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Supersymmetric regularized path-integral measure in x space

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The complete supersymmetric Liouville action for a scalar multiplet coupled to supergravity in two-dimensional x space is obtained by integrating *all* anomalies, and not only the conformal anomaly. The resulting action is supersymmetric, but supersymmetry is derived, and not merely used as a device to extend the integrated conformal anomaly to the complete supersymmetric Liouville action. The integration variables are obtained from the matter variables not only (as usual) by multiplication by powers of $\det e(x)$, but also by *addition* of terms containing products of matter fields and supergravity fields. The dependence of the action on $\det e$, $\gamma\cdot\psi$, *and* S is removed by rescaling and shifting the new matter variables. As a regulator we use a matrix operator which contains the D'Alembertian in each diagonal entry and which is obtained from the matter action by squaring and introducing a "twist" operator. Because the regulator contains off-diagonal elements, all three anomalies (Weyl, super-Weyl, and "auxiliary") contribute.

I. INTRODUCTION

In Fujikawa's approach,[1] anomalies arise in quantum field theory when the classical action has a symmetry but the measure of the path integral is not invariant under that symmetry. The anomaly is then the product of the Jacobians at all points of spacetime. One may regulate this infinite product with the exponential of a negative-definite operator R/M^2 with eigenvalues $-\lambda^2/M^2$, taking the limit $M^2\rightarrow\infty$ at the end. To fix the measure, one requires that it is invariant under some preferred symmetries; these are then the symmetries without anomalies. Having fixed the measure in this way, one may proceed to calculate the anomalies for other symmetries. The choice of preferred symmetries is a matter of physical prejudice. Different choices for the measure, i.e., different preferred symmetries, in general, lead to different anomalies for the nonpreferred symmetries.

In this paper we discuss the measure for supersymmetric matter in two-dimensional x space, considering ordinary supersymmetry as a preferred symmetry, on the same footing as general coordinate invariance. In superspace, local ordinary (i.e., nonconformal) supersymmetry transformations and general x-space coordinate transformations constitute together general supercoordinate transformations. Applying the same arguments in superspace as in x space, one finds that the supercoordinate-invariant measure in superspace is the direct analogue of the general coordinate-invariant measure in x space. Specifically, one only needs to replace the determinant of the ordinary vielbein in the measure by that of the supervielbein.[2] One of our problems will thus be to translate this superspace measure to x space. Another problem to be solved is the construction of a supersymmetric regulator for the x-space supersymmetric matter model. Having determined the supersymmetric measure and corresponding supersymmetric regulator, we can then solve an outstanding problem in string theory. Namely, we show that the complete supersymmetric Liouville action can be viewed as an integration of several x-space anomalies. Previous determinations of the effective action in x space using the measure[3,4] only kept track of the leading bosonic term in the supersymmetric Liouville action, which is due to Weyl transformations, and completed the effective action afterward by requiring that it be supersymmetric.[4] We compute the various anomalies separately, integrate them, and *verify* afterward that the resulting effective action is indeed supersymmetric.

We consider the matter part of the action and not the ghost part. Consequently, we shall determine the terms proportional to d in $(d-10)$ times the integrated anomalies. Our procedure should also be applicable straightforwardly to the ghost sector, and should then yield the same integrated anomalies times -10.

In the following sections we discuss these issues in detail, but since the presentation is rather technical, we first give a qualitative overview of which problems arise and how we solve them. In the process of determining an invariant regulator we employ methods developed for heat kernels. Although our results are self-contained and do not necessarily need the general heat-kernel formalism, we often give parallel derivations using the heat-kernel formalism in order to clarify the algebraic manipulations.

The measure for a real scalar field $A(x)$ in x space that is free from general coordinate anomalies is

$$D[\tilde{A}]=D[A\,\det e^{1/2}]\ . \qquad (1.1)$$

It is easy to prove coordinate invariance as follows. Expanding $\tilde{A}(x)$ as $\sum a_n\phi^n(x)$ [where $\phi^n(x)$ are a complete orthonormal set of eigenfunctions of a positive-definite Hermitian operator, such as $\tilde{\Box}\equiv(-g)^{-1/4}\partial_\mu(-g)^{1/2}g^{\mu\nu}$

$\times\partial_\nu(-g)^{-1/4}]$, the measure $\prod D\tilde{A}(x)$ becomes $\prod da_n$. (Actually, we *define* the measure by $\prod da_n$.) Under an infinitesimal general coordinate transformation, $\delta\tilde{A}=\xi^\lambda\partial_\lambda\tilde{A}+\frac{1}{2}(\partial_\lambda\xi^\lambda)\tilde{A}$. Hence, using the orthonormality of ϕ^m,

$$\begin{aligned}\delta a_m &= \int dx\,\phi^m(x)[\delta\tilde{A}]\,,\\ J &= 1+\sum_m \frac{\partial\delta a_m}{\partial a_m}\\ &= 1+\sum_m \int dx\,\partial_\lambda[\phi^m(x)\xi^\lambda\phi^m(x)]\,.\end{aligned}\tag{1.2}$$

Assuming that fields fall off sufficiently fast at infinity, the measure in (1.1) is thus indeed naively general coordinate invariant, where by naive invariance we mean invariance before regularization. Regulating the divergent sum in (1.2) by multiplying with $\exp(-\lambda_m^2/M^2)$ we can bring these exponentials inside the total derivative. Since even after regularization we have a total derivative, we thus have genuine invariance. By replacing ξ^λ by $C^\lambda\Lambda$, where Λ is the Becchi-Rouet-Stora-Tyutin (BRST) parameter and C^λ the coordinate ghost, we also obtain a genuinely BRST coordinate-invariant measure. The measure in (1.1) has led to the correct axial and trace anomalies and critical dimensions, for ordinary field theories as well as for higher-derivative field theories.[5]

Suppose one would like to construct a supersymmetric measure, e.g., for a scalar multiplet $\Sigma=(A,\chi^\alpha,F)$ in $(d=2)$-dimensional spacetime. Since such a multiplet contains only coordinate scalars, one might be inclined to take as a measure the natural extension of (1.1):

$$D[\tilde{A}]D[\tilde{\chi}^\alpha]D[\tilde{F}]\,. \tag{1.3}$$

Indeed, this measure is naively supersymmetric, since the unregulated Jacobian for an infinitesimal supersymmetry transformation equals unity:[6]

$$\frac{\partial\delta\tilde{A}}{\partial\tilde{A}}-\sum_{\alpha=1}^{2}\frac{\partial\delta\tilde{\chi}^\alpha}{\partial\tilde{\chi}^\alpha}+\frac{\partial\delta\tilde{F}}{\partial\tilde{F}}=0\,. \tag{1.4}$$

This follows easily from the following supersymmetry transformation:

$$\begin{aligned}&\delta A=\tfrac{1}{2}\bar{\epsilon}\chi,\quad \delta\chi=\tfrac{1}{2}(\partial_\mu A-\tfrac{1}{2}\bar{\psi}_\mu\chi)\gamma^\mu\epsilon+\tfrac{1}{2}F\epsilon\,,\\ &\delta F=\tfrac{1}{2}\bar{\epsilon}\gamma^\mu[D_\mu\chi-\tfrac{1}{2}\gamma^\nu\psi_\mu(\partial_\nu A-\tfrac{1}{2}\bar{\psi}_\nu\chi)-\tfrac{1}{2}F\psi_\mu]\,,\\ &\delta(\det e)=\tfrac{1}{2}\bar{\epsilon}\gamma\cdot\psi\det e\,,\end{aligned}\tag{1.5}$$

where ψ_μ is the gravitino and e_μ^m the vielbein field. The contributions from $\delta(\det e)$ cancel even separately from the contribution due to $\partial\delta A/\partial A$, etc., but this does not hold in general for covariantly quantized gauge theories.[4] However, the measure in (1.3) is not genuinely supersymmetric, because different fields in general have different regulators, thus upsetting the cancellation in (1.4). For general coordinate transformations, the problem of justifying the correct regulated BRST-invariant measure was only recently solved in Ref. 7. In this paper we solve the corresponding problem for local supersymmetry.

Thus, it is clear that the problem of a genuinely supersymmetric measure cannot be detached from the problem of finding a supersymmetric regulator; in fact, one of our basic problems is to define what is meant by a supersymmetric regulator. In a previous study[2] the superspace measure for a scalar superfield $\phi(x,\theta)$ was taken to be

$$D[\tilde{\phi}]=D[\phi(\mathrm{sdet}E)^{1/2}]\,, \tag{1.6}$$

where sdet$E$ is the superdeterminant of the supervielbein, while the regulator was the iterated action in the superconformal gauge

$$R=[(\mathrm{sdet}E)^{-1/2}D^\alpha D_\alpha(\mathrm{sdet}E)^{-1/2}]^2\,. \tag{1.7}$$

Expansion of $\phi\,\mathrm{sdet}E^{1/2}$ in powers of θ suggests that the correct x-space measure should be of the form

$$D[\tilde{A}]D[\tilde{\chi}+\gamma\cdot\psi\tilde{A}]D[\tilde{F}+\cdots]\,. \tag{1.8}$$

Thus, we see that the correct integration variables $(\tilde{A},\tilde{\chi},\tilde{F})$ are obtained from the original variables (A,χ,F) not only by multiplication with factors $(\det e)^{1/2}$ but also by *addition* of terms with the same weight. Each of these measures, separately, is still BRST coordinate invariant as long as all terms have the same factor $(\det e)^{1/2}$, because then each term will separately yield a total derivative. However, to study genuine (i.e., after regularization) supersymmetry invariance of the measure, one should consider the total Jacobian for $(\tilde{A},\tilde{\chi},\tilde{F})$ *including off-diagonal terms,* and use a regulator which is supersymmetric and which also has off-diagonal elements. We construct a regulator denoted by $(\tilde{O}\,\tilde{O})$ in the space $(\tilde{A},\tilde{\chi},\tilde{F})$ which transforms under local supersymmetry as

$$\delta(\tilde{O}\,\tilde{O})=[\tilde{K},\tilde{O}\,\tilde{O}]\,, \tag{1.9}$$

where $\tilde{K}(x)$ has the property

$$\begin{aligned}&\tilde{K}(x)\delta(x-x')=-T^{-1}\tilde{K}(x')^T T\delta(x-x')\,,\\ &T=\begin{bmatrix}0&0&1\\0&C&0\\1&0&0\end{bmatrix}\,.\end{aligned}\tag{1.10}$$

This means that this regulator is a supersymmetric as well as a coordinate-invariant operator. Consequently, the regulated Jacobian for both general coordinate and local supersymmetry variations equals unity. We can then extend previous determinations of the critical dimension which were based on keeping track of only the bosonic terms in the Liouville action, to a complete analysis and deduce the complete supersymmetric Liouville action from the integration of various x-space anomalies. At the end we verify that the effective action is indeed supersymmetric, but note that this is a result (obtained from our choice of measure and regulator) and not input by hand.

II. A NEW DEFINITION OF JACOBIAN AND AN ALGORITHM FOR THE CONSTRUCTION OF THE REGULATOR

In this section we discuss the definition of the Jacobian to be used in path integrals for generic systems, and the question of how to regulate these Jacobians. In Sec. IV we apply these general considerations to the case of local supersymmetry, and show that there is no local supersymmetry anomaly in the scalar multiplet coupling to $d=2$ supergravity.

Our main result is a new definition of the path-integral Jacobian. Given a set of fields $\tilde{\phi}$ and a transformation law

$$\delta\tilde{\phi}=\tilde{K}\,\tilde{\phi} \tag{2.1}$$

it would be most natural to define the Jacobian matrix by

$$J(\text{naive})=\tilde{K}(x)\delta(x-x')\ . \tag{2.2}$$

Yet, as we shall argue, this is not the correct definition. Rather, in path integrals one needs a symmetrized Jacobian, to be derived below, which in some sense is the sum of the Jacobian for the bras and the Jacobian for the kets.

Given a transformation law as in (2.1), there are, in general, many dynamical systems which are invariant under it. The infinitesimal anomaly is in each case the supertrace of the regulated infinitesimal Jacobian matrix, but the regulator varies from system to system as it depends on the action of the particular model whose anomaly is calculated. Preferred symmetries are symmetries without anomalies at the quantum level, independently of any particular dynamical model. In order that this notion of preferred symmetries makes sense as a purely kinematical concept, we are led to require that the Jacobian matrix should be identically the unit matrix for all preferred symmetries before regularization. Obviously, any sensible regularization scheme then gives zero for the regularized anomaly. There are also classical models which are invariant under certain symmetry transformations, such that the path-integral Jacobian matrices are not the unit matrix but for which the regulated anomaly happens to vanish if one employs a particular regulator (or class of regulators) for this model. Such absence of anomalies is not generic but rather accidental. We can, therefore, summarize the preceding discussion as follows.

(i) Preferred symmetries are symmetries for which the regulated anomaly vanishes for any sensible regulator. As a consequence, the unregulated infinitesimal Jacobian matrix must vanish identically.

(ii) Accidental symmetries are symmetries of a particular model such that for a particular (class of) regulators the regulated anomaly vanishes.

A clear example of an accidental symmetry is chiral symmetry in an anomaly-free SU(5) theory where the anomaly cancellation occurs between traces of two subblocks of the Jacobian that have different dimensions because they came from different SU(5) representations. Similarly, according to our definition, chiral symmetry in a vectorlike theory is accidental (although in this case, because the cancellation occurs between identical blocks in the subclass of vectorlike theories, one could modify the definition of the Jacobian so that chiral symmetry becomes preferred).

