General coupling of strings to the low-energy effective theory

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Starting from the field equations for supergravity coupled to super-Yang-Mills theory, we derive the classical equations of motion for spinning strings in an arbitrary background. These equations are also obtained from an action principle based on a generalized Wess-Zumino term.

I. INTRODUCTION

Considerable attention has been given to the low-energy effective-field-theory formulation of superstrings. Most notably, the arguments of anomaly cancellations[1] are based on such a formulation.

The effective field theory can be described in terms of an action S_0, which gives the low-energy dynamics for the massless modes of the superstring. The dynamics is essentially that of supergravity coupled to a super-Yang-Mills theory in ten dimensions.[2]

In principle, S_0 can be derived starting from a first-quantized description of superstrings. The procedure involves integrating out all of the massive modes for the string and performing a derivative expansion. From the couplings of a string to a background field, it was shown[3] that the field equations for the massless modes can be obtained upon demanding conformal invariance in the quantum theory.

Conversely, in this paper we shall be interested in deriving the dynamics of spinning strings in an arbitrary background, starting from the field equations of the effective field theory. Our approach shall be to introduce "stringlike" sources to (the bosonic sector of) the effective theory. Various identities (e.g., Bianchi identities) for the fields can then be exploited to constrain the dynamics of the sources. A minimal set of equations of motion for a spinning string in a supergravity background result. They reduce to standard string equations of motion in the absence of a nontrivial background.

The above procedure of deriving the equations of motion from the field equations is a very old one. It was most notably used to derive the geodesic equation for a test particle in general relativity, starting from the Einstein field equations.[4] The procedure was also adapted to finding the motion of particles in a Yang-Mills field,[5] as well as extended to the dynamics of spinning particles in a gravitational field.[6]

Long ago, the equations of motion for a Nambu string coupled to general relativity were derived, starting from the Einstein equations, by Gürses and Gürsey.[7] Our work is a generalization of this. The procedure is discussed in Sec. II. We attach five different quantities to the string. They are the standard momentum P^α ($\alpha=0,1$) and spin angular momentum S^α currents, as well as a "Yang-Mills" current I^α and scalar densities χ and c which couple to the dilaton and antisymmetric tensor field, respectively. Identities on the fields give a minimal set of equations of motion for these quantities.

The dynamics of these quantities, however, is not completely determined by the field equations. With this in mind, we formulate an action principle for the coupled system in Sec. III. The procedures is not unique. Our approach is based on a generalized Wess-Zumino term,[8] as well as a free-string-action formulation given by Balachandran, Lizzi, Sparano, and Sorkin.[9] The generalized Wess-Zumino term is invariant under Yang-Mills transformations, as well as local Lorentz transformations. (The Yang-Mills-invariant expression was written down by Nepomechie.[10]) The generalized Wess-Zumino term contains the couplings to all the massless modes, except for the vielbein fields and the dilaton. The interactions are uniquely given by the symmetries of the theory, and are characterized by a single dimensionless coupling constant, which is quantized in the quantum theory. Another feature of this action is that because of the specific form for the currents, additional novel constraints (not present in the minimal equations) are obtained. One of them implies that the string behaves as a superconductor. This property has been noted previously.[11]

II. STRING EQUATIONS OF MOTION FROM SUPERGRAVITY FIELD EQUATIONS

We first briefly review the bosonic sector of the low-energy effective theory, i.e., supergravity coupled to super-Yang-Mills theory in ten dimensions.[2] The fields of the theory are the following: the vielbein $e^A{}_M$, the spin connection ω_M [taking values in the Lorentz algebra SO(9,1)], the Yang-Mills connection A_M (taking values in some Lie algebra $\hat{G}$), the antisymmetric tensor field B_{MN}, and the scalar dilaton ϕ. Here, $M,N,\ldots$ denote space-time indices, while $A,B,\ldots$ are tangent or flat-space indices. The low-energy effective Lagrangian L_0 is given by

$$e^{-1}L_0=-\frac{1}{2\kappa^2}R_{MN}^{AB}e^M{}_Ae^N{}_B-\frac{1}{\kappa^2\phi^2}\partial_M\phi\partial^M\phi-\frac{1}{4\kappa^{3/2}\phi}\mathrm{Tr}F_{MN}F^{MN}-\frac{3}{8\kappa\phi^2}H_{MNP}H^{MNP}\ , \tag{2.1}$$

