Notes on the Hawking effect in de Sitter space. I

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(Received 6 July 1987)

We consider the Hawking effect for a quantized massless conformal scalar field in two- and four-dimensional de Sitter space. The relation among Bogoliubov coefficients is investigated without explicit integration of Klein-Gordon products. Our method presents a clear view of the property of Bogoliubov coefficients. In the two-dimensional case the thermal distribution is exactly derived. The application to the four-dimensional case is not straightforward, but we can derive the same result with some techniques.

I. INTRODUCTION

The Hawking effect[1] is one of the most interesting features in quantum field theory in curved space, and its investigation will also shed new light on the quantum theory of gravity. In the last decade, there have been various attempts to understand the Hawking effect more deeply.[2] It has been revealed that the Hawking effect has an intimate relation to the event horizon or the spacetime structure. There is, however, not a clear understanding of the origin of the Hawking effect as yet.

de Sitter space is an important example which has an event horizon. By various approaches, many authors have concluded that a comoving observer in de Sitter space would find himself in a thermal state.[2] In some methods, for example, the Unruh detector method[2,3] or path-integral method,[4] the Hawking effect is derived for, at least, a massless conformal field without any approximation. On the other hand, by the mode-mixing method[5-7] which is one of the standard approaches in quantum field theory, the Hawking effect is obtained only under some approximations. For example, it is often assumed that only a portion of the mode functions near the event horizon contributes to the Hawking effect.

In this paper we discuss whether the mode-mixing method based on the exact mode functions gives the same result as the other methods. We concentrate on the study of a massless conformal scalar field in de Sitter space. Following other works, we choose two different charts,[8] the static chart and the full covering chart, on which two different vacua are defined. One is a no-particle state associated to an observer at the origin, which is defined on the static chart, and the other is the conformal vacuum on the full covering chart. We discuss how this observer will appreciate the conformal vacuum. The Bogoliubov coefficients play an important role in connecting the above two states and by a survey of these coefficients we can learn the spectrum of the particle number distribution which the conformal vacuum has. Our concern at present is its thermal structure. The explicit calculation of each coefficient is not necessary for this purpose, and we show the thermality in integral form.

The plan of the paper is as follows. First we give a brief review of the Bogoliubov transformation. In Sec. III the Hawking effect in two-dimensional de Sitter space is investigated. The four-dimensional case is studied in Sec. IV. Finally in Sec. V, conclusions and discussions are given.

II. BOGOLIUBOV TRANSFORMATION

In quantum field theory, generally, one can consider two different complete orthonormal sets φ_n and ψ_k as mode solutions of the field equation and the pictures of particles are defined in association with these modes. The field operator Ψ is expanded with these sets as

$$\Psi=\sum_n (a_n\varphi_n+a_n^\dagger\varphi_n^*)=\sum_k (b_k\psi_k+b_k^\dagger\psi_k^*)\ , \tag{2.1}$$

where $a_n^\dagger,b_k^\dagger$ and a_n,b_k are creation and annihilation operators, respectively. These operators are connected by the Bogoliubov transformation

$$b_k=\sum_n(\alpha_{kn}a_n+\beta_{kn}a_n^\dagger)\ ,\qquad b_k^\dagger=\sum_n(\alpha_{kn}^*a_n^\dagger+\beta_{kn}^*a_n)\ . \tag{2.2}$$

The Bogoliubov coefficients α_{kn},β_{kn} possess the following properties:

$$\sum_n(\alpha_{kn}\alpha_{k'n}^*-\beta_{kn}\beta_{k'n}^*)=\delta_{kk'}\ ,\qquad \sum_n(\alpha_{kn}\beta_{k'n}-\beta_{kn}\alpha_{k'n})=0\ . \tag{2.3}$$

We can define two different vacua $|0_a\rangle$ and $|0_b\rangle$ associated with each particle notion in (2.1):

$$|0_a\rangle:\ a_n|0_a\rangle=0\ \text{ for all } n\ ,\qquad |0_b\rangle:\ b_k|0_b\rangle=0\ \text{ for all } k\ . \tag{2.4}$$

If we select $|0_b\rangle$ as a natural vacuum, then $|0_a\rangle$ is generally considered as a many-particle state. Its expectation value of the particle number operator of the ψ_k mode is

$$\langle 0_a | b_k^\dagger b_k | 0_a \rangle = \sum_n |\beta_{kn}|^2 . \quad (2.5)$$

To find the particle number distribution the Bogoliubov coefficients must be calculated. However, we can know its spectrum without explicit calculation. For example, if the relation

$$|\alpha_{kn}|^2 = e^{\beta\omega_k} |\beta_{kn}|^2 \quad (2.6)$$

for any n,k is established, we can say that the spectrum is a thermal one with temperature $\beta(=1/k_B T)$ due to the orthonormal completeness (2.3) of the Bogoliubov coefficients.