We consider general coordinate and local supersymmetry as preferred symmetries. Thus, we wish to find a general definition of the infinitesimal Jacobian matrix in terms of $\tilde{K}(x)$, such that this matrix vanishes *identically* in the case of general coordinate transformations, *without the usual total derivative terms.* Consider, for example, a scalar field $S(x)$. As quantum variable we consider $\tilde{S}(x)=e^{\alpha}S(x)$, where $e=\det e_{\mu}^{m}$ and α is a constant. The naive Jacobian is then

$$J(\text{naive})=\left[\xi^{\lambda}(x)\frac{\partial}{\partial x^{\lambda}}+\alpha\left[\frac{\partial}{\partial x^{\lambda}}\xi^{\lambda}(x)\right]\right]\delta(x-x')$$
$$\equiv\tilde{K}(x)\delta(x-x')\ . \tag{2.3}$$

Clearly, J (naive) does not vanish. For $\alpha=\frac{1}{2}$, it becomes a total derivative in some sense, as discussed in the Introduction, but even this is not enough. Let us now observe that we can construct an identity involving $\tilde{K}(x)$. We begin with some simple manipulations, suppressing the spacetime index λ to emphasize the essential steps

$$\begin{aligned}\{\xi(x)\partial^{x}+\alpha[\partial^{x}\xi(x)]\}\delta(x-y)&=\{-\xi(x)\partial^{y}+\alpha[\partial^{y}\xi(y)]\}\delta(x-y)\\&=\{-\partial^{y}\xi(x)+\alpha[\partial^{y}\xi(y)]\}\delta(x-y)\\&=\{-\xi(y)\partial^{y}-(1-\alpha)[\partial^{y}\xi(y)]\}\delta(x-y)\ .\end{aligned} \tag{2.4}$$

For $\alpha=\frac{1}{2}$ we find an identity

$$[\tilde{K}(x)+\tilde{K}(y)]\delta(x-y)=0\ . \tag{2.5}$$

Let us reinterpret this simple result by first defining a matrix $\hat{\tilde{K}}(x)$ by

$$[\hat{\tilde{K}}(x)+\tilde{K}(y)]\delta(x-y)=0\ . \tag{2.6}$$

The path-integral Jacobian is then defined by

$$J=\left[\frac{-\hat{\tilde{K}}(x)+\tilde{K}(x)}{2}\right]\delta(x-y)\ . \tag{2.7}$$

Obviously, this "matrix" J vanishes for general coordinate symmetry. Moreover, the correct integration variable is found to be $Se^{1/2}$, which is the same result as in previous derivations, but deduced by a different line of reasoning.

The results obtained for general coordinate transformations are so trivial that they may not shed light on the direction in which we are moving. Let us now consider the case of local supersymmetry. In principle, there is no difference between local supersymmetry and general coordinate symmetry, and we perform the same steps, but in practice the algebra is much less trivial. We begin by making the observation that the naive Jacobian for local supersymmetry never vanishes. To see this, consider the scalar multiplet in $d=2$ coupled to supergravity for which the transformation law of a scalar A into a spinor χ reads $\delta A=\epsilon\chi$. No matter how one chooses the correct integration variables

$$\tilde{A}=e^{\alpha}A,\quad \tilde{\chi}=e^{\beta}\chi+e^{\gamma}\gamma\cdot\psi A \tag{2.8}$$

(there are no other possible terms on dimensional grounds), the matrix $\tilde{K}$ never vanishes identically. For example, $\tilde{K}_{12}=\bar{\epsilon}e^{\alpha-\beta}\neq 0$. As in the case of general coordinate invariance, we look for a new definition of the Jacobian J, such that J again vanishes identically. We could perhaps find (2.7) by purely kinematical methods, but it is easier to use properties of the action to determine the matrix $\hat{\tilde{K}}(x)$. We, therefore, now make a brief analysis of the symmetry properties of invariant actions, and then we will come back to the definition of the path-integral Jacobian and its associated regulator(s).

Consider an action quadratic in quantum variables ϕ,

$$\mathcal{L}=\phi^T(x)TO(x)\phi(x)\ . \tag{2.9}$$

For example, for a scalar multiplet in $d=2$ coupled to supergravity, $\phi(x)=\{A(x),\chi(x),F(x)\}$ and in the conformal gauge [for quantizing and gauge fixing the general coordinate symmetry, one may choose the conformal gauge $e_\mu^m=\delta_m^n(\det e)^{1/2}$ if the gravitational field is external] one has

$$TO=\frac{1}{2}\begin{bmatrix}\partial_\mu\eta^{\mu\nu}\partial_\nu & 0 & 0\\ & -e^{1/2}\gamma^m\delta_m^\mu\partial_\mu & 0\\ 0 & 0 & e\end{bmatrix}+\text{more}\ . \tag{2.10}$$

This action is invariant under simultaneous transformation of the matter fields, $\delta\phi=K\phi$, and of the supergravity background fields contained in $O(x)$. Thus

$$\delta\mathcal{L}=\delta\phi^T TO\phi+\phi^T T\delta O\phi+\phi^T TO\delta\phi=\partial_\mu k^\mu\ . \tag{2.11}$$

In $\delta\phi^T=\phi^T\overleftarrow{\mathbf{K}}^T$, where T denotes matrix transposition, there are derivatives acting to the left (to ϕ^T). We now insert a δ function and partially integrate and then replace $\partial^{x'}$ by ∂^x. This allows us to let the derivatives in K^T act to the right *without change in sign.* (There may be terms in K with derivatives and terms without derivatives. Clearly, it would break the general discussion in terms of K only, if only the terms with a derivative would acquire a minus sign.) We thus, obtain

$$\delta\mathcal{L}=\int dx'\phi^T(x')T\{T^{-1}[K^T(x)\delta(x-x')]TO(x)+O(x')K(x')\delta(x-x')\}\phi(x)$$
$$+\phi^T T\delta O\phi(x)=\text{total derivative}\ . \tag{2.12}$$

The square brackets around $[K^T(x)\delta(x-x')]$ mean that x derivatives in $K^T(x)$ do *not* act to the right on $O(x)\phi(x)$. Let us now define a matrix $\hat{K}(x')$ by

$$\{[\hat{K}(x')+T^{-1}K^T(x)T]\delta(x-x')\}=0\ , \tag{2.13}$$

where again all derivatives stop at $\delta(x-x')$. Then we find

$$\int dx'\phi^T(x')T[-\hat{K}(x')\delta(x'-x)O(x)+O(x')K(x')\delta(x-x')]\phi(x)$$
$$+\phi^T(x)T\delta O(x)\phi(x)=\text{total derivative}\ . \tag{2.14}$$

Now, however, we do not need square brackets around $\hat{K}(x')\delta(x'-x)$ and thus, we can perform the x' integration. We find

$$\delta O(x)=\hat{K}(x)O(x)-O(x)K(x)\ . \tag{2.15}$$

[Strictly speaking, one cannot consider $\phi^T(x)$ and $\phi(x)$ as independent fields. However, one can use "the doubling trick" of 't Hooft and Veltman with two actions, for ϕ_1 and ϕ_2. Then $\phi=\phi_1+i\phi_2$ and $\delta\phi=K\phi$, but now ϕ^* and ϕ can be varied independently.]

In the path integral, the correct integration variables are not, in general, ϕ but rather

$$\tilde{\phi}=E^{1/2}\phi\ , \tag{2.16}$$

where $E^{1/2}$ is a matrix to be defined. The same arguments as before show us that

$$\begin{aligned}&\mathcal{L}=\tilde{\phi}^T T\tilde{O}\tilde{\phi},\quad \tilde{O}=T^{-1}E^{-1/2,T}TOE^{-1/2}\ ,\\ &\delta\tilde{\phi}=\tilde{K}\tilde{\phi},\quad \tilde{K}=-E^{1/2}\delta E^{-1/2}+E^{1/2}KE^{-1/2}\ ,\\ &\{[\hat{\tilde{K}}(x')+T^{-1}\tilde{K}^T(x)T]\delta(x-x')\}=0\ ,\\ &\delta\tilde{O}(x)=\hat{\tilde{K}}(x)\tilde{O}(x)-\tilde{O}(x)\tilde{K}(x)\ .\end{aligned} \tag{2.17}$$

We can express $\hat{\tilde{K}}(x)$ in terms of $\hat{K}(x)$ at the same point [but not in terms of $K(x)$ at the same point] by using the relation between $\hat{K}(x')$ and $K(x)$. One finds

$$\begin{aligned}&\hat{\tilde{K}}=(\delta\hat{E}^{-1/2})\hat{E}^{1/2}+\hat{E}^{-1/2}\hat{K}\hat{E}^{1/2}\ ,\\ &\hat{E}^{-1/2}=T^{-1}E^{-1/2,T}T\ .\end{aligned} \tag{2.18}$$

We now link these considerations concerning the action and the basis choice determined by $E^{-1/2}$ to the question of how to define the Jacobian and what to choose as regulator.

We rewrite the transformation rule of $\tilde{O}(x)$ as

$$\delta\tilde{O}(x)=[\tilde{K}(x),\tilde{O}(x)]+[\hat{\tilde{K}}(x)-\tilde{K}(x)]\tilde{O}(x)\ . \tag{2.19}$$

Preferred symmetries are now by definition symmetries for which

$$\hat{\tilde{K}}(x)-\tilde{K}(x)=0\ \text{ as a matrix}\ . \tag{2.20}$$

Using our previous expression for $\tilde{K}$ and $\hat{\tilde{K}}$, we find the following "master equation" for $E^{-1/2}$ and T:

$$-E^{1/2}(\delta E^{-1/2})+E^{1/2}KE^{-1/2}=(\delta\hat{E}^{-1/2})\hat{E}^{1/2}+\hat{E}^{-1/2}\hat{K}\hat{E}^{1/2}\ . \tag{2.21}$$

This equation selects the basis ($E^{-1/2}$ and T) one must choose for certain preferred symmetries. We have here an equation for the matrices $E^{-1/2}$ and T. In our example of local supersymmetry we are able to completely determine T by other means, and hence $\hat{\hat{K}}=\tilde{K}$ is then an equation for $E^{-1/2}$; i.e., it will determine the correct integration variables in the path integral. (Those variables which lead to vanishing anomalies of the preferred symmetries.) More generally, *we define the path-integral Jacobian by*

$$J=\left[\frac{-\hat{\hat{K}}(x)+\tilde{K}(x)}{2}\right]\delta(x-x') . \tag{2.22}$$

We will shortly discuss the regulator, but it may be useful to first check our results for $\delta\tilde{O}$ in a simple case. Consider again a scalar field S and define $\tilde{S}=Sg^{\alpha}$. Then $TO=\partial_\mu g^{\mu\nu}\sqrt{-g}\,\partial_\nu$ and since the supercharge-conjugation matrix T is unity in this simple case,

$$O=\partial_\mu\sqrt{-g}\,g^{\mu\nu}\partial_\nu,\quad \tilde{O}=E^{-1/2}OE^{-1/2} . \tag{2.23}$$

To verify that $\hat{\hat{K}}=\tilde{K}$, we must show that

$$\delta\tilde{O}=[\tilde{K},\tilde{O}] . \tag{2.24}$$

This will fix $E^{-1/2}$. We will assume that $E^{-1/2}=g^{-\alpha}$, and compute $\delta\tilde{O}$. From

$$\begin{aligned}
&\delta\tilde{O}=\delta(g^{-\alpha}\partial_\mu\sqrt{-g}\,g^{\mu\nu}\sqrt{-g}\,\partial_\nu g^{-\alpha}) ,\\
&\delta g^{\beta}=[\xi^\lambda\partial_\lambda,g^{\beta}]+2\beta(\partial_\lambda\xi^\lambda)g^{\beta} ,\\
&\delta g^{\mu\nu}=[\xi^\lambda\partial_\lambda,g^{\mu\nu}]-g^{\mu\lambda}(\partial_\lambda\xi^\nu)-g^{\nu\lambda}(\partial_\lambda\xi^\mu) ,\\
&\delta\partial_\mu=[\xi^\lambda\partial_\lambda,\partial_\mu]+(\partial_\mu\xi^\lambda)\partial_\lambda ,
\end{aligned} \tag{2.25}$$

we find that

$$\delta\tilde{O}=[\xi^\lambda\partial_\lambda+\tfrac{1}{2}(\partial_\lambda\xi^\lambda),\tilde{O}]+(\tfrac{1}{2}-2\alpha)\{(\partial_\lambda\xi^\lambda),\tilde{O}\} . \tag{2.26}$$

Clearly $\delta\tilde{O}=[\tilde{K},\tilde{O}]$ if $\alpha=\frac{1}{4}$ and the correct integration variable is, therefore, in this case

$$\tilde{S}=E^{1/2}S=e^{1/2}S . \tag{2.27}$$

We now turn to the question of which regulator one should use to regulate this Jacobian.

As regulator we propose

$$\exp(R/M^2)=\exp(\tilde{O}\,\tilde{O}/M^2) . \tag{2.28}$$

(Note that we do not use the iterated action $\tilde{T}\tilde{O}\tilde{T}\tilde{O}$, for reasons to be explained.) The regulated anomalies are then defined by

$$J^{\rm reg}=\int dx\,{\rm str}\left[\frac{-\hat{\hat{K}}(x)+\tilde{K}(x)}{2}\right]\exp[\tilde{O}\,\tilde{O}(x)/M^2]\times\delta(x-x') . \tag{2.29}$$

The anomaly is closely related to the gauge invariance of the regulator. Under a symmetry

$$\delta\tilde{O}(x)=[\hat{\hat{K}}(x)-\tilde{K}(x)]\hat{O}(x)+[\tilde{K}(x),\tilde{O}(x)] \tag{2.30}$$

the supertrace of the regulator varies into

$$\delta\,{\rm str}e^{\tilde{O}\,\tilde{O}/M^2}={\rm str}2(\hat{\hat{K}}-\tilde{K})\frac{\tilde{O}\,\tilde{O}e^{\tilde{O}\,\tilde{O}/M^2}}{M^2} . \tag{2.31}$$

Hence, preferred symmetries are also gauge invariances of the regulator and vice versa. In fact, this regulator is the effective action because integrating over $\tilde{\phi}$ in the path integral yields[8]

$$({\rm det}T\tilde{O})^{-1/2}=({\rm det}T)^{-1/2}({\rm det}\tilde{O}\,\tilde{O})^{-1/4} \tag{2.32}$$

and exponentiating we find for the effective action

$$\exp(i\Gamma)=\exp\left[i\left[-\frac{i}{4}\int_0^\infty\frac{d\tau}{\tau}\int d^2x[{\rm str}e^{i\tau\tilde{O}\,\tilde{O}(x)}\delta(x-x')]\,|_{x=x'}\right]\right] . \tag{2.33}$$

[Use $A^{-1}=-i\int_0^\infty d\tau\exp(i\tau A)$ and integrate over A.] This result yields in the case of a Dirac field the Laplace-Beltrami operator $\tilde{O}\,\tilde{O}=\not{D}\,\not{D}$ and by analogy one expects that one should use as regulator $\tilde{O}\,\tilde{O}$ and not $\tilde{T}\tilde{O}\tilde{T}\tilde{O}$.