where κ is the gravitational coupling constant in ten dimensions. (The Yang-Mills coupling constant has been absorbed in the definition of ϕ.) $e^M{}_A$ denotes the inverse vielbein and $e=\det(e^A{}_M)$. The Lorentz and Yang-Mills curvature two-forms are defined as usual, i.e., $R=d\omega+\omega^2$ and $F=dA+A^2$, respectively. The three-form $H=\frac{1}{6}H_{MNP}dx^M\wedge dx^N\wedge dx^P$ contains the Lorentz and Yang-Mills Chern-Simons forms, i.e.,

$$H=dB-\omega_3^L+\omega_3^{\mathrm{YM}},\quad B=\tfrac{1}{2}B_{MN}dx^M\wedge dx^N\ ,$$
$$\omega_3^L=\mathrm{Tr}(\omega\wedge R-\tfrac{1}{3}\omega^3),\quad \omega_3^{\mathrm{YM}}=\mathrm{Tr}(A\wedge F-\tfrac{1}{3}A^3)\ . \tag{2.2}$$

H is invariant under Yang-Mills transformations, as well as local Lorentz transformations. Under an infinitesimal Yang-Mills transformation,

$$\delta A=d\Lambda+[A,\Lambda],\quad \delta B=-\mathrm{Tr}A\wedge d\Lambda\ , \tag{2.3}$$

where the infinitesimal parameter Λ takes values in the Lie algebra $\hat{G}$. Under an infinitesimal local Lorentz transformation,

$$\delta\omega=d\Theta+[\omega,\Theta],\quad \delta B=\mathrm{Tr}\omega\wedge d\Theta\ , \tag{2.4}$$

where the infinitesimal parameter Θ takes values in the Lorentz algebra.

The field equations resulting from variations in B_{NP}, A_P, ω_P, $e^C{}_P$, and ϕ are

$$(\beta_B)^{NP}\equiv\partial_M\left[\frac{e}{\phi^2}H^{MNP}\right]=0\ , \tag{2.5}$$

$$(\beta_A)^P\equiv D_M{}^{(A)}\left[\frac{e}{\phi}F^{MP}\right]-\frac{9}{2}\frac{\kappa^{1/2}e}{\phi^2}F_{MN}H^{MNP}=0\ , \tag{2.6}$$

$$(\beta_\omega)^P\equiv D_M{}^{(\omega)}(eE^{MP})-\frac{9}{2}\frac{\kappa e}{\phi^2}R_{MN}H^{MNP}=0\ , \tag{2.7}$$

$$(\beta_e)^P{}_C\equiv-\tfrac{1}{2}ee^M{}_C(E^{PN})_{AB}R_{MN}^{AB}-\frac{\kappa^2}{2}e^P{}_CL_0=0\ , \tag{2.8}$$

$$\beta_\phi\equiv\frac{1}{e}\partial_M(e\partial^M\phi)-\frac{1}{\phi}(\partial_M\phi)^2+\frac{\kappa^{1/2}}{8}\mathrm{Tr}F_{MN}F^{MN}+\frac{3\kappa}{8\phi}H_{MNP}H^{MNP}=0\ , \tag{2.9}$$

respectively. Here $(E^{MN})_{AB}=\frac{1}{2}(e^M{}_Ae^N{}_B-e^N{}_Ae^M{}_B)$ and $D^{(A)}$ and $D^{(\omega)}$ are the covariant derivatives associated with Yang-Mills and local Lorentz transformation, respectively. When acting on matrices in the adjoint representation of the appropriate group, $D_M{}^{(A)}=\partial_M+[A_M,\ ]$ and $D_M{}^{(\omega)}=\partial_M+[\omega_M,\ ]$.

Now consider introducing sources into the field equations. Since there are five field equations, we can define a set of five sources. We thus replace the zeros on the right-hand side of Eqs. (2.5)–(2.9) with currents, denoted by j^{MN}, ι^M, s^M, $t^M{}_A$, and ρ, respectively,

$$(\beta_B)^{NP}=j^{NP}\ , \tag{2.5'}$$
$$(\beta_A)^P=\iota^P\ , \tag{2.6'}$$
$$(\beta_\omega)^P=s^P\ , \tag{2.7'}$$
$$(\beta_e)^P{}_C=t^P{}_C\ , \tag{2.8'}$$
$$\beta_\phi=\rho\ . \tag{2.9'}$$

The currents ι^M (s^M) are $\hat{G}$ valued (Lorentz-algebra valued), and transform covariantly under the action of the Yang-Mills group (local Lorentz group). The remaining quantities $j^{MN}=-j^{NM}$, $t^M{}_A$, and ρ are invariant under Yang-Mills transformations.

