Power-law-lapse time gauges

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The choice of time function for cosmological solutions of gravitational field equations is related to the action of the group of independent scale transformations of the unit of length along orthogonal spatial directions. This is accomplished by the introduction of lapse functions which depend explicitly on the spatial metric in an appropriately defined power-law fashion. The resulting power-law lapse time gauges are the key to producing nearly all exact solutions of the class of models for which the field equations reduce to ordinary differential equations.

I. INTRODUCTION

With the near exhaustion of the production of new exact cosmological solutions of gravitational field equations in four spacetime dimensions has come the explosion of the industry into higher dimensions, where the possibilities are much richer in almost every way. Unfortunately, most people working in the field are too busy turning the crank on the exact-solution machine to examine the machine itself. The present article analyzes one aspect of this machine which may seem trivial and yet is still not commonly understood. It is also important in appreciating the qualitative behavior of classes of models which do not admit exact solutions, as well as in quantum calculations which involve manipulations of the classical field equations and Hamiltonian.

This paper deals with the choice of time variable used in discussing the field equations and in obtaining exact or approximate solutions. The power-law-lapse time gauges to be described below are so natural that some solutions are still being produced without the realization that such a gauge choice has been made. Indeed it is the key to the existence of those very solutions which have been found essentially by trial and error.

The choice of power-law exponents which fixes the power-law-lapse time gauge is crucial in decoupling the field equations,[1–4] a necessary step in obtaining exact solutions. For a decoupled field equation for a single dependent variable, having fixed the independent variable (the time), one may often introduce a new dependent variable of "power-law type" which considerably simplifies the equation, in many cases leading to familiar elementary functions or less familiar, but known, special functions. The decoupled equations may be reinterpreted in terms of one-dimensional scattering problems. In the most favorable case, one may redefine the dependent variable so that the scattering potential is polynomial and of low order.[4] For potentials which are linear or quadratic, the familiar elementary functions appear. For third- or fourth-degree potentials, elliptic functions result. After this, any claim to familiarity is no longer possible.

These remarks will refer mostly to the case of a spatially homogeneous metric, but apply also to certain classes of spatially inhomogeneous spacetimes where the field equations are reducible to ordinary differential equations with respect to the time, with spatial variables appearing in them essentially as parameters. The spherically symmetric dust spacetimes and Szekeres dust spacetimes in four dimensions are examples of this latter type.[5] A similar discussion holds for spacetimes where the field equations are reducible to ordinary differential equations with respect to a spacelike variable, as occurs in static spacetimes, for example. In both cases it is the conformal properties of the spacetime curvature which are being exploited.

Note that in the spatially homogeneous case, the slicing of spacetime by the family of homogeneous spacelike hypersurfaces determines a preferred class of time functions which differ only in the way they parametrize the natural slicing. However, this reparametrization freedom is not trivial and can play a very useful role in studying the dynamics of such models, most notably through power-law-lapse gauge conditions. The slicing itself is a constant-mean-curvature slicing,[6] first advocated by York[7] as a preferred slicing condition due to the resulting simplifications of the initial-value problem. In the inhomogeneous case, the power-law-lapse gauge freedom affects the time slicing as well, and the gauge is no longer compatible with the constant-mean-extrinsic-curvature slicing condition.

In the context of higher-dimensional theories, Weyl transformations of the spacetime metric often occur involving either a metric variable or a Brans-Dicke-type scalar field, which is essentially equivalent to a metric variable on a still higher-dimensional spacetime, or both.[8] In practice the choice of "conformal gauge" associated with this Weyl freedom reduces to an anisotropic power-law lapse condition when viewed appropriately. This choice is made on the basis of the scaling properties of the spacetime curvature under constant rescalings of the metric. From the point of view of the time gauge, power-law lapses extend this freedom to more general scalings.

II. SPATIALLY HOMOGENEOUS SPACETIMES

Consider a spacetime of dimension $D+1$ whose manifold is the product of the real line R with a D-dimensional manifold Σ equipped with a spatially homogeneous Lorentz metric

$$^{D+1}g = -N^2 dt\otimes dt + g_{ab}(\omega^a + N^a dt)\otimes(\omega^b + N^b dt)\ , \tag{2.1}$$

where $\{w^a\}$ is a symmetry-adapted dual frame on Σ, dual to a frame $\{e_a\}$. The lapse function N and metric components g_{ab} are assumed to depend only on the time coordinate t, with (g_{ab}) a positive-definite matrix. This is possible if the "spatial frame" $\{e_a\}$ is chosen to be compatible with the symmetry and if the shift vector field $N^a e_a$ is then chosen to preserve this compatibility.

The spatial homogeneity of the metric means that Σ is either a Lie group manifold, the case of simple transitivity or a specialization thereof by imposing additional symmetry, or that it is a nontrivial coset space with a multiply transitive symmetry group. In the case of simple transitivity, one may assume a left-invariant frame on the group manifold and the metric components are not constrained. Imposing additional symmetry on such metrics is accomplished by subjecting the metric component matrix to linear constraints. These kinds of constraints are necessary in the case of a nontrivial coset space, where one may choose a natural frame which is as invariant as possible under the natural left action of the symmetry group on the coset space.

Most discussions assume zero shift vector field and unit lapse function, so that the time lines are orthogonal to the spatial sections and the time function is the synchronous proper time measured along the geodesic congruence normal to the natural slicing by orbits of the symmetry group, or homogeneous hypersurfaces. However, any smooth reparametrization of the time variable is also a valid choice. There is no more reason to insist on using the proper time than there is to use a proper radial coordinate in problems with spherical, cylindrical, or axial symmetry. Indeed, one rarely uses such a coordinate in those problems and for good reason. A proper radial coordinate complicates the field equations. It is more natural to consider such proper-distance coordinate functions as geometrical properties of the spacetime that one can consider evaluating once the metric is known in some convenient coordinate gauge. This perspective is essential to making progress in cosmology.