Up to this point we have not yet discussed in detail the properties of the supercharge conjugation matrix T. We only mentioned that the equation $\hat{\hat{K}}=\tilde{K}$ was an equation for E involving T and claimed that in actual problems, T can often be determined independently from E. This is the case in local symmetry, the subject to which we (finally) turn.

In local supersymmetry, the coupling of a scalar multiplet to supergravity has a kinetic term of the form

$$TO=\frac{1}{2}\begin{bmatrix}\partial^\mu\partial_\mu & 0 & 0\\ 0 & -e^{1/2}\partial^\mu C\gamma_\mu & 0\\ 0 & 0 & e\end{bmatrix}+\text{more} . \tag{2.34}$$

Clearly, $(TO)^2$ would not be a good regulator as it would overregulate in the A sector, and underregulate in the F sector. Some sort of "twist" is needed to equally distribute the D'Alembertians along the diagonal of the regulator.

A dimensional argument brings further insight. The dimensions of the fields are as follows:

$$[A]=0,\quad [\chi]=\tfrac{1}{2},\quad [F]=1 . \tag{2.35}$$

Clearly, the dimensions of the (ij) entry of the Jacobian is

$$[J_{ij}]=\tfrac{1}{2}(i-j) . \tag{2.36}$$

The entries of the regulator should have the same dimensional character, in order that all terms in the supertrace of J times $\exp(R)$ have the same dimension. The exponent of the regulator must be a square of matrices in order that the variation of this square be a commutator under the preferred symmetries. Thus, the regulator must be of the form

$$R = N(T\tilde{O})N(T\tilde{O}) , \tag{2.37}$$

where $T\tilde{O}$ is the field operator on the basis $\tilde{\phi}$ and N some as yet undetermined matrix. The dimension of the entries of $T\tilde{O}/M$ is $-\frac{1}{2}(i+j)$. It follows that the dimension of the entries $(NT\tilde{O})^j_i/M$ must satisfy the condition

$$[N_{ik}(T\tilde{O})_{kj}] = \tfrac{1}{2}(i-j) = 2 - \tfrac{1}{2}(k+j) . \tag{2.38}$$

Clearly

$$i = k = 4 , \tag{2.39}$$

which means that N has the form

$$N = \begin{bmatrix} 0 & & a \\ & bC^{-1} & \\ d & & 0 \end{bmatrix} \tag{2.40}$$

for a, b, d dimensionless constants. Furthermore, Lorentz covariance requires that C is the charge-conjugation matrix in spinor space. By rescaling the fields in $\tilde{\phi}$, we can arrange that $a = b = d = 1$. (If $\tilde{\phi}' = D\tilde{\phi}$, then $\phi' = E'^{-1/2}\phi'$ with $E'^{-1/2} = DE^{-1/2}D^{-1}$ and $\phi' = D\phi$. We can assume that we have chosen the scale of ϕ such that $a = b = d = 1$ and call E' henceforth E.)

The field operator F was written as $T\tilde{O}$. Calling $N = T^{-1}$, we thus find for the regulator $\exp(\tilde{O}\tilde{O}/M^2)$ where $T\tilde{O}$ is the field operator, and T is given by

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & C & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R = \tilde{O}\,\tilde{O}/M^2 . \tag{2.41}$$

Note that in this form, $\tilde{O}\,\tilde{O}$ has the following properties.

(i) It contains $\Box$ in the diagonal; hence, the "twist" matrix T has eliminated the over-regulation/under-regulation problem: the regulator R indeed regulates all sectors.

(ii) Gauge invariance: the supertrace of the variations of R vanishes for preferred symmetries.

(iii) For other systems, such as the coordinate and supersymmetry ghosts and antighosts of the string, one has used in the past the operators $O^\dagger O$ and $OO^\dagger$ as regulators for the conformal symmetries.[4] When one chooses a basis of (anti)ghost fields such that $\delta\tilde{O} = [\tilde{K}, \tilde{O}]$ for the preferred symmetries, one finds also $\delta O^\dagger = [\tilde{K}, O^\dagger]$ as long as $\tilde{K}$ is anti-Hermitian, and the same analysis would seem to go through.

Our new definition of Jacobian in (2.22) coincides with the naive definition in the case of chiral symmetry since $C\gamma_5 C^{-1} = \gamma_5^T$. Also for algebraic symmetries (symmetries whose transformation laws contain no derivatives) the old and new definitions coincide. Examples are local Weyl invariance and local conformal supersymmetry. The new definition can only differ from the old one whenever a nontrivial matrix T and/or derivatives appear [the latter prevent x' from being equivalent to x in $\hat{K}(x) = -K(x')$]. Local supersymmetry is a case where both effects are present but it is a preferred symmetry. It would be nice to have an example of a nonpreferred symmetry where our new definition of Jacobian would not be equivalent to the old definition. In the absence of such examples the results of the general formalism developed in this section are only that they justify the choice of regulator, in general, and allow us to prove in Sec. IV that local supersymmetry is a preferred symmetry.

III. THE SUPERSYMMETRIC x-SPACE AND MEASURE AND GAUGE COMPLETION

In this section we construct the measure by expanding $\phi(\mathrm{sdet}E)^{1/2}$ in powers of θ. The dependence of the scalar superfield ϕ and the superdeterminant of the supervielbein, sdet$E$, on the fields of the x-space theory is determined by the program of gauge completion. One identifies at order $\theta = 0$ certain superparameters and superfields with corresponding x-space fields and x-space parameters, and then compatibility requirements determine the content of superparameters and superfields at higher orders in θ. Thus, gauge completion is an initial-value problem. Of importance is the question of uniqueness: is the mapping from x space into superspace unique or are there arbitrary integration "constants" (where the "constants" are constants with respect to θ but fields with respect to x space). For a discussion of the gauge completion program we refer the reader to Refs. 9 and 2, but here we start from the compatibility equations and solve them. (Instead of gauge completion, one can begin in superspace and fix enough gauge freedoms to reduce the system to a Wess-Zumino gauge: this is entirely equivalent, and the integration "constants" in the gauge completion program should correspond to the freedom in making gauge choices.[2])

At order $\theta = 0$, the most natural identification is as follows:

$$\phi_| = A, \quad \mathrm{sdet}E_| = \det e, \quad \Xi^\mu_| = \xi^\mu, \quad \Xi^\alpha_| = \epsilon^\alpha , \tag{3.1}$$

where the vertical bar denotes the $\theta = 0$ projection. The compatibility equation for ϕ that the superspace and x-space transformations agree reads

$$\Xi^\Lambda \partial_\Lambda \phi = \delta\phi , \tag{3.2}$$

where δ are x-space variations with $\xi^\mu(x)$ and $\epsilon^\alpha(x)$. At order $\theta = 0$, from (1.5) one has

$$\Xi^\mu_| \partial_\mu \phi_| + \Xi^\alpha_| (\partial_\alpha \phi)_| = \xi^\mu \partial_\mu A + \tfrac{1}{2}\bar{\epsilon}\chi . \tag{3.3}$$

Clearly

$$\phi = A + \tfrac{1}{2}\bar{\theta}\chi + \bar{\theta}\theta \text{ terms} . \tag{3.4}$$

At the next level in θ we need Ξ^μ and Ξ^α to order θ. These are obtained from the compatibility equation for the superparameters.

The compatibility requirement that the gauge algebras in superspace and in x space agree reads

$$\Xi_2^\Pi \partial_\Pi \Xi_1^\Lambda + \delta_1(\Xi_2^\Lambda) - (1 \leftrightarrow 2) = \Xi_{12}^\Lambda , \tag{3.5}$$

where Ξ^{Λ}_{12} equals $\Xi^{\Lambda}(\epsilon=\epsilon_{12},\xi=\xi_{12},\lambda=\lambda_{12})$ with $\epsilon_{12},\xi_{12},\lambda_{12}$ the composite gauge parameters of the x-space gauge algebra. For $d=2$, $N=1$ supergravity, these composite gauge parameters follow from (1.5) and are given by[10]

$$\begin{aligned}\epsilon_{12}&=\xi_2^{\nu}\partial_\nu\epsilon_1+\tfrac{1}{4}\lambda_2^{mn}\gamma_{mn}\epsilon_1-\tfrac{1}{4}\bar\epsilon_2\gamma^{\mu}\epsilon_1\psi_\mu-(1\leftrightarrow2)\ ,\\ \xi^{\mu}_{12}&=\xi_2^{\nu}\partial_\nu\xi_1^{\mu}+\tfrac{1}{4}\bar\epsilon_2\gamma^{\mu}\epsilon_1-(1\leftrightarrow2)\ ,\\ \lambda_{12}^{mn}&=\xi_2^{\nu}\partial_\nu\lambda_1^{mn}+\lambda_2^{mk}\lambda_{1k}{}^{n}+\tfrac{1}{4}\bar\epsilon_2\gamma^{\mu}\epsilon_1\omega_\mu^{mn}(e,\psi)\\ &\quad+\tfrac{1}{8}(\bar\epsilon_2\gamma^{mn}\epsilon_1)S-(1\leftrightarrow2)\ .\end{aligned}\tag{3.6}$$

We first solve the compatibility equation for the parameters in (3.5) at successive θ levels, taking the cases $\Lambda=\mu$ and $\Lambda=\alpha$ separately. (The index μ is a curved bosonic index and α a curved fermionic index in superspace.) Afterward we return to the superfields ϕ and sdet$E$.

At $\theta=0$ one finds, for $\Lambda=\mu$,

$$\Xi_2^{\nu}|\,\partial_\nu\Xi_1^{\mu}|+\Xi_2^{\alpha}|(\partial_\alpha\Xi_1^{\mu})|-(1\leftrightarrow2)=\Xi^{\mu}_{12}|=\xi^{\mu}_{12}\ .\tag{3.7}$$

There is no term due to $\delta_1(\Xi_2^{\mu})$ since $\Xi_2^{\mu}|=\xi_2^{\mu}$ is field independent. With the identification in (3.1) and the explicit expression in (3.6) for ξ^{μ}_{12} we easily find a particular solution: namely, $\Xi^{\mu}=\frac{1}{4}\bar\theta\gamma^{\mu}\epsilon$. However, the homogeneous equation for $\Xi^{\mu}(\theta)$,

$$\epsilon_2^{\alpha}\partial_\alpha\Xi_1^{\mu}(\theta)-\epsilon_1^{\alpha}\partial_\alpha\Xi_2^{\mu}(\theta)=0\ ,$$

also has a solution: namely, $\Xi^{\mu}(\theta)=\bar\epsilon\theta V^{\mu}$ with V^{μ} arbitrary. The "integration constant" $\bar\epsilon\theta V^{\mu}$ satisfies $\epsilon_2^{\alpha}\partial_\alpha\Xi_1^{\mu}-1\leftrightarrow2=0$, because $\bar\epsilon_2\epsilon_1-1\leftrightarrow2$ vanishes indentically. Thus, the general solution of Ξ^{μ} to order θ reads

$$\Xi^{\mu}=\xi^{\mu}+\tfrac{1}{4}(\bar\theta\gamma^{\mu}\epsilon)+(\bar\theta\epsilon)V^{\mu}+O(\theta^2)\ .\tag{3.8}$$

The corresponding fermionic problem concerns the $\Lambda=\alpha$ part of (3.5) at $\theta=0$. One now has

$$\Xi_2^{\nu}|\,\partial_\nu\Xi_1^{\alpha}|+\Xi_2^{\beta}|(\partial_\beta\Xi_1^{\alpha})|-(1\leftrightarrow2)=\Xi^{\alpha}_{12}|=\epsilon^{\alpha}_{12}\ .\tag{3.9}$$

The solution of this fermionic partial differential equation is

$$\begin{aligned}\Xi^{\alpha}&=\epsilon^{\alpha}+\tfrac{1}{4}\bar\epsilon\gamma^{\mu}\theta\psi_\mu^{\alpha}-\tfrac{1}{4}\lambda^{kl}(\gamma_{kl}\theta)^{\alpha}+(\bar\theta\epsilon)V^{\alpha}\\ &\quad+O(\theta^2)\ ,\end{aligned}\tag{3.10}$$

where $V^{\alpha}(x)$ is again an arbitrary fermionic integration "constant."

The integration constants $V^{\mu}(x)$ and $V^{\alpha}(x)$ must have particular dimensions. Since the dimensions of the various fields and parameters are given by

$$\begin{aligned}&[e_\mu^m]=0,\ [\psi_\mu]=\tfrac{1}{2},\ [S]=1,\ [\partial_\mu]=1\ ,\\ &[\xi^{\mu}]=-1,\ [\epsilon]=-\tfrac{1}{2},\ [\lambda]=0,\ [\theta]=-\tfrac{1}{2}\ ,\end{aligned}\tag{3.11}$$

it follows that V^{μ} has dimension zero and V^{α} dimension $+\frac{1}{2}$. There is no Lorentz-covariant candidate for V^{μ}, so $V^{\mu}=0$, but for V^{α} we have a possibility: namely,

$$V^{\alpha}=k(\gamma\cdot\psi)^{\alpha}(\det e)^{t}\ ,\tag{3.12}$$

where k and t are constants to be determined at levels which are of higher orders in θ.