III. TWO-DIMENSIONAL CASE

The treatment of quantum fields is much simpler in two dimensions than in four. We consider a two-dimensional toy model of de Sitter space. Because of its simplicity, the evaluation of the Bogoliubov coefficient is easy and presents a clear view for the four-dimensional treatment.

Two-dimensional de Sitter space is represented as the hyperboloid

$$z_0^2 - z_1^2 - z_2^2 = -a^2 \quad (3.1)$$

embedded in three-dimensional Minkowski space.[9] On this manifold we employ two different charts.

(1) Full covering chart (τ, χ):

$$\begin{aligned} z_0 &= a \sinh(\tau/a) , \\ z_1 &= a \cosh(\tau/a)\cos\chi , \\ z_2 &= \cosh(\tau/a)\sin\chi . \end{aligned} \quad (3.2)$$

The metric is

$$\begin{aligned} ds^2 &= d\tau^2 - a^2\cosh^2(\tau/a) d\chi^2 \quad (-\pi < \chi < \pi) \\ &= \frac{a^2}{\sin^2\eta}(d\eta^2 - d\chi^2) \quad (0 < \eta < \pi) , \end{aligned} \quad (3.3)$$

where $\eta = 2\arctan[\exp(\tau/a)]$, called the conformal time.

(2) Static chart (t, r):

$$\begin{aligned} z_0 &= (a^2 - r^2)^{1/2}\sinh(t/a) , \\ z_1 &= (a^2 - r^2)^{1/2}\cosh(t/a) , \\ z_2 &= r . \end{aligned} \quad (3.4)$$

The metric is

$$\begin{aligned} ds^2 &= \left[1 - \frac{r^2}{a^2}\right] dt^2 - \frac{1}{1 - \frac{r^2}{a^2}} dr^2 \quad (-a < r < a) \\ &= \left[1 - \frac{r^2}{a^2}\right] (dt^2 - dr_*^2) \quad (-\infty < r_* < \infty) , \end{aligned} \quad (3.5)$$

where $r_* = (a/2)\ln[(a+r)/(a-r)]$. It is an essential feature that the metric (3.5) possesses event horizons at $r = \pm a$, and the no-particle state associated with the observer at $r = 0$ must be defined on this chart.

By using these charts we can easily solve the field equation

$$\Box \Psi = 0 . \quad (3.6)$$

On the full chart (3.3), the mode solutions are

$$\varphi_n = \frac{1}{\sqrt{4\pi |n|}} e^{-i|n|\eta + in\chi}, \quad n = \text{integer} \quad (3.7)$$

and on the static chart (3.5)

$$\psi_k = \begin{cases} \dfrac{1}{\sqrt{4\pi |k|}} e^{-i|k|t - ikr_*} & (R\text{-}1), k = \text{real} , \\ 0 \quad (R\text{-}2) . \end{cases} \quad (3.8)$$

The mode functions (3.8) are the complete set on the chart (3.5) but not on the chart (3.3). To construct the complete set on (3.3) we need $\tilde{\psi}_k$ whose support is on R-2 (Fig. 1). But we do not describe them because their expressions are not necessary.