The currents are subject to constraints which follow from identities on the fields. For example, from $\partial_N\partial_M(eH^{MNP}/\phi^2)\equiv0$ and (2.5′), we have

$$\partial_Nj^{NP}=0\ . \tag{2.10}$$

The identity $D_P{}^{(A)}D_M{}^{(A)}(eF^{MP}/\phi)=0$ plus (2.6′) yields

$$D_N{}^{(A)}\iota^N+\tfrac{9}{2}\kappa^{1/2}F_{MN}j^{MN}=0\ . \tag{2.11}$$

When we apply $D_P{}^{(\omega)}$ to (2.7′), we find

$$D_N{}^{(\omega)}s^N+\tfrac{9}{2}\kappa R_{MN}j^{MN}+\frac{e}{2}[R_{MN},E^{MN}]=0\ . \tag{2.12}$$

The last term in (2.12) can be written in terms of the current $t^M{}_A$, since, from (2.8′),

$$\frac{e}{2}[R_{MN},E^{MN}]_{AB}=e_{AM}t^M{}_B-e_{BM}t^M{}_A\ . \tag{2.13}$$

A further condition on the currents results from the requirement of local coordinate transformation invariance. Under an infinitesimal coordinate transformation (parametrized by ϵ^N),

$$\begin{aligned}&\delta A_M=\partial_NA_M\epsilon^N+A_N\partial_M\epsilon^N\ ,\\&\delta\omega_M=\partial_N\omega_M\epsilon^N+\omega_N\partial_M\epsilon^N\ ,\\&\delta e^A{}_M=\partial_Ne^A{}_M\epsilon^N+e^A{}_N\partial_M\epsilon^N\ ,\\&\delta B_{MP}=\partial_NB_{MP}\epsilon^N+B_{NP}\partial_M\epsilon^N+B_{MN}\partial_P\epsilon^N\ ,\\&\delta\phi=\partial_N\phi\epsilon^N\ .\end{aligned} \tag{2.14}$$

Requiring that L_0 is invariant means that (up to a total divergence)

$$\mathrm{Tr}\left[\frac{1}{\kappa^{3/2}}(\beta_A)^M-\frac{9}{2\kappa}(\beta_B)^{PM}A_P\right]\delta A_M-\mathrm{Tr}\left[\frac{1}{\kappa^2}(\beta_\omega)^M-\frac{9}{2\kappa}(\beta_B)^{PM}\omega_P\right]\delta\omega_M+\frac{9}{4\kappa}(\beta_B)^{MP}\delta B_{MP}+\frac{2e}{\kappa^2\phi^2}\beta_\phi\delta\phi-\frac{2}{\kappa^2}(\beta_e)^P{}_C\delta e^C{}_P=0\ . \tag{2.15}$$

For transformations (2.14), the invariance condition reads

$$[D_M^{(\omega)}(\beta_e)^M{}_A]e^A{}_N+(\beta_e)^M{}_A\tau^A{}_{MN}+\frac{e}{\phi^2}\beta_\phi\partial_N\phi+\frac{\kappa^{1/2}}{2}\mathrm{Tr}[(\beta_A)^M F_{NM}]-\tfrac{1}{2}\mathrm{Tr}[(\beta_\omega)^M R_{NM}]+\tfrac{9}{8}\kappa(\beta_B)^{MP}H_{NMP}=0\ , \quad (2.16)$$

where $\tau^A{}_{MN}$ is the torsion tensor, i.e., $\tau^A{}_{MN}=D_M{}^{(\omega)}e^A{}_N-D_N{}^{(\omega)}e^A{}_M$. In deriving (2.16), we have used the previously discussed identities. Equation (2.16) is itself an identity for sourceless supergravity [from Eqs. (2.5)–(2.9)]. However, when sources are present, it leads to the following condition on the currents:

$$(D_M{}^{(\omega)}t^M{}_A)e^A{}_N+t^M{}_A\tau^A{}_{MN}+\frac{e}{\phi^2}\rho\partial_N\phi+\frac{\kappa^{1/2}}{2}\mathrm{Tr}(\iota^M F_{NM})-\tfrac{1}{2}\mathrm{Tr}(s^M R_{NM})+\tfrac{9}{8}\kappa j^{MP}H_{NMP}=0\ . \quad (2.17)$$

