalies (even if they cancel) are directly related to the existence of these extended objects.

Furthermore, in superstring theories, the necessity to quantize in ten dimensions is attributed to the infinite number of beads, as they, in turn, imply the infinite number of modes of oscillation of the string. Each mode gives rise to a time oscillation needing to be "suppressed" in the quantum theory. The constraints which must exist to accomplish this, in turn, lead to algebras which can be closed only upon the introduction of extra spatial dimensions.[1,2]

Based on this, an idea immediately suggests itself: namely, to circumvent the problem of an infinite number of modes of oscillation by going back to a particle description. In a given Lorentz frame, a particle traveling with infinite velocity back and forth between two points describes a "string." It maps out a two-dimensional surface in space-time just as the "physically extended" string. Transforming to other Lorentz frames also yields world sheets, although the particle interpretation will be different. Such a particle does not correspond to a physically observable particle, so it is natural to regard it, for example, as a quark confined inside of a hadron.[3] The hope is then to construct a theory based on these particles such that the composite particles are well behaved (e.g., slower than light).

The most obvious candidate to provide the framework for such a theory is an action-at-a-distance theory. The particles (of real mass) whose trajectories can bend outside the light cone, previously thought to be the nemesis of such theories, are ready made to become such quarks. It then remains to formulate a theory in which they remain confined as constituents of "well-behaved," i.e., observable, particles. Since such a theory is formulated without fields or other extended objects, it holds the promise of being quantized self-consistently and without anomalies in four-dimensional space-time.

This paper is the third in a series[4,5] which represents an effort to construct such a theory. The first two papers were concerned with the classical Lagrangian formulation of action-at-a-distance models of constituent particles. Here, one of those models is reformulated in the Hamiltonian formalism and quantized in four dimensions. In addition to providing a workable example of the ideas outlined in the preceding paragraphs, there is a further aim to the paper. For the constituent model is not just a reshuffling of old ideas; it involves a reformulation of the description of a system of N free particles. The constituents, after all, to remain unobservable, must remain forever confined to the system.

The constituent model is intended to be a cosmological approach to modeling the Universe in the sense articulated by Hoyle and Narlikar.[16] That is to say, the existence of every observable quantity arises only because of its interactions with the whole of the Universe. In this context, there is no such thing as a free *independent* particle.

Particles may be free in the sense that they describe straight-line trajectories in some region of space-time, but only because the action of the Universe upon them remains constant in that region. The measurable quantities we associate with a particle, such as its mass, position, and velocity, are determined by the whole action of the Universe.

The description of a free particle has, historically, been a very useful place to start. However, in order to construct a theory of interacting particles (or fields, or strings, etc.), there must be more than one particle in the system. Implicitly or explicitly, it has been conventionally assumed that if the interactions between particles are "turned off," the theory reverts back to the description of free and *independent* particles [see, for example, (2.1)]. It is at this point that this work parts company with conventional theories. The "center-of-mass" Lagrangian introduced in Sec. III forms the basis of all the constituent models of this and the earlier papers.[4,5]

In this paper, interactions are introduced in the most simplified version of the constituent model capable of realistic application, denoted as the $4N$ model. The $4N$ model contains N each of four kinds of constituents and is chosen to be a "first approximation" to particle interactions. The constituent solutions fall into two groups. Within each group, it is possible to form composite two-particle systems whose internal state is described by a single oscillatory term. The two systems are interpreted as "lepton" and "baryon" composite particles, respectively.

The model is first formulated in terms of a set of generalized normal coordinates. The Lagrangian displays a gauge symmetry in the parameter s. The model is quantized by defining a Dirac Hamiltonian and assuming quantum conditions for the normal coordinates and their conjugate momenta. In close analogy to the relativistic free string model, a set of orthonormal invariant supplementary conditions are applied to the normal coordinates. Then the model is reinterpreted in terms of constituent and composite particles. For any composite of real mass, it is shown that in its rest frame the time oscillations of its constituents vanish and its internal orbital angular momentum is a linear function of the $(\text{mass})^2$.

Once these general results are obtained, the problem is specialized to systems corresponding to $N=2$. In this case, there are eight constituents. The final interpretation of the constituents is in terms of "strings," which are obtained by assuming that the physical eigenstates of the Hamiltonian correspond to zero *Poincaré* four-momentum of the total system. As this limit is approached, it is shown that the oscillator frequency increases without bound, thus yielding constituent "strings." By imposing initial conditions in s, we study systems in which the constituent strings pair up asymptotically to form free composite particles. In the "first approximation" represented by the $4N$ model, the "lepton" and "baryon" composites decouple. In addition, there are no couplings between composites of real and imaginary masses. The invariant scattering amplitudes for two-particle "meson" scattering is calculated and takes the form

$$A(s,t,u)\sim \prod_{I=1}^{4}\delta(P_I^2-M^2c^2)\delta(P_1+P_2+P_3+P_4)\times[D(s)+D(t)+D(u)]\ , \qquad (1.1)$$

TABLE I. Comparison of free string and constituent models.

Free string	Constituent model
1. Lagrangian with gauge invariance in parameters τ and σ.	1. Lagrangian with gauge invariance in parameter s.
2. Orthonormal supplementary conditions applied. Implies a more limited gauge invariance, namely, conformal invariance.	2. Orthonormal supplementary conditions applied. Still invariant under transformations in s.
3. Two constraints (rather, two infinite sets of constraints).	3. One constraint.
4. Solution in general contains infinite number of oscillatory modes.	4. Solutions contain only one oscillatory mode (nonlinear differential equations).
5. Therefore, there are an infinite number of oscillations in time	5. Therefore, there is only one oscillation in time.
6. Choice of gauge $x_0=\tau$ (x_μ=c.m. vector), plus the above subsidiary conditions implies no internal time oscillations in the rest frame of the string.	6. Choice of gauge $W_{10}=A_0 s$ (W_μ=c.m. vector of system), plus the above subsidiary conditions implies no internal time oscillations in the rest frame of a composite.

where

$$D(z)=\sum_{n=-\infty}^{\infty}\delta(\alpha(z)-n)\ . \tag{1.2}$$

The approach resembles the treatment of a classical relativistic string, as outlined in Table I. However, quantization is closely analogous to nonrelativistic quantum mechanics as formulated, for example, by Dirac.[7]

Since the presentation in terms of the generalized set of normal coordinates precedes the physical interpretation of the model, it may be helpful for the reader to examine Fig. 1. The diagrams of the figure schematically illustrate the scattering of two composite particles, and are intended to be taken much more literally than former "duality" diagrams.[8] Constituent trajectories, defined in terms of the normal coordinates in Sec. VI, are hyperbolic functions of an unmeasurable parameter s (the oscillatory nature of the constituents has been suppressed in the diagrams), with the arrows indicating increasing s. If the time axis is taken vertically, it is clear that s is not identified with any kind of physical time. The constituents "pair up" to form composite particles at $s=\pm\infty$. In Figs. 1(b) and 1(c), two of the constituents turn around in time.

It is worth emphasizing that in the $4N$ model, all constituents oscillate with the same frequency and there is no "Fourier" decomposition of the solutions. In other words, since there is only one "bead" making up the string, it has only one mode of oscillation. The substitution of a single "bead" for the infinite number of beads of the conventional free-string model[9] is equivalent to eliminating the parameter σ (see also the further remarks in Sec. VI).

This paper has been limited to the simple $4N$ model. As a result, the strings of this paper are in reality more like "rigid rods."

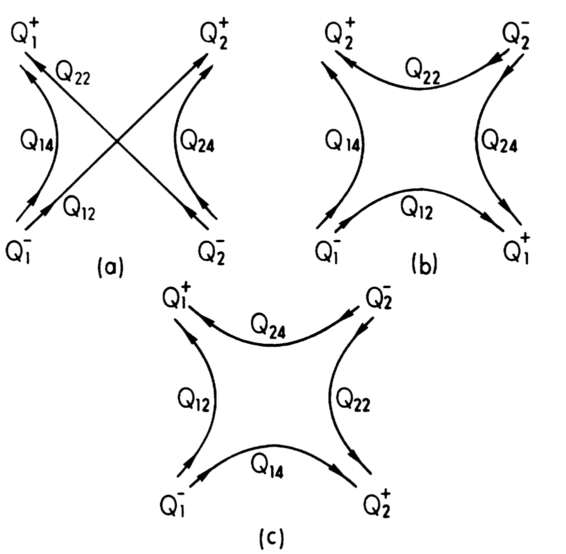

FIG. 1. Elastic scattering of two X-composite particles. (a) s channel; (b) t channel; (c) u channel.

Sections II and III are concerned with a reformulation of the Lagrangian for a system of N free particles. The reader may wish to go directly to the development of the $4N$ model by starting with Sec. IV.

II. ROLE OF FREE PARTICLES IN CONSTRUCTION OF PARTICLE THEORIES

Historically, an observable particle has been treated as an entity which can be isolated from outside influences, in which case it becomes "free." Properties such as mass, spin, and charge are treated as inherent to the particle, and independent of the rest of the system, even though we know there must be some underlying reason why such properties come as multiples of basic units (albeit roughly for mass). The "free" particle has traditionally provided the starting point in the construction of interacting particles, whether classically, in quantum mechanics, in field theories, or in string theories. Interactions are introduced in such a way that if the interaction is "turned off," the description of a system of free particles is retrieved, taking the form (if there is a Lagrangian)

$$L=\sum_I L_I\ , \tag{2.1}$$

where L_I is a Lagrangian for the Ith free particle (or field, string, system, etc.).

Alternatively, "cosmological" action-at-a-distance[6] theories suggest that every particle owes its existence, i.e., its trajectory, mass, charge, spin, etc., to the existence of all the other particles in the Universe. In this case, the particle becomes "free" in certain regions of space-time where the influences of the Universe remain effectively constant. This viewpoint is attractive because it fits with our experience of the Universe, but such an approach implies we must *begin* with a description of the entire Universe, a formidable task, indeed.

Nevertheless, the hope is to catch a glimpse of the structure of the Universe by constructing models of smaller, more tractable systems. Such systems ought to reflect the dependence of each particle on the whole, even when the interaction becomes vanishingly small. Thus we are led to examine more closely possible alternative descriptions of a system of noninteracting particles. One such alternative to the Lagrangian (2.1) is discussed in the next section. This "center-of-mass" Lagrangian has formed the basis of the constituent models in this and the previous papers.[4,5]

III. ALTERNATIVE LAGRANGIANS FOR A SYSTEM OF N FREE PARTICLES

Special relativity is assumed to hold, i.e., the velocity of light is the same in all inertial frames, and it is assumed that the Lagrangian must be invariant under Lorentz transformations.

If a particle describes a straight-line trajectory in some region of Minkowski space, we shall define it as a free particle in the region. That is, its trajectory can be parametrized by

$$x_{I\mu}(s)=a_{I\mu}s+b_{I\mu}\ , \tag{3.1}$$

where I labels the particle, and s is an "evolution" parameter, varying in some range $s_0 < s < s_1$. If the particle is free in the whole of space-time, we set $s_0 = -\infty$, and $s_1 = +\infty$.

From (3.1) it follows that

$$\mathbf{x}_I(s) = (\mathbf{v}_I/c)x_{I0}(s) + [\mathbf{b}_I + (b_{I0}/a_{I0})\mathbf{a}_I] , \tag{3.2}$$

with $\mathbf{v}_I/c = \mathbf{a}_I/a_{I0}$, implying that s may be replaced by any monotonically increasing function of s, $f(s)$, without affecting the trajectory in space-time.

Historically in nonrelativistic classical mechanics, it was assumed that the measurable quantities of a particle are its mass, its position, and its velocity (at a given time). It was observed experimentally that certain quantities appear to be conserved before and after (conservative) forces act upon the particle. These quantities are defined as the energy

$$E = \tfrac{1}{2}mv^2 \tag{3.3}$$

and the momentum

$$\mathbf{p} = m\mathbf{v} . \tag{3.4}$$

Subsequently, E and $\mathbf{p}$ have often been alternatively treated as the measurable properties of the particle. In those cases, the mass and velocity were derivable from (3.3) and (3.4). The two descriptions have been used interchangeably.