The simplest case of (2.1) occurs when the dual frame $\{\omega^a\}$ and initial data may be chosen so that the spatial metric component matrix is diagonal in the zero shift spatial gauge, the so-called diagonalizable case. The nondiagonal case may be handled by assuming a spatial gauge with nonzero shift which leads to at least a block-diagonal form for the metric and then using the block determinants as the factors in the power-law behavior to be discussed for the diagonal case below. This is very similar to the generalization of the diagonal exact power-law metrics[9] to the nondiagonal case in four spacetime dimensions,[10] where a completely diagonal spatial gauge always exists. Although the choice of time gauge is equally important in the nondiagonal case, only the diagonal case will be considered here for simplicity.

For the diagonal case, which may or may not be compatible with the symmetry, one may assume the matrix of spatial metric components to be diagonal,

$$(g_{ab}) = e^{2\boldsymbol{\beta}},\quad \boldsymbol{\beta} = \mathrm{diag}(\beta^{(1)}, \ldots, \beta^{(D)})\ . \tag{2.2}$$

Constant translations in the natural logarithmic variables $\{\beta^{(a)}\}$ lead to independent rescaling of the unit of length along the orthogonal frame vectors $\{e_a\}$. It is convenient to split the action of this Abelian group of scale transformations into isotropic and anisotropic parts by introducing a basis of the Lie algebra of diagonal matrices adapted to the direct sum into pure trace and trace-free matrices[11]

$$\begin{aligned}&\boldsymbol{\beta} = \beta^A \mathbf{e}_A,\quad A = 0, 1, , \ldots, D-1\ ,\\ &\mathbf{e}_0 = \mathbf{1},\quad \mathrm{Tr}\,\mathbf{e}_A = D\delta^0{}_A\ .\end{aligned} \tag{2.3}$$

It is further convenient to choose a basis orthogonal with respect to the Lorentz DeWitt[12] inner product on the space of square matrices:

$$\langle \mathbf{A}, \mathbf{B}\rangle_{\mathrm{DW}} = \mathrm{Tr}\,\mathbf{AB} - \mathrm{Tr}\,\mathbf{A}\,\mathrm{Tr}\mathbf{B}\ , \tag{2.4}$$

subject to the normalization

$$\langle \mathbf{e}_A, \mathbf{e}_B\rangle_{\mathrm{DW}} = D(D-1)\eta_{AB}\ , \tag{2.5}$$

where $(\eta_{AB}) = (\eta^{AB}) = \mathrm{diag}(-1, 1, \ldots, 1)$ and the normalization factor is just the DeWitt inner product of the unit matrix with itself, reversed in sign. The purely isotropic scale transformations correspond to conformal rescaling of the spatial metric. The notation $g = \det(g_{ab}) = e^{2D\beta^0}$ will be used for the spatial metric determinant.

As is well known, the Lorentzian DeWitt metric on the metric configuration space plays a fundamental role in gravitational dynamics. Apart from a normalization factor, the coordinates β^A are orthonormal coordinates on the flat diagonal configuration space of diagonal positive-definite matrices. The matrices $\mathbf{e}_A$ may be identified with tangent vectors to this space if one identifies the tangent space to the configuration space with the space of symmetric tensors; this makes the matrices $\mathbf{e}_A$ elements of an orthonormal frame (modulo normalization). The variable $\beta^0 = (\ln \det g)/2D$ is a natural time coordinate on the metric configuration space, and a translation along this coordinate alone corresponds to a uniform rescaling of the unit of spatial length. The quantity $R = e^{\beta^0}$ is an average scale factor for the spatial metric, while the other exponentials e^{β^A}, $A\neq 0$ represent the relative anisotropies among the individual scale factors $R_{(a)} = e^{\beta^{(a)}}$ along the orthogonal axes. Translations along the spacelike coordinates β^A, $A\neq 0$ describe anisotropic rescalings of the units of spatial length.

Often the D-dimensional spatial tangent space is decomposed into the direct sum of two subspaces of dimensions d_1 and d_2, with $d_1 + d_2 = D$. It is natural to adapt the metric variables to this decomposition by introducing new orthonormal coordinates (again modulo normalization). Let the notation $\mathrm{diag}(a;b)=\mathrm{diag}(a,\ldots,a,b,\ldots,b)$ stand for a diagonal $D\times D$ matrix with identical diagonal values on each individual subspace.

First choose a basis $\{\mathbf{e}_A\}$ with the first two elements

$$\mathbf{e}_0=\mathrm{diag}(1;1)\,,\qquad \mathbf{e}_+=[(D-1)/d_1d_2]^{1/2}\mathrm{diag}(d_2;-d_1)\,, \tag{2.6}$$

and the remaining trace-free elements describing the internal anisotropies within each subspace. As long as $d_1>1$, as will be assumed, the first two basis elements are, respectively, timelike and spacelike with respect to the DeWitt inner product. Next, consider boosting this pair of basis vectors by the following Lorentz transformation:

$$\begin{aligned}
\begin{bmatrix}\bar{\mathbf{e}}_0\\ \bar{\mathbf{e}}_+\end{bmatrix}&=\gamma_{\mathrm{boost}}\begin{bmatrix}1 & \beta_{\mathrm{boost}}\\ \beta_{\mathrm{boost}} & 1\end{bmatrix}\begin{bmatrix}\mathbf{e}_0\\ \mathbf{e}_+\end{bmatrix}
=\begin{bmatrix}\zeta\,\mathrm{diag}(1;0)\\ \chi\,\mathrm{diag}[d_2;-(d_1-1)]\end{bmatrix},\\
\gamma_{\mathrm{boost}}&=\left[\frac{d_1(D-1)}{D(d_1-1)}\right]^{1/2}=\frac{d_1}{D}\zeta\,,\\
\beta_{\mathrm{boost}}&=\left[\frac{d_2}{d_1(D-1)}\right]^{1/2}=\frac{d_2}{D}\chi\,,\\
\zeta&=\left[\frac{D(D-1)}{d_1(d_1-1)}\right]^{1/2},\quad \chi=\left[\frac{D}{d_1d_2(D-1)}\right]^{1/2}.
\end{aligned} \tag{2.7}$$