Finally, let the current sources originate from strings; i.e., we require j^{MN}, ι^M, s^M, $t^M{}_A$, and ρ to have support on a two-dimensional surface M, which we parametrize by σ^α, $\alpha=0,1$. σ^0 is a time parameter. (For simplicity, we shall only consider closed strings, so M is topologically $R\times S^1$.) We denote the string space-time coordinates by $z^M=z^M(\sigma)$. Additional variables must be defined on the string surface if the sources are to be nonzero. For this purpose we introduce the scalar densities $c(\sigma)$ and $\chi(\sigma)$, a $\hat{G}$-valued vector $I^\alpha(\sigma)$ on M, a Lorentz-algebra-valued vector $S^\alpha(\sigma)$ on M, and the set of vectors $P_A{}^\alpha(\sigma)$. Now set

$$j^{MN}(x)=\kappa\int_M d^2\sigma\,\delta^2(x-z(\sigma))c(\sigma)\epsilon^{\alpha\beta}\partial_\alpha z^M\partial_\beta z^N\ , \quad (2.18)$$

$$\iota^M(x)=\kappa^{3/2}\int_M d^2\sigma\,\delta^2(x-z(\sigma))I^\alpha(\sigma)\partial_\alpha z^M\ , \quad (2.19)$$

$$s^M(x)=\kappa^2\int_M d^2\sigma\,\delta^2(x-z(\sigma))S^\alpha(\sigma)\partial_\alpha z^M\ , \quad (2.20)$$

$$t^M{}_A(x)=\kappa^2\int_M d^2\sigma\,\delta^2(x-z(\sigma))P_A{}^\alpha(\sigma)\partial_\alpha z^M\ , \quad (2.21)$$

$$\rho(x)=\kappa^2\int_M d^2\sigma\,\delta^2(x-z(\sigma))\chi(\sigma)\ , \quad (2.22)$$

where $\epsilon^{01}=-\epsilon^{10}=1$ and $\partial_\alpha\equiv\partial/\partial\sigma^\alpha$. The vectors I^α, S^α, and $P_A{}^\alpha$ can be interpreted as "Yang-Mills," spin, and momentum currents, respectively.

The dynamics of the above string variables are constrained by Eqs. (2.10)–(2.12) and (2.17). Equation (2.10) implies that $c(\sigma)=c$ is a constant. From Eqs. (2.11) and (2.12), we get

$$D_\alpha{}^{(A)}I^\alpha+9cF(\sigma)=0\ , \quad (2.23)$$

$$(D_\alpha{}^{(\omega)}S^\alpha)_{AB}+9cR(\sigma)_{AB}+(e_{AM}P_B{}^\alpha-e_{BM}P_A{}^\alpha)\partial_\alpha z^M=0\ , \quad (2.24)$$

respectively. The covariant derivatives on M are defined according to

$$D_\alpha{}^{(A)}\equiv\partial_\alpha z^M D_M^{(A)},\quad D_\alpha{}^{(\omega)}\equiv\partial_\alpha z^M D_M{}^{(\omega)}\ , \quad (2.25)$$

while the contracted Yang-Mills and Lorentz curvatures are

$$F(\sigma)\equiv\tfrac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha z^M\partial_\beta z^N F_{MN}(z),\quad R(\sigma)\equiv\tfrac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha z^M\partial_\beta z^N R_{MN}(z)\ . \quad (2.26)$$

The final condition (2.17) leads to

$$(D_\alpha{}^{(\omega)}P_A{}^\alpha)e^A{}_N+P_A{}^\alpha\partial_\alpha z^M\tau^A{}_{MN}+\frac{e}{\phi^2}\chi\partial_N\phi+\tfrac{1}{2}\mathrm{Tr}(I^\alpha F_{NM})\partial_\alpha z^M-\tfrac{1}{2}\mathrm{Tr}(S^\alpha R_{NM})\partial_\alpha z^M+\frac{9c}{8}\epsilon^{\alpha\beta}\partial_\alpha z^M\partial_\beta z^N H_{NMP}=0\ . \quad (2.27)$$

Equations (2.23), (2.24), and (2.27) comprise a minimal set of equations of motion for a string coupled to an arbitrary effective-field-theory background. Since they involve divergences they do not completely specify the dynamics of the system. To remedy this situation we shall formulate an action principle for strings in the next section. Upon extremizing the action we will then recover Eqs. (2.23)–(2.25).