The Bogoliubov coefficients among mode solutions (3.7) and (3.8) are given by

$$\begin{aligned} \alpha_{kn} &= i \int_\Sigma \varphi_n^* \overleftrightarrow{\nabla}_\mu \psi_k d\Sigma^\mu = \frac{1}{4\pi\sqrt{|n||k|}} \int_{-\pi/2}^{\pi/2} d\chi \, e^{-n\chi} \left[\frac{1+\sin\chi}{1-\sin\chi}\right]^{iak/2} \left[\frac{a|k|}{\cos\chi} + n\right] \\ &= \frac{1}{4\pi\sqrt{|n||k|}} e^{i(n+|n|)\pi/2} \int_0^\pi d\chi \, e^{-in\chi} \left[\tan\frac{\chi}{2}\right]^{iak} \left[\frac{a|k|}{\sin\chi} + n\right] , \end{aligned} \quad (3.9a)$$

$$\beta_{kn} = -i \int_\Sigma \varphi_n \overleftrightarrow{\nabla}_\mu \psi_k d\Sigma^\mu = -\frac{1}{4\pi\sqrt{|n||k|}} e^{-i(n+|n|)\pi/2} \int_0^\pi d\chi \, e^{in\chi} \left[\tan\frac{\chi}{2}\right]^{iak} \left[\frac{a|k|}{\sin\chi} - n\right] . \quad (3.9b)$$

In Eqs. (3.9a) and (3.9b) the Cauchy surface Σ was taken to be a spacelike surface $\tau = 0$ where $r = a \sin\chi$, and then the integral variable χ was transformed into $\chi + \pi/2$.

In the following we treat the case of positive n. The integrals over real interval are replaced with the integrals over paths $C1$ and $C2$ on a complex plane by the Cauchy integral theorem. We choose the path $C1$ for the integral of α_{kn} and $C2$ for β_{kn} (Fig. 2). Then Eqs. (3.9a) and (3.9b) become

$$\alpha_{kn}=iN_{kn}e^{\pi ak/2}\left[-e^{-in\pi}\int_0^\infty ds\, e^{-ns}\left[\tanh\frac{s}{2}\right]^{-iak}\left[i\frac{a\,|k|}{\sinh s}-n\right]\right.$$
$$\left.+\int_0^\infty ds\, e^{-ns}\left[\tanh\frac{s}{2}\right]^{iak}\left[-i\frac{a\,|k|}{\sinh s}-n\right]\right], \tag{3.10a}$$

$$\beta_{kn}=i\tilde{N}_{kn}e^{-\pi ak/2}\left[-e^{-in\pi}\int_0^\infty ds\, e^{-ns}\left[\tanh\frac{s}{2}\right]^{-iak}\left[i\frac{a\,|k|}{\sinh s}-n\right]\right.$$
$$\left.+\int_0^\infty ds\, e^{-ns}\left[\tanh\frac{s}{2}\right]^{iak}\left[-i\frac{a\,|k|}{\sinh s}-n\right]\right], \tag{3.10b}$$

where $|N_{kn}|=|\tilde{N}_{kn}|$ and the following relation can be derived:

$$|\alpha_{kn}|^2=e^{2\pi ak}|\beta_{kn}|^2\ . \tag{3.11}$$

It is noticed that the power of the exponential of (3.11) is not $2\pi a\,|k|$ but $2\pi ak$. Equation (3.11) suggests the Hawking effect only for positive k and not for negative k. In the latter case we should notice the fact that the integrand of (3.10a) becomes a total divergence:

$$\alpha_{kn}=iN_{kn}e^{-\pi ak/2}\left\{-e^{-in\pi}\int_0^\infty ds\frac{d}{ds}\left[e^{-ns}\left[\tanh\frac{s}{2}\right]^{-iak}\right]+\int_0^\infty ds\frac{d}{ds}\left[e^{-ns}\left[\tanh\frac{s}{2}\right]^{iak}\right]\right\}=0\ . \tag{3.12}$$

Therefore, we can say that relation (2.6) is exactly held for positive n.

The above procedure can be applied to the case of negative n also, and the same relation is derived. As the result we conclude that the Hawking effect is established exactly in two dimensions, that is, the conformal vacuum is an exact thermal equilibrium state for the comoving observer.

IV. FOUR-DIMENSIONAL CASE

Four-dimensional de Sitter space is the main problem in this paper. In this case we can also go on with the same calculation as the two-dimensional case. However, some techniques are needed to perform the calculation because the mode functions are very complicated.

In four dimensions two charts are also employed as before.