Returning to the relativistic particle, we know that there is no unique choice of Lagrangian from which to derive the trajectory (3.1). Similarly, there is no unique Lagrangian for a system of N noninteracting particles. Below, we review the traditionally used Lagrangian formulation, and then consider an alternative description.

A. "Center-of-energy" Lagrangian

The Lagrangian for N free particles is traditionally given by

$$L = -c \sum_{I=1}^{N} m_I [\dot{x}_I^2(s)]^{1/2} , \tag{3.5}$$

where an overdot denotes d/ds; the metric is $\dot{x}^2 = -g_{\mu\nu}\dot{x}^\mu x^\nu$, $g_{00} = -1$, $g_{ii} = 1$, and m_I is the mass of the Ith particle. Parametric invariance of the action leads to N constraints, and thus to N gauge choices which can be taken to be

$$x_{I0} = a_{I0}s, \quad I = 1, 2, \ldots, N , \tag{3.6}$$

where a_{I0} are arbitrarily chosen constants (and therefore do not completely specify the gauges). With these conditions, it is straightforward to derive the equations of motion

$$\ddot{x}_{I\mu}(s) = 0 , \tag{3.7}$$

with solutions

$$x_{I\mu}(s) = a_{I\mu}s + b_{I\mu} . \tag{3.8}$$

Conjugate momenta are defined by

$$p_{I\mu} \equiv \partial L/\partial \dot{x}_I^\mu \tag{3.9}$$

or

$$p_{I\mu} = m_I c \dot{x}_{I\mu}/(\dot{x}_I^2)^{1/2} . \tag{3.10}$$

From (3.10) it follows that the N constraints take the form

$$p_I^2 = m_I^2 c^2 . \tag{3.11}$$

The total momentum of the system is defined as

$$P_\mu = \sum_{I=1}^{N} p_{I\mu} , \tag{3.12}$$

and is therefore associated with the motion of the center-of-energy vector of the system.

In the nonrelativistic approximation, i.e., when $|\mathbf{v}_I/c| \ll 1$, the conjugate momentum (3.10) becomes

$$cp_{I0} \approx m_I c^2 + \tfrac{1}{2}m_I \mathbf{v}_I^2 , \tag{3.13a}$$

$$\mathbf{p}_I \approx m_I \mathbf{v}_I . \tag{3.13b}$$

Thus one obtains the identification of the fourth component of the conjugate momentum with the energy of the particle, and the spatial part $\mathbf{p}_I$ with the "physical" three-momentum defined by (3.4). Traditionally, the conjugate four-momentum is defined as the "physical momentum" and considered to be a measurable quantity. Considered as such, it is a real quantity, so that for real m_I, the velocity of the particle is necessarily less than the speed of light.

One begins to run into difficulty with such an interpretation when one recalls that there is no unique Lagrangian which will yield the equations of motion (3.7) for the N free particles. For every Lagrangian, the conjugate four-momenta defined by (3.9) will, in principle, be different. Thus, it becomes slippery footing to attempt to treat any or all such conjugate momenta as directly measurable quantities. For this reason we adopt the viewpoint that the measurable properties of a particle are the positions, velocity, and mass.

B. "Center-of-mass" Lagrangian

Consider now the Lagrangian defined by

$$L = -\left\{ Mc^2 \sum_{I=1}^{N} m_I [\dot{x}_I(s)]^2 \right\}^{1/2} , \tag{3.14}$$

where $M \equiv \sum_I m_I$. Parametric invariance of the action leads to only one constraint in this case, allowing us to choose one gauge, which we take to be

$$X_0 = A_0 s , \tag{3.15}$$

where X_0 is the fourth component of the c.m. vector:

$$X_\mu = \sum_{I=1}^{N} m_I x_{I\mu}/M . \tag{3.16}$$

The constant A_0 is an arbitrarily chosen constant, so (3.15) does not completely fix the gauge. With the condition (3.15), the equations of motion become

$$\ddot{x}_{I\mu}=0 \ , \tag{3.17}$$

with solutions

$$x_{I\mu}=a_{I\mu}s+b_{I\mu} \ , \tag{3.18}$$

where $b_{I0}=0$ and

$$\sum_{I=1}^{N} m_I a_{I0}=MA_0 \ . \tag{3.19}$$

Although (3.19) appears, at first glance, to imply that the N trajectories are not all independent, recall that A_0 is an arbitrary gauge constant. The solutions can always be regauged, so they are, in effect, independent. Another way of stating this, is that the physically measurable quantities are gauge independent.

From the Lagrangian (3.14) and the definition of conjugate momentum (3.9), it follows that the conjugate momenta are given by

$$p_{I\mu}=m_I c\dot{x}_{I\mu}\Bigg/\left[\sum_{J=1}^{N} m_J\dot{x}_J^{\,2}/M\right]^{1/2} . \tag{3.20}$$

The single constraint is given by

$$\sum_{I=1}^{N}(p_I^2/m_I)=Mc^2 \ . \tag{3.21}$$

The total four-momentum of the system is defined as

$$P_\mu=\sum_{I=1}^{N} p_{I\mu} \ , \tag{3.22}$$

and is thus associated with the center-of-mass four-vector (3.16).

It is clear from (3.20) that the conjugate momenta depend on the masses and trajectories of all N particles of the system. Thus they cannot be given the same physical interpretation as the conjugate momenta of Sec. III A for the center-of-mass Lagrangian. The $p_{I\mu}$ of (3.20) cannot be calculated from physical measurements made on the Ith particle alone.

It is the center-of-mass Lagrangian which provides the basis of the constituent model.

C. Other gauge choices

Later, two-constituent systems will be identified as free composite particles in certain regions of Minkowski space. As we shall see, the composite particle four-vectors are not linear functions of the evolution parameter s. Therefore, it is worth emphasizing that the gauge choices (3.6) and (3.15) were made for convenience only.

Let us consider an alternative derivation of the trajectory equation. For simplicity, take the one-particle Lagrangian

$$L=-mc[\dot{x}^2(s)]^{1/2} \ . \tag{3.23}$$

The equations of motion are

$$d[\dot{x}_\mu/(\dot{x}^2)^{1/2}]/ds=0 \ . \tag{3.24}$$

Providing $\dot{x}^2\neq 0$, (3.24) can be reexpressed as

$$(1/\dot{x}_\mu)d(\dot{x}_\mu)=(1/2\dot{x}^2)d(\dot{x}^2) \ . \tag{3.25}$$

Integrating (3.25) we obtain

$$\ln\dot{x}_\mu=\ln[a_\mu(\dot{x}^2)^{1/2}] \tag{3.26}$$

or

$$\dot{x}_\mu=a_\mu(\dot{x}^2)^{1/2} \ . \tag{3.27}$$

Note that here the gauge choice $x_0=a_0 s$ implies that $(\dot{x}^2)^{1/2}=1$. Instead of specifying the gauge, use (3.27) to derive the trajectory equation

$$\dot{\mathbf{x}}=\mathbf{a}\dot{x}_0/a_0 \ , \tag{3.28}$$

which, upon integration, yields

$$\mathbf{x}=(\mathbf{a}/a_0)x_0+\mathbf{b} \ , \tag{3.29}$$

with $\mathbf{b}$ a second arbitrary constant.

The conjugate momentum is

$$\begin{aligned} p_\mu&=\partial L/\partial\dot{x}_\mu=mc\dot{x}_\mu/(\dot{x}^2)^{1/2} \\ &=mc\dot{x}_\mu a_0/\dot{x}_0 \ . \end{aligned} \tag{3.30}$$

Now consider the gauge choice

$$x_0=a_0C\sinh Ds, \quad C \text{ and } D\text{const} \ . \tag{3.31}$$

Then the solution becomes

$$x_\mu=a_\mu C\sinh Ds+b_\mu \tag{3.32}$$

and the conjugate momentum is

$$p_\mu=mca_\mu \ . \tag{3.33}$$

The mass-shell constraint $p^2=m^2c^2$, following from (3.30), implies that

$$a^2=a_\mu a^\mu=1 \ . \tag{3.34}$$

IV. LAGRANGIAN FOR THE 4*N* MODEL IN NORMAL COORDINATES

The $4N$ model was first described in Ref. 5. It consists of a system of $4N$ constituent particles; there are N each of four distinct varieties. Later we shall make a correspondence between the four types and a set of quantum numbers, but for now let it suffice that they couple to each other in different ways. The $4N$ Lagrangian was first given in terms of a set of generalized coordinates and velocities:

$$x_{IA}(s),\dot{x}_{IA}(s) \ , \tag{4.1}$$

where $A=1,2,3,4$; and $I=1,2,\ldots,N$. The evolution parameter s is discussed in detail in the earlier work;[4,5] we remark here again that it is not to be confused with physical time. The Lagrangian takes the form

$$L=-\left[4Nm^2c^2V(x_{11},x_{12},x_{13},\ldots)\sum_{I=1}^{N}\sum_{A=1}^{4}(\dot{x}_{IA})^2\right]^{1/2} , \tag{4.2}$$

which is seen to be a simple generalization of *the c.m. Lagrangian* (3.14) for N free particles. The potential V is a harmonic-oscillator potential of the form

$$V = 1 + \text{const} \times \sum_{I,J}^{N} \sum_{A,B}^{4} (\text{coupling})^2_{AB} (x_{IA} - x_{JB})^2 . \tag{4.3}$$

The explicit form need not concern us here[5] except that the couplings are for both attractive and repulsive harmonic-oscillator forces.

Lorentz invariance of the Lagrangian yields ten constants in s, which are

$$P_\mu = \sum_{I}^{N} \sum_{A}^{4} p_{IA\mu} , \tag{4.4a}$$

$$J^{\mu\nu} = \sum_{I}^{N} \sum_{A}^{4} (x^\mu_{IA} p^\nu_{IA} - x^\nu_{IA} p^\mu_{IA}) , \tag{4.4b}$$

where the conjugate momenta are defined by

$$p_{IA} \equiv \partial L / \partial \dot{x}_{IA} . \tag{4.5}$$

With a transformation to a new set of generalized coordinates, the Lagrangian can be expressed in terms of a set of "normal" coordinates and their velocities.[5] The resulting expression will be considered the starting point in this paper.

The set of normal coordinates are taken to be

$$y_{IA}(s), \quad W_A(s) , \tag{4.6}$$

where the y_{IA} are not all independent, but obey

$$\sum_{I=1}^{N} y_{IA} = 0 . \tag{4.7}$$

(In this way we retain the correct number of degrees of freedom.) The normal coordinates are related to the generalized coordinates x_{IA} as[5]

$$x_{IA} = y_{IA} + \{\gamma W\}_A ,$$

where

$$\gamma \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} .$$

However, we shall not be concerned with the coordinates x_{IA} in this paper.

The Lagrangian is given by[5]

$$L = -\left[4Nm^2c^2V \left[\sum_{I}^{N} \sum_{A}^{4} \dot{y}^2_{IA} + 4N \sum_{A}^{4} \dot{W}^2_A \right] \right]^{1/2} . \tag{4.8}$$

where

$$V = 1 + (4N/c^2) \left\{ \alpha^2(0) \sum_{I}^{N} \sum_{A}^{4} y^2_{IA} + 4N\alpha^2(0) W_2^2 - 4Nw^2(0) \sum_{A=3,4} W_A^2 \right\} . \tag{4.9}$$

The couplings $\alpha^2(0)$ and $w^2(0)$ are taken to be positive and correspond to repulsive and attractive harmonic-oscillator forces, respectively. The normalization factors are included for convenience. Note the absence of W_1 from the potential V.