The two coordinates β^0 and β^+ undergo the inverse boost. Translations in the new coordinate $\bar\beta^0$ lead to a uniform rescaling of the metric components associated with the first subspace alone:

$$\begin{aligned}
e^{\beta^0\mathbf{e}_0+\beta^+\mathbf{e}_+}&=e^{\bar\beta^0\bar{\mathbf{e}}_0+\bar\beta^+\bar{\mathbf{e}}_+}\\
&=\mathrm{diag}(e^{\zeta\bar\beta^0+d_2\chi\bar\beta^+};e^{-(d_1-1)\chi\bar\beta^+})\,,\\
g^{1/2}&=e^{D\beta^0}=e^{d_1\zeta\bar\beta^0+d_2\chi\bar\beta^+}\,.
\end{aligned} \tag{2.8}$$

Seen in the context of the full metric, this redefinition of the coordinates corresponds to a Weyl transformation of the metric associated with the first subspace. The coordinate $\zeta\bar\beta^0$ plays the role of the logarithmic average scale factor associated with the first subspace before the Weyl transformation of the metric coefficients associated with that subspace by the square of the factor $e^{\psi}=e^{d_2\chi\bar\beta^+}$ to yield the actual metric coefficients. [If one lets ${}^{(2)}g$ be the metric determinant associated with the second subspace, this conformal factor can be written $e^{\psi}={}^{(2)}g^{-1/2(d_1-1)}$, which reduces to ${}^{(2)}g^{-1/4}$ for the case $d_1=3$ corresponding to three "outer space" dimensions.] For convenience, the metric variables associated with the first subspace before the active Weyl transformation which yields the actual metric coefficients will be referred to as the Weyl rescaled variables. These are orthogonal to those of the second subspace.

III. POWER-LAW LAPSES

The power-law-lapse time gauges are those for which the lapse function is explicitly equal to a product of powers of the diagonal metric components and a normalization constant $N_{(0)}$:

$$N^2=N_{(0)}^2\prod_{a=1}^{D}g_{aa}{}^{Q_a}$$

or

$$N=N_{(0)}\exp(Q_a\beta^{(a)})=N_{(0)}\exp(Q_A\beta^A)\,, \tag{3.1}$$

where Q_A are the components with respect to the basis $\{\mathbf{e}_A\}$ of the covector Q_a on the space of diagonal matrices. Symmetry considerations (diagonalizability or coset space restrictions) may warrant linear constraints on the exponents Q_a reflecting possible linear constraints on the diagonal metric components. These exponents characterize the particular choice of power-law-lapse time gauge. The normalization constant $N_{(0)}$ is unimportant but often convenient to choose equal to either 1 or $2D(D-1)$; the latter value will be assumed here unless otherwise specified.

Well-known examples of power-law-lapse time gauges are the trivial case of proper-time time gauge with $Q_a=0$ and $N_{(0)}=1$ and the nontrivial case of Taub[13] time gauge $N=N_{(0)}g^{1/2}$ with $Q_a=1$. Each of these is an isotropic power-law-lapse time gauge in which all the exponents are equal $Q_a=Q$ and the lapse is a power of the determinant of the entire spatial metric component matrix: $N=N_{(0)}g^{Q/2}$. One may also consider cases in which subsets of the exponents are equal, leading to partially isotropic power-law-lapse time gauges, with only the determinant of the corresponding submatrices of the spatial metric component matrix entering into the lapse and no ratios of the metric components associated with those subspaces.

Under constant uniform scale transformations of the limit of spacetime length under which the frame one-forms are assumed to be invariant (dimension 0), the spatial metric undergoes an isotropic scale transformation (dimension 2) fixing the power-law-lapse dimension to be $\sum_{a=1}^{D}Q_a$. This in turn determines the dimension of the time variable as $1-\sum_{a=1}^{D}Q_a$. Only when this vanishes is the time variable "scale invariant" in the usual sense. The proper time carries dimension 1, for example, and the Taub time dimension $1-D$. The proper time may be defined as a function of any other time by the integral

$$\tau=\int_{t_0}^{t}N\,dt\,. \tag{3.2}$$

Conventionally, this time is chosen to vanish at the initial singularity $g=0$ (or equivalently $R=0$ or $\beta^0\to-\infty$) in initially expanding models. A well-known example of an isotropic scale-invariant time is conformal time, for which $N=N_{(0)}e^{\beta^0}$, i.e., $Q_a=1/D$. An often used scale-invariant time adapted to a particular spatial direction, say the last, corresponds to the lapse $N=N_{(0)}e^{\beta^{(D)}}$. In four spacetime dimensions such a gauge was used by Siklos[14] to study Killing horizons. One may also choose a scale-invariant time by using the average scale factor for a subspace of any dimension between 1 and D. For an isotropic subspace this is equivalent to a Siklos-type time gauge.

To discuss the field equations it is convenient to introduce a lapse density (weight -1) by dividing the lapse by the Taub value,

$$x^{-1}=N/N_{\text{Taub}}=Ng^{-1/2}/2D(D-1)$$
$$=[N_{(0)}/2D(D-1)]\exp\left[\sum_{a=1}^{D}(Q_a-1)\beta^{(a)}\right]. \quad (3.3)$$

Assuming the choice $N_{(0)}=2D(D-1)$ eliminates the constant factor. Letting $\tilde{Q}_{(a)}=Q_a-1$ and letting $\tilde{Q}_A=Q_A-D\delta^0_A$ be its components with respect to the basis $\{\mathbf{e}_A\}$ leads to the expressions

$$x^{-1}=e^{\tilde{Q}_{(a)}\beta^{(a)}}=e^{\tilde{Q}_A\beta^A}. \quad (3.4)$$

This quantity describes the time rate of change of the Taub time t_{Taub} in the same sense that the lapse describes the time rate of change of the proper time

$$d\tau/dt=N=N_{(0)}e^{Q_A\beta^A},$$
$$dt_{\text{Taub}}/dt=x^{-1}=e^{\tilde{Q}_A\beta^A}. \quad (3.5)$$

Note that moving in the direction of the vector $Q^A=\eta^{AB}Q_B$ in $\boldsymbol{\beta}$ space decreases the amount of coordinate time which elapses with respect to proper time thus slowing down the coordinate time, while moving in the opposite direction speeds up the coordinate time. In particular, in the limit of large displacements in the given direction, the coordinate time comes to a halt, while in the opposite limit the coordinate time speeds up to try to make an infinite amount of coordinate time elapse during a finite amount of proper time. For example, for an isotropic power-law lapse $N=N_{(0)}g^{Q/2}$ with $Q>0$, coordinate time speeds up in the limit $g\to 0$ but slows down in the limit $g\to\infty$, while switching the sign of Q reverses these behaviors. Whether a finite or infinite amount of coordinate time elapses during a finite amount of proper time depends on the field equations.