If we turn off the external fields ($A_M=\omega_M=B_{MN}=0$, $\phi=$const) and set $e^A{}_M=\delta^A{}_M$, the system of equations reduces to the following:

$$\partial_\alpha I^\alpha=0\ , \quad (2.28)$$

$$\partial_\alpha S^\alpha{}_{AB}+\partial_\alpha z_A P_B{}^\alpha-\partial_\alpha z_B P_A{}^\alpha=0\ , \quad (2.29)$$

$$\partial_\alpha P_A{}^\alpha=0\ . \quad (2.30)$$

Equation (2.28) states that the Yang-Mills current is conserved, while Eq. (2.30) gives the conservation of the momentum current on the string. Upon using (2.30), we see that (2.29) corresponds to the conservation of the total angular momentum current on the string. Equations (2.29) and (2.30) are the standard equations of motion for a free spinning string.[12]

III. ACTION PRINCIPLE

We shall now rederive equations of motion (2.23), (2.24), and (2.27) starting from an action principle. We begin by examining the Wess-Zumino term[8] for a string in the absence of external fields:

$$S_{\rm WZ}^{\rm YM}=\frac{n}{12\pi}\int_N {\rm Tr}(dg\,g^{-1})^3\ , \tag{3.1}$$

where we have introduced a dynamical variable $g=g(\sigma^0,\sigma^1,\sigma^2)$, which takes values in the Yang-Mills group G. n is a dimensionless constant which is required to have integer values for a consistent quantum theory. N is a three-dimensional disc (parametrized by $\sigma^0,\sigma^1,\sigma^2$), with $\partial N=M$. The equation of motion resulting from infinitesimal left variations on g (i.e., $\delta g=-\Lambda g$) in Eq. (3.1) is

$$d(dg\,g^{-1})=0 \quad \text{on } M\ . \tag{3.2}$$

On comparing Eq. (3.2) with Eq. (2.28), we see that we can identify the Yang-Mills current according to $I^\alpha \sim \epsilon^{\alpha\beta}\partial_\beta g g^{-1}$.

This result is modified when external fields are present. Nepomechie[10] wrote down the coupling to Yang-Mills potentials. It necessarily contains a coupling to the antisymmetric tensor field B_{MN} as well. Along with the infinitesimal Yang-Mills transformation (2.3), one includes $\delta g=-\Lambda g$. Then the sum of Eq. (3.1),

$$S_I^{\rm YM}=\frac{n}{4\pi}\int_M {\rm Tr}(dg\,g^{-1}\wedge A) \quad\text{and}\quad S^B=\frac{n}{4\pi}\int_M B(z)\ , \tag{3.3}$$

is invariant under such a combined transformation.

Although the sum of the three terms $S_{\rm WZ}^{\rm YM}$, $S_I^{\rm YM}$, and S^B is gauge invariant, it is not invariant under local Lorentz transformations. This is because B_{MN} transforms nontrivially under the latter [cf. Eq. (2.4)]. In order to rectify this situation we need to add the following two terms to Nepomechie's action:

$$S_I^L=-\frac{n}{4\pi}\int_M {\rm Tr}(dh\,h^{-1}\wedge\omega),$$
$$S_{\rm WZ}^L=-\frac{n}{12\pi}\int_N {\rm Tr}(dh\,h^{-1})^3\ . \tag{3.4}$$

Here we have introduced a new dynamical variable $h=h(\sigma^0,\sigma^1,\sigma^2)$, in analogy to g, where h takes values in the Lorentz group. To the local Lorentz transformations defined in Eqs. (2.4), we attach $\delta h=-\Theta h$. Then the total action

$$S_{\rm WZ}=S_{\rm WZ}^{\rm YM}+S_{\rm WZ}^L+S_I^{\rm YM}+S_I^L+S^B$$

is invariant under both Yang-Mills and local Lorentz transformations. This invariance can be made explicit by writing $S_{\rm WZ}$ in the form

$$S_{\rm WZ}=\frac{n}{4\pi}\int_N L_{\rm WZ}\ ,$$
$$L_{\rm WZ}=H(z)-{\rm Tr}[D^{(A)}gg^{-1}\wedge F-\tfrac{1}{3}(D^{(A)}gg^{-1})^3]$$
$$+{\rm Tr}[D^{(\omega)}hh^{-1}\wedge R-\tfrac{1}{3}(D^{(\omega)}hh^{-1})^3]\ , \tag{3.5}$$

where $D^{(A)}g=dg+Ag$ and $D^{(\omega)}h=dh+\omega h$. Now using $dH={\rm Tr}F^2-{\rm Tr}R^2$, it is easy to see that $L_{\rm WZ}$ is a closed (but not exact) three-form. This is a generic property for Wess-Zumino terms.