Parametric invariance of the action implies a constraint and a choice of gauge. By choosing the form of the gauge as $\dot{W}_{10} = \text{const}$ in s, it follows that

$$L/V = \text{const in } s . \tag{4.10}$$

Let us express this constraint by defining a constant w proportional to L/V:

$$w/w(0) = \left[8N \left[\sum_{IA} \dot{y}^2_{IA} + 4N \sum_{A} \dot{W}^2_A \right] \Big/ Vc^2 \right]^{1/2} = \text{const} . \tag{4.11}$$

The momenta conjugate to the normal coordinates are defined by

$$p_{IA} \equiv \partial L / \partial \dot{y}_{IA} = 4Nm [w(0)/w] \dot{y}_{IA} , \tag{4.12a}$$

$$p_A \equiv \partial L / \partial \dot{W}_A = (4N)^2 m [w(0)/w] \dot{W}_A , \tag{4.12b}$$

where by (4.7) the p_{IA} satisfy

$$\sum_{I}^{N} p_{IA} = 0 . \tag{4.13}$$

The momentum p_1 is equal to the total momentum P given by (4.4a).[5]

Equations (4.11) and (4.12) allow the constraint to be given in the form

$$\phi = \left[4N \sum_{IA} p^2_{IA} + \sum_{A} p^2_A - (4N)^2 m^2 Vc^2 \right] = 0 . \tag{4.14}$$

The parametric invariance of the action implies that the conventional Hamiltonian vanishes, i.e.,

$$H = \sum_{IA} \dot{y}_{IA} (\partial L / \partial \dot{y}_{IA}) + \sum_{A} \dot{W}_A (\partial L / \partial \dot{W}_A) - L = 0 . \tag{4.15}$$

According to Dirac's theory of constraints,[10] the Hamiltonian in this case has the form

$$H = v\phi, \quad v = \text{const in } s . \tag{4.16}$$

Here, H is not identified as the energy; it is the generator of the evolution of the system in terms of the parameter s. Although $H=0$, the functional dependence of ϕ on y_{IA}, W_A, and their conjugate momenta allows us to calculate the equations of motion in the canonical way; i.e.,

$$\dot{y}_{IA} = \partial H / \partial p_{IA}, \quad \dot{p}_{IA} = -\partial H / \partial y_{IA} , \tag{4.17a}$$

$$\dot{W}_A = \partial H / \partial p_A, \quad \dot{p}_A = -\partial H / \partial W_A . \tag{4.17b}$$

The choice of the constant v is equivalent to choosing a gauge. We shall choose

$$v = [1/2m(4N)^2] w / w(0) , \tag{4.18}$$

V. HAMILTONIAN APPROACH AND QUANTUM CONDITIONS

With the choice of gauge (4.18), the Hamiltonian (4.16) becomes

$$H=[1/2m(4N)^2][w/w(0)]\phi\ , \tag{5.1}$$

where $w/w(0)$ is given by (4.11) and ϕ by (4.14). Substituting the expression for V given by (4.9) into ϕ, we obtain the explicit expression for H in terms of the normal coordinates and their conjugate momenta:

$$H=[1/2m(4N)^2][w/w(0)]\left[p_1^2+[p_2^2-(4N)^4m^2\alpha^2(0)W_2^2]+\sum_{A=3,4}[p_A^2+(4N)^4m^2w^2(0)W_A^2]\right.$$
$$\left.+4N\sum_I\sum_A[p_{IA}^2-(4N)^2m^2\alpha^2(0)y_{IA}^2]-(4Nmc)^2\right]\ , \tag{5.2}$$

where $\sum_I^N y_{IA}=\sum_I^N p_{IA}=0$. Here, the nature of the harmonic-oscillator forces are clear: W_3 and W_4 describe attractive harmonic-oscillator motion while W_2 and y_{IA} correspond to repulsive harmonic oscillators.

We shall quantize by imposing quantum conditions on the *normal coordinates and their conjugate momenta:*

$$[y_{IA\mu},p_{IA\nu}]=i\hbar g_{\mu\nu},\quad [W_{A\mu},p_{A\nu}]=i\hbar g_{\mu\nu}\ . \tag{5.3}$$

Now define the operators $\tilde{a}_2$, $\tilde{b}_2$, $\tilde{a}_{IA}$, $\tilde{b}_{IA}$, $\tilde{a}_A$, and $\tilde{a}_A^\dagger$ through the relations

$$W_2=[1/4N\alpha(0)][\hbar w(0)/2m]^{1/2}(\tilde{a}_2+\tilde{b}_2)\ , \tag{5.4a}$$

$$p_2=4N[m\hbar w(0)/2]^{1/2}(\tilde{a}_2-\tilde{b}_2)\ , \tag{5.4b}$$

$$y_{IA}=[1/4N\alpha(0)][\hbar w(0)/2m]^{1/2}(\tilde{a}_{IA}+\tilde{b}_{IA})\ , \tag{5.4c}$$

$$p_{IA}=[m\hbar w(0)/2]^{1/2}(\tilde{a}_{IA}-\tilde{b}_{IA})\ , \tag{5.4d}$$

$$W_A=[1/4Nw(0)][\hbar w(0)/2m]^{/2}(\tilde{a}_A+\tilde{a}_A^\dagger),\quad A=3,4\ , \tag{5.4e}$$

$$p_A=-i4N[m\hbar w(0)/2]^{1/2}(\tilde{a}_A-\tilde{a}_A^\dagger),\quad A=3,4\ . \tag{5.4f}$$

The quantum conditions yield the commutation relations

$$\begin{aligned}&[\tilde{a}_{A\mu},\tilde{a}_{A\nu}^\dagger]=g_{\mu\nu},\quad [\tilde{a}_{2\mu},\tilde{b}_{2\nu}]=-i[\alpha(0)/w(0)]g_{\mu\nu}\ ,\\ &[\tilde{a}_{IA\mu},\tilde{b}_{IA\nu}]=-i4N[\alpha(0)/w(0)]g_{\mu\nu}\ .\end{aligned} \tag{5.5}$$

Substituting (5.4) into the expression for H in (5.2) yields

$$H=\tfrac{1}{2}[w/w(0)]\left[[p_1^2/m(4N)^2]-mc^2-\hbar w(0)\left[(\tilde{a}_2\cdot\tilde{b}_2+\tilde{b}_2\cdot\tilde{a}_2)-\sum_{A=3,4}(\tilde{a}_A\cdot\tilde{a}_A^\dagger+\tilde{a}_A^\dagger\cdot\tilde{a}_A)\right.\right.$$
$$\left.\left.+(1/4N)\sum_{IA}(\tilde{a}_{IA}\cdot\tilde{b}_{IA}+\tilde{b}_{IA}\cdot\tilde{a}_{IA})\right]\right]\ . \tag{5.6}$$

By now, the reader may be concerned because of the appearance of the Hermitian operators $\tilde{a}_2,\tilde{b}_2$ and $\tilde{a}_{IA},\tilde{b}_{IA}$, in the repulsive harmonic-oscillator terms in H. The commutation relations (5.5) present a difficulty, since $\tilde{a}_{IA},\tilde{b}_{IA}$ (a_2,b_2) generate eigenstates of $\tilde{a}_{IA}\cdot\tilde{b}_{IA}+\tilde{b}_{IA}\cdot\tilde{a}_{IA}$ $(\tilde{a}_2\cdot\tilde{b}_2+\tilde{b}_2\cdot\tilde{a}_2)$ with complex eigenvalues. We avoid the difficulty by putting

$$\alpha(0)/w(0)\equiv\epsilon^2,\quad \epsilon^2\ \text{infinitesimal}\ . \tag{5.7}$$

We shall return to this assumption later. For the time being we continue the *formal* development of the model.

In the Heisenberg picture, the equation of motion for an operator O is given by

$$\dot{O}=(1/i\hbar)[O,H]\ . \tag{5.8}$$

With (5.6) and (5.8), along with the commutation relations (5.5), we obtain the solutions

$$p_1=(4N)^2m[w(0)/w]A=\text{const in } s\ , \tag{5.9a}$$

$$\tilde{a}_2=a_2\exp(\epsilon^2ws),\quad \tilde{b}_2=b_2\exp(-\epsilon^2ws)\ , \tag{5.9b}$$

$$\tilde{a}_A=a_A\exp(-iws),\quad \tilde{a}_A^\dagger=a_A^\dagger\exp(iws),\quad A=3,4\ , \tag{5.9c}$$

$$\tilde{a}_{IA}=a_{IA}\exp(\epsilon^2ws),\quad \tilde{b}_{IA}=b_{IA}\exp(-\epsilon^2ws)\ , \tag{5.9d}$$

where a_2, b_2, a_A, a_{IA}, b_{IA} are constant operators.

Thus, the Hamiltonian (5.6) can be expressed as

$$H=\tfrac{1}{2}[w/w(0)]\left[(4N)^2m[w(0)/w]^2A^2-mc^2-\hbar w(0)\left[(a_2\cdot b_2+b_2\cdot a_2)-\sum_{A=3,4}(a_A\cdot a_A^\dagger+a_A^\dagger\cdot a_A)+(1/4N)\sum_{IA}(a_{IA}\cdot b_{IA}+b_{IA}\cdot a_{IA})\right]\right] . \quad (5.10)$$

It follows from $\sum_I y_{IA}=\sum_I p_{IA}=0$ that

$$\sum_I^N a_{IA}=\sum_I^N b_{IA}=0 . \quad (5.11)$$

The commutation relations (5.5) become

$$[a_{A\mu},a_{A\nu}^\dagger]=g_{\mu\nu},\quad [a_{2\mu},b_{2\nu}]=-i[\alpha(0)/w(0)]g_{\mu\nu} ,$$
$$[a_{IA\mu},b_{IA\nu}]=-i4N[\alpha(0)/w(0)]g_{\mu\nu} . \quad (5.12)$$

In the classical theory, the constraint $\phi=0$ implies that $H=0$. In the quantized theory, the constraint is not applied to the operators, but as a condition for the allowed states. That is, the allowed eigenstates of H must satisfy

$$H\,|\,\bar{\Psi}\rangle=0 . \quad (5.13)$$

The Hamiltonian can be expressed as[11]

$$H=H_0+\sum_A^4 H_A+\sum_I^N\sum_A^4 H_{IA} , \quad (5.14)$$

with

$$H_0=-\tfrac{1}{2}[w/w(0)]mc^2 , \quad (5.15a)$$

$$H_1=[1/2m(4N)^2][w/w(0)]p_1^2 , \quad (5.15b)$$

$$H_2=-\tfrac{1}{2}\hbar w(a_2\cdot b_2+b_2\cdot a_2) , \quad (5.15c)$$

$$H_A=\tfrac{1}{2}\hbar w(a_A\cdot a_A^\dagger+a_A^\dagger\cdot a_A),\quad A=3,4 , \quad (5.15d)$$

$$H_{IA}=-\left[\frac{1}{2\times 4N}\right]\hbar w(a_{IA}\cdot b_{IA}+b_{IA}\cdot a_{IA}) . \quad (5.15e)$$

It follows that the eigenstates $|\,\bar{\Psi}\rangle$ can be expressed as

$$|\,\Psi(H_0,H_A,H_{IA})\rangle=\prod_{A,I}|\,H_0\rangle\,|\,H_A\rangle\,|\,H_{IA}\rangle , \quad (5.16)$$

and, from (5.16), the *eigenvalues* of H_0, H_A, and H_{IA} must satisfy

$$H_0+\sum_A^4 H_A+\sum_I^N\sum_A^4 H_{IA}=0 . \quad (5.17)$$

Note that calculating the eigenvalues of H_A and H_{IA} allows us to use (5.15) and (5.17) to obtain $w/w(0)$ in terms of the *eigenvalues*:

$$w/w(0)=4N[mc^2/\hbar w(0)]^{1/2}(A^2/c^2)^{1/2}\Big/\left[[mc^2/\hbar w(0)]+(a_2\cdot b_2+b_2\cdot a_2)-\sum_{A=3,4}(a_A\cdot a_A^\dagger+a_A^\dagger\cdot a_A)+(1/4N)\sum_{IA}(a_{IA}\cdot b_{IA}+b_{IA}\cdot a_{IA})\right]^{1/2} . \quad (5.18)$$

The operators H_0, H_A, H_{IA} are a set of commuting operators, but not the complete set which would include angular momentum. Angular momentum is considered in Sec. VII.