We now turn to the order θ level. For $\Lambda=\mu$ the compatibility equation (3.5) becomes

$$\begin{aligned}&\Xi_2^{\nu}(\theta)\partial_\nu\xi_1^{\mu}+\xi_2^{\nu}\partial_\nu\Xi_1^{\mu}(\theta)+\Xi_2^{\beta}(\theta)[\partial_\beta\Xi_1^{\mu}(\theta)]|\\ &\qquad+\epsilon_2^{\beta}\partial_\beta\Xi_1^{\mu}(\theta^2)+\delta_1(\Xi_2^{\mu}(\theta)=\tfrac{1}{4}\bar\theta\gamma^{\mu}\epsilon_2)-(1\leftrightarrow2)\\ &\qquad=\Xi^{\mu}_{12}(\theta)=\tfrac{1}{4}\bar\theta\gamma^{\mu}\epsilon_{12}\ .\end{aligned}\tag{3.13}$$

Now the $\delta_1(\Xi_2^{\Lambda})$ terms contribute because Ξ_2^{Λ} to order θ is x-space field dependent. A particular solution of the inhomogeneous equation for $\Xi_1^{\Lambda}(\theta^2)$ is easily found, although the algebra is somewhat lengthy. The homogeneous equation reads

$$\epsilon_2^{\beta}\partial_\beta\Xi_1^{\mu}(\theta^2)-\epsilon_1^{\beta}\partial_\beta\Xi_2^{\mu}(\theta^2)=0$$

and can be solved by noting that, in general, $\Xi_1^{\mu}(\theta^2)=\bar\theta\theta(\bar\epsilon_1W^{\mu})$. Substitution into the homogeneous equation yields $(\bar\epsilon_1O\epsilon_2-\bar\epsilon_2O\epsilon_1)\ \bar\theta OW^{\mu}=0$ for $O=1,\gamma^{\lambda},\tau_3$; hence, $\gamma^{\lambda}W^{\mu}=\tau_3W^{\mu}=0$. Thus, $W^{\mu}=0$, and the solution for the complete $\Xi^{\mu}(\theta)$ is unique except for the dependence on V^{α}. The ambiguity in $\Xi_1^{\alpha}(\theta)$ leads to an extra term in $\Xi_1^{\mu}(\theta^2)$ given by $\frac{1}{8}\bar\theta\theta(\bar\epsilon\gamma^{\mu}V)$. The complete Ξ^{μ} is then

$$\Xi^{\mu}(\theta^2)=\xi^{\mu}+\tfrac{1}{4}\bar\theta\gamma^{\mu}\epsilon+\tfrac{1}{16}(\bar\theta\gamma^{\mu}\psi^{\nu})(\bar\epsilon\gamma_\nu\theta)+\tfrac{1}{8}\bar\theta\theta(\bar\epsilon\gamma^{\mu}V)\ .\tag{3.14}$$

The analogous analysis at order θ level for $\Lambda=\alpha$ in (3.5) leads to the equation

$$\begin{aligned}&\Xi_2^{\nu}(\theta)\partial_\nu\epsilon_1^{\alpha}+\xi_2^{\nu}\partial_\nu\Xi_1^{\alpha}(\theta)+\Xi_2^{\beta}(\theta)\partial_\beta\Xi_1^{\alpha}(\theta)+\epsilon_2^{\beta}\partial_\beta\Xi_1^{\alpha}(\theta^2)+\delta_1(\Xi_2^{\alpha}(\theta)=\tfrac{1}{4}\bar\epsilon_2\gamma^{\mu}\theta\psi_\mu-\tfrac{1}{4}\lambda_2^{kl}\gamma_{kl}\theta+\bar\theta\epsilon_2V^{\alpha})-(1\leftrightarrow2)\\ &\qquad=\Xi^{\alpha}_{12}(\theta)=\tfrac{1}{4}(\bar\epsilon_{12}\gamma^{\mu}\theta)\psi_\mu^{\alpha}-\tfrac{1}{4}\lambda_{12}^{kl}(\gamma_{kl}\theta)^{\alpha}+(\bar\theta\epsilon_{12})V^{\alpha}\ .\end{aligned}\tag{3.15}$$

We discuss in turn the homogeneous solution, a particular solution and the dependence on V^{α}. We claim that the homogeneous equation

$$\epsilon_2^{\beta}\partial_\beta\Xi_1^{\alpha}(\theta^2)-\epsilon_1^{\beta}\partial_\beta\Xi_2^{\alpha}(\theta^2)=0\tag{3.16}$$

has no solution. Indeed $\Xi_2^{\alpha}(\theta^2)=\bar\theta\theta(M\epsilon_2)^{\alpha}$ must satisfy $M\gamma^{\lambda}\theta=M\tau^3\theta=0$ for arbitrary θ, which implies $M=0$. The particular solution is found as follows. All terms involving ξ^{μ} and λ^{mn} cancel separately. The remaining terms involve only ϵ and, for $V^{\alpha}=0$, a solution is

$$\Xi^{\alpha}(\theta^2)=\tfrac{1}{16}[(\bar\theta\gamma^{\mu}\psi^{\nu})(\bar\theta\gamma_\nu\epsilon)\psi_\mu^{\alpha}+(\bar\epsilon\gamma^{\mu}\theta)\omega_\mu^{mn}(e,\psi)(\gamma_{mn}\theta)^{\alpha}]\ .\tag{3.17}$$

In particular, no S terms survive. They appear as

$$S(\bar{\epsilon}_2\gamma^\mu\theta)\gamma_\mu\epsilon_1+\tfrac{1}{2}S\bar{\epsilon}_2\gamma^{kl}\epsilon_1\gamma_{kl}\theta$$

and this expression vanishes in $d=2$. [In $d=4$ it does not vanish, and one indeed finds S terms in $\Xi_1^\alpha(\theta^2)$ (Ref. 9).]

To investigate the dependence of $\Xi^\alpha(\theta^2)$ on V^α, we retain all terms from (3.15) which depend on V^α. (Since all equations are linear in Ξ^Λ one can solve the problem first for $V^\alpha=0$ and then for $V^\alpha\neq 0$.) They read

$$\xi_2^\nu\partial_\nu(\bar{\theta}\epsilon_1V^\alpha)+(\bar{\theta}\epsilon_2)V^\beta\partial_\beta(\bar{\theta}\epsilon_1V^\alpha)+(\bar{\theta}\epsilon_2)V^\beta\partial_\beta\tfrac{1}{4}[\bar{\epsilon}_1\gamma^\mu\theta\psi_\mu^\alpha-\lambda_1^{kl}(\gamma_{kl}\theta)^\alpha]$$
$$+\tfrac{1}{4}(\bar{\epsilon}_2\gamma^\mu\theta\psi_\mu^\beta-\lambda_2^{kl}\gamma_{kl}\theta^\beta)\partial_\beta(\bar{\theta}\epsilon_1V^\alpha)+\epsilon_2^\beta\partial_\beta\Xi_1^\alpha(\theta^2)+\delta_1(\bar{\theta}\epsilon_2V^\alpha)-(1\leftrightarrow 2)=(\bar{\theta}\epsilon_{12})V^\alpha\ . \quad (3.18)$$

The terms with $\xi^\nu(x)$ cancel

$$\xi_2^\nu\partial_\nu(\bar{\theta}\epsilon_1V^\alpha)+(\bar{\theta}\epsilon_2)(\delta V^\alpha=\xi_1^\nu\partial_\nu V^\alpha)-(1\leftrightarrow 2)=\bar{\theta}(\epsilon_{12}=\xi_2^\nu\partial_\nu\epsilon_1)V^\alpha-(1\leftrightarrow 2)\ . \quad (3.19)$$

The terms with $\lambda^{mn}(x)$ are

$$(\bar{\theta}\epsilon_2)(-\tfrac{1}{4}\lambda_1^{kl}\gamma_{kl}V)-\tfrac{1}{4}\bar{\epsilon}_1\lambda_2^{kl}\gamma_{kl}\theta V^\alpha+(\bar{\theta}\epsilon_2)(\tfrac{1}{4}\lambda_1^{kl}\gamma_{kl}V^\alpha)-(1\leftrightarrow 2)=\bar{\theta}(\tfrac{1}{4}\lambda_2^{mn}\gamma_{mn}\epsilon_1)V^\alpha-(1\leftrightarrow 2) \quad (3.20)$$

and cancel too. The remaining terms yield

$$(\bar{\theta}\epsilon_2)(\bar{\epsilon}_1V)V^\alpha+\tfrac{1}{4}(\bar{\theta}\epsilon_2)(\bar{\epsilon}_1\gamma^\mu V)\psi_\mu+\tfrac{1}{4}(\bar{\epsilon}_2\gamma^\mu\theta)(\bar{\epsilon}_1\psi_\mu)V^\alpha+\epsilon_2^\beta\partial_\beta\Xi_1^\alpha(\theta^2)+(\bar{\theta}\epsilon_2)\delta(\epsilon_1)V^\alpha-(1\leftrightarrow 2)$$
$$=-\tfrac{1}{4}(\bar{\theta}\psi_\mu)(\bar{\epsilon}_2\gamma^\mu\epsilon_1)-(1\leftrightarrow 2)\ . \quad (3.21)$$

Since $V^\alpha\sim\gamma\cdot\psi$ while $\delta(\epsilon)\gamma\cdot\psi\sim\partial\epsilon$, we see that

$$V^\alpha=0\ . \quad (3.22)$$

Hence we have found the general solution for the supercoordinate parameters:

$$\Xi^\mu(x_1\theta)=\xi^\mu+\tfrac{1}{4}(\bar{\theta}\gamma^\mu\epsilon)+\tfrac{1}{16}(\bar{\theta}\gamma^\mu\psi^\nu)(\bar{\epsilon}\gamma_\nu\theta)\ , \quad (3.23)$$

$$\Xi^\alpha(x_1\theta)=\epsilon^\alpha-\tfrac{1}{4}(\bar{\theta}\gamma^\mu\epsilon)\psi_\mu-\tfrac{1}{4}\lambda^{kl}\gamma_{kl}\theta+\tfrac{1}{16}(\bar{\theta}\gamma^\mu\psi^\nu)(\bar{\theta}\gamma_\nu\epsilon)\psi_\mu-\tfrac{1}{16}(\bar{\theta}\gamma^\mu\epsilon)\omega_\mu{}^{kl}(e,\psi)\gamma_{kl}\theta\ . \quad (3.24)$$

The interested reader may discover regularities. (Higher orders in θ follow from $\xi^\mu\to\frac{1}{4}\bar{\theta}\gamma^\mu\epsilon$, $\epsilon\to-\frac{1}{4}\bar{\theta}\gamma^\mu\epsilon\psi_\mu$, and $\lambda^{kl}\to\frac{1}{4}\bar{\theta}\gamma^\mu\epsilon\omega_\mu{}^{kl}$, while the transition from Ξ^μ to Ξ^α is effected by $e_m^\mu\to -e_m^\mu\psi_\mu^\alpha$.) These regularities convince us that the results for Ξ^Λ to all orders in θ are correct.

We now return to the gauge completion for ϕ and sdet$E$. From $\Xi^\Lambda\partial_\Lambda\phi=\delta\phi$ at level θ we get

$$\Xi^\mu(\theta)\partial_\mu A+\xi^\mu\partial_\mu(\tfrac{1}{2}\bar{\theta}\chi)+\Xi^\alpha(\theta)\partial_\alpha(\tfrac{1}{2}\bar{\theta}\chi)$$
$$+\epsilon^\alpha\partial_\alpha\phi(\theta^2)=\tfrac{1}{2}\bar{\theta}\delta\chi\ . \quad (3.25)$$

We easily deduce

$$\phi=A+\tfrac{1}{2}\bar{\theta}\chi+\tfrac{1}{8}\bar{\theta}\theta F\ . \quad (3.26)$$

As a check we note that at order θ^2 one finds the requirement that

$$(-\tfrac{1}{32}\bar{\theta}\theta)(\bar{\epsilon}\gamma^\nu\gamma^\mu\psi_\nu)(\partial_\mu A)+\tfrac{1}{8}(\bar{\theta}\gamma^\mu\epsilon)(\bar{\theta}\partial_\mu\chi)$$
$$-\tfrac{1}{16}(\bar{\theta}\psi_\mu)(\bar{\theta}\gamma^\mu\epsilon)F+\tfrac{1}{2}\bar{\chi}\Xi(\theta^2)$$
$$=\tfrac{1}{8}\bar{\theta}\theta(\delta F=\tfrac{1}{2}\bar{\epsilon}\not{D}^{\,\mathrm{cov}}\chi)\ , \quad (3.27)$$

which is indeed satisfied, confirming both the results for Ξ^Λ in (3.23) and (3.24) and those for ϕ in (3.26).

The last field to be determined by gauge completion is sdet$E$. The compatibility equation for this case reads

$$\partial_\lambda(\Xi^\Lambda\mathrm{sdet}E)(-)^\Lambda=\delta(\mathrm{sdet}E)\ , \quad (3.28)$$

where the left-hand side contains the transformation rule of a density in superspace. At $\theta=0$, we set sdet$E_|$ $=$dete. (If we were to do the much harder problem of determining E_Λ^M, we would begin by setting $E_{\mu|}^m=e_\mu^m$ and $E_{\mu|}^a=\psi_\mu^a$. We would then find that $E_{\alpha|}^m=0$ and $E_{\alpha|}^a=\delta_\alpha^a$. Thus, we would indeed find sdet$E_|$ $=$dete.) The compatibility equation becomes

$$\partial_\mu(\xi^\mu\mathrm{det}e)-\partial_\alpha[\Xi^\alpha(\theta)\mathrm{det}e+\epsilon^\alpha\mathrm{sdet}E(\theta)]$$
$$=\delta(\mathrm{det}e)=\partial_\mu(\xi^\mu\mathrm{det}e)+\tfrac{1}{2}\bar{\epsilon}\gamma\cdot\psi\,\mathrm{det}e\ . \quad (3.29)$$

This equation reduces to

$$\epsilon^\alpha\partial_\alpha\mathrm{sdet}E(\theta)+\tfrac{1}{4}\bar{\epsilon}\gamma\cdot\psi\,\mathrm{det}e=\tfrac{1}{2}\bar{\epsilon}\gamma\cdot\psi(\mathrm{det}e)\ ,$$

so that

$$\mathrm{sdet}E=\mathrm{det}e[1+\tfrac{1}{4}\theta\gamma\cdot\psi+O(\theta^2)]\ . \quad (3.30)$$

At order θ, the compatibility equation leads to

$$\partial_\mu\left[\Xi^\mu(\theta)\mathrm{det}e+\xi^\mu\frac{e}{4}\bar{\theta}\gamma\cdot\psi\right]-\partial_\alpha[\Xi^\alpha(\theta^2)\mathrm{det}e$$
$$+\Xi^\alpha(\theta)\tfrac{1}{4}\bar{\theta}\gamma\cdot\psi+\epsilon^\alpha\mathrm{sdet}E(\theta^2)]$$
$$=\delta(\tfrac{1}{4}\bar{\epsilon}\gamma\cdot\psi\,\mathrm{det}e)\ . \quad (3.31)$$