For spacetimes which contain either an initial or final singularity ($R=0$) or both, one can follow Gotay and Demaret[15] in classifying the time variable t as "fast" or "slow" depending on whether an infinite amount of coordinate time t elapses during a finite interval $[0,R_{(0)}]$ of values of the scale factor at a singularity in which the scale factor vanishes. This idea of fast or slow times at $R=0$ may be extended to the limit $R\to\infty$, if it exists, by requiring the same conditions for a finite interval of values $[0,R_{(0)}^{-1}]$ of the inverse scale factor. For this extended usage, the qualifier "at $R\to\infty$" will be used.

For the decomposition (2.6)–(2.8), the reciprocal of the lapse density is

$$x=2D(D-1)e^{d_1\zeta\bar{\beta}^0+d_2\chi\bar{\beta}^+}/N$$
$$=e^{d_1\zeta\bar{\beta}^0+d_2\chi\bar{\beta}^+-Q_A\beta^A} \quad (3.6)$$

The power-law lapse can always be factored into two factors $N^{(1)}$ and $N^{(2)}$, the first of which depends only on $\bar{\beta}^0$ and the internal anisotropies of the first subspace, i.e., on the Weyl transformed metric associated with this subspace, and the second of which depends only on the remaining variables. By choosing $N^{(2)}$ to equal the Weyl scale factor $e^{\psi}=e^{d_2\chi\bar{\beta}^+}$ and allowing $N^{(1)}$ to remain arbitrary, only the factor in (3.6) associated with the second subspace is absorbed into the lapse and x will have the value determined by the factor $N^{(1)}$ alone, involving only the Weyl rescaled metric associated with the first subspace. The square of the Weyl scale factor e^{ψ} then becomes an overall scale factor of the Lorentz block of the metric associated with the time variable and the first subspace, i.e., one has a Weyl transformation in the usual sense of the term. Such a time gauge will be referred to as a Weyl gauge. The Taub time gauge with respect to the whole space also looks like Taub time gauge with respect to the Weyl transformed Lorentz metric, while the choice $N^{(1)}=1$ looks like a proper-time time gauge with respect to this partial metric.

For the special case in which $d_2=1$ so that $d_1=D-1$, the power-law lapse obtained in this way with $N^{(1)}=1$ is explicitly

$$N=N_{(0)}(e^{\beta^0-(D-1)\beta^+})^{-1/D-2}$$
$$=N_{(0)}e^{-\beta^{(D)}/D-2}. \quad (3.7)$$

In three spatial dimensions this yields the value $N=e^{-\beta^{(3)}}$ described by the power-law exponents $(Q_1,Q_2,Q_3)=(0,0,-1)$ originally used by Misner to study the Taub solutions in four spacetime dimensions.[13,16] The two equal exponents Q_1 and Q_2 correspond to the locally rotationally symmetric two-dimensional subspace of the decomposition of the tangent space.

Such a gauge has been used by Gibbons and Wiltshire[17] to discuss higher-dimensional Einstein-Maxwell theory in the context of static cylindrically symmetric solutions, where the same ideas may be applied to the ordinary differential equations for which the spacelike radial coordinate ρ is the independent variable. The Misner time gauge for the Taub solution arose from its link with static solutions in which the corresponding gauge is natural, again for reasons of conformal rescaling properties connected with the field equations. In the context of four-dimensional static black-hole solutions, this radial gauge is just the usual one. Gibbons and Maeda[18] have applied these ideas in studying higher-dimensional static solutions.

IV. FIELD EQUATIONS

The terms which appear in spatially homogeneous gravitational field equations can be classified either as kinetic or potential terms, depending on whether or not they involve time derivatives. Most interesting cosmological scenarios involve potential terms which at least in the diagonal case have a power-law dependence on the spatial metric components or on suitably defined scalar fields. These are exponential in terms of the natural logarithmic variables. Furthermore, scalar fields often contribute to the kinetic terms in a way equivalent to metric coefficients associated with additional spatial dimensions and a compensating Weyl transformation. One thus often has evolution equations with a typical gravitational kinetic sector and a sum of power-law potentials which are exponential in the natural logarithmic variables.[19] Of course in some cases the original scalar fields are already natural variables and one must deal with power-law potentials rather than exponential potentials, leading to a modification of the present discussion.

For example, in discussing higher-dimensional theories involving form-field source fields, Freund-Rubin-Englert-[20,21] type conditions, or others are often imposed which enable the form field to be expressed in terms of the metric and constants of the motion, at least in the terms which enter into the evolution equations for the remaining variables. This generalizes the four-dimensional behavior of the electromagnetic Taub solutions first studied by Brill[22] and that of all locally rotationally symmetric spatially homogeneous electromagnetic field sources in that dimension.[23] As another example, an isotropic perfect fluid obeying an equation of state $p=(\gamma-1)\rho$ and flowing orthogonally to the spatially homogeneous slicing contributes terms which depend on powers of g, even in the nondiagonal case.