Let us summarize the problem so far. The model was first formulated in terms of a set of generalized normal coordinates $y_{IA}(s)$ and $W_A(s)$. The coordinate $W_{1\mu}$ is linear in s, and its conjugate momentum $P_{1\mu}$ corresponds to the total "four-momentum" p_μ of the system in terms of the original generalized coordinates $x_{IA\mu}$; i.e., it yields four of the ten Poincaré constants of the system. The remaining normal coordinates W_A, $A=2,3,4$, y_{IA} are the solutions of various attractive and repulsive harmonic-oscillator differential equations. It is important to realize that *these equations are not a set of decoupled equations.* (We have used the term "normal coordinates" because the differential equations in matrix form becomes diagonal in terms of them.)

The model was next quantized by assuming canonical quantum conditions for the normal coordinates and their momenta. Following this, the Hamiltonian was reexpressed in terms of p_1^2, and the set of generators a_2, b_2, a_{IA}, b_{IA}, a_A, $a_A^\dagger$, $A=3,4$. Eigenstates of H must satisfy $H\,|\,\Psi\rangle=0$.

As yet, nothing has been said concerning the physical interpretation of these eigenstates. Thus far they are only the formal eigenstates of the Hamiltonian describing a system of $4N$ normal coordinates. Later we shall show that each eigenstate can be reinterpreted in terms of *all* the four-momenta of observable composite particles in the asymptotic regions corresponding to physical times $t=\pm\infty$. The state vector will be interpreted in a way somewhat analogous to a bound-state problem, e.g., eigenstates for given energy levels of the hydrogen atom. The reason for this is because we are working in the Heisenberg picture, and the state vector is a stationary state in s.

Before coming to this description, it is necessary to add invariant supplementary conditions, which, as we shall see are very similar to the set of invariant conditions placed on solutions for a free relativistic string (the "gauge" conditions). The supplementary conditions in the constituent model also consist of imposing a set of orthonormal relations.

VI. INVARIANT SUPPLEMENTARY CONDITIONS

We have stated that the $4N$ model is developed along lines similar to the free relativistic string model. Let us outline very briefly this development. For a free string described by the function $x_\mu(\tau,\sigma)$, the classical action is given by[9]

$$I=-A\int d\tau\int_0^{\sigma_0} d\sigma[(\dot{x}x')^2-\dot{x}^2x'^2]^{1/2}\ , \quad (6.1)$$

where A is a constant, $\dot{x}_\mu=dx_\mu/d\tau$, and $x'_\mu=dx_\mu/d\sigma$. The action is invariant under a change of variables

$$\tau\rightarrow f(\tau,\sigma),\quad \sigma\rightarrow h(\tau,\sigma)\ , \quad (6.2)$$

where f and h are arbitrary functions. In order to define an orthonormal coordinate system on the two-dimensional surface generated in space-time by the string, invariant supplementary conditions are applied:

$$(\dot{x}\pm x')^2=0\ . \quad (6.3)$$

With these conditions the action is still invariant under conformal transformations of τ and σ. It can be shown[9] that the general solution to the problem has the form

$$x_\mu(\tau,\sigma)=r_\mu\tau+f_\mu(\tau+\sigma)+f_\mu(\tau-\sigma)\ , \quad (6.4)$$

where r_μ is a constant vector and $f_\mu(u)$ is a differentiable vector function. From the boundary conditions at $\sigma=0$ and σ_0, and from (6.3), it can be shown that

$$f_\mu(u)=f_\mu(u+2\sigma_0)\ , \quad (6.5a)$$

$$(r_\mu+2df_\mu/d\sigma)^2=0\ . \quad (6.5b)$$

The conformal gauge symmetries lead to two constraints in the theory, which, because the constraints must hold over the entire range of σ, actually represent an infinite number of constraints. These, plus the (infinite number of) conditions (6.5) are sufficient to suppress the (infinite number of) time oscillations. If we choose the gauge $x_0=\tau$, the time-oscillations vanish in the rest frame of the string.

In the case of the constituent model, there is only one "bead" on the string, which is equivalent to eliminating the parameter σ. Therefore, we shall *impose* a set of invariant supplementary conditions analogous to (6.5). We first define the functions

$$r_{IA}^{\pm}=\pm\exp(\mp\beta s)(d/ds)[y_{IA}(s)\pm(1/\beta)\dot{y}_{IA}(s)]\ , \quad (6.6a)$$

$$r_2^{\pm}=\pm\exp(\mp\beta s)(d/ds)[W_2(s)\pm(1/\beta)\dot{W}_2(s)]\ . \quad (6.6b)$$

From (5.4) and (5.9),

$$r_{IA}^{+}=(1/4N)[w/w(0)][\hbar w(0)/2m]^{1/2}\times\begin{cases}a_{IA}\\ b_{IA}\end{cases} \quad (6.7a)$$

and

$$r_2^{\pm}=(1/4N)[w/w(0)][\hbar w(0)/2m]^{1/2}\times\begin{cases}a_2\\ b_2\end{cases}\ . \quad (6.7b)$$

Now impose the conditions

$$(r_{IA}^{\pm}+dW/ds)^2=0,\quad A=1,3\ , \quad (6.8)$$

$$[(r_{IA}^{\pm}+r_2^{\pm})+dW/ds]^2=0,\quad A=2,4\ . \quad (6.9)$$

The analogy to (6.5b) will become clearer after the constituent solutions are defined in (7.1).

In terms of the constant operators a_{IA}, b_{IA}, etc., (6.8) and (6.9) imply the relations

$$a_3^2=0\ , \quad (6.10a)$$

$$a_{IA}\cdot a_3=b_{IA}\cdot a_3=0,\quad A=1,3\ , \quad (6.10b)$$

$$a_{IA}^2=b_{IA}^2=-(a_3\cdot a_3^\dagger+a_3^\dagger\cdot a_3),\quad A=1,3\ , \quad (6.10c)$$

and

$$a_4^2=0\ , \quad (6.11a)$$

$$a_2\cdot a_4=b_2\cdot a_4=0\ , \quad (6.11b)$$

$$a_{IA}\cdot a_4=b_{IA}\cdot a_4=0,\quad A=2,4\ , \quad (6.11c)$$

$$(a_{IA}+a_2)^2=(b_{IA}+b_2)^2=-(a_4\cdot a_4^\dagger+a_4^\dagger\cdot a_4),\quad A=2,4\ . \quad (6.11d)$$

These can be rephrased as a set of orthonormal relations for Hermitian operators by defining, e.g., $a_A=M_A+iN_A$.

To this we shall add one more condition which will allow for later interpretation of composites and anticomposites in terms of increasing and decreasing s (Ref. 5):

$$r_2^{+}+r_2^{-}=\epsilon^2\ . \quad (6.12)$$

This in turn implies

$$a_2=-b_2+\epsilon^2\ . \quad (6.13)$$

The conditions (6.8)–(6.13) are all to be understood as operators acting on the state vector $|\Psi\rangle$ as in (5.13).

The next step is to reformulate the model in terms of constituent and composite particles.

VII. CONSTITUENT PARTICLES AND THE FORMATION OF COMPOSITE PARTICLES

Following earlier work[5] we *define* the physical constituent four-vectors by

$$Q_{IA}=y_{IA}+\{\delta W\}_A,\quad \sum_I^N y_{IA}=0\ , \quad (7.1)$$

where

$$\delta=\begin{vmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{vmatrix}=2\delta^{-1}\ . \quad (7.2)$$

The inverse relations are

$$W_A=(1/N)\left[\delta^{-1}\sum_{I=1}^{N}Q_I\right]_A\ ,$$
$$y_{IA}=Q_{IA}-(1/N)\sum_J^N Q_{IA}\ . \quad (7.3)$$

Note that the coordinates W_A are shared by all constituents. For this reason, the W_A will be denoted as the "intrinsic" coordinates of the system. Shortly, they will be related to internal quantum numbers of the composite particles. For fixed W_A, the kinematic behavior of the various constituents is determined by the y_{IA}.

The constituent solutions written out in full are

$$Q_{I1}=[1/4N\alpha(0)][\hbar w(0)/2m]^{1/2}[a_{I1}\exp(\beta s)+b_{I1}\exp(-\beta s)]+As+B$$
$$+[1/4Nw(0)][\hbar w(0)/2m]^{1/2}[a_3^\dagger\exp(iws)+a_3\exp(-iws)]\ , \tag{7.4a}$$

$$Q_{I3}=[1/4N\alpha(0)][\hbar w(0)/2m]^{1/2}[a_{I3}\exp(\beta s)+b_{I3}\exp(-\beta s)]+As+B$$
$$-[1/4Nw(0)][\hbar w(0)/2m]^{1/2}[a_3^\dagger\exp(iws)+a_3\exp(-iws)]\ , \tag{7.4b}$$

$$Q_{I2}=[1/4N\alpha(0)][\hbar w(0)/2m]^{1/2}[(a_{I2}+a_2)\exp(\beta s)+(b_{I2}+b_2)\exp(-\beta s)]$$
$$+[1/4Nw(0)][\hbar w(0)/2m]^{1/2}[a_4^\dagger\exp(iws)+a_4\exp(-iws)]\ , \tag{7.4c}$$

$$Q_{I4}=[1/4N\alpha(0)][\hbar w(0)/2m]^{1/2}[(a_{I4}+a_2)\exp(\beta s)+(b_{I4}+b_2)\exp(-\beta s)]$$
$$-[1/4Nw(0)][\hbar w(0)/2m]^{1/2}[a_4^\dagger\exp(iws)+a_4\exp(-iws)]\ , \tag{7.4d}$$

and we have defined

$$\beta\equiv[\alpha(0)/w(0)]w=\epsilon^2 w\ . \tag{7.5}$$

The four-vector W_A will be associated with the intrinsic properties of the constituents in the following way.[5] Denote the coefficients of W_1 and W_2 as the P number and X number, respectively; and the coefficients of W_3 and W_4 as the L number and B number, respectively. Then the constituents are characterized by

$$Q_{I1}:\ P=1,\ L=1\ ,\qquad Q_{I3}:\ P=1,\ L=-1\ ; \tag{7.6}$$

$$Q_{I2}:\ X=1,\ B=1\ ,\qquad Q_{I4}:\ X=1,\ B=-1\ . \tag{7.7}$$

Although the $4N$ model has been chosen mainly for heuristic purposes, it is also intended to provide a first approximation to particle interactions. The quantum numbers are intended to have the following connotations: P for "pointlike;" X for "extended;" L for "lepton;" and B for "baryon." The reasons for this will become clear later on.

It now remains to formulate the criteria for the interpretation of two constituents as a composite particle in a given region of space-time. For the $4N$ model, this means the formation of asymptotic-free composites. Therefore, as a first criterion, let us specify that the total four-vector of the two constituents must describe a straight-line trajectory in space-time. Second, their relative vector must be localized in space-time. Therefore, we shall require that their relative vector contain oscillatory terms only. Third, the internal motion of the two constituents must be independent of kinematics. Fourth, the composite four momentum must satisfy a relation of the form $P^2=M^2c^2$. And, finally, we shall demand that only one angular momentum be associated with each composite, which is equivalent to saying that the degeneracy of the composite mass operator be removed.

Examination of the constituent solutions (7.4) shows that, potentially, such composite particles can be formed from a Q_{I1} and Q_{I3}, or from a Q_{I2} and a Q_{I4}. Thus, we arrive at the following condition (which is necessary but not sufficient).

Two-body X composite particles are constructed from constituents of like X number and opposite B number, while P composites consist of constituents of like P number and opposite L number. Thus, the two-body X and P composites are characterized by zero B and L number, respectively.