(1) Full covering chart (τ,χ,θ,ϕ):

$$ds^2=d\tau^2-a^2\cosh^2(\tau/a)(d\chi^2-\sin^2\chi\, d\Omega^2)$$
$$=\frac{a^2}{\sin^2\eta}(d\eta^2-d\chi^2-\sin^2\chi\, d\Omega^2)\quad(0<\chi<\pi)\ . \tag{4.1}$$

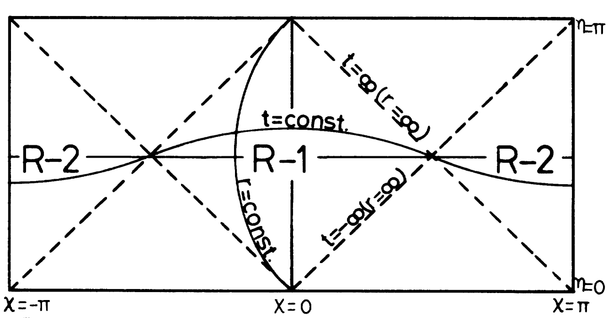

FIG. 1. The Penrose diagram of two-dimensional de Sitter space. $\chi=-\pi$ and $\chi=\pi$ are identified. The observer is supposed to exist on the solid line at $\chi=0$ $(r=0)$. The dashed lines are event horizons. R-1 is the center region with a diamond shape, and R-2 is the opposite region of R-1.

(2) Static chart (t,r,θ,ϕ):

$$ds^2=\left[1-\frac{r^2}{a^2}\right]dt^2-\frac{1}{1-\frac{r^2}{a^2}}dr^2-r^2d\Omega^2\quad(0<r<a)\ . \tag{4.2}$$

The massless conformal field equation in four dimensions is

$$\left[\Box+\frac{R}{6}\right]\Psi=0\ , \tag{4.3}$$

where R is the scalar curvature. In chart (1) we can derive the well-known mode solutions by the hyperspherical harmonics[2]

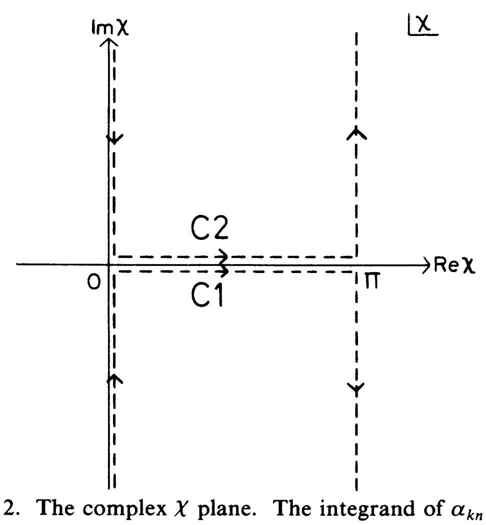

FIG. 2. The complex χ plane. The integrand of α_{kn} (β_{kn}) is analytic on the lower (upper) half plane, and then the contour C1 (C2) can be used.

$$\varphi_{nlm} = \sin\eta e^{-n\eta} \sin^l \chi C_{n-l-1}^{l+1}(\cos\chi) Y_{lm}(\theta,\phi) , \quad n > 0, \text{ integer} . \tag{4.4}$$

In chart (2), because of conformal masslessness, the exact mode solutions are described by the Legendre functions of the second kind:[3,6,10]

$$\psi_{\omega lm} = \begin{cases} e^{-i\omega t} \frac{1}{r} Q_l^{i\omega a}\left(\frac{a}{r}\right) Y_{lm}(\theta,\phi) & \left(0<\chi<\frac{\pi}{2}\right), \ \omega>0, \text{ real} , \\ 0 & \left(\frac{\pi}{2}<\chi<\pi\right) , \end{cases} \tag{4.5}$$

which are regular at $r=0$.

The Bogoliubov coefficients are given by

$$\alpha_{\omega n l l' m m'} = i \int_\Sigma \varphi^*_{nl'm'} \overleftrightarrow{\nabla}_\mu \psi_{\omega lm} d\Sigma^\mu , \quad \beta_{\omega n l l' m m'} = -i \int_\Sigma \varphi_{nl'm'} \overleftrightarrow{\nabla}_\mu \psi_{\omega lm} d\Sigma^\mu . \tag{4.6}$$