Excluding "higher derivative" theories, assume that the field equations for the metric can be written in the form

$$^{D+1}G^{\alpha}{}_{\beta}=\kappa T^{\alpha}{}_{\beta} , \tag{4.1}$$

where the remaining fields and possible cosmological constant term are lumped into the energy-momentum tensor. These equations may also be written in Ricci form:

$$^{D+1}R^{\alpha}{}_{\beta}=\kappa E^{\alpha}{}_{\beta}=\kappa[T^{\alpha}{}_{\beta}-(D-1)^{-1}T^{\gamma}{}_{\gamma}\delta^{\alpha}{}_{\beta}] . \tag{4.2}$$

Greek indices assume the values $0,1,\ldots,D$, where the 0 index refers to the time-coordinate index; the index $\perp$ will refer instead to a component along the unit normal $e_{\perp}$ to the homogeneous hypersurfaces.

The Ricci form of the evolution equations for the diagonal metric take the form[24]

$$x^{-1}(x\dot{\beta}^{(a)})^{\cdot}=F^{(a)} , \tag{4.3}$$

where a dot indicates the time derivative and the driving force arises from the spatial curvature and the spatial energy-momentum tensor

$$F^{(a)}=-[2D(D-1)x^{-1}]^2 g(^{D}R^{a}{}_{a}-\kappa E^{a}{}_{a}) . \tag{4.4}$$

With the assumptions made above the driving terms can be written as a linear combination of products of powers of the diagonal metric coefficients

$$F^{(a)}=-\sum_i x^{-2}c_i^{(a)}\left[\prod_{b=1}^{D} g_{bb}{}^{s_{i(b)}^{(a)}}\right]$$

$$=-\sum_i x^{-2}c_i^{(a)}\exp(2s_{i(b)}^{(a)}\beta^{(b)}) . \tag{4.5}$$

These equations may be reexpressed in terms of the matrix basis $\{\mathbf{e}_A\}$:

$$x^{-1}(x\dot{\beta}^{A})^{\cdot}=F^{A} . \tag{4.6}$$

With $N_{(0)}=2D(D-1)$ and noting (3.4) the kinetic terms have the structure

$$x^{-1}(x\dot{\beta}^{A})^{\cdot}=\ddot{\beta}^{A}-\dot{\beta}^{A}(\tilde{Q}_B\dot{\beta}^{B}) . \tag{4.7}$$

If there are real scalar fields involved as is often the case, the combined evolution equations in most cases are still of this form but with more logarithmic variables such as β^A associated with the scalar fields; these may be accommodated by extending the range of values of the index A. Of course in this case the power-law-lapse discussion takes place in the context of the fake fiber symmetric higher-dimensional theory which allows one to incorporate the scalar fields into the total spacetime metric. This means that two separate splittings of the spacetime tangent space are often relevant—a splitting of the real spacetime together with its imbedding in the artificial higher-dimensional theory necessary to accommodate the scalar fields.[8]

The first splitting is often imposed in considering cosmological solutions of the bosonic sector of higher-dimensional theories. One essentially views the spacetime of the higher-dimensional theory as a fiber bundle with a four-dimensional spacetime as a base space, then imposing symmetry along the fibers to obtain an effective low-energy four-dimensional theory. Usually this is done by assuming a product manifold, so one deals with the so-called "warped product" spacetimes. This allows the possibility of redefining the scale of the four-dimensional metric by a factor depending on the fiber metric, i.e., Weyl transformations come into play. Such transformations lead to changes in the time gauge through the scaling of the lapse function. The second splitting may be understood in terms of the extended DeWitt metric on the configuration space of metric and scalar field variables. The choice of new variables which often mix metric and scalar fields is dictated by the diagonalization of this extended DeWitt metric.

The importance of the power-law-lapse time gauges is that they allow one to rescale the potential terms in the field equations by absorbing an overall common factor of all those terms into the definition of the lapse function; which factor is useful is often best seen from the Hamiltonian or Lagrangian which generates the field equations. This freedom allows one to simplify the field equations and possibly decouple some of the variables. The Taub gauge, for example, absorbs the metric determinant factor in the kinetic terms into the lapse, thus leading to the simplest form for those terms.[3] A Weyl gauge leads to the same form of the field equations for the terms involving only the variables associated with the Weyl rescaled partial Lorentz metric; the coupling to the remaining variables in that sector of the field equations then occurs indirectly through the additional mixed terms. A number of examples of this occur in the literature in various gauges with respect to the Weyl rescaled partial metric; some use proper-time time gauge,[8,25–28] some Taub time gauge,[29,30] and some use gauges adapted to certain potential terms.[8,26] Lorentz has made use of some scale-invariant time gauges.[31,32] Hanquin and Demaret have considered adapting the gauge to various one-form field potentials.[33] It would be difficult to survey the numerous applications of power-law lapses which occur in the literature, some of which are not even recognizable; these often occur disguised as a parametric representation of the proper time and metric variables.[34] The first reference to power-law-lapse time gauges and their relation to the field equations seems to be the discussion by Bonanos[1,2] of the vacuum Taub solution.

As is well known, the evolution equations must be supplemented by the remaining Einstein equations which act as constraints on the solutions of the evolution equations. The supermomentum constraints are often identically satisfied for the diagonal case or impose linear constraints on the diagonal metric components. For the present discussion they will be ignored. The super-Hamiltonian constraint requires the vanishing of the total gravitational Hamiltonian H:

$$\begin{aligned} L&=T-U=L_{\mathrm{ADM}}, \quad H=T+U=N\mathcal{H} , \\ T&=\tfrac{1}{2}x\eta_{AB}\dot\beta^A\dot\beta^B=\tfrac{1}{2}x^{-1}\eta^{AB}p_Ap_B , \\ U&=-2D(D-1)x^{-1}g({}^DR+2\kappa T^{\perp}{}_{\perp}) , \end{aligned} \tag{4.8}$$

where R is the spatial curvature scalar and $\mathcal{H}$ is the total super-Hamiltonian:

$$\mathcal{H}=2g^{1/2}({}^{D+1}G^{\perp}{}_{\perp}-\kappa T^{\perp}{}_{\perp}) . \tag{4.9}$$