Consider a specific example in which two constituents Q_{I2} and Q_{J4} have solutions such that $a_{I2}=a_{J4}$. Then, as $s\to\infty$, a free two-body system is formed from these two constituents, described by the four-vector

$$Q^+(IJ)=\tfrac{1}{2}(Q_{I2}+Q_{J4})\,|_{s\to+\infty}=[1/4N\alpha(0)][\hbar w(0)/2m]^{1/2}(a_{I2}+a_2)\exp(\beta s)\ . \tag{7.8}$$

Note that this solution describes a straight-line in Minkowski space. The internal state of this system is described by the relative vector

$$q(IJ)=(Q_{I2}-Q_{J4})\,|_{s\to\infty}=[1/2Nw(0)][\hbar w(0)/2m]^{1/2}[a_4\exp(-iws)+a_4^\dagger\exp(iws)]\ , \tag{7.9}$$

which satisfies the second condition. The third criterion for $Q^\dagger(IJ)$ to be a composite particle implies that the frequency w must be independent of the terms $a_{IA}\cdot b_{IA}+b_{IA}\cdot a_{IA}$. We will return to this requirement in Sec. IX when the initial-value problem is considered. For the present, assume that these terms vanish.

From the discussion of Sec. III we must distinguish the composite's physical four-momentum from the canonical four-momentum. In order to have a measurable four-momentum which corresponds to an isolated free composite, it will be defined in analogy to (3.33). We take the physical four-momentum of the two-body system to be

$$P^+(IJ)=\sqrt{2}\,mc\,(a_{I2}+a_2)\equiv P_I^\dagger\ . \tag{7.10}$$

(An arbitrary overall multiplicative constant is allowed in defining the momenta, since only relative masses can be measured.) From (6.11c),

$$(P_I^\dagger)^2=-\tfrac{1}{2}(2mc)^2(a_4\cdot a_4^\dagger+a_4^\dagger\cdot a_4)\ . \tag{7.11}$$

We therefore identify the composite mass as

$$M_{24}^2=-\tfrac{1}{2}(2m)^2(a_4\cdot a_4^\dagger+a_4^\dagger\cdot a_4)\ . \tag{7.12}$$

Intuitively we could have arrived at this expression, since the term H_4 in H describes the internal motion of the composite's constituent particles.

In a similar manner, a composite $Q^-(IJ)$ can be formed, having the four-vector

$$Q^-(IJ)=[1/4N\alpha(0)][\hbar w(0)/2m]^{1/2}(b_{I2}+b_2)\exp(-\beta s)\ , \tag{7.13}$$

and four-momentum

$$P_I^-=\sqrt{2}\,mc(b_{I2}+b_2)\ . \tag{7.14}$$

From (6.11c) it follows that this composite also has mass M_{24}.

P composites can be defined in the analogous way and have mass M_{13} given by

$$M_{13}^2=-\tfrac{1}{2}(2m)^2(a_3\cdot a_3^\dagger+a_3^\dagger\cdot a_3)\ . \tag{7.15}$$

Thus, for a given solution W_3 (or W_4), all two-body P (or X) composites have the same mass, respectively, which can be real or imaginary. This will not present a problem, as we shall later see that real and imaginary masses are decoupled. However, the *existence* of imaginary mass composites is necessary in order to have a crossing-symmetric scattering amplitude, as will be seen in Sec. XI.

For convenience some choices of notation will be made. Define the number operators

$$n_{Ai}\equiv a_{Ai}^\dagger a_{Ai},\quad n_{A0}\equiv a_{A0}^\dagger a_{A0}\ , \tag{7.16a}$$

$$n_A\equiv -a_A^\dagger\cdot a_A=\sum_i n_{Ai}-n_{A0}\ . \tag{7.16b}$$

Then the mass formulas (7.12) and (7.15) can be expressed as

$$M_{BA}^2=(2m)^2(n_A+1)\ . \tag{7.17}$$

Consider now the case when the composite mass is real, i.e., $M_{BA}^2>0$. Then it is possible to make a Lorentz transformation to the rest frame of the composite, in which case (6.10) or (6.11) imply

$$a_{A0}\,|\,\Psi\rangle=0 \tag{7.18}$$

and

$$\mathbf{a}_A^2\,|\,\Psi\rangle=0\ . \tag{7.19}$$

Then, for states satisfying (7.18) and (7.19),

$$n_A=\sum_i n_{Ai}\ , \tag{7.20}$$

and n_A takes on the values 0,1,2,. . . . Thus we see that in the 4N model, the choice of gauge $W_{10}=A_0 s$ [which led to solutions in the form (5.9)] and the supplementary conditions (6.8) and (6.9) eliminate time oscillations in the rest frame of the composite particle. (See also Table I.)

The remaining condition for the formation of a composite particle is that only one angular momentum be associated with each composite particle of real mass. The mass formula (7.17) is highly degenerate, even for real mass. But the supplementary conditions have not yet been fully applied.

In analogy to the angular momentum for the three-dimensional harmonic oscillator, define the internal orbital angular momentum of the composite's constituents by

$$\mathbf{l}_A=i\hbar\mathbf{a}_A^\dagger\times\mathbf{a}_A\ . \tag{7.21}$$

Using the commutation relations $[a_A,a_A^\dagger]=1$ we obtain, from (7.21),

$$|\,\mathbf{l}_A\,|^2=\hbar^2[-a_{A1}^{\dagger 2}(a_{A2}^2+a_{A3}^2)-a_{A2}^{\dagger 2}(a_{A1}^2+a_{A3}^2)-a_{A3}^{\dagger 2}(a_{A1}^2+a_{A2}^2)$$
$$+2(n_{A1}n_{A2}+n_{A1}n_{A3}+n_{A2}n_{A3})+2(n_{A1}+n_{A2}+n_{A3})]\ . \tag{7.22}$$

It follows that for composites of real mass satisfying (7.19), that in the composite rest frame

$$|\,\mathbf{l}_A\,|^2\,|\,\Psi\rangle=\hbar^2 n_A(n_A+1)\,|\,\Psi\rangle\ . \tag{7.23}$$

Then, for these states, we can identify the angular momentum quantum number with n_A, i.e.,

$$l_A=n_A\ . \tag{7.24}$$

Substituting (7.24) into (7.23) and solving for l_A,

$$l_A=(M_{BA}/2m)^2-1\ . \tag{7.25}$$

If, for X composites (mesons), we assume that the M_{BA} lie on the leading Regge trajectory, and we identify l_A with the internal orbital angular momentum of the nonrelativistic quark model, then comparing (7.25) to the known resonances in Table II indicates good agreement with experiment. We will return to further consideration of (7.25) in Sec. XI where the X-composite scattering amplitude is calculated.

VIII. SOLUTIONS CORRESPONDING TO $p_{1\mu}=0$: CONSTITUENTS AS "STRINGS"

Looking ahead, we will later reformulate the constituent description of the system in terms of composite particles. That is, the constituent description will be replaced by a composite-particle description in which *each* state vector will be an eigenvector of the four-momenta of all the (asymptotic) composites of the system. However, it is clear from the expressions for composite four-momenta given in (7.10) and (7.14), and from the commutation relations (5.12), that such cannot be the case at present. For $|\Psi\rangle$ cannot be an eigenstate of $P_I^{\pm}$ unless $[a_2,b_2]=0$ and $[a_{IA},b_{IA}]=0$. Therefore, let us return now to examine the commutation relations (5.12):

$$\begin{aligned}&[a_{2\mu},b_{2\nu}]=-i\epsilon^2 g_{\mu\nu}\ ,\\&[a_{IA\mu},b_{IA\nu}]=-i4N\epsilon^2 g_{\mu\nu}\ .\end{aligned}\tag{5.12}$$

By allowing ϵ^2 to vanish, the a_2, b_2, a_{IA}, b_{IA} will become commuting operators, in which case the $P_I^{\pm}$ commute with H. At the same time, by (7.8) and (7.13), the Q_{IJ} become singular and therefore unmeasurable. Considerable care must be exercised in taking this limit, however, for if the frequency w is finite, then

$$\exp(\pm\beta s)=\exp(\pm\epsilon^2 ws)\underset{\epsilon^2\to 0}{\sim}1\pm\epsilon^2 ws\ .\tag{8.1}$$

In the limit $\epsilon^2=0$, all possible composite four-vectors become linear in s, *for all s*. In other words, the system consists of free, noninteracting composites.

Therefore, we shall assume that at the same time as ϵ^2 tends to zero, the frequency w tends to infinity, that is,

$$w\underset{\epsilon^2\to 0}{\sim}1/\epsilon\ ,\tag{8.2}$$

so that

$$\beta=\epsilon^2 w\underset{\epsilon^2\to 0}{\sim}\epsilon\ .\tag{8.3}$$

Then β tends to zero less rapidly than ϵ^2. *Therefore, to first order, we shall treat the a_2, b_2, a_{IA}, b_{IA} as commuting operators, but retain β as nonzero until the end of the calculation.*

All of this is accomplished by postulating that the physical solutions of the system correspond to the Poincaré constants $p_{1\mu}=0$. That is, the eigenstates of H must also satisfy

$$p_{1\mu}|\Psi\rangle=\text{const}\times\epsilon|\Psi\rangle\ .\tag{8.4}$$

(Recall that $p_{1\mu}=P_\mu$, the total "four-momentum" of the system in terms of the *original* generalized coordinates $x_{IA\mu}$.[5]) Therefore, the symmetry of the system is considerably enlarged.

From (8.4) it follows that $p_1^2\sim\epsilon^2$, and thus, from the solution (5.9a) for $p_{1\mu}$, that $w/w(0)\sim 1/\epsilon$, as long as the constant A is finite. Then, since H must vanish, Eq. (5.10) for H implies that

$$\left[[mc^2/\hbar w(0)]+a_2\cdot b_2+b_2\cdot a_2-\sum_{A=3,4}(a_A\cdot a_A^\dagger+a_A^\dagger\cdot a_A)+\sum_{IA}(a_{IA}\cdot b_{IA}+b_{IA}\cdot a_{IA})\right]\sim\epsilon^2\ ,\tag{8.5}$$

and, thus,

$$H\sim\epsilon\ .\tag{8.6}$$

That is, we must relax the condition $H|\Psi\rangle=0$ to

$$H|\Psi\rangle=\text{const}\times\epsilon|\Psi\rangle\ .\tag{8.7}$$

The eigenstates of H are now taken to be eigenstates also of the operators a_{IA}, b_{IA}, a_2, b_2, and $a_A^\dagger a_A$, $A=3,4$.

The eigenvectors $|\Psi\rangle$ of H are conveniently expressed as

$$|\Psi\rangle=|n_3\rangle|n_4\rangle|a_2\rangle|b_2\rangle\prod_I|a_{IA}\rangle|b_{IA}\rangle\ ,\tag{8.8}$$

or, equivalently,

$$|\Psi\rangle=|n_3;n_4;a_2;b_2;\prod_{IA}a_{IA};b_{IA}\rangle\ ,\tag{8.9}$$

where the following are to be understood:

$$|n_3\rangle=|n_{31}\rangle|n_{32}\rangle|n_{33}\rangle|n_{30}\rangle,\ \text{etc.}\tag{8.10}$$

The states are assumed to be orthonormal, i.e.,

$$\langle n'_A|n_A\rangle=\delta_{n'_A n_A}\ ,\tag{8.11a}$$

$$\langle a'_2|a_2\rangle=\delta(a'_2-a_2)\ ,\tag{8.11b}$$

$$\langle b'_2|b_2\rangle=\delta(b'_2-b_2)\ ,\tag{8.11c}$$

$$\langle a'_{IA}|a_{IA}\rangle=\delta(a'_{IA}-a_{IA})\ ,\tag{8.11d}$$

$$\langle b'_{IA}|b_{IA}\rangle=\delta(b'_{IA}-b_{IA})\ .\tag{8.11e}$$

Recalling that the $a_{IA}(b_{IA})$ are not all independent, we shall express the general eigenstates $|\Psi\rangle$ as

$$|\Psi\rangle=\prod_{I=1}^{N}\prod_{A=1}^{4}\delta\left[\sum_J a_{JA}\right]\delta\left[\sum_J b_{JA}\right]|n_3;n_4;a_2;b_2;a_{IA};b_{IA}\rangle\ .\tag{8.12}$$

These states must satisfy (8.7). Since H is constant in s, the state vector in the Schrödinger picture is related to the state vector in the Heisenberg picture by

$$|\Psi_S\rangle = \exp(-iHs)|\Psi_H\rangle \ . \tag{8.13}$$

That is, $|\Psi_S\rangle$ is a stationary state in s, and in the limit $\epsilon \to 0$, $|\Psi_S\rangle = |\Psi_H\rangle$.