We note that Eq. (3.5) contains couplings to all fields but the vielbein and the dilaton. From the coupings to A_M, ω_M, and B_{MN}, we ascertain

$$I^\alpha=\frac{n}{4\pi}\epsilon^{\alpha\beta}D_\beta{}^{(A)}gg^{-1}\ ,$$
$$S^\alpha=\frac{n}{4\pi}\epsilon^{\alpha\beta}D_\beta{}^{(\omega)}hh^{-1}\ , \tag{3.6}$$
$$c=-\frac{n}{18\pi}\ .$$

The string equation of motion (2.23) can now be easily recovered from the action (3.5). Variations in g, $\delta g=-\Lambda g$, lead to

$$(D^{(A)}gg^{-1})^2-F=0\ . \tag{3.7}$$

Now use the identity

$$D^{(A)}(D^{(A)}gg^{-1})=(D^{(A)}gg^{-1})^2+F \tag{3.8}$$

and the identifications (3.6) to obtain the result.

An analogous equation is obtained by performing variations in h. This however does not lead to the desired result, because the last term in Eq. (2.24) is not recovered. That is, the string described by only the Wess-Zumino term (3.5) has no momentum current $P_A{}^\alpha$. From Eqs. (2.8′) and (2.21), the momentum current couples to the vielbein field. Since the Wess-Zumino term contains no such coupling, an additional term $S_{\rm KE}$ must be included in the action.

$S_{\rm KE}$ should reduce to the free string action for a trivial vielbein field. Furthermore, it must be a functional of the group variable h, as well as z^M (and $e^A{}_M$), in order that we recover Eq. (2.24) upon minimizing with respect to h. *We note that this requires a nonstandard formulation for the free string action.*

Let us first examine the limit of no external fields (and flat space). In this limit, Eq. (2.24) reduced to the angular-momentum-conservation equation (2.29). We wish to obtain the latter by varying h in some action containing $S_{\rm WZ}^L$. The appropriate action was given by Balachandran, Lizzi, Sparano, and Sorkin.[9] It is, namely, $S_{\rm WZ}^L$ plus

$$S_{\rm KE}[h,z]=\int_m \tfrac{1}{2}\Sigma_{AB}dZ^A\wedge dZ^B \tag{3.9}$$

where

$$\Sigma_{AB}=\frac{1}{2\pi\alpha'}(hT_{09}h^{-1})_{AB} \tag{3.10}$$

and T_{AB} are SO(9,1) generators. α' is the usual Regge slope parameter.

For $n=0$, corresponding to no Wess-Zumino term, it was shown[9] that the action (3.9) is equivalent to the Nambu-Goto string action. [The proof involves the elimination of h (which is an auxiliary variable when $n=0$) from the action.] In addition, upon replacing T_{09} in Eq. (3.9) by other Lorentz generators new classes of strings were found.[9]

Now consider $n\neq 0$. By minimizing the action

$S_{KE}+S^L_{WZ}$ (in the flat-space limit) with respect to h, we find

$$0=\frac{n}{4\pi}(dh\,h^{-1})^2{}_{AB}-\tfrac{1}{2}\Sigma_{BC}dz^C\wedge dz_A$$
$$+\tfrac{1}{2}\Sigma_{AC}dz^C\wedge dz_B\ . \quad (3.11)$$

Equation (3.11) is equivalent to (2.29) after making the identification

$$P_A{}^\alpha=\tfrac{1}{2}\Sigma_{AB}\epsilon^{\alpha\beta}\partial_\beta z^B\ . \quad (3.12)$$

The momentum current is conserved as a result of minimizing the action in z^A.

The generalization of Eq. (3.9) to curved space is straightforward. Coordinate transformation invariance is ensured upon including the vielbein fields according to

$$S_{KE}[h,z,e]=\int_M \tfrac{1}{2}\Sigma_{AB}e^A\wedge e^B\ , \quad (3.13)$$

where $e^A=e^A{}_M(z)dz^M$. From the field equation (2.8′), obtained by minimizing with respect to $e^A{}_M$, and Eq. (2.21), we now get the following assignment for the momentum current:

$$P_A{}^\alpha=\tfrac{1}{2}\Sigma_{AB}e^B{}_M\epsilon^{\alpha\beta}\partial_\beta z^M\ . \quad (3.14)$$

Equation (3.14) reduces to (3.12) in the flat-space limit.

Now minimizing the total action $S_{KE}[h,z,e]+S_{WZ}$, with respect to h, gives

$$0=\frac{n}{4\pi}[(D^{(\omega)}hh^{-1})^2-R]_{AB}$$
$$+\tfrac{1}{2}(\Sigma_{AC}e^C\wedge e_B-\Sigma_{BC}e^C\wedge e_A)\ . \quad (3.15)$$

Upon using the identity $D^{(\omega)}(D^{(\omega)}hh^{-1})=(D^{(\omega)}hh^{-1})^2+R$, and the definition (3.14) for the momentum current, we recover the string equation of motion (2.24).