The metric/scalar "evolution equations" are not well defined but may always be changed by adding multiples of the super-Hamiltonian. This changes the quadratic first-time-derivative terms and rearranges the potential terms. If one is explicitly solving these equations, the choice of evolution equations is crucial to decoupling the variables. This freedom is also useful in simplifying a qualitative analysis of the field equations. The Lagrangian or Hamiltonian equations of motion which follow from (4.8) (modified by a nonpotential force when necessary[24]) are not equivalent to (4.6) except in the Taub time gauge. In practice, to obtain decoupled equations one must take new linear combinations of the equations of motion and the super-Hamiltonian constraint.[3,4]

The DeWitt metric on the space of diagonal metric matrices is explicitly[24]

$$\mathcal{G}=4D(D-1)e^{D\beta^0}\eta_{AB}d\beta^A\otimes d\beta^B \tag{4.10}$$

but it is the rescaled metric $\frac{1}{2}N^{-1}\mathcal{G}=x\eta_{AB}d\beta^A\otimes d\beta^B$ which gives the kinetic energy of the system as half the square of the velocity vector $\dot\beta^A$. The freedom of choice of the power-law lapse is thus equivalent to a conformal rescaling freedom for the DeWitt metric as it appears in the kinetic energy.[35] In Taub time gauge $x=1$, the resulting metric is flat and convenient to use when discussing the Lorentz geometry of the space; it is assumed that this metric remains flat when extended to the scalar fields of the theory. In other power-law lapse time gauges the metric is only conformally flat but not flat, with the gauge variable x representing the rescaling factor with respect to the flat metric.

If a constant vector C_A exists for which $C_AF^A=0$, then a direction exists along which the driving force has no component and the variable $\beta=C_A\beta^A$ obeys the equation of motion $\dot p=(xC_A\dot\beta^A)^{\cdot}=0$ characteristic of free motion with respect to the conformally rescaled DeWitt metric. In Taub time gauge this leads to linear dependence on the time corresponding to a constant velocity solution. In other time gauges, one must integrate the equation of conservation of momentum along this direction:

$$\beta=p\int x^{-1}dt, \quad p=xC_A\dot\beta^A=\eta^{AB}C_Ap_A . \tag{4.11}$$

If β is a non-null variable then one may adopt the inertial coordinates to it using a Lorentz transformation and the cyclic variable β leaves behind an effective potential in the kinetic energy $x\dot\beta^2\sim x^{-1}p^2$, where $\sim$ means equal to within a constant. In the Taub time gauge this is just a constant.

To integrate (4.11) for the free motion, one needs only an expression for x. This variable has a simple equation of motion as indirectly noted by Lorentz-Petzold;[32] from (3.4) and (4.6) one obtains

$$x^{-1}\ddot x=-\tilde Q_AF^A . \tag{4.12}$$

In those special cases where the right-hand side depends only on x, this equation decouples from the remaining field equations. For example, if the right-hand side is proportional to x^{-1}, one obtains a quadratic expression for x if the constant of proportionality is nonzero and linear solutions if not. If the right-hand side is instead constant, one obtains either hyperbolic, exponential, or trigonometric solutions depending on the sign of this constant.

Such decoupled equations are the key to all exact solutions of the field equations. A variable $\theta=S_A\beta^A$ has a decoupled equation of motion in a given time gauge if one can add a suitable multiple of the Hamiltonian to its equation of motion so that no other variables appear in the result. For example, starting from the Ricci evolution equation for θ,

$$\ddot\theta-\dot\theta(\tilde Q_B\dot\beta^B)-S_AF^A=0 , \tag{4.13}$$

and adding a certain constant times $x^{-1}H$ to it, may lead to a result of the form

$$\ddot\theta+\delta\dot\theta^2+f(e^\theta)=0 . \tag{4.14}$$

When this is possible, the variable θ decouples from the remaining variables and its evolution is governed by an equivalent one-dimensional scattering problem. Its equation of motion has the first integral

$$E_\theta=\tfrac{1}{2}e^{2\delta}\dot\theta^2+g(e^\theta), \quad g'(z)=f(z) , \tag{4.15}$$

which is more natural to interpret in terms of the power variable $u=e^{\delta\theta}$ when $\delta\neq0$:

$$E_\mu=\delta^2E_\theta=\tfrac{1}{2}\dot u^2+\delta^2g(u^{1/\delta}) . \tag{4.16}$$

The integral E_θ when $\delta=0$ and E_μ when $\delta\neq0$ are the energy functions for a one-dimensional scattering problem for the potential function g and δ^2g, respectively. The energy integral can always be formally integrated to yield the time as a function of the dependent variable, but when the potential function is simple enough, one obtains explicit exact solutions for θ or u as a function of the time.

If only one variable decouples, one is free to rescale the lapse by a factor depending on that variable without losing the decoupling. This is useful to simplify the equivalent scattering problem. If more than one variable decouples, this rescaling with respect to one decoupled variable may break the decoupling of another variable. Clearly the most attractive situation would be the existence of a time gauge which admits decoupled equations for each variable, reducing the problem to a collection of one-dimensional problems, with the optimal case yielding each decoupled variable as an explicit function of the time. The individual problems are related by the super-Hamiltonian constraint; some multiple of the Hamiltonian will be a constant for the resulting set of decoupled evolution equations and must be chosen to vanish. One can calculate which multiple using the Bianchi identities and the conservation equations satisfied by the source energy-momentum tensor.[2] Of course such a situation does not often occur except in very simple models. More common for models which allow explicit exact solution (as opposed to implicit exact solution) is that one variable decouples, but the remaining equations although depending explicitly on the decoupled variable can still be integrated. In realistic models, neither of these situations occur. However, limiting behavior of the solutions can often be described by solutions of simplified systems in which certain potential terms are neglected and it is this possibility which gives some credibility to the search for exact solutions. With more variables involved, another possibility which may occur is that a linear subspace of logarithmic variables decouples from the remaining variables even though coupling within the subspace occurs. This is relevant for models for which a decomposition of the form (2.3)–(2.8) is natural.