Now we consider the constituent trajectories as w tends to infinity. Each constituent four-vector has an oscillatory term in s, given by either W_3 or W_4. That is, the constituent trajectory in space-time bears a resemblance to a "slinky" toy (the oscillatory terms have been suppressed in the schematic diagrams of Fig. 1). As $w \to \infty$, these oscillations in Minkowski space become infinitely rapid, and in the limit, the constituent trajectories actually map out two-dimensional surfaces in space-time (see also, Ref. 6). In other words, the constituents can be reinterpreted as strings.

A word of caution is due here. The oscillations in s, given by the terms $\exp(\pm iws)$, should not be confused with the oscillatory modes of a physical string. In the $4N$ model, for instance, a better name than "string" might be a "rigid rod," or "rigid string," as the "string" itself is in a zero-frequency mode.

It is important to recognize that the "strings" of the constituent theory are not equivalent to conventional strings defined, for example, in superstring theories. Because the conventional string is really an infinite number of "beads," it has an infinite number of modes of oscillation. That is, in general, a string solution can be decomposed into a Fourier series. The constituent "string" solution cannot.

IX. REFORMULATION IN TERMS OF COMPOSITE PARTICLES

A great deal of groundwork has been done, first in terms of the normal coordinates, then the commuting operators a_2, b_2, a_{IA}, b_{IA}, $a_A^\dagger a_A$, and finally in terms of the constituent particles. We are now in a position to consider the physical interpretation of the model. In doing so, several questions must be answered. (1) Under what conditions are composite particles formed in some region of Minkowski space? (2) How may we explicitly demonstrate that couplings between composites of real and imaginary mass are ruled out? (3) Can we eliminate solutions in which constituents can be isolated from the remainder of the system?

We shall now consider these questions in detail for systems corresponding to $N=2$. The generalization to arbitrary N will be discussed later. For $N=2$, there are just eight constituents:

$$Q_{11}, Q_{13}, Q_{21}, Q_{23} (P \text{ constituents}) \tag{9.1a}$$

and

$$Q_{12}, Q_{14}, Q_{22}, Q_{24} (X \text{ constituents}) \ . \tag{9.1b}$$

Their solutions take the form

$$Q_{I1} = [a_{I1}\exp(\beta s) + b_{I1}\exp(-\beta s)] + (As + B) + W_3 \ , \tag{9.2a}$$

$$Q_{I3} = [a_{I3}\exp(\beta s) + b_{I3}\exp(-\beta s)] + (As + B) - W_3 \ , \tag{9.2b}$$

$$Q_{I2} = [(a_{I2} + a_2)\exp(\beta s) + (b_{I2} + b_2)\exp(-\beta s)] + W_4 \ , \tag{9.2c}$$

$$Q_{I4} = [(a_{I4} + a_2)\exp(\beta s) + (b_{I4} + b_2)\exp(-\beta s)] - W_4 \ . \tag{9.2d}$$

P composites can be formed from a Q_{I1} and a Q_{J3}, while the X composites can be formed from a Q_{I2} and a Q_{J4} (see Sec. VI). From the mass formulas (7.12) and (7.15) it follows that composites can have both real and imaginary masses.

In order to keep in mind the physical picture that will emerge, it may be helpful to consult the scattering diagrams of Fig. 1. There, the scattering of two X composites is schematically illustrated. The arrows on the trajectories indicate increasing s. At $s = -\infty$, Q_{12} and Q_{22} pair up to form composite Q_1^-, and Q_{22} and Q_{24} pair up to form composite Q_2^-. Similarly, at $s = +\infty$, Q_{12} and Q_{24} pair up to form Q_1^+, while Q_{14} and Q_{22} pair up to form Q_2^+. If the time axis is taken vertically, the diagrams can be interpreted as follows. In Fig. 1(a) two composites come together and exchange a constituent such that two new composites emerge. In Figs. 1(b) and 1(c), the two incoming (outgoing) composites "share" a constituent (possible because the constituent turns around in time). These last two diagrams can be reinterpreted as illustrating constituent pair annihilation followed by pair creation.

We shall now *define* composite and anticomposite particles as having four-vectors with time components increasing or decreasing with increasing s, respectively. Thus, Fig. 1(a) represents the scattering of two composites or two anticomposites, while Figs. 1(b) and 1(c) represent composite-anticomposite scattering. Note that in the constituent model, a composite (meson) is *not* constructed from a constituent (quark) and an anticonstituent (antiquark). The composite contains two constituents of opposite B number (see Sec. VI).

Returning now to the formation of composite particles we consider the initial-value problem in s. A distinction must be made between initial conditions in s and the physical initial conditions set or observed in the laboratory at an initial "time." The latter consist of specifying what composite particles exist and their four-momenta, at time, say, $t = -\infty$. We will come to the physical initial problem in Sec. X. At present we are concerned with initial conditions (or, equivalently, boundary conditions) in s, which will be regarded as inherent to the system.

Although the equations of motion for the normal coordinates are nonlinear (linear superpositions of solutions are not solutions), with a little algebra one is easily convinced that specifying, say,

$$y_{IA}(s_0), \quad \dot{y}_{IA}(s_0), \quad W_A(s_0), \quad \text{and} \quad \dot{W}_A(s_0) \tag{9.3}$$

will uniquely determine the solution. Alternatively, the same is true if we specify the normal coordinates at two different values of s:

$$y_{IA}(s_0),\quad y_{IA}(s_1),\quad W_A(s_0),\quad \text{and } W_A(s_1)\ . \tag{9.4}$$

(In fact, the boundary conditions of the earlier work in Refs. 4 and 5 take the form of the above with $s_0=-\infty$, and $s_1=+\infty$.)

We do not completely specify the initial conditions, but, instead, impose

$$r_{I1}^{\pm}(s)=\mp r_{I3}^{\pm}(s)\ , \tag{9.5a}$$

$$r_{I2}^{\pm}(s)=\mp r_{I4}^{\pm}(s)\ . \tag{9.5b}$$

These conditions still allow a very wide class of solutions. The conditions (9.5) have been imposed in order to force the constituents to pair up into composite particles at $s=\pm\infty$; they do not eliminate composites of imaginary mass.

Now consider the consequences of the conditions (9.5) on the solutions for the $N=2$ system. Substitution of the normal coordinates given by (5.4) and (5.9) into (9.5a)–(9.5c) yields

$$\begin{aligned} &a_{11}=-a_{13},\quad a_{21}=-a_{23}\ ,\\ &b_{11}=b_{13},\quad b_{21}=b_{23}\ ,\end{aligned} \tag{9.6}$$

$$\begin{aligned} &a_{12}=-a_{14},\quad a_{22}=-a_{24}\ ,\\ &b_{12}=b_{14},\quad b_{23}=b_{24}\ ,\end{aligned} \tag{9.7}$$

$$a_2=-b_2\ . \tag{9.8}$$

Substituting (9.6)–(9.8) into the expressions for the constituent four-vectors (9.5) suggests that the constituents pair up into composite particles at $s=\pm\infty$. At $s=-\infty$, define composite four-vectors as

$$Q_1^-(1,3)\equiv\tfrac{1}{2}(Q_{11}+Q_{13})\,|_{s\to-\infty} =b_{11}\exp(-\beta s)\ , \tag{9.9a}$$

$$Q_2^-(1,3)\equiv\tfrac{1}{2}(Q_{21}+Q_{23})\,|_{s\to-\infty} =-b_{11}\exp(-\beta s)\ , \tag{9.9b}$$

$$Q_1^-(2,4)\equiv\tfrac{1}{2}(Q_{12}+Q_{14})\,|_{s\to-\infty} =(b_{12}-a_2)\exp(-\beta s)\ , \tag{9.9c}$$

$$Q_2^-(2,4)\equiv\tfrac{1}{2}(Q_{22}+Q_{24})\,|_{s\to-\infty} =(-b_{12}-a_2)\exp(-\beta s)\ . \tag{9.9d}$$

Note that these solutions describe straight-line trajectories in Minkowski space.

The internal states of these two-constituent systems are described by the corresponding relative vectors:

$$q_1^-(1,3)\equiv\tfrac{1}{2}(Q_{11}-Q_{13})\,|_{s\to-\infty}=W_3\ , \tag{9.10a}$$

$$q_2^{-1}(1,3)\equiv\tfrac{1}{2}(Q_{21}-Q_{23})\,|_{s\to-\infty}=W_3\ , \tag{9.10b}$$

$$q_1^-(2,4)\equiv\tfrac{1}{2}(Q_{12}-Q_{14})\,|_{s\to-\infty}=W_4\ , \tag{9.10c}$$

$$q_2^-(2,4)\equiv\tfrac{1}{2}(Q_{22}-Q_{24})\,|_{s\to-\infty}\equiv W_4\ . \tag{9.10d}$$

We now note the important fact that the relations (9.6) and (9.7) imply that

$$\sum_I\sum_A a_{IA}\cdot b_{IA}=0\ , \tag{9.11}$$

which, in turn, implies that the *frequency w is free of kinematics.* From the expression (5.18) it follows that w takes the form

$$w/w(0)=4N[mc^2/\hbar w(0)]^{1/2}(A^2/c^2)^{1/2}\left[[mc^2/\hbar w(0)]+(a_2\cdot b_2+b_2\cdot a_2)-\sum_{A=3,4}(a_A\cdot a_A^{\dagger}+a_A^{\dagger}\cdot a_A)\right]^{1/2}\ . \tag{9.12}$$

Following Sec. VII and the relations (9.6)–(9.8), the physical momenta of the composites at $s=-\infty$ are given by

$$P_1^-(1,3)=\sqrt{2}\,mcb_{11}\ , \tag{9.13a}$$

$$P_2^-(1,3)=-\sqrt{2}\,mcb_{11}\ , \tag{9.13b}$$

$$P_1^-(2,4)=\sqrt{2}\,mc(b_{12}-a_2)\ , \tag{9.13c}$$

$$P_2^-(2,4)=\sqrt{2}\,mc(-b_{12}-a_2)\ . \tag{9.13d}$$

The momenta satisfy the mass-shell constraint

$$(P_I^{\pm})^2=M_{BA}^2c^2\ . \tag{9.14}$$

A similar picture is obtained at $s\to+\infty$. In that asymptotic region the following composites are formed:

$$Q_1^+(1,3)\equiv\tfrac{1}{2}(Q_{11}+Q_{23})\,|_{s\to+\infty} =a_{11}\exp(\beta s)\ , \tag{9.15a}$$

$$Q_2^+(1,3)\equiv\tfrac{1}{2}(Q_{21}+Q_{13})\,|_{s\to+\infty} =-a_{11}\exp(\beta s)\ , \tag{9.15b}$$

$$Q_1^+(2,4)\equiv\tfrac{1}{2}(Q_{12}+Q_{24})\,|_{s\to+\infty} =(a_{12}+a_2)\exp(\beta s)\ , \tag{9.15c}$$

$$Q_2^+(2,4)\equiv\tfrac{1}{2}(Q_{22}+Q_{14})\,|_{s\to+\infty} =(-a_{12}+a_2)\exp(\beta s)\ . \tag{9.15d}$$

As before, the internal states are described by W_3 and W_4 for the P and X composites, respectively. The corresponding composite four-momenta are

$$P_1^+(1,3)=\sqrt{2}\,mca_{11}\ , \tag{9.16a}$$

$$P_2^+(1,3)=-\sqrt{2}\,mca_{11}\ , \tag{9.16b}$$

$$P_1^+(2,4)=\sqrt{2}\,mc(a_{12}+a_2)\ , \tag{9.16c}$$

$$P_2^+(2,4)=\sqrt{2}\,mc(-a_{12}+a_2)\ . \tag{9.16d}$$

And again, the composite momenta satisfy (9.14).