It remains to obtain Eq. (2.27). For this purpose, let us vary z^P in $S_{KE}[h,z,e]+S_{WZ}$. The resulting equation of motion is

$$\partial_\alpha\left[\frac{\delta L_{KE}}{\delta\partial_\alpha z^P}\right]-\frac{\delta L_{KE}}{\delta z^P}=\frac{n}{4\pi}\epsilon^{\alpha\beta}\partial_\alpha z^M\Bigg[\mathrm{Tr}(R_{PM}D_\beta{}^{(\omega)}hh^{-1})-\mathrm{Tr}(F_{PM}D_\beta{}^{(A)}gg^{-1})+\tfrac{1}{2}\partial_\beta z^N H_{MNP}$$
$$-\frac{4\pi}{n}(\omega_P)^{AB}\Sigma_{AC}e^C{}_M e_{BN}\partial_\beta z^N\Bigg]$$
$$=\mathrm{Tr}(R_{PM}S^\alpha)\partial_\alpha z^M-\mathrm{Tr}(F_{PM}I^\alpha)\partial_\alpha z^M-\frac{9c}{4}\epsilon^{\alpha\beta}H_{MNP}\partial_\alpha z^M\partial_\beta z^N-2P_A{}^\alpha(\omega_P)^{AB}e_{BM}\partial_\alpha z^M\ , \quad (3.16)$$

where $S_{KE}=\int_M d^2\sigma\, L_{KE}$, and we have substituted equations of motion (3.7) and (3.15) into the right-hand side of (3.16). Equation (3.16) differs from (2.27) only by the fact that the former contains no coupling to the dilaton field. The standard interaction with the dilaton was proposed by Fradkin and Tseytlin.[13] Here we suggest an alternative. We simply replace L_{KE} by

$$\frac{g^2_{YM}}{\kappa^{3/2}\phi(z)}L_{KE}\ ,$$

where g_{YM} is the Yang-Mills coupling constant.

The above prescription is equivalent to multiplying the Regge slope parameter α' by $\phi(z)\kappa^{3/2}/g^2_{YM}$. The latter reduces to one when ϕ attains its vacuum value. We note that we recover (in a simple manner) the known result[14] that a rescaling of ϕ by a constant is identical to a redefinition of the Regge slope parameter.

With the above choice of coupling to the dilaton field, we identify the scalar density $\chi(\sigma)$ in Eq. (2.22) according to

$$\chi(\sigma)=\frac{\kappa^{1/2}g^2_{YM}}{2e(z)}L_{KE}\ . \quad (3.17)$$

Now upon varying z^P in

$$\int_M d^2\sigma\frac{g^2_{YM}}{\kappa^{3/2}\phi(z)}L_{KE}+S_{WZ}\ , \quad (3.18)$$

we obtain the equation of motion (2.27). Thus from the action (3.18), we recover the full set of minimal equations: (2.23), (2.24), and (2.27).

On the other hand, the particular form for the currents [cf. Eq. (3.6)] leads to some extra conditions, not present in the minimal set. Let us first examine the Yang-Mills current defined in Eq. (3.6). Starting from the identity

$$\mathrm{Tr}(\epsilon^{\alpha\beta}D_\alpha{}^{(A)}gg^{-1}D_\beta{}^{(A)}gg^{-1}D_\gamma{}^{(A)}gg^{-1})\equiv 0\ ,$$
$$\alpha,\beta,\gamma=0,1 \quad (3.19)$$

and the equation of motion (3.7), we arrive at the condition

$$\mathrm{Tr}F(\sigma)I^\gamma(\sigma)=0 \quad (3.20)$$

which is not present in the minimal equations. This condition was previously found for strings in interaction with Yang-Mills fields.[11]

The interpretation of Eq. (3.20) is as follows: If at a point $z(\sigma)$ on the string a component $I^\alpha(\sigma)$ of the current is nonzero, it defines a direction in the Lie algebra $\hat{G}$. By Eq. (3.20), the components of $F(\sigma)$ vanish in this direction. There are at most two such directions in $\hat{G}$, corresponding to $\alpha=0,1$. Furthermore, these directions change from point to point on the string. If, on the other hand, $I^0(\sigma)=I^1(\sigma)=0$, there are no restrictions on the field F at the point σ.