This equation reduces to

$$\partial_\mu\left[\frac{e}{4}\bar\theta\gamma^\mu\epsilon\right]+\frac{e}{8}(\bar\epsilon\gamma^\mu\theta)(\bar\psi_\mu\gamma\cdot\psi)+\epsilon^\alpha\partial_\alpha\mathrm{sdet}E\,(\theta^2)$$

$$+\frac{e}{16}\bar\epsilon\gamma^\mu\omega_\mu^{kl}(e,\psi)\gamma_{kl}\theta=-\frac{e}{16}(\bar\theta\gamma^\nu\psi^\mu)(\bar\epsilon\gamma_\mu\psi_\nu)$$

$$+\frac{e}{16}(\bar\epsilon\gamma\cdot\psi)(\bar\theta\gamma\cdot\psi)+\frac{e}{4}\bar\theta\gamma^\mu(D_\mu\epsilon+\tfrac{1}{4}S\gamma_\mu\epsilon)\ . \qquad (3.32)$$

Using that

$$\partial_\mu(ee_m^\mu)=-\omega_{\mu m}{}^n(e)(e_n^\mu e)$$

and

$$\omega_\mu^{mn}(e,\psi)-\omega_\mu^{mn}(e)=\tfrac{1}{4}(\bar\psi_\mu\gamma^m\psi^n-\bar\psi_\mu\gamma^n\psi^m+\bar\psi^m\gamma_\mu\psi^n)\ ,$$

only covariant terms remain, and from here on, one easily obtains

$$\mathrm{sdet}E=\mathrm{det}e\,[1+\tfrac{1}{4}\bar\theta\gamma\cdot\psi+\tfrac{1}{16}(\tfrac{1}{2}\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu+S)\bar\theta\theta]\ . \qquad (3.33)$$

As a check we note that $\int d^2\theta\, d^2x\ \mathrm{sdet}E$ is indeed proportional to the supercosmological term

$$\mathcal{L}=\frac{e}{2}\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu+eS\ , \qquad (3.34)$$

which should be, and indeed is, invariant under the following local supersymmetry transformations:

$$\delta\psi_\mu=D_\mu\epsilon+\tfrac{1}{4}S\gamma_\mu\epsilon,\quad \delta e_\mu^m=\tfrac{1}{2}\bar\epsilon\gamma^m\psi_\mu\ ,$$
$$\delta S=-\tfrac{1}{4}\bar\epsilon\gamma\cdot\psi S+\bar\epsilon\gamma^{\mu\nu}D_\mu\psi_\nu\ . \qquad (3.35)$$

Having obtained ϕ and sdet$E$, it is straightforward to compute $\phi\,\mathrm{sdet}E^{1/2}$. We find

$$\phi(\mathrm{sdet}E)^{1/2}=e^{1/2}A+\tfrac{1}{2}e^{1/2}\bar\theta(\chi+\tfrac{1}{4}\gamma\cdot\psi A)$$
$$+\frac{1}{8}e^{1/2}\bar\theta\theta(F+\tfrac{1}{4}\bar\psi\cdot\gamma\chi+\tfrac{1}{4}AS$$
$$+\tfrac{1}{8}\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu A-\tfrac{1}{32}\bar\psi\cdot\gamma\gamma\cdot\psi A)\ . \qquad (3.36)$$

Taking the coefficients of the various powers in θ as the fields which determine the measure, the measure in (1.8) is fixed, and in particular the terms to be added to $\tilde F$ are now known.

Let us discuss whether these results are unique. The gauge completion program yielded unique results once the initial data at $\theta=0$ were given. However, these initial data are not unique and an important other set of initial data is the one compatible with the superconformal gauge in superspace. As discussed in detail in Ref. 2 this gauge requires the initial condition

$$\mathrm{sdet}E_|=(\mathrm{det}e)^{1/2} \qquad (3.37)$$

rather than sdet$E_|$ =det$e$. The identification of the parameters at $\theta=0$ is also different. In the superconformal gauge, the supervielbein assumes the form

$$\mathcal{E}_M^A=\begin{bmatrix}\delta_m^\mu e^{2\psi} & -i\gamma_m^{ab}e^\psi D_b^0e^\psi\\ 0 & \delta_a^\alpha e^\psi\end{bmatrix}, \qquad (3.38)$$

where $E_M^\Lambda\partial_\Lambda=\mathcal{E}_M^A D_A^0$ with D_A^0 the covariant derivatives of rigid supersymmetry. Since sdet$E$=sdet$\mathcal{E}^{-1}=e^{-2\psi}$, and this is equal to $(\mathrm{det}e)^{1/2}$ at $\theta=0$, one may determine sdet$E$ from the initial condition sdet$E_|$ =det$e^{1/2}$. The final result is

$$\phi=A+\tfrac{1}{2}e^{1/4}\bar\theta\chi+\tfrac{1}{8}e^{1/2}\bar\theta\theta F\ , \qquad (3.39)$$

$$\mathrm{sdet}E=e^{1/2}\left[1+\tfrac{1}{4}e^{1/4}\bar\theta\gamma\cdot\psi+\frac{e^{1/2}}{16}S\bar\theta\theta+\frac{e^{1/2}}{32}\bar\theta\theta\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu\right]. \qquad (3.40)$$

These results can be explained by noting that a scalar field in superspace must also be a scalar in x space since $\Delta\phi=\xi^\mu\partial_\mu\phi+\cdots$ while from the dimensional analysis in (3.11) it follows that $\Xi^\mu_|=\xi^\mu(\mathrm{det}e)^s$. Compatibility of the parameter composition law at $\theta=0$ for Ξ_{12} then reveals that $s=0$ (Ref. 2). Hence, $\phi(\theta=0)$ is indeed an x-space scalar. Furthermore, the $\bar\theta\theta$ term in sdet$E$ is again proportional to the supercosmological constant, as it should since $\int d^2\theta\, d^2x$ sdet$E$ is gauge invariant. In fact, the results in ϕ and sdet$E$ differ from these derived previously only by a conformal rescaling $\theta\to e^\alpha\theta$.

We can, in fact, consider a one-parameter class of measures which includes both the usual gauge completion result and the superconformal gauge completion result. It is given by

$$\phi(\mathrm{sdet}E)^{1/2}=e^{1/2-\alpha}A+\tfrac{1}{2}e^{1/2}\bar\theta(\chi+\tfrac{1}{4}\gamma\cdot\psi A)$$
$$+\tfrac{1}{8}e^{1/2+\alpha}\bar\theta\theta(F+\tfrac{1}{4}\bar\psi\cdot\gamma\chi+\tfrac{1}{4}AS$$
$$+\tfrac{1}{8}A\,\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu$$
$$-\tfrac{1}{32}A\,\bar\psi\cdot\gamma\gamma\cdot\psi)\ . \qquad (3.41)$$

This is the measure we consider henceforth. Since the square of this superfield is a density, its $\int d^2\theta$ integral should give an action that is α independent, and this explains why the $\bar\theta\theta$ term has a factor $e^{1-\alpha}$ if the θ^0 term contains a factor e^α, and why the θ term is α independent.

One could try to find the most general solution to the gauge-completion program, i.e., all possible ways of mapping the x-space theory into superspace. Below, we instead confine our attention to (3.41), which contains the two superspace gauges most often used.

IV. LOCAL SUPERSYMMETRY AS A PREFERRED SYMMETRY

We apply the general formalism developed in Sec. II for Jacobians of preferred symmetries to local supersymmetry transformations in two dimensions (the spinning string). The method can, in this instance, be regarded as a generalization of that used in Ref. 11 for the computation of spinor current anomalies.

The action in two dimensions of a scalar multiplet $(-A,\chi,F)$ in a supergravity background (e,ψ,S) can be written in matrix form

$$I=\int d^2x\, \tfrac{1}{2}\bar{\rho}O\rho,\quad \rho=\begin{pmatrix}-A\\ \chi\\ F\end{pmatrix},\qquad \bar{\rho}=\rho^T T,\quad T=\begin{pmatrix}0&0&1\\0&C&0\\1&0&0\end{pmatrix}, \tag{4.1}$$

where the matrix operator O is given by

$$O=\begin{pmatrix}0 & 0 & e\\ -\frac{e}{2}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu & -e\not{D}+\frac{e}{8}\gamma^\mu\gamma^\nu\psi_\mu\bar{\psi}_\nu+\frac{e}{8}\psi_\nu\bar{\psi}_\mu\gamma^\nu\gamma^\mu & 0\\ e\Box & \frac{e}{2}\partial_\nu\bar{\psi}_\mu\gamma^\nu\gamma^\mu & 0\end{pmatrix}. \tag{4.2}$$

In the terminology of Sec. II, ρ transforms under a local supersymmetry transformation according to

$$\delta\rho=K\rho\ , \tag{4.3}$$

where the matrix operator K can be read off from (1.5):

$$K=\begin{pmatrix}0 & -\frac{1}{2}\bar{\epsilon} & 0\\ -\frac{1}{2}\gamma_\mu\epsilon\partial^\mu & -\frac{1}{4}\gamma^\mu\epsilon\bar{\psi}_\mu & \frac{1}{2}\epsilon\\ \frac{1}{4}\bar{\epsilon}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu & \frac{1}{2}\bar{\epsilon}\not{D}^{\,0}-\frac{1}{16}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu\bar{\epsilon} & -\frac{1}{4}\bar{\epsilon}\gamma^\mu\psi_\mu\end{pmatrix}. \tag{4.4}$$

From K as given in (4.4), and (2.13) that defines the matrix operator $\hat{K}$ in general, and the specific form of the matrix T given in (4.1), one determines $\hat{K}$ in this specific case

$$\hat{K}=\begin{pmatrix}\frac{1}{4}\bar{\epsilon}\gamma^\mu\psi_\mu & -\frac{1}{2}\bar{\epsilon} & 0\\ -\frac{1}{2}\not{D}\epsilon+\frac{1}{16}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu\epsilon & -\frac{1}{4}\psi_\mu\bar{\epsilon}\gamma^\mu & \frac{1}{2}\epsilon\\ \frac{1}{4}\partial_\nu\bar{\psi}_\mu\gamma^\nu\gamma^\mu\epsilon & \frac{1}{2}\partial^\mu\bar{\epsilon}\gamma_\mu & 0\end{pmatrix}. \tag{4.5}$$

$\hat{K}$ is obviously not equal to K. Thus, in order to avoid the breaking of local supersymmetry at the one-loop level, one has to introduce tilde fields $\tilde{\rho}$ according to Sec. II:

$$\tilde{\rho}=E^{1/2}\rho \tag{4.6}$$

with $E^{1/2}$ a matrix to be determined. Obviously

$$\bar{\tilde{\rho}}=\bar{\rho}\hat{E}^{\,1/2}\ , \tag{4.7}$$

where

$$\hat{E}^{\,1/2}=T^{-1}(E^{1/2})^T T \tag{4.8}$$

and the tilde operator $\tilde{O}$ is

$$\tilde{O}=\hat{E}^{\,-1/2}OE^{-1/2}\ . \tag{4.9}$$

According to Sec. II, the condition for local supersymmetry being a preferred symmetry is

$$\hat{\tilde{K}}=\tilde{K}\ , \tag{4.10}$$

where $\tilde{K}$ is introduced for $\tilde{\rho}$ in analogy to (4.3):

$$\delta\tilde{\rho}=\tilde{K}\tilde{\rho}\ , \tag{4.11}$$

and where $\hat{\tilde{K}}$ is determined by

$$[\hat{\tilde{K}}(x)+T^{-1}\tilde{K}^{\,T}(x')T]\delta(x-x')=0\ . \tag{4.12}$$

Combination of (4.12) with

$$\tilde{K}=-E^{1/2}(\delta E^{-1/2})+E^{1/2}KE^{-1/2} \tag{4.13}$$

following directly from (4.3), (4.6), and (4.11) leads to Eq. (2.21) for the determination of $E^{-1/2}$:

$$\begin{aligned}\tilde{K}&=(\delta\hat{E}^{\,-1/2})\hat{E}^{\,1/2}+\hat{E}^{\,-1/2}\hat{K}\hat{E}^{\,1/2}\\ &=-E^{1/2}(\delta E^{-1/2})+E^{1/2}KE^{-1/2}\ .\end{aligned} \tag{4.14}$$

If (4.14) is supplemented with the condition

$$\hat{E}^{\,-1/2}=E^{-1/2}\ , \tag{4.15}$$

one obtains, as we shall see, a unique solution that, in fact, is the Wess-Zumino gauge value of $E^{-1/2}$. It is possible also to have more general solutions, where (4.15) is not valid; we shall comment on this point below.

We now turn to a direct, entry-by-entry determination of $E^{-1/2}$ by combination of (4.4), (4.5), (4.14), and (4.15).