Recall now that "constant" up to now has meant constant in the parameter s, not physical time. Therefore, it is necessary to verify the conservation laws of four-momentum and angular momentum. Examination of (9.13) and (9.15) shows that the total four-momentum is separately conserved for the P composites and the X composites:

$$P_1^-(1,3)+P_2^-(1,3)+P_1^+(1,3)+P_2^+(1,3)=0 , \quad (9.17)$$

$$P_1^-(2,4)+P_2^-(2,4)+P_1^+(2,4)+P_2^+(2,4)=0 . \quad (9.18)$$

Each incoming (outgoing) composite has zero orbital angular momentum, since its velocity is directed along its position vector. Also, since the internal states determine the internal angular momenta of the composites, total internal angular momentum is conserved for the P (Q) system.

Thus, we conclude that the P composites are decoupled kinematically from the X composites in the 4N model. This is a desirable feature for a future interpretation of the P and X composites as decoupling strongly.

Let us consider the P composite system more carefully. Equations (9.13) and (9.16) imply, in fact, that

$$P_1^+=-P_2^+, \quad P_1^-=-P_2^- . \quad (9.19)$$

In other words, the P composite solution describes the elastic scattering of two composites in the forward-backward direction.[5,6] In this simple model, the P composites act as point particles. We suggest that in this "first approximation" the P composites bear a similarity to leptons, which do not interact strongly.

For the X composite system, however, the situation is quite different. In spite of the initial and final composites having zero-orbital angular momentum, scattering can take place at arbitrary angles (hence the terminology "X" for "extended"). The scattering angle depends on the eigenvalue of a_2^2, so that a_2 acts as a kind of "impact parameter." We will return to this interpretation in Sec. XI where the X-composite scattering amplitude is calculated.

Let us now summarize the results of this section. By partially specifying the initial conditions we limit the solutions to systems in which the constituents pair up asymptotically into composite particles. As a result we obtain independent solutions describing elastic scattering of two X composites and of two P constituents, respectively. The usual space-time conservation laws hold separately for both systems, in the sense that we calculate total momentum and angular momentum at $t=-\infty$ and $t=+\infty$ (since the composites lose their identities for finite s, it is meaningless to define "energy," "momentum," etc., except in the asymptotic regions). It is clear from the mass formulas that all X (P) composites have the same masses. Therefore, there is no coupling between real and imaginary mass particles.

Before continuing to the physical interpretation of the state vectors and the calculation of the X-composite scattering amplitude, a few remarks are in order concerning the limitations of the 4N model. Within this model, only two-body composite particles can be formed, as an attempt to define, say, four-body composite particles would lead to overlapping constituent trajectories, a situation we reject as unphysical. Further, the simple 4N model is incapable of describing interactions between composites of different mass, since the a_A and $a_A^\dagger$ are fixed in a given solution. In addition, the forward-backward scattering of the P composites is due to the lack of a term in the solution analogous to a_2.

X. INTERPRETATION OF THE PHYSICAL STATE VECTORS

We are finally in a position to interpret the state vectors $|\Psi\rangle$ in terms of physical composite states (we continue to limit the discussion to $N=2$ systems). Before doing so, let us summarize the conditions so far obtained on the eigenvalues of the commuting set of operators. From (9.8), i.e., $a_2=-b_2$, so the label b_2 can be dropped from the state vectors in (8.12). From the relations between the eigenvalues in (9.6) and (9.7), and the vanishing of H and p_1^2, we obtain, from (8.5),

$$a_2^2-[mc^2/2\hbar w(0)]+n_3+n_4+4\sim\epsilon^2 , \quad (10.1)$$

where $n_A=-a_A^\dagger\cdot a_A$.

Two new operators are now introduced:

$$n\equiv n_3+n_4+4 \quad (10.2)$$

and

$$\bar{\alpha}(a_2^2)\equiv a_2^2-[mc^2/2\hbar w(0)] . \quad (10.3)$$

Since the P-composite system corresponds to forward-backward scattering of P composites, and is decoupled from the X-composite system, we shall drop the corresponding kinematic labels from the state vectors and assume the P-composite system remains unobserved. The state vectors satisfying the conditions (6.11c), (9.7), and (10.1) can be expressed as

$$|\Psi\text{phys}\rangle=\delta(\bar{\alpha}(a_2^2)-n)\prod_{I=1}^{2}\prod_{A=2,4}\delta\left[\sum_J^2 a_{JA}\right]\delta\left[\sum_J^2 b_{JA}\right]\delta(a_{I2}+a_{I4})\delta(b_{I2}-b_{I4})$$
$$\times\delta(2(a_{IA}+a_2)^2-(n_4+1))\delta(2(b_{IA}-a_2)^2-(n_4+1))\,|n_3;n_4;a_2;a_{IA};b_{IA}\rangle . \quad (10.4)$$

We see that the state vectors $|\Psi\text{phys}\rangle$ have a very different interpretation than state vectors generally associated with scattering problems. Typically, two complete sets of state vectors are employed—the "incoming" states and the "outgoing" states of the "system," where the "system" is defined, at $t=-\infty$, as the incoming particles, and at $t=+\infty$, as the outgoing particles. One goes from the "out" basis to the "in" basis by means of a unitary transformation, namely, the S matrix. A particular "transition" is characterized by an S-matrix element, e.g.,

$$\langle \xi \text{ out} \mid \xi' \text{ in} \rangle = \langle \xi \text{ out} \mid S \mid \xi' \text{ out} \rangle = S_{\xi\xi'} \ , \tag{10.5}$$

where $\xi'(\xi)$ stand for all the quantum numbers necessary to describe the state $|\xi' \text{ in}\rangle(|\xi \text{ out}\rangle)$ as $t \to -\infty$ $(t \to +\infty)$.

In the constituent model, $|\Psi\text{phys}\rangle$ contains all the quantum numbers of all the composites, regardless of whether they exist at $t = +\infty$ or $t = -\infty$. It must be remembered that $|\Psi\text{phys}\rangle$ is a stationary state of the system, in terms of the evolutionary parameter s. The system is the totality of the $4N$ constituents, and a given solution of the equations of motion for the system yields all the trajectories of the constituents in space-time, i.e., their "orbits." We may draw an analogy, for example, to the nonrelativistic hydrogen atom. Stationary eigenstates describe the electron in particular "orbits," depending on the energy level of H (and other quantum numbers).

Therefore, it is misleading to speak of the "transition" of the composites at $t = -\infty$ to the composites at $t = +\infty$. A particular solution $|\Psi\text{phys}\rangle$ is characterized by the eigenvalues of n_3, n_4, a_2, a_{IA}, b_{IA}, which completely specify what composites exist at both $t = \pm\infty$ and what their momenta are.

Why, then, does the repeated scattering experiment of two given hadrons, for example, continually yield different results? After all, the hydrogen atom always exhibits the same characteristics in a given energy level. The answer is that not all the eigenvalues n_3, n_4, a_2, a_{IA}, b_{IA} correspond to physically measurable quantities. All that we can measure at $t = \pm\infty$ are (1) which composites are present and (2) the composite four momenta. We cannot know which pairs of constituents have combined to form each composite, and even if we did, we could only measure, for the X composites, the quantities n_4, $a_{IA} + a_2$, and $b_{IA} - a_2$. From this point of view, at least some of the eigenvalues become hidden variables.

Summarizing, the physical "initial" conditions at $t = -\infty$ do not determine the eigenstate of the system uniquely. In the language of "multiple universes," the initial conditions do not tell us what universe we are in. Measurements made, say, on one of the "final" composites at $t = +\infty$ will further reduce the number of possibilities, but still will not determine the eigenstate uniquely. It is not the *measurement* made, say, on one of the final composites which forces a second composite, a large distance away in space, to behave in a given way. it is simply the additional information gained about the possible eigenstates of the total system that allows for the deduction.

XI. SCATTERING AMPLITUDE FOR TWO-COMPOSITE ELASTIC SCATTERING

Now we shall calculate the amplitude for the elastic scattering of the two X composites. For such processes, there are four constituents involved, Q_{12}, Q_{14}, Q_{22}, and Q_{24}. The possible scattering configurations are shown in Fig. 1. These diagrams correspond to the initial conditions [see (9.7)]

$$\begin{aligned} &a_{12} = a_{24}, \quad a_{22} = a_{14} \ , \\ &b_{12} = b_{14}, \quad b_{22} = b_{24} \ . \end{aligned} \tag{11.1}$$

The composites Q_1^+, Q_1^-, Q_2^+, Q_2^- are formed as $s \to \pm\infty$, as indicated in the diagrams. Their four-vectors are defined by (9.9c) and (9.9d), and their four-momenta satisfy (9.18), i.e.,

$$P_1^+ + P_2^+ + P_1^- + P_2^- = 0 \ . \tag{9.19}$$

Recall that the signs of the momenta depend on the interpretation of the composites at time $t = -\infty$. For example, in Fig. 1(a), if the time axis is taken vertically with the time increasing upward, the $b_{120} - a_{20}$ and $b_{220} - a_{20}$ are both negative, and the composites Q_1^- and Q_2^- are interpreted to have physical momentum $-P_1^-$ and $-P_2^-$, respectively.

The physical state vectors are given by (10.4), or

$$|\Psi\text{phys}\rangle = \delta(\bar{\alpha}(a_2^2) - n) \prod_{I=1,2} \prod_{A=2,4} \delta(a_{12} - a_{24})\delta(b_{12} - b_{14})\delta(2(a_{IA} + a_2)^2 - (n_4 + 1))\delta(2(b_{IA} - a_2)^2 - (n_4 + 1))$$
$$\times \delta(a_{1A} + a_{2A})\delta(b_{1A} + b_{2A}) \mid n_3; n_4; a_2; a_{IA}; b_{IA}\rangle \ . \tag{11.2}$$

The integers n_3 and n_4 can take on both positive and negative values, so the same is true for $n = n_3 + n_4 + 4$. In the present case X composites are assumed to be observable, i.e., the states above are restricted to values of n_4 corresponding to positive (mass), i.e., $n_4 = 0, 1, 2, \ldots$.

Specifying the physical initial conditions at $t = -\infty$, i.e., the incoming composite momenta, does not determine which of the diagrams in Fig. 1 apply (assuming anticomposites are not distinguishable from composites in this model). Thus, the state $|\Psi\text{phys}\rangle$ which satisfies the initial conditions must be expressed as a linear combination of states describing all possible diagrams. Assume that each diagram in Fig. 1 has an equal probability of occurring, and that each diagram is, in principle, distinguishable from all others. Then the state is the sum (or integration) over states describing each process.

Consider first Fig. 1(a), the "s channel," or direct channel. Denote the incoming momenta as P_1 and P_2. Then $P_1 = P_1^-$ and $P_2 = P_2^-$, or vice versa. The permutations are easily added in, so the first choice will be used. From (7.14) this implies

$$\frac{P_1}{\sqrt{2}\,mc}-b_{12}+a_2=0,\quad \frac{P_2}{\sqrt{2}\,mc}+b_{12}+a_2=0\ , \tag{11.3}$$

where we have used the relations $b_{22}=-b_{12}$ and $b_2=-a_2$.