From the above, if the current $I^1(\sigma)$ is not zero, the component of the "electric field" $F(\sigma)$ parallel to $I^1(\sigma)$

vanishes at σ. The string can be said to *superconduct.* Superconducting strings [with $G=\mathrm{U}(1)$] are of interest in the cosmological context.[15] It is thus conceivable that the action (3.18) (rewritten in four space-time dimensions) might be suitable for discussing such macroscopic, as well as microscopic, strings.

By introducing additional terms involving g in the action, we will modify the form of the current I^α in Eq. (3.6), and hence the condition (3.20). One such term which is commonly included is the nonlinear σ-model action

$$S_\sigma=\frac{1}{4\lambda^2}\int_M d^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}\mathrm{Tr}(D_\alpha{}^{(A)}g^{-1})(D_\beta{}^{(A)}g)\ , \quad (3.21)$$

where $\gamma_{\alpha\beta}=\partial_\alpha z^M\partial_\beta z_M$ is the induced metric on the string, with $\gamma^{\alpha\rho}\gamma_{\rho\beta}=\delta^\alpha{}_\beta$ and $\gamma=\det[\gamma_{\alpha\beta}]$. λ can be determined by demanding that, in the absence of external fields, the quantized theory has a zero beta function. In that case, it was found that[16] $\lambda^2=\pm 4\pi/n$. Indeed, it may be argued that a consistent quantum theory of the string requires the term (3.21). We note that the inclusion of (3.21) will alter the definition of the momentum current, as well as the Yang-Mills current.

In addition to the constraint (3.20) on I^α, we can similiarly derive a constraint on the spin current S^α based on its form [cf. Eq. (3.6)]. From the identity

$$\mathrm{Tr}(\epsilon^{\alpha\beta}D_\alpha{}^{(\omega)}hh^{-1}D_\beta{}^{(\omega)}hh^{-1}D_\gamma{}^{(\omega)}hh^{-1})\equiv 0 \quad (3.22)$$

and the equation of motion (2.24), we get

$$\frac{n}{4\pi}\mathrm{Tr}R(\sigma)S^\beta=2e(z)P_A{}^\alpha\partial_\alpha z_M e^M{}_B(S^\beta)^{AB}\ . \quad (3.23)$$

In flat space, this condition reduces to

$$(S^\beta)^{AB}\partial_\alpha(z_A P_B{}^\alpha - z_B P_A{}^\alpha)=0\ . \quad (3.24)$$

Equation (3.24) can be interpreted as follows: If the spin current $S^\alpha(\sigma)$ is not zero, the orbital angular momentum current and spin current are separately conserved, at the point σ, in the direction in the Lorentz algebra parallel to S^α.

A final condition not present in the minimal equations of motion is

$$2e(z)\chi(\sigma)=\kappa^{1/2}g^2_{\mathrm{YM}}P_A{}^\alpha e^A{}_M\partial_\alpha z^M\ , \quad (3.25)$$

which follows from (3.17). The physical relevance of Eqs. (3.24) and (3.25) is not clear to us.

IV. CONCLUDING REMARKS

We have derived a minimal set of equations that a classical superstring must satisfy when interacting with the zero-mass modes of the effective theory. We further realized these equations by proposing an action principle (3.18) based on the Wess-Zumino term (3.5). The latter potentially offers a bosonic description of the heterotic superstring. The standard σ-model action for the heterotic string coupled to the background of massless particles is known to be anomalous with regard to Yang-Mills and local Lorentz transformations.[17] Counterterms were introduced in order to recover these symmetries. For us, the effects of the anomalies are already contained in the action S_{WZ} and no counterterms are necessary. It remains to show that the action presented here can lead to a conformally invariant quantum theory. One can further ask if the effective theory equations of motion can be recovered by demanding that relevant β functions vanish for the coupled Wess-Zumino action. We would then have the situation where the classical string action leads to the effective-field-theory equations of motion, while the effective-field-theory action leads to the classical string equation of motion.

We remark that although our starting point was ten-dimensional supergravity, the resulting equations of motion and action principle is easily generalized to an arbitrary number of space-time dimensions. Thus, the action (3.18) may also prove useful for the description of a four-dimensional macroscopic cosmic string. This classical string has the novel features of possessing spin, as well as being superconducting. Regarding the former, some surprising consequences have already been noted.[18] Equation (3.5) gives a unique coupling of the string to A_μ, ω_μ, and $B_{\mu\nu}$ (the latter being interpreted as the axion field) which may lead to interesting consequences.

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