The hypersurface Σ is taken to be the $\tau=0$ surface in the same way as Sec. III. By the orthogonality of spherical harmonics we can omit the indices l',m,m'. Then Eqs. (4.6) are written as

$$\alpha_{\omega nl} = N_{\omega n} \int_0^{\pi/2} d\chi \sin^{l+1}\chi C_{n-l-1}^{l+1}(\cos\chi) \times Q_l^{i\omega a}\left(\frac{1}{\sin\chi}\right)\left(\frac{a\omega}{\cos\chi}+n\right) ,$$
$$\beta_{\omega nl} = \tilde{N}_{\omega n} \int_0^{\pi/2} d\chi \sin^{l+1}\chi C_{n-l-1}^{l+1}(\cos\chi) \times Q_l^{i\omega a}\left(\frac{1}{\sin\chi}\right)\left(\frac{a\omega}{\cos\chi}-n\right) . \tag{4.7}$$

First, we note that the case of $l=0$ is reduced to the two-dimensional one directly, and the relations (2.6) are satisfied (Appendix A). For nonzero l the calculation is not as straightforward as before, but we can use the relation (Appendix B)

$$\begin{pmatrix} \alpha_{\omega n(l+1)} \\ \beta_{\omega n(l+1)} \end{pmatrix} = N_{(n+1)} \begin{pmatrix} \alpha_{\omega(n+1)l} \\ \beta_{\omega(n+1)l} \end{pmatrix} + N_{(n-1)} \begin{pmatrix} \alpha_{\omega(n-1)l} \\ \beta_{\omega(n-1)l} \end{pmatrix} . \tag{4.8}$$

Hence, if $|\alpha_{\omega nl}| = e^{\pi a\omega}|\beta_{\omega nl}|$ holds, then

$$\begin{aligned} \alpha_{\omega n(l+1)} &= N_{(n+1)}\alpha_{\omega(n+1)l} + N_{(n-1)}\alpha_{\omega(n-1)l} \\ &= e^{\pi a\omega}\tilde{N}(N_{(n+1)}\beta_{\omega(n+1)l} + N_{(n-1)}\beta_{\omega(n-1)l}) \\ &= e^{\pi a\omega}\tilde{N}\beta_{\omega n(l+1)} , \end{aligned} \tag{4.9}$$

where $|\tilde{N}|=1$. By mathematical induction the final result

$$|\alpha_{\omega nl}|^2 = e^{2\pi a\omega}|\beta_{\omega nl}|^2 \tag{4.10}$$

is derived for any ω, n, and l.

According to the discussion in Sec. II, we conclude that the Hawking effect is exactly confirmed in four dimensions also, and the observer inhabiting de Sitter space will perceive particles with the thermal distribution.

V. SUMMARY AND DISCUSSION

In this paper we have shown that the particle distribution observed by a free-falling observer at $r=0$ in de Sitter space becomes an exactly thermal one. Though many authors have already concluded the exact thermality of the spectrum by other methods, no one ever has derived the same result exactly by the mode-mixing method. Furthermore, our result gives not only an evidence of the Hawking effect, but also more information. One of the important features of our result is the existence of relation (4.10) for every quantum number, while the thermality requires only the relation among $\sum_n |\alpha_{kn}|^2$ and $\sum_n |\beta_{kn}|^2$. Therefore, relation (4.10) may give us different information about the Hawking effect, such as the possibility of determining whether this thermal-like state is truly a mixed state or not,[11] but it is a future problem. Moreover, the relationship between $|\alpha_{\omega kl}|$ and $|\beta_{\omega kl}|$ has been found without explicit integration. We will hope that this similar procedure may be applied to other cases, such as, for example, a black hole.

We have treated the stationary de Sitter space and have studied how a free-falling observer appreciates the conformal vacuum associated with the full covering chart. Our results, however, do not always show that the observer in the de Sitter-type phase finds himself in a thermal equilibrium state. For example, it is often discussed that the de Sitter phase appears in the early Universe,[12] where the Universe develops into de Sitter space dynamically. Such a case must be treated with taking into account the time development of quantum fields. With respect to this subject, further investigation is necessary.

ACKNOWLEDGMENTS

We would like to thank our colleagues in the Elementary Particle Physics group and Theoretical Physics group.

APPENDIX A

In the following calculation, Appendixes A and B, irrelevant multiplication factors are included in $N, N_1, N_2, \ldots$.