$E^{-1/2}$ will be a polynomial in the background gravitino field ψ_μ, and its occurrence is restricted by the following relation discussed in Sec. II:

$$\dim(E^{-1/2})_{ij}=\tfrac{1}{2}(i-j) . \quad (4.16)$$

From (4.16) follows that $E^{-1/2}$ has only diagonal and subdiagonal entries. This circumstance simplifies (4.14) enormously. Likewise K, $\hat{K}$, and $\tilde{K}$ are polynomials in ψ_μ and ϵ linear in ϵ, with

$$\dim K_{ij}=\dim\hat{K}_{ij}=\dim\tilde{K}_{ij}=\tfrac{1}{2}(i-j) . \quad (4.17)$$

It is convenient to start the determination of $E^{-1/2}$ with the part of (4.14) of zero order in ψ_μ, i.e., the 21 and 32 entries. Taking first the 21 component we get

$$\tilde{K}_{21}=(\delta E^{-1/2})_{21}e^{1/2}-\tfrac{1}{2}\not{D}^{0}\epsilon$$
$$\simeq-(\delta E^{-1/2})_{21}e^{1/2}-\tfrac{1}{2}\gamma^\mu\epsilon\partial_\mu , \quad (4.18)$$

whence, by $\delta\psi_\mu\simeq D^0_\mu\epsilon$,

$$(E^{-1/2})_{21}\simeq e^{-1/2}\tfrac{1}{4}\gamma_\mu\psi^\mu , \quad (4.19)$$

$$\tilde{K}_{21}\simeq-\tfrac{1}{2}\gamma^\mu\epsilon\partial_\mu-\tfrac{1}{4}(\not{D}^{0}\epsilon) . \quad (4.20)$$

Likewise for the 32 component,

$$\tilde{K}_{32}\simeq(\delta E^{-1/2})_{32}e^{1/2}+\tfrac{1}{2}\partial_\mu\bar{\epsilon}\gamma^\mu$$
$$\simeq-(\delta E^{-1/2})_{32}e^{1/2}+\tfrac{1}{2}\bar{\epsilon}\not{D}^{0} , \quad (4.21)$$

whence

$$(E^{-1/2})_{32}\simeq-e^{-1/2}\tfrac{1}{4}\bar{\psi}^\mu\gamma_\mu , \quad (4.22)$$

$$\tilde{K}_{32}\simeq\tfrac{1}{2}\partial_\mu\bar{\epsilon}\gamma^\mu-\tfrac{1}{4}(\bar{\epsilon}\overleftarrow{\not{D}}^{0}) . \quad (4.23)$$

We then turn to the diagonal elements of $\tilde{K}$ to first order in ψ_μ, where we verify

$$(E^{-1/2})_{11}=(E^{-1/2})_{22}=(E^{-1/2})_{33}=e^{-1/2} . \quad (4.24)$$

The transformation formula for $e^{-1/2}$ under local supersymmetry transformations is

$$\delta e^{-1/2}=-\tfrac{1}{4}\bar{\epsilon}\gamma_\mu\psi^\mu . \quad (4.25)$$

By means of this, in connection with the explicit expression for K and $\hat{K}$ in (4.4) and (4.5) and the two entries of $E^{-1/2}$ determined above, one obtains, from (4.14) and (4.15),

$$\tilde{K}_{11}=\tfrac{1}{8}\bar{\epsilon}\gamma^\mu\psi_\mu ,$$
$$\tilde{K}_{22}=-\tfrac{1}{8}\tau_3\bar{\epsilon}\tau_3\gamma^\mu\psi_\mu+\tfrac{1}{8}\gamma_\lambda\bar{\epsilon}\gamma^\lambda\gamma^\mu\psi_\mu , \quad (4.26)$$
$$\tilde{K}_{33}=-\tfrac{1}{8}\bar{\epsilon}\gamma^\mu\psi_\mu .$$

For $\tilde{K}_{31}$, one gets by means of (4.4), (4.5), (4.19), and (4.22), the following expressions to first order in ψ_μ:

$$[(\delta E^{-1/2})E^{1/2}]_{31}+(E^{-1/2}\hat{K}E^{1/2})_{31}$$
$$\simeq(\delta E^{-1/2})_{31}e^{1/2}+\tfrac{1}{16}(\bar{\epsilon}\overleftarrow{\not{D}}^{0})\partial^\mu\psi_\mu+\tfrac{1}{4}\partial_\nu\bar{\psi}_\mu\gamma^\nu\gamma^\mu\epsilon$$
$$+\tfrac{1}{8}\bar{\psi}_\mu\gamma^\mu\not{D}^{0}\epsilon-\tfrac{1}{8}\partial_\mu\bar{\epsilon}\gamma^\mu\gamma^\nu\psi_\nu , \quad (4.27)$$
$$-[E^{1/2}(\delta E^{-1/2})]_{31}+(E^{1/2}KE^{-1/2})_{31}$$
$$\simeq-(\delta E^{-1/2})_{31}e^{1/2}-\tfrac{1}{16}\bar{\psi}_\mu\gamma^\mu(\not{D}^{0}\epsilon)+\tfrac{1}{4}\bar{\epsilon}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu$$
$$-\tfrac{1}{8}\bar{\psi}_\mu\gamma^\mu\gamma^\nu\epsilon\partial_\nu+\tfrac{1}{8}\bar{\epsilon}\not{D}^{0}\gamma^\mu\psi_\mu \quad (4.28)$$

that are equal for

$$(\delta E^{-1/2})_{31}e^{1/2}\simeq-\tfrac{1}{16}\bar{\psi}_\mu\gamma^\mu(\not{D}^{0}\epsilon)-\tfrac{1}{8}(\partial_\nu\bar{\psi}_\mu\gamma^\nu\gamma^\mu\epsilon)$$
$$+\tfrac{1}{8}\bar{\epsilon}\gamma^\nu\gamma^\mu(D^0_\nu\psi_\mu) , \quad (4.29)$$

whence

$$\tilde{K}_{31}\simeq\tfrac{1}{4}\bar{\epsilon}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu+\tfrac{1}{8}(\partial_\nu\bar{\epsilon}\gamma^\mu\gamma^\nu\psi_\mu) , \quad (4.30)$$

while (4.29) is solved by

$$(E^{-1/2})_{31}=-\tfrac{1}{16}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu-\tfrac{1}{32}\bar{\psi}_\mu\gamma^\mu\gamma^\nu\psi_\nu+\tfrac{1}{4}S , \quad (4.31)$$

where the auxiliary field S transforms under local supersymmetry transformations according to

$$\delta S\simeq-\tfrac{1}{2}\bar{\epsilon}[\gamma^\mu,\gamma^\nu]D_\nu\psi_\mu . \quad (4.32)$$

So far, all components of $E^{-1/2}$ have been determined. It remains to be found that some terms of $\tilde{K}$ that are quadratic in ψ_μ and to verify that this can be done consistently on the basis of (4.14) and (4.15).

To do so, one needs the complete transformation formula of the background gravitino field under local supersymmetry transformations,

$$\delta\psi_\mu=D_\mu\epsilon+\tfrac{1}{4}\gamma_\mu\epsilon S,\quad D_\mu=D^0_\mu-\tfrac{1}{4}\tau_3\bar{\psi}_\lambda\gamma^\lambda\tau_3\psi_\mu \quad (4.33)$$

and the vielbein

$$\delta e_{a\mu}=\tfrac{1}{2}\bar{\epsilon}\gamma_a\psi_\mu \quad (4.34)$$

as well as the complete transformation formula of the auxiliary field S:

$$\delta S=-\tfrac{1}{2}\bar{\epsilon}[\gamma^\mu,\gamma^\nu]D_\nu\psi_\mu-\tfrac{1}{16}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu\bar{\psi}_\lambda\gamma^\lambda\epsilon+\tfrac{1}{4}\bar{\epsilon}\gamma^\lambda\psi_\lambda S . \quad (4.35)$$

After straightforward but laborious calculations, one determines the three entries in the lower-left corner of $\tilde{K}$ and verifies their consistency with (4.14) and (4.15).

The final result for the matrix operator $\tilde{K}$ for local supersymmetry transformations for tilde fields is

$$\tilde{K}=\begin{bmatrix} \frac{1}{8}\bar{\epsilon}\gamma^{\mu}\psi_{\mu} & -\frac{1}{2}\bar{\epsilon} & 0 \\ -\frac{1}{4}(\gamma^{\mu}D^{0}_{\mu}\epsilon)-\frac{1}{2}\gamma^{\mu}\epsilon D^{0}_{\mu}+\frac{1}{32}\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\psi_{\nu}\epsilon & \frac{1}{8}\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\epsilon\gamma_{\nu}-\frac{1}{8}\bar{\psi}_{\mu}\gamma^{\mu}\tau_{3}\epsilon\tau_{3} & \frac{1}{2}\epsilon \\ \frac{1}{4}\xi^{\nu}\partial_{\nu}+\frac{1}{8}(\partial_{\nu}\xi^{\nu}) & \frac{1}{4}(D^{0}_{\mu}\bar{\epsilon}\gamma^{\mu})+\frac{1}{2}\bar{\epsilon}\not{D}^{0}-\frac{1}{32}\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\psi_{\nu}\bar{\epsilon} & -\frac{1}{8}\bar{\epsilon}\gamma^{\mu}\psi_{\mu} \end{bmatrix} \tag{4.36}$$

with

$$\xi^{\nu}=\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\epsilon \ , \tag{4.37}$$

while the outcome for $E^{-1/2}$ is

$$E^{-1/2}=e^{-1/2}\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4}\gamma^{\mu}\psi_{\mu} & 1 & 0 \\ -\frac{1}{16}\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\psi_{\nu}-\frac{1}{32}\bar{\psi}_{\nu}\gamma^{\nu}\gamma^{\mu}\psi_{\mu}+\frac{1}{4}S & -\frac{1}{4}\bar{\psi}_{\mu}\gamma^{\mu} & 1 \end{bmatrix} . \tag{4.38}$$

Even cursory inspection of (4.36) displays striking regularities. To explain these [and as a check on the correctness of (4.36)] we note that $\tilde{K}$ satisfies (4.10), which means that $\tilde{K}$ is antisymmetric with respect to reflection about the minor diagonal (up to charge conjugation) except that free derivatives are treated as follows: they stand in $\tilde{K}$ always on the far right but for (4.10) one should move them to the far left and add an extra − sign. Thus

$$\tfrac{1}{2}(\partial_{\mu}a^{\mu})+a^{\mu}\partial_{\mu}\rightarrow -\tfrac{1}{2}(\partial_{\mu}a^{\mu})+\partial_{\mu}a^{\mu} \tag{4.39}$$

which is equal to itself. For example, for nonderivative terms one has

$$-\partial\delta\tilde{A}/\tilde{\chi}=\tilde{K}_{12}=-\partial\delta\bar{\bar{\tilde{\chi}}}/\partial\tilde{F}=-\overline{\partial\delta\tilde{\chi}/\partial\bar{\tilde{F}}}=-\bar{\bar{\tilde{K}}}_{23} \ , \tag{4.40}$$

while for the derivative terms one has, for example,

$$\begin{aligned} \delta\tilde{\chi}&=+\tfrac{1}{2}\gamma^{\mu}\epsilon D^{0}_{\mu}\tilde{A}+\tfrac{1}{4}(\not{D}^{0}\epsilon)\tilde{A}-\tfrac{1}{32}\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\psi_{\nu}\epsilon\tilde{A}+\cdots \ , \\ \delta\tilde{F}&=\tfrac{1}{2}\bar{\epsilon}\gamma^{\mu}D^{0}_{\mu}\tilde{\chi}+\tfrac{1}{4}(D^{0}_{\mu}\bar{\epsilon}\gamma^{\mu})\tilde{\chi}-\tfrac{1}{32}\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\psi_{\nu}\bar{\epsilon}\tilde{\chi}+\cdots \ , \\ -\partial\delta\tilde{\chi}/\partial\tilde{A}&=\tilde{K}_{21} \\ &=-\tfrac{1}{2}\gamma^{\mu}\epsilon D^{0}_{\mu}-\tfrac{1}{4}(\gamma^{\mu}D^{0}_{\mu}\epsilon)+\tfrac{1}{32}\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\psi_{\nu}\epsilon \ , \\ \partial\delta\tilde{F}/\partial\tilde{\chi}&=\tilde{K}_{32} \\ &=\tfrac{1}{2}\bar{\epsilon}\partial^{\mu}D^{0}_{\mu}+\tfrac{1}{4}(D^{0}_{\mu}\bar{\epsilon}\gamma^{\mu})-\tfrac{1}{32}\bar{\psi}_{\mu}\partial^{\nu}\partial^{\mu}\psi_{\nu}\bar{\epsilon} \ , \end{aligned} \tag{4.41}$$

$\partial\delta F/\partial\bar{\bar{\tilde{\chi}}}=\overline{(\partial\delta F/\partial\tilde{\chi})}$ after partial integration. Hence, indeed (4.39) holds, and $\tilde{K}_{32}$ and $\tilde{K}_{21}$ are equal up to a partial integration.

Next, we verify by direct computation that the result for $E^{-1/2}$ when inserted into (4.6), for a local supersymmetry transformation indeed produces $\tilde{K}$ in (4.36).

For local supersymmetry variations we find, after straightforward but rather laborious computations for $\delta\tilde{A}$ and $\delta\tilde{\chi}^{\alpha}$,

$$\delta\tilde{A}=\tfrac{1}{8}\bar{\epsilon}\gamma^{\mu}\psi_{\mu}\tilde{A}+\tfrac{1}{2}\bar{\epsilon}\tilde{\chi} \ , \tag{4.42}$$

$$\begin{aligned} \delta\tilde{\chi}=&\tfrac{1}{2}\gamma^{\mu}\epsilon e^{1/2}\partial_{\mu}(\tilde{A}e^{-1/2})+\tfrac{1}{4}\gamma^{\mu}(D^{0}_{\mu}\epsilon)\tilde{A} \\ &-\tfrac{1}{32}\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\psi_{\nu}\tilde{A}\epsilon+\tfrac{1}{8}(\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\epsilon)\gamma_{\nu}\tilde{\chi} \\ &-\tfrac{1}{8}(\bar{\psi}_{\mu}\gamma^{\mu}\tau_{3}\epsilon)\tau_{3}\tilde{\chi}+\tfrac{1}{2}F\epsilon \ . \end{aligned} \tag{4.43}$$

Not only have the torsion terms in $D_{\mu}(\omega)\epsilon$ canceled, but since $(e^{1/2}\partial_{\mu}e^{-1/2})=-\frac{1}{2}\Gamma^{\nu}_{\mu\nu}$ where $\Gamma^{\rho}_{\mu\nu}$ is the Christoffel symbol, we can combine the first two terms in $\delta\tilde{\chi}^{\alpha}$ as $\frac{1}{4}[D^{0}_{\mu}(\gamma^{\mu}\epsilon)]\tilde{A}'+\frac{1}{2}\gamma^{\mu}\epsilon D^{0}_{\mu}\tilde{A}'$ where D^{0}_{μ} is the covariant derivative without torsion.

The computation of $\delta\tilde{F}$ is quite tedious but the final result is very simple and reads

$$\begin{aligned} \delta\tilde{F}=&-\tfrac{1}{4}\xi^{\nu}\partial_{\nu}\tilde{A}-\tfrac{1}{8}(\partial_{\nu}\xi^{\nu})\tilde{A}+\tfrac{1}{4}(D^{0}_{\mu}\bar{\epsilon}\gamma^{\mu})\tilde{\chi}+\tfrac{1}{2}\bar{\epsilon}\not{D}\tilde{\chi} \\ &-\tfrac{1}{32}(\bar{\epsilon}\tilde{\chi})(\bar{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\psi_{\nu})-\tfrac{1}{8}\bar{\epsilon}\gamma^{\mu}\psi_{\mu}\tilde{F} \end{aligned} \tag{4.44}$$

with ξ^{ν} given in (4.37).