V. A SIMPLE EXAMPLE

For the sake of an explicit example, consider a spatially homogeneous model neglecting for the moment any anisotropic potential terms in the field equations, i.e., $F^A = F^0 \delta^A{}_0$. This means that all of the anisotropy variables β^A, $A \neq 0$ undergo free motion in the sense of (4.11) and one has $\sum_{A\neq 0} (\dot\beta^A)^2 = x^{-2}\Sigma^2$ for some constant $\Sigma \geq 0$ which may be interpreted in terms of shear. Putting in a cosmological constant, an isotropic spatial curvature, and a spatially homogeneous perfect fluid obeying an equation of state $p=(\gamma-1)\rho$ and moving orthogonally to the homogeneous hypersurfaces (one finds from the conservation equations that $\rho=\rho_{(0)} g^{-\gamma/2}$), one obtains the generalized Friedmann equation

$$H = -\tfrac{1}{2} x \dot\beta^{0\,2} + \tfrac{1}{2} x^{-1}\Sigma^2 + 2D(D-1)x^{-1}[2\kappa\rho_{(0)} e^{D(2-\gamma)\beta^0} - D(D-1)k e^{2(D-1)\beta^0} + 2\Lambda e^{2D\beta^0}] = 0 . \tag{5.1}$$

The most general such equation is of the form

$$H = -\tfrac{1}{2} x\dot\beta^{0\,2} + \tfrac{1}{2}x^{-1}(A_1 e^{q_1\beta^0} + A_2 e^{q_2\beta^0} + A_3 e^{q_3\beta^0} + \cdots) = 0 ,$$
$$0 \leq q_1 < q_2 < q_3 < \cdots . \tag{5.2}$$

Comparing this to (5.1), one may introduce equivalent perfect-fluid parameters for each potential term by

$$q_i = (2-\gamma_i)D, \quad 8D(D-1)\kappa\rho_{(0)i} = A_i . \tag{5.3}$$

A physical perfect fluid must have parameters which satisfy $\rho_{(0)} > 0$ and $\gamma \in [1,2]$. Table I shows the values of these parameters for possible terms in the generalized Friedmann equation.

As long as x is chosen to depend only on β^0, then β^0 decouples from the remaining variables and the result is a one-dimensional problem in the variable β^0 with a linear combination of exponential potentials whose exponents depend on the choice of x. Suppose just one potential term is present:

$$H = -\tfrac{1}{2} x\dot\beta^{0\,2} + \tfrac{1}{2} x^{-1} A_i e^{q_i\beta^0} , \tag{5.4}$$

which requires $A_i > 0$. There are two obvious choices for x which simplify the dynamics associated with this term. One is to choose x so that the term is reduced to a constant (the absolute choice), and the other is to choose x so that x becomes an overall conformal factor of the kinetic term and this potential term (the conformal choice):

absolute choice,

$$x = e^{q_i\beta^0} = e^{D(2-\gamma_i)\beta^0} = g^{(2-\gamma_i)/2} ,$$
$$N/N_{(0)} = e^{(q_i - D)\beta^0} = e^{D(\gamma_i - 1)} = g^{(\gamma_i-1)/2} ;$$

conformal choice , (5.5)

$$x = e^{q_i\beta^0/2} = e^{D(2-\gamma_i)\beta^0/2} = g^{(2-\gamma_i)/4} ,$$
$$N/N_{(0)} = e^{(q_i/2 - D)\beta^0} = e^{D\gamma_i/2} = g^{\gamma_i/4} .$$

These choices for each of the possible terms in the generalized Friedmann equation are summarized in Table I.

When $q_i = 0$ $(\gamma_i = 2)$, both choices reduce to the Taub time gauge $x=1$ and the Hamiltonian describes one-dimensional Euclidean motion in a constant potential (the negative of the actual potential term). When $q_i \neq 0$ $(\gamma \neq 2)$, these choices are distinct and lead to a one-dimensional problem with a constant potential or a parabolic potential, respectively, when reexpressed in terms of the "power variable" $u = x^{1/2}$:

TABLE I. Interpretation of the values of the equation-of-state parameter γ for possible terms in the generalized Friedmann equation characterized by the Taub time-gauge potential exponent q, together with both natural choices of time gauge adapted to the associated potential term. These exponents are defined by $N/N_{(0)}=e^{Q_0\beta^0}$ and $x^{-1}=e^{\tilde{Q}_0\beta^0}$. The final column lists exact solutions when only the given potential term is present. Note that the values $0,1,D$ of Q_0 correspond to proper time, conformal time, and Taub time, respectively.

	γ	q	Interpretation	Conformal choice Q_0	Conformal choice $-\tilde{Q}_0$	Absolute choice Q_0	Absolute choice $-\tilde{Q}_0$	Exact solution
Λ	0	2D	Cosmological constant	0	D	$-D$	$2D$	DeSitter ($\Lambda>0$)
k	$2/D$	$2D-2$	Isotropic curvature	1	$D-1$	$-(D-2)$	$2(D-1)$	Minkowski ($k<0$)
$\rho_{(0)}$	$\gamma\in[1,2]$	$D(2-\gamma)$	Perfect fluid	$D\gamma/2$	$D(2-\gamma)/2$	$D(\gamma-1)$	$D(2-\gamma)$	$k=0$ FRW ($\rho_{(0)}>0$)
	1	D	Dust	$D/2$	$D/2$	0	D	
	$1+1/D$	$D-1$	Radiation	$(D+1)/2$	$(D-1)/2$	1	$D-1$	
(Σ)	2		Stiff/scalar/shear	D	0	D	0	(Kasner)

$$\text{absolute choice,}\quad -H\sim\tfrac{1}{2}\dot{u}^{2}-B_i=0\ ,$$
$$\text{conformal choice,}\quad -H\sim\tfrac{1}{2}\dot{u}^{2}-B_iu^{2}=0\ , \tag{5.6}$$

where B_i differs from A_i only by a positive factor. In the first case, one finds linear solutions for u and in the second, exponential solutions. The absolute choice of power-law-lapse time gauge adapted to a physical perfect fluid was first introduced for all spatially homogeneous spacetimes by Bogoyavlensky and Novikov.[36] For such a perfect fluid flowing orthogonally to the homogeneous hypersurfaces, the Hamiltonian equations of motion in this time gauge are identical with those in the absence of the fluid since the Hamiltonian only differs by a constant.