Here it is shown that the Bogoliubov coefficients in four dimensions are the same as two dimensions when

$l=0$. First we extend the intergration range from $[0,\pi/2]$ to $[-\pi/2,\pi/2]$. Because of the formula[13]

$$Q_\nu^\mu(z)=\frac{\pi e^{\mu\pi i}}{2\sin\mu\pi}\left[P_\nu^\mu(z)-\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P_\nu^{-\mu}(z)\right], \quad \mu\neq\text{integer},\ |z|>1,$$

$$P_\nu^{-\mu}(z)=(-1)^\nu\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}P_\nu^\mu(-z), \quad \nu=\text{integer},\ |z|>1,$$

$\alpha_{\omega n0},\beta_{\omega n0}$ are rewritten as

$$\begin{aligned}\begin{bmatrix}\alpha_{\omega n0}\\ \beta_{\omega n0}\end{bmatrix}&=N\int_0^{\pi/2}d\chi\,\sin\chi C_{n-1}^1(\cos\chi)Q_0^{i\omega a}\left[\frac{1}{\sin\chi}\right]\left[\frac{a\omega}{\cos\chi}\pm n\right]\\&=N_1\int_0^{\pi/2}d\chi\,\sin\chi C_{n-1}^1(\cos\chi)\left[P_0^{i\omega a}\left[\frac{1}{\sin\chi}\right]-P_0^{i\omega a}\left[-\frac{1}{\sin\chi}\right]\right]\left[\frac{a\omega}{\cos\chi}\pm n\right]\\&=N_1\int_{-\pi/2}^{\pi/2}d\chi\,\sin\chi C_{n-1}^1(\cos\chi)P_0^{i\omega a}\left[\frac{1}{\sin\chi}\right]\left[\frac{a\omega}{\cos\chi}\pm n\right].\end{aligned}$$

Using the explicit forms of $C_{n-1}^1(\cos\chi)$ and $P_0^{i\omega a}(1/\sin\chi)$

$$\begin{aligned}\begin{bmatrix}\alpha_{\omega n0}\\ \beta_{\omega n0}\end{bmatrix}&=N_2\int_{-\pi/2}^{\pi/2}d\chi\,\sin n\chi\left[\frac{1+\sin\chi}{1-\sin\chi}\right]^{i\omega a/2}\left[\frac{a\omega}{\cos\chi}\pm n\right]\\&=N_3\left[\int_{-\pi/2}^{\pi/2}d\chi\,e^{in\chi}\left[\frac{1+\sin\chi}{1-\sin\chi}\right]^{i\omega a/2}\left[\frac{a\omega}{\cos\chi}\pm n\right]-\int_{-\pi/2}^{\pi/2}d\chi\,e^{-in\chi}\left[\frac{1+\sin\chi}{1-\sin\chi}\right]^{i\omega a/2}\left[\frac{a\omega}{\cos\chi}\pm n\right]\right].\end{aligned}$$

This equation is that of two dimensions. Therefore, following the same discussion as Sec. III, we can conclude that

$$|\alpha_{\omega n0}|^2=e^{2\pi a\omega}|\beta_{\omega n0}|^2 .$$

APPENDIX B

We show that the Bogoliubov coefficients for $l+1$ can be reduced to those for l. First we rewrite the Gegenbauer polynomial with the associated Legendre function

$$\begin{aligned}\begin{bmatrix}\alpha_{\omega(n+1)(l+1)}\\ \beta_{\omega(n+1)(l+1)}\end{bmatrix}&=N\int_0^{\pi/2}d\chi\,\sin^{l+2}\chi C_{n-l}^{l+2}(\cos\chi)Q_{l+1}^{i\omega a}\left[\frac{1}{\sin\chi}\right]\left[\frac{a\omega}{\cos\chi}\pm(n+1)\right]\\&=N_1\int_0^{\pi/2}d\chi\sqrt{\sin\chi}P_{n+1/2}^{-l-3/2}(\cos\chi)Q_{l+1}^{i\omega a}\left[\frac{1}{\sin\chi}\right]\left[\frac{a\omega}{\cos\chi}\pm(n+1)\right],\end{aligned}\tag{B1}$$

where we use the formula

$$C_n^\nu(x)=\frac{\Gamma(n+2\nu)\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu)n!2^{1/2-\nu}}(x^2-1)^{1/4-\nu/2}P_{n+\nu-1/2}^{1/2-\nu}(x), \quad |x|<1 .$$