If the constraint (4.15) is relaxed, more solutions of (4.14) exist. Thus, instead of (4.24) we can allow

$$(E^{-1/2})_{11}=e^{-1/2+\alpha}, \quad (E^{-1/2})_{33}=e^{-1/2-\alpha} \ , \tag{4.45}$$

with α an arbitrary parameter, since

$$\begin{aligned} (\hat{E}^{-1/2})_{11}&=(E^{-1/2})_{33} \ , \\ (\hat{E}^{-1/2})_{33}&=(E^{-1/2})_{11} \ , \end{aligned} \tag{4.46}$$

so the α-dependent parts of $(\delta\hat{E}^{-1/2})\hat{E}^{1/2}$ and $-E^{1/2}(\delta E^{-1/2})$ are equal. The constraint (4.15) can thus, be considered a gauge-fixing constraint selecting the Wess-Zumino gauge.

We conclude this section by discussing the regulator $e^{i\tau\tilde{O}\tilde{O}}$. The explicit form of the operator $\tilde{O}\tilde{O}$ is $\Box$ + more:

$$\tilde{O}\tilde{O}=\begin{bmatrix} \Box-\frac{1}{8}\bar{\psi}\not{D}\psi+\frac{1}{8}S^{2} & \frac{1}{2}\#\bar{\psi}-\frac{1}{2}\bar{\psi}\not{D} & 2\#+\frac{1}{16}\bar{\psi}\psi \\ \frac{1}{4}\not{D}\not{D}\psi+\frac{1}{4}\psi\Box & \not{D}\not{D}-\frac{1}{8}\psi\bar{\psi}\not{D}-\frac{1}{8}\not{D}\psi\bar{\psi} & -\frac{1}{2}\not{D}\psi+\frac{1}{2}\psi\# \\ \Box\#+\#\Box+\frac{1}{16}\bar{\psi}\not{D}\not{D}\psi+\frac{1}{32}S^{3} & \frac{1}{4}\Box\bar{\psi}+\frac{1}{4}\bar{\psi}\not{D}\not{D} & \Box-\frac{1}{8}\bar{\psi}\not{D}\psi+\frac{1}{8}S^{2} \end{bmatrix} , \tag{4.47}$$

$\psi \equiv \gamma\cdot\psi$; $\bar{\psi} \equiv -\bar{\psi}\cdot\gamma$, $\#\equiv \frac{1}{4}S+\frac{1}{32}\bar{\psi}\psi$. Although it is not difficult to evaluate $\tilde{O}\,\tilde{O}$ in general, we have only recorded it in the gauge $\psi_\mu=\frac{1}{2}\gamma_\mu\psi$ because that is what we need in Sec. V.

The choice of $\tilde{O}\,\tilde{O}$ is also suggested by superspace where one uses as regulator

$$R=\exp[(\mathcal{E}_t^{-1/2}D^2\mathcal{E}_t^{-1/2})^2/M^2] \tag{4.48}$$

in the superconformal gauge.[2] Here ϵ is the superdeterminant of the supervielbein and $\mathcal{E}_t\equiv(\mathcal{E})^{-t}$, while D_α is the covariant derivative of rigid supersymmetry and $D^2\equiv D^\alpha D_\alpha$. This regulator has a term $\Box$ and equals the iterated action in superspace on a tilde basis. It should be possible to translate this superspace regulator to x space directly, thus picking up the matrices O along the way, but we have not succeeded in this. Rather, we have argued directly in x space that the regulator in (4.47) is the correct one.

V. THE EFFECTIVE x-SPACE ACTION FOR A SCALAR MULTIPLET

In this section we come to the heart of the matter, and compute the various terms appearing in the supersymmetric Liouville action. We use two methods which give the same answer.

(i) The Fujikawa method, in which one uses the path integral and rescales and shifts fields such that the original action becomes a free field action, while the sum of the infinitesimal Jacobians becomes the effective action.

(ii) We start directly from the heat-kernel representation of the effective action, compute its variation and show that this variation is proportional to the variation of the Liouville action.

We begin with the first method. We use the measure and the regulator previously obtained and compute the effective action for the scalar multiplet in a background supergravitational field in the superconformal gauge. We compute this effective action as follows: First we write the action in the form

$$I=\int d^2x(\tilde{F}\,\tilde{\bar{\chi}}-\tilde{A})E^{-1/2}OE^{-1/2}\begin{pmatrix}-\tilde{A}\\ \tilde{\chi}\\ \tilde{F}\end{pmatrix}. \tag{5.1}$$

Then we rescale $(-\tilde{A},\tilde{\chi},\tilde{F})$ in little steps by factors $E^{+\delta t/2}$. This removes the supergravitational dependence from the action. The final action is the standard Wess-Zumino model in flat spacetime, and describes a free field theory, but the Jacobians for these rescalings are nonzero, and summing these infinitesimal Jacobians (i.e., integrating over δt from $t=0$ to 1) yields the effective action for the supergravitational modes. One expects that this effective action again describes a rigidly supersymmetric scalar multiplet because the original action was rigidly supersymmetric, but we want to see this emerging as a result, not by imposing it *a priori* as a property of the effective action. In this sense we are really computing the various anomalies having to do with the change of integration variables $(-A,\chi,F)\to(-\tilde{A},\tilde{\chi},\tilde{F})$.

The fact that one can find a transformation of the basic variables such that (A,χ,F) is rescaled by a multiplicative factor depending on e_μ^m, ψ_μ^a, and S is nontrivial, and is only true in the superconformal gauge (because one cannot remove the vielbein from the D'Alembertian by a simple rescaling of A, for example). Moreover, in this superconformal gauge, if one puts $\psi_\mu=S=0$, the action in terms of variables rescaled by powers of (det$e$) is free. The removal of these det$e$ factors from the action leads to an effective action proportional to $\sigma\Box\sigma$ where $\det e=e^\sigma$; we refer to Refs. 2–4 for a further discussion. Here we look at the complement of this problem, namely, what happens after the det$e$ factors have been scaled away, or, computationally equivalently, we consider vielbeins satisfying det$e=1$. In this case O in (5.1) is field independent.

If at a given moment the rescalings have decreased $E^{-1/2}$ to $E^{(1-t)/2}$, then the infinitesimal Jacobian for a further rescaling is given by [see (2.22) and (4.15)]

$$\begin{aligned}J&=E^{\delta t/2}=1+\tfrac{1}{2}\delta t\,\ln E\\ &=1+\tfrac{1}{2}\delta t\,\ln e-\tfrac{1}{4}\delta t\begin{bmatrix}0&0&0\\ \psi&0&0\\ S&\bar{\psi}&0\end{bmatrix},\end{aligned} \tag{5.2}$$

where $\gamma\cdot\psi=\psi$ and $\bar{\psi}\equiv-\bar{\psi}\cdot\gamma$. As regulator we use e^{R_t/M^2} with

$$R_t\equiv\tilde{O}_t\tilde{O}_t,\quad \tilde{O}_t\equiv E_t^{-1/2}OE_t^{-1/2},\quad E_t^{-1/2}\equiv E^{-(1-t)/2}. \tag{5.3}$$

Since

$$E_t^{-1/2}=E^{-(1-t)/2}=\begin{bmatrix}1&0&0\\ \frac{1}{4}(1-t)\psi&1&0\\ \#_t&\frac{1}{4}(1-t)\bar{\psi}&1\end{bmatrix}(\det e)^{-(1-t)/2},\quad \#_t=(1-t)(\tfrac{1}{4}S-\tfrac{1}{16}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu)-\tfrac{1}{32}(1-t)^2\bar{\psi}\cdot\gamma\gamma\cdot\psi, \tag{5.4}$$

it follows that *in the superconformal gauge* $E_t^{-1/2}$ is related to $E^{-1/2}$ by rescaling $\psi\to(1-t)\psi$, $S=(1-t)S$, and $\sigma\to(1-t)\sigma$ where $\sigma\equiv\ln\det e$. Thus, the superconformal fields are multiplicatively modified by the same factor $(1-t)$, just as the scalar superfield ϕ is in superspace. Outside the superconformal gauge the term $\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu$ violates this simple pattern, but one could still compute anomalies in this general case. (Anomalies use the regulator at $t=0$, and are from this point of view the first infinitesimal Jacobian. Effective actions use the regulator at t and are the sum of infinitesimal Jacobians).

From now on we set $\det e = 1$; the regulated infinitesimal Jacobian at t is thus given by

$$J-1=\text{"tr"}\left[\frac{-\delta t}{4}\right]\begin{bmatrix}0&0&0\\ \psi&0&0\\ S&\bar\psi&0\end{bmatrix}e^{\tilde O_t\tilde O_t/M^2}\equiv\left[\frac{-\delta t}{4}\right]\int d^2x\int\frac{d^2k}{(2\pi)^2}e^{-ik\cdot x}\text{"tr"}\begin{bmatrix}0&0&0\\ \psi&0&0\\ S&\bar\psi&0\end{bmatrix}e^{\tilde O_t\tilde O_t/M^2}e^{ik\cdot x}\,. \tag{5.5}$$

Bringing the plane wave $\exp(ik\cdot x)$ to the left is equivalent to replacing ∂_μ in $\tilde O_t$ by $\partial_\mu+ik_\mu$. Next we rescale k_μ by $1/M$, i.e., we replace k_μ by $\hat k_\mu M$. The Jacobian has a factor M^2 and expanding $\tilde O_t\tilde O_t$ in terms of $\hat k_\mu$ we arrive at

$$J-1=\left[\frac{-\delta t}{4}\right]\int d^2x\,M^2\int\frac{d^2\hat k}{(2\pi)^2}\begin{bmatrix}0&0&0\\ \psi&0&0\\ S&\bar\psi&0\end{bmatrix}\exp\left[-\hat k^2(1+L)+i\frac{\hat k_\mu R^\mu_{(1)}}{M}+\frac{R_{(2)}}{M^2}\right], \tag{5.6}$$

where L, $R_{(1)}$, and $R_{(2)}$ are given below.

We now use the Baker-Campbell-Hausdorff theorem to decompose $\exp(\tilde O\,\tilde O/M^2)$ into a Gaussian regulator $\exp[-\hat k^2(1+L)]$ times another exponential which we expand and of which only a few terms survive in the limit $M\to\infty$. Then we perform the $\hat k$ integration.

Using

$$e^{a+b}=e^a\exp(b-\tfrac12[a,b]+\tfrac16[a,[a,b]]-\tfrac1{12}[b,[b,a]]+\cdots)$$

we have, with $a=-\hat k^2(1+L)$ and $b=i\hat k_\mu R^\mu_{(1)}/M+R_{(2)}/M^2$, using symmetric integration and retaining only terms which do not vanish in the limit $M\to\infty$,

$$e^{\tilde O_t\tilde O_t/M^2}=e^{-\hat k^2(1+L)}\left\{1+\frac{R_{(2)}}{M^2}+\frac12\frac{[\hat k^2L,R_{(2)}]}{M^2}+\frac16\frac{[\hat k^2L,[\hat k^2L,R_{(2)}]]}{M^2}\right.$$
$$\left.-\frac1{24}\frac{\hat k^4}{M^2}[R^\mu_{(1)},[R^\mu_{(1)},L]]-\frac14\hat k^2\left[\frac{R^\mu_{(1)}}{M}+\frac12\left[\hat k^2L,\frac{R^\mu_{(1)}}{M}\right]\right]^2\right\}. \tag{5.7}$$

Next we expand $\exp[-\hat k^2(1+L)]$ as $\exp(-\hat k^2)(1-\hat k^2L\cdots)$ and find

$$\int dk^2e^{-k^2(1+L)}(k^2)^p=(1+L)^{-(p+1)}\Gamma(p+1)\,. \tag{5.8}$$

In this way we obtain

$$e^{\tilde O_t\tilde O_t/M^2}=\frac{1}{M^2}e^{-\hat k^2}\left[\frac{M^2}{1+L}+\frac{1}{1+L}R_{(2)}+\tfrac12(1+L)^{-2}[L,R_{(2)}]-\tfrac14(1+L)^{-2}(R^\mu_{(1)})^2\right.$$
$$+\tfrac13(1+L)^{-3}[L,[L,R_{(2)}]]-\tfrac1{12}(1+L)^{-3}[R^\mu_{(1)},[R^\mu_{(1)},L]]$$
$$\left.-\tfrac14(1+L)^{-3}\{R^\mu_{(1)},[L,R^\mu_{(1)}]\}-\tfrac38(1+L)^{-4}[L,R^\mu_{(1)}][L,R^\mu_{(1)}]\right]. \tag{5.9}$$

In the supertrace only few terms contribute because $\ln E$ and L are "lowering operators":

$$-2\ln E=\begin{bmatrix}0&0&0\\ \psi&0&0\\ S&\bar\psi&0\end{bmatrix},\quad L=\frac{1-t}{2}\begin{bmatrix}0&0&0\\ \psi&0&0\\ S+\frac{1-t}{4}\bar\psi\psi&\bar\psi&0\end{bmatrix}. \tag{5.10}$$

For example, the divergent term $M^2(1+L)^{-1}$ cancels as expected for a supersymmetric system, while in $(1+L)^{-1}R_{(2)}$ only $R_{(2)}-LR_{(2)}$ contribute, and in $(1+L)^{-2}[L,R_{(2)}]$ we may even drop the whole factor $(1+L)^{-2}$. A further simplification arises because $R^{(\mu)}_{(1)}$ has no (1,3) entry (no derivatives appear in the right upper entry of $\tilde O\,\tilde O$). Hence, we may replace $(1+L)^{-2}(R^\mu_{(1)})^2$ by $(1-2L)(R^\mu_{(1)})^2$ and drop the $[L,R^\mu_{(1)}][L,R^\mu_{(1)}]$ term.