Summing all states in (11.2) which satisfy (11.3), we write

$$\begin{aligned}|\Psi_{IN}^a\rangle=\sum_{n_3=-\infty}^{\infty}\prod_{\substack{I=1,2\\A=2,4}}\int d^4a_2\int d^4a_{IA}\int d^4b_{IA}\,\delta(a_{12}-a_{24})\delta(b_{12}-b_{14})\\
\times\delta(2(a_{IA}+a_2)^2-(n_4+1))\delta(2(b_{IA}-a_2)^2-(n_4+1))\\
\times\delta(a_{1A}+a_{2A})\delta(b_{1A}+b_{2A})\delta(\bar{\alpha}(a_2^2)-n)\\
\times\delta\left[\frac{P_1}{\sqrt{2}\,mc}-b_{12}+a_2\right]\delta\left[\frac{P_2}{\sqrt{2}\,mc}+b_{12}+a_2\right]|n_3;n_4;a_2;a_{IA};b_{IA}\rangle\ .\end{aligned} \tag{11.4}$$

The momenta of the composites at $t=+\infty$ will be labeled P_3 and P_4. For the diagram of Fig. 1(a) we put

$$\frac{P_3}{\sqrt{2}\,mc}-a_{12}-a_2=0,\quad \frac{P_4}{\sqrt{2}\,mc}+a_{12}-a_2=0\ , \tag{11.5}$$

where we have used $a_{22}=-a_{12}$. Then, in analogy to (11.4), define the state $\langle\Psi_{\rm out}^a|$ by

$$\begin{aligned}\langle\Psi_{\rm out}^a|=\sum_{n_3=-\infty}^{\infty}\prod_{\substack{I=1,2\\A=2,4}}\int d^4a_2\int d^4a_{IA}\int d^4b_{IA}\,\delta(a_{12}-a_{24})\delta(b_{12}-b_{14})\\
\times\delta(2(a_{IA}+a_2)^2-(n_4+1))\delta(2(b_{IA}-a_2)^2-(n_4+1))\\
\times\delta(a_{1A}+a_{2A})\delta(b_{1A}+b_{2A})\delta(\bar{\alpha}(a_2^2)-n)\\
\times\delta\left[\frac{P_3}{\sqrt{2}\,mc}-a_{12}+a_2\right]\delta\left[\frac{P_4}{\sqrt{2}\,mc}+a_{12}-a_2\right]\langle n_3;n_4;a_2;a_{IA};b_{IA}|\ .\end{aligned} \tag{11.6}$$

Let us continue to calculate the states corresponding to the "t channel" of Fig. 1(b). Corresponding to $t=-\infty$, set

$$\frac{P_1}{\sqrt{2}\,mc}-b_{12}+a_2=0,\quad \frac{P_2}{\sqrt{2}\,mc}-a_{12}-a_2=0\ . \tag{11.7}$$

Then the state $|\Psi_{IN}^b\rangle$ is defined analogously to (11.4) with the replacement

$$\frac{P_2}{\sqrt{2}\,mc}+b_{12}+a_2\rightarrow\frac{P_2}{\sqrt{2}\,mc}-a_{12}-a_2\ .$$

At $t=+\infty$, set

$$\frac{P_3}{\sqrt{2}\,mc}+b_{12}+a_2=0,\quad \frac{P_4}{\sqrt{2}\,mc}+a_{12}-a_2=0\ . \tag{11.8}$$

Then $\langle\Psi_{\rm out}^b|$ may be obtained from (11.6) by the replacement

$$\frac{P_3}{\sqrt{2}\,mc}-a_{12}-a_2\rightarrow\frac{P_3}{\sqrt{2}\,mc}+b_{12}+a_2\ .$$

From these expressions we can calculate

$$\langle\Psi_{\rm out}^a|\Psi_{\rm in}^b\rangle=\langle\Psi_{\rm out}^b|\Psi_{\rm in}^a\rangle=0\ , \tag{11.9}$$

$$\langle\Psi_{\rm out}^a|\Psi_{\rm in}^a\rangle=\delta_{n_4'n_4}\prod_{I=1}^{4}\delta(P_I^2-M^2c^2)\delta(P_1+P_2+P_3+P_4)\sum_{n=-\infty}^{\infty}\delta\left[\bar{\alpha}\left[\frac{(P_1+P_2)^2}{4M^2c^2}\right]-n\right]\ , \tag{11.10}$$

$$\langle \Psi^b_{\rm out} \mid \Psi^b_{\rm in} \rangle = \delta_{n'_4 n_4} \prod_{I=1}^{4} \delta(P_I^2 - M^2c^2)\delta(P_1+P_2+P_3+P_4) \sum_{n=-\infty}^{\infty} \delta\left[\bar{\alpha}\left[\frac{(P_1+P_3)^2}{4M^2c^2}\right] - n\right] . \quad (11.11)$$

From the similar calculation for the "u channel" of Fig. 1(c), one finds that the states $|\Psi^c_{\rm in}\rangle$ and $\langle \Psi^c_{\rm out}|$ are orthogonal to the $|\Psi^a\rangle$ and the $|\Psi^b\rangle$ states and satisfy

$$\langle \Psi^c_{\rm out} \mid \Psi^c_{\rm in} \rangle = \delta_{n'_4 n_4} \prod_{I=1}^{4} \delta(P_I^2 - M^2c^2)\delta(P_1+P_2+P_3+P_4) \sum_{n=-\infty}^{\infty} \delta\left[\bar{\alpha}\left[\frac{(P_1+P_4)^2}{4M^2c^2}\right] - n\right] . \quad (11.12)$$

The above results are combined to give the scattering amplitude

$$A = \mathcal{N}(\langle \Psi^a_{\rm out} \mid \Psi^a_{\rm in} \rangle + \langle \Psi^b_{\rm out} \mid \Psi^b_{\rm in} \rangle + \langle \Psi^c_{\rm out} \mid \Psi^c_{\rm in} \rangle) , \quad (11.13)$$

where $\mathcal{N}$ is a normalization constant.

To express A in terms of invariant parameters s, t, and u, define

$$s = (P_1+P_2)^2, \quad t = (P_1+P_3)^2, \quad u = (P_1+P_4)^2 . \quad (11.14)$$

Also, define a "trajectory" function $\alpha(z)$ by

$$\alpha(z) \equiv \bar{\alpha}(z/4M^2c^2) = (1/4M^2c^2)z - mc^2/2\hbar w(0) . \quad (11.15)$$

Then A takes the form

$$A(s,t,u) = \mathcal{N}\delta_{n'_4 n_4} \prod_{I=1}^{4} \delta(P_I^2 - M^2c^2)\delta(P_1+P_2+P_3+P_4)[D(s)+D(t)+D(u)] , \quad (11.16)$$

where

$$D(z) = \sum_{n=-\infty}^{\infty} \delta(\alpha(z) - n) \quad (11.17)$$

and, by (7.17),

$$\alpha(z) = \frac{1}{n_4+1}\frac{z}{4(2mc)^2} - \alpha_0 , \quad (11.18)$$

with, we recall,

$$\alpha_0 \equiv mc^2/2\hbar w(0) . \quad (11.19)$$

In the s-t-u plot of Fig. 2 (the plotted regions indicate the physical regions), the solid lines correspond to nonzero terms of $D(s)$, $D(t)$, or $D(u)$. Their intersections, represented by the heavy dots, then correspond a threefold increase in scattering amplitude.

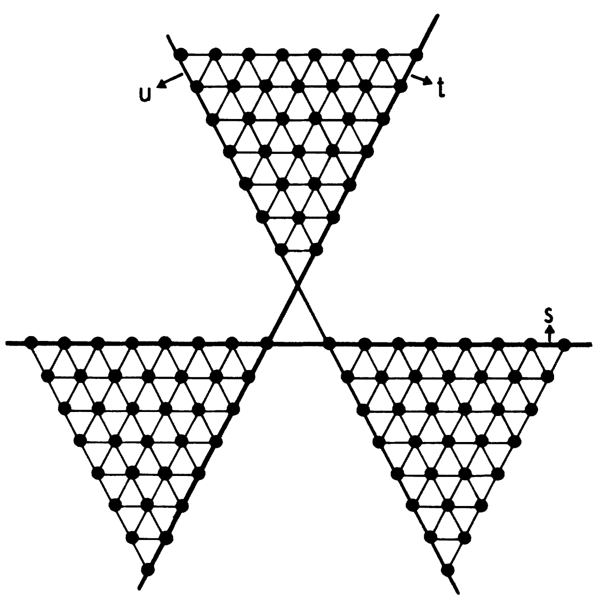

FIG. 2. Contributing physical values of s, t, and u to the invariant scattering amplitude for two-composite scattering.

In the center-of-energy system of the incoming (or outgoing) composites,

$$s = (E_{\rm CE}/c)^2 , \quad (11.20)$$

where $E_{\rm CE}$ is the energy in the CE system,

$$t = -2q^2(1-\cos\theta) , \quad (11.21)$$

where q and θ are the CE three-momentum and scattering angle, respectively, and

$$u = -2q^2(1+\cos\theta) . \quad (11.22)$$

Note that we may now allow $\epsilon \to 0$ without affecting the results.

From (10.3), (11.14), and (11.15), it follows that a_2 plays the role of an impact parameter. When $a_2 = 0$, scattering occurs only in the forward or backward direction. It should be noted that, in general, scattering can occur at arbitrary angles *even though the relative orbital angular momentum is zero for the incoming and outgoing composites.*[5] Thus, an observer unable to measure the initial (zero) orbital angular momentum may interpret the scattering as due to the composites having an interaction range and a nonzero impact parameter.

The internal angular momenta of the composites have not played a direct role in the interactions of the $4N$ model. However, the solutions to the trajectory equation

$$\alpha(s_n) = n \quad (11.23)$$

are generally interpreted as resonances of mass $\sqrt{s_n}$ and angular momentum $\alpha(s_n) = n$. Therefore, let us tentatively identify $\alpha(s_n)$ as the orbital angular momentum of the

TABLE II. Observed singlet mesons (based on quark-antiquark model) and typical masses (Refs. 12 and 13).

$q\bar{q}$ orbital angular momentum	$q\bar{q}$ spin	J^{PC} nonet	Typical mass (MeV)
$L=0$	$S=0$	0^{-+}	500
$L=1$	$S=0$	1^{+-}	1250
$L=2$	$S=0$	2^{-+}	1680

nonrelativistic quark model. In Table II, the observed mesons in the states L_J, based on the quark-antiquark models[12,13] are given along with the typical meson masses. The spacing in energy between mesons is generally taken to be about 600 MeV (Ref. 13). We have already assumed the states 0^{-+}, 1^{+-}, and 2^{-+} lie on the leading Regge trajectory defined by (7.25). This suggests that we assume (11.23) yields trajectory equations for lower-lying (unstable) resonances. This would yield

$$[M(\text{res})/2m]^2=(n_4+1)(l+\alpha_0) . \tag{11.24}$$

Figure 3 shows a plot of the leading trajectory and the next few lower trajectory resonances, where we have set $\alpha_0=1$. The figure illustrates the paucity of lower-lying meson states, a fact that seems in accord with existing experimental evidence. However, we caution that the results of this simple version of the constituent model should not be taken too seriously.

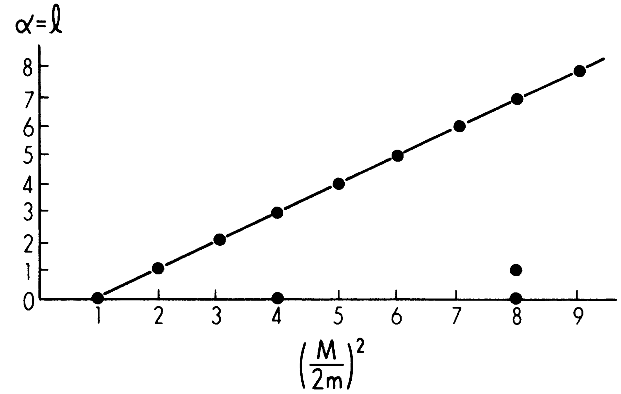

FIG. 3. Plot of resonance mass squared vs orbital angular momentum.

XII. DISCUSSION

To be sure, the constituent model is a limited model. Much remains to be understood even within the context of the simple 4N version. Invariant supplementary conditions are imposed in addition to the Lagrangian to eliminate time oscillations and unwanted solutions. The model seems to yield local interactions between composite particles even though it is based on action-at-a distance. The classical version of the model contains aspects of the quantum theory and of quantum field theory. At the very least, it requires us to rethink creation and annihilation of particles in classical theories. The analytical properties of the scattering amplitudes need to be understood, since the usual partial-wave expansions do not apply. Quite apart from such fundamental questions, the question arises of how to extend the model to more realistic problems, for example, to composites of differing masses and spins, to the inclusion of other quantum numbers and other interactions (Ref. 5 contains an initial investigation of color in the classical model), to multiple scattering, etc. Nevertheless, it appears to contain features which suggest it could lead to an alternative approach to understanding particle interactions without many of the burdens that plague conventional theories. In addition, the inadequacy of the physical initial conditions to uniquely determine the solutions (even classically) and the nonlocal nature of the equations suggest a bearing on the foundations of quantum mechanics.

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