By the formulas

$$Q_{\nu+1}^\mu(z)=\frac{1}{\nu-\mu+1}\left[(z^2-1)\frac{dQ_\nu^\mu}{dz}+(\nu+1)zQ_\nu^\mu(z)\right], \quad (1-x^2)\frac{dP_\nu^\mu}{dx}=-\mu xP_\nu^\mu(x)-\sqrt{1-x^2}P_\nu^{\mu+1}(x), \quad |x|<1$$

and partial integration, Eq. (B1) is

$$\begin{bmatrix}\alpha_{\omega(n+1)(l+1)}\\ \beta_{\omega(n+1)(l+1)}\end{bmatrix}=\left\{\int_0^{\pi/2}d\chi\sqrt{\sin\chi}\frac{l+1}{\sin\chi}P_{n+1/2}^{-l-3/2}Q_l^{i\omega a}\left[\frac{a\omega}{\cos\chi}\pm(n+1)\right]\right.$$

$$\left.+\int_0^{\pi/2}d\chi\, Q_l^{i\omega a}\frac{d}{d\chi}\left[\sqrt{\sin\chi}\cos\chi P_{n+1/2}^{-l-3/2}\left[\frac{a\omega}{\cos\chi}\pm(n+1)\right]\right]\right\}$$

$$=N_2\left[\int_0^{\pi/2}d\chi\sqrt{\sin\chi}\sin\chi P_{n+1/2}^{-l-3/2}Q_l^{i\omega a}\left[(l+1)\frac{a\omega}{\cos\chi}\pm l(n+1)\right]\right.$$

$$\left.-\int_0^{\pi/2}d\chi\sqrt{\sin\chi}\cos\chi P_{n+1/2}^{-l-1/2}Q_l^{i\omega a}\left[\frac{a\omega}{\cos\chi}\pm(n+1)\right]\right]. \tag{B2}$$

Furthermore, we use the recursion formulas

$$(\nu+\mu)(1-x^2)^{1/2}P_\nu^{\mu-1}(x)=P_{\nu+1}^{\mu}(x)-xP_\nu^{\mu}(x)\ , \tag{B3}$$

$$(2\nu+1)xP_\nu^\mu(x)=(\nu-\mu+1)xP_{\nu+1}^{\mu}(x)+(\nu+\mu)P_{\nu-1}^{\mu}(x)\ . \tag{B4}$$

By Eq. (B3), Eq. (B2) is

$$\begin{bmatrix}\alpha_{\omega(n+1)(l+1)}\\ \beta_{\omega(n+1)(l+1)}\end{bmatrix}=N_2\left[\frac{1}{n-l}\int_0^{\pi/2}d\chi\sqrt{\sin\chi}P_{n+3/2}^{-l-1/2}Q_l^{i\omega a}\left[(l+1)\frac{a\omega}{\cos\chi}\pm l(n+1)\right]\right.$$

$$\left.-\frac{n+1}{n-l}\int_0^{\pi/2}d\chi\sqrt{\sin\chi}\cos\chi P_{n+1/2}^{-l-1/2}Q_l^{i\omega a}\left[\frac{a\omega}{\cos\chi}\pm n\right]\right],$$

and, by Eq. (B4),

$$=N_2\left[-\tfrac{1}{2}\int_0^{\pi/2}d\chi\sqrt{\sin\chi}P_{n+3/2}^{-l-1/2}Q_l^{i\omega a}\left[\frac{a\omega}{\cos\chi}\pm(n+2)\right]-\tfrac{1}{2}\int_0^{\pi/2}d\chi\sqrt{\sin\chi}P_{n-1/2}^{-l-1/2}Q_l^{i\omega a}\left[\frac{a\omega}{\cos\chi}\pm n\right]\right].$$

The first term is the same integration as the Bogoliubov coefficient for energy $n+2$ and angular momentum l, and the second is that for n and l. Therefore, we derive

$$\begin{bmatrix}\alpha_{\omega(n+1)(l+1)}\\ \beta_{\omega(n+1)(l+1)}\end{bmatrix}=N_{(n+2)}\begin{bmatrix}\alpha_{\omega(n+2)l}\\ \beta_{\omega(n+2)l}\end{bmatrix}+N_{(n)}\begin{bmatrix}\alpha_{\omega nl}\\ \beta_{\omega nl}\end{bmatrix}.$$

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