All terms rearrange themselves into the symmetrized expression

$$R_{(2)}-\tfrac12\{R_{(2)},L\}-\tfrac14(R^\mu_{(1)})^2+\tfrac16(R^\mu_{(1)}R^\mu_{(1)}L+R^\mu_{(1)}LR^\mu_{(1)}+LR^\mu_{(1)}R^\mu_{(1)})\,. \tag{5.11}$$

Moreover, the last term in (5.11) also vanishes in the supertrace since it is proportional to $(\psi)^3$. The final infinitesimal Jacobian to be evaluated is, thus,

$$J-1=-\frac{\delta t}{16\pi}\int d^2x\,\mathrm{str}\begin{bmatrix}0&0&0\\ \psi&0&0\\ S&\bar{\psi}&0\end{bmatrix}[R_{(2)}-\tfrac{1}{2}\{L,R_{(2)}\}-\tfrac{1}{4}(R^{\mu}_{(1)})^2]\ . \tag{5.12}$$

For the final computations we record the relevant parts of $R_{(1)}$ and $R_{(2)}$:

$$R^{\mu}_{(1)}=\begin{bmatrix}2\partial_\mu & \frac{-(1-t)}{2}\bar{\psi}\gamma^\mu & 0\\ 0 & 2\partial_\mu-\frac{(1-t)^2}{8}\{\gamma^\mu,\psi\bar{\psi}\} & \frac{-(1-t)}{2}\gamma^\mu\psi\\ 0&0&2\partial_\mu\end{bmatrix}+\cdots\ , \tag{5.13}$$

$$R_{(2)}=\frac{1-t}{2}\begin{bmatrix}0 & -\bar{\psi}\partial\!\!\!/+\frac{1-t}{4}S\bar{\psi} & S+\frac{1-t}{4}\bar{\psi}\psi\\ 0&0&-\partial\!\!\!/\psi+\frac{1-t}{4}S\psi\\ 0&0&0\end{bmatrix}+\cdots\ . \tag{5.14}$$

Straightforward matrix evaluation yields

$$R_{(2)}-\tfrac{1}{2}\{R_{(2)},L\}-\tfrac{1}{4}(R^{\mu}_{(1)})^2=\frac{1-t}{2}\begin{bmatrix}0&\frac{1}{2}(\partial_\mu\bar{\psi})\gamma^\mu & S\\ 0&0&-\frac{1}{2}(\partial\!\!\!/\psi)\\ 0&0&0\end{bmatrix}+\cdots\ . \tag{5.15}$$

As in the analogous computation in superspace,[2] all $(1-t)^2$ terms cancel, and the derivative terms combine to commutators $[\partial_\mu,\psi]$, i.e., no free derivatives are left.

The final step is to take the supertrace. We obtain

$$J-1=\int d^2x\left[\frac{-\delta t}{16\pi}\right]\left[\frac{1-t}{2}\right](S^2-\bar{\psi}\partial\!\!\!/\psi)\ . \tag{5.16}$$

After integration over t from 0 to 1 we find the effective action

$$I_{\mathrm{eff}}=\frac{1}{64\pi}\int(\bar{\psi}\partial\!\!\!/\psi-S^2)\quad(\psi=\gamma\cdot\psi)\ . \tag{5.17}$$

If one adds to this result the purely bosonic part of the Liouville action as obtained in Refs. 2–4

$$I^{\mathrm{Bose}}_{(\mathrm{eff})}=-\frac{1}{16\pi}\int\sigma\Box\sigma\quad(\det e^{\mu}_{m}=e^{2\sigma}) \tag{5.18}$$

then one obtains indeed a supersymmetric result, since σ, $\frac{1}{2}\gamma\cdot\psi$ and $\frac{1}{2}S$ form a scalar multiplet of rigid supersymmetry.[4]

The effective action can also be calculated using a heat-kernel approach.[11] The starting point is the representation (2.33) of the one-loop-induced action Γ:

$$\Gamma=-\frac{i}{4}\int_0^\infty\frac{d\tau}{\tau}\int d^2x\,\mathrm{str}(e^{i\tau\tilde{O}\tilde{O}})(x)\delta(x-x')\,|_{x'=x}\ . \tag{5.19}$$

Varying the matrix $E^{-1/2}$ one obtains, using the cyclicity of the supertrace, the following result, valid when O is field-independent

$$\begin{aligned}\delta\Gamma&=\tfrac{1}{2}\int_0^\infty d\tau\int d^2x\,\mathrm{str}[(\delta E^{-1/2})E^{1/2}\tilde{O}\tilde{O}+\tilde{O}\tilde{O}E^{1/2}(\delta E^{-1/2})](e^{i\tilde{O}\tilde{O}})(x)\delta(x-x')\,|_{x'=x}\\ &=\frac{i}{2}\lim_{\tau\to 0}\int d^2x\,\mathrm{str}[(\delta E^{-1/2})E^{1/2}+E^{1/2}(\delta E^{-1/2})](e^{i\tau\tilde{O}\tilde{O}})(x)\delta(x-x')\,|_{x'=x}\ .\end{aligned} \tag{5.20}$$

This expression obviously has the same structure as the Jacobian encountered in Sec. II and can itself be interpreted as a Jacobian.

In the superconformal gauge ($\psi_\mu=\frac{1}{2}\gamma_\mu\psi$) where $\tilde{O}\tilde{O}$ is given by (4.47), we have, using (4.38),

$$(\delta E^{-1/2})E^{1/2}+E^{1/2}(\delta E^{-1/2})=\frac{1}{2}\begin{bmatrix}0&0&0\\ \delta\psi&0&0\\ \delta S&\delta\bar{\psi}&0\end{bmatrix} \tag{5.21}$$

(cf. the definition above of the matrix $\ln E$), which inserted into (5.20) leads to

$$\delta\Gamma=\frac{i}{4}\lim_{\tau\to 0}\int d^2x[\delta S(e^{i\tau\tilde{O}\tilde{O}})_{13}-\mathrm{tr}\delta\psi(e^{i\tau\tilde{O}\tilde{O}})_{12}+\delta\bar{\psi}(e^{i\tau\tilde{O}\tilde{O}})_{23}]_{(x)}\delta(x-x')\,|_{x'=x}\ . \tag{5.22}$$

This expression can be evaluated by means of the explicit expression for $\tilde{O}\,\tilde{O}$, writing

$$\tilde{O}\,\tilde{O}=\begin{bmatrix}\Box & 0 & 0\\ 0 & \not{D}\not{D} & 0\\ 0 & 0 & \Box\end{bmatrix}+\Delta\tilde{O}\,\tilde{O} \tag{5.23}$$

and treating $\Delta\tilde{O}\,\tilde{O}$ as a perturbation. Only a few terms will be nonvanishing for $\tau\to 0$. The expansion in terms of ΔOO is performed by using $\delta e^{A}=\int_0^1 d\alpha\, e^{\alpha A}(\delta A)e^{(1-\alpha)A}$ repeatedly. For example, $(1/2!)\delta\delta e^{A}$ yields two terms which, after suitable redefinitions of integrations variables, yields

$$\frac{1}{2!}\delta\delta e^{A}=\int_0^1\int_0^1\int_0^1 d\alpha\, d\beta\, d\gamma\ \delta(1-\alpha-\beta-\gamma)e^{\alpha A}\delta A e^{\beta A}\delta A e^{\gamma A}\ . \tag{5.24}$$

First consider the term containing δS; it becomes

$$\begin{aligned}
&\frac{i}{4}\lim_{\tau\to 0}\int d^2x\ \delta S(e^{i\tau\tilde{O}\tilde{O}})_{13}(x)\delta(x-x')\,|_{x'=x}\\
&\quad=-\tfrac{1}{4}\lim_{\tau\to 0}\tau\int d^2x\ \delta S\int_0^1 d\alpha[e^{i\alpha\tau\Box}(\tfrac{1}{2}S+\tfrac{1}{8}\bar{\psi}\psi)e^{i(1-\alpha)\tau\Box}]_{(x)}\delta(x-x')\,|_{x'=x}\\
&\qquad-\frac{i}{16}\lim_{\tau\to 0}\tau^2\int d^2x\ \delta S\int_0^1\int_0^1\int_0^1 d\alpha\, d\beta\, d\gamma\ \delta(1-\alpha-\beta-\gamma)(e^{i\alpha\tau\Box}\bar{\psi}\not{D}e^{i\beta\tau\not{D}^2}\not{D}\psi e^{i\gamma\tau\Box})(x)\delta(x-x')\,|_{x'=x}\ .
\end{aligned}\tag{5.25}$$

The right-hand side of (5.25) is evaluated by means of the plane-wave representation of the δ function. In this way (5.25) becomes

$$-\tfrac{1}{4}\tau\int d^2x\ \delta S(\tfrac{1}{2}S+\tfrac{1}{8}\bar{\psi}\psi)_{(x)}\int\frac{d^2k}{(2\pi)^2}e^{i\tau k^2}-\tfrac{1}{2}\tfrac{1}{16}\tau^2\int d^2x\ \delta S\ \bar{\psi}\psi(x)\int\frac{d^2k}{(2\pi)^2}k^2e^{i\tau k^2}=-\frac{1}{32\pi}\int d^2x\,(\delta S)S\ . \tag{5.26}$$

The term in (5.22) involving $\delta\psi$ is found in the same way, by applying the plane-wave representation of the δ function in the expression

$$\begin{aligned}
&-\frac{i}{4}\lim_{\tau\to 0}\int d^2x\ \mathrm{tr}\delta\psi(e^{i\tau\tilde{O}\tilde{O}})_{12}(x)\delta(x-x')\,|_{x'=x}\\
&\quad=\tfrac{1}{4}\lim_{\tau\to 0}\tau\int d^2x\ \mathrm{tr}\delta\psi\int_0^1 d\alpha[e^{i\alpha\tau\Box}(-\tfrac{1}{2}\bar{\psi}\not{D}+\tfrac{1}{8}\bar{\psi}S)e^{i(1-\alpha)\tau\not{D}^2}]_{(x)}\delta(x-x')\,|_{x'=x}\\
&\qquad+\frac{i}{4}\lim_{\tau\to 0}\tau^2\int d^2x\ \mathrm{tr}\delta\psi\int_0^1\int_0^1\int_0^1 d\alpha\, d\beta\, d\gamma\ \delta(1-\alpha-\beta-\gamma)\\
&\qquad\qquad\times[e^{i\alpha\tau\Box}(\tfrac{1}{2}S+\tfrac{1}{8}\bar{\psi}\psi)e^{i\beta\tau\Box}(\tfrac{1}{4}\Box\bar{\psi}+\tfrac{1}{4}\bar{\psi}\not{D}^2)e^{i\gamma\tau\not{D}^2}]_{(x)}\delta(x-x')\,|_{x'=x}\\
&\qquad-\tfrac{1}{4}\lim_{\tau\to 0}\tau^3\int d^2x\ \mathrm{tr}\delta\psi\int_0^1\int_0^1\int_0^1\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4\delta\left[1-\sum_{i=1}^{4}\alpha_i\right]\\
&\qquad\qquad\times[e^{i\alpha_1\tau\Box}(-\tfrac{1}{2}\bar{\psi}\not{D})e^{i\alpha_2\tau\not{D}^2}(-\tfrac{1}{2}\not{D}\psi)\\
&\qquad\qquad\times e^{i\alpha_3\tau\Box}(\tfrac{1}{4}\Box\bar{\psi}+\tfrac{1}{4}\bar{\psi}\not{D}^2)e^{i\alpha_4\tau\not{D}^2}]_{(x)}\delta(x-x')\,|_{x'=x}\\
&\qquad+\frac{i}{4}\lim_{\tau\to 0}\tau^2\int d^2x\ \mathrm{tr}\delta\psi\int_0^1\int_0^1\int_0^1 d\alpha\, d\beta\, d\gamma\ \delta(1-\alpha-\beta-\gamma)[e^{i\alpha\tau\Box}(-\tfrac{1}{2}\bar{\psi}\not{D})e^{i\beta\tau\not{D}^2}\\
&\qquad\qquad\times(-\tfrac{1}{8}\psi\bar{\psi}\not{D}-\tfrac{1}{8}\not{D}\psi\bar{\psi})e^{i\gamma\tau\not{D}^2}]_{(x)}\delta(x-x')\,|_{x'=x}\ .
\end{aligned}\tag{5.27}$$

Here all terms containing three ψ's vanish because of the identity

$$\psi\bar{\psi}\psi=0 \tag{5.28}$$

obtained by a Fierz transformation, while the two terms containing S cancel each other, so (5.27) reduces to

$$\tfrac{1}{4}\lim_{\tau\to 0}\tau\int d^2x\,\mathrm{tr}\delta\psi\int_0^1 d\alpha[e^{i\alpha\tau\Box}(-\tfrac{1}{2}\bar{\psi}\not{D})e^{i(1-\alpha)\tau\not{D}^2}]_{(x)}\delta(x-x')\,|_{x'=x}$$

$$=-\tfrac{1}{8}\lim_{\tau\to 0}i\tau^2\int d^2x\,\mathrm{tr}\delta\psi\int_0^1\alpha\,d\alpha 2(\partial_\mu\bar{\psi})\gamma_\nu\int\frac{d^2k}{(2\pi)^2}k^\mu k^\nu e^{i\tau k^2}=-\frac{1}{64\pi}\int d^2x\,\bar{\psi}\overleftarrow{\not{\partial}}\delta\psi\ . \quad (5.29)$$

The part of (5.22) containing $\delta\bar{\psi}$ contributes equally to $\delta\Gamma$ as that containing $\delta\psi$, so we obtain

$$\delta\Gamma=\frac{1}{32\pi}\int d^2x(\bar{\psi}\not{\partial}\delta\psi-S\delta S)\ . \quad (5.30)$$

By integration one obtains the same result as before.

We conclude with a short discussion of the "auxiliary field anomaly." In the basis of (A,χ,F), the action is independent of the auxiliary field S of the $d=2$ supergravity multiplet, but in the tilde basis $\tilde{F}$ contains a term $\frac{1}{4}e^{1/2}AS$. It follows that $\delta\tilde{F}=\Lambda(x)e^{1/2}A$ is a local symmetry. The Jacobian is proportional to

$$\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ \Lambda(x) & 0 & 0\end{bmatrix} \quad (5.31)$$

and the infinitesimal anomaly is thus proportional to $\Lambda(x)S$. This is in agreement with the result that the variation of the effective action is proportional to the variation of the Liouville action. Hence, the S^2 term of the Liouville action can be interpreted as the integrated auxiliary field anomaly.

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