Of course if one makes an arbitrary isotropic power-law-lapse choice $x=e^{-\tilde{Q}_0\beta^0}$ ($\tilde{Q}_0\neq0$) when $q_i\neq0$, one ends up with an arbitrary power in the potential for the power variable $u=x^{1/2}$:

$$-H=\tfrac{1}{2}\dot{u}^{2}-B_iu^{-2(1+q_i/\tilde{Q}_0)}=0\ , \tag{5.7}$$

but this does not turn out to be very useful since only very special cases admit exact solutions even in the absence of other potential terms. (The first power case leads to quadratic solutions for u, while the inverse second power leads to quadratic solutions for $U=u^2$.) For a similar reason, the absolute choice of time gauge with respect to a given potential term is less useful than the conformal choice for the isotropic models being considered.

Suppose two potential terms are present in the generalized Friedmann equation. If one makes the absolute choice with respect to one of them and chooses the power variable $u=x^{1/2}$ as before, one is left with nonzero energy scattering in a potential with a general power which can be integrated explicitly only in a few special cases. On the other hand, since there is only one independent variable, one can forget the second-order equation of motion and integrate the energy constraint directly, so one may throw away an overall conformal factor multiplying the Hamiltonian. This allows one the freedom to choose the power variable u in order to get an equivalent scattering problem which is exactly integrable, described by a Hamiltonian with kinetic term $\frac{1}{2}\dot{u}^2$ and a rescaled potential function reversed in sign. In this way any two-potential-term case can be reduced in several different ways to a problem with nonzero energy scattering in a potential which is either linear or quadratic in the dependent variable. In the same way any three-potential-term case with equally spaced exponents can be reduced in many ways to an equivalent scattering problem in a quadratic potential. Four or five equally spaced exponents (some that may be missing terms in the potential) lead to third- and fourth-degree polynomial potentials and elliptic function solutions. These cases are considered in detail elsewhere.[4]

As an example, consider the two-potential-term case

$$H=-\tfrac{1}{2}x\dot{\beta}^{-2}+\tfrac{1}{2}x^{-1}(A_1e^{q_1\beta^0}+A_2e^{q_2\beta^0}) \tag{5.8}$$

but without assuming (q_1,q_2) are ordered. By assuming the conformal choice $x=e^{q_1\beta^0/2}$ with respect to the first term and introducing the power variable $u=e^{-q\beta^0/2}$, where $q=q_2-q_1$, one obtains an equivalent scattering problem with a parabolic potential in u:

$$-x^{-1}H=\tfrac{1}{2}\dot{\beta}^{02}-(A_1+A_2e^{q\beta^0})=0$$
$$\leftrightarrow\tfrac{1}{2}\dot{u}^{2}-(q^2/8)(A_1u^2+A_2)=0\ , \tag{5.9}$$

which has either hyperbolic or trigonometric sine and cosine solutions, depending on the signs of the parameters. If (q_1,q_2) are ordered, then $q>0$ and one obtains one time gauge; if they are not, then reversing the order leads to $q<0$ and another time gauge.

Consider the case in which a cosmological constant ($q_1=2D$) and shear term ($q_2=0$) are present, appropriate for a spatially flat vacuum model. The conformal choice of power-law lapse for the cosmological constant term is proper-time time gauge apart from a normalization constant, and the power variable leading to (5.9) is $u=e^{D\beta^0}=x$. One finds solutions of the form

$$u\sim\delta^{-1}\sinh\delta t,\quad \exp\left[\int x^{-1}dt\right]\sim v^{1/\Sigma}\ ,$$
$$v=\delta^{-1}\tanh(\delta t/2)\ . \tag{5.10}$$

With the definition $p_a - 1/D = \sum_{A\neq 0} p_A e_A{}^a{}_a/\Sigma$, so that p_a satisfy the usual Kasner relations, one finds the individual scale factors

$$e^{\beta(a)} \sim u^{1/D} v^{p_a - 1/D} \sim (\delta^{-1}\sinh\delta t)^{1/D}(\delta^{-1}\tanh\delta t/2)^{p_a - 1/D} \sim (\cosh\delta t/2)^{2/D - p_a}(\delta^{-1}\sinh\delta t/2)^{p_a} . \quad (5.11)$$

The last form of these scale factors was given by Sato,[37] but the single decoupled mode and the free modes are all mixed up in his representation. The $t \to 0$ and $t \to \infty$ limits in the present representation more clearly show the connection of the decoupling with the Kasner and DeSitter behavior, respectively. One may repeat this analysis in quite a few power-law-lapse time gauges.

Once the scattering problem (5.2) is reexpressed in terms of a suitable power variable u and the Hamiltonian rescaled so that the kinetic term is $\frac{1}{2}\dot{u}^2$, a graph of the resulting potential alone allows one to classify the kinds of solutions which are possible, the simplest being monotonically expanding or contracting and bounce and recollapse solutions. As discussed by Harrison,[38] more complicated potentials lead to oscillating, static, and asymptotic solutions. He considers a three-potential-term case, where one may reduce the potential in a number of different ways to a linear potential plus a power, the power being two in the case of equally spaced values of the exponents.

Whether a certain time gauge is fast or slow at $R=0$ or $R \to \infty$ depends only on the limiting behavior of the potential, which is dominated by the first and the last term of (5.2), respectively, in these limits. If the time is adapted to the first term using the conformal choice (5.5), then as $\alpha \to -\infty$, the solution approaches flat free motion in α; the exponential dependence of R on the time then makes the time gauge fast. On the other hand, as $\alpha \to \infty$ (if such a limit exists), then the last term (assuming there are more than one) dominates and leads to solutions for which R approaches an inverse power of $|t_\infty - t|$. This implies that the time gauge is slow at $R \to \infty$.

If the time is not adapted to the first term in the potential, then R behaves like a positive power of $|t - t_{(0)}|$ as $R \to 0$ and the time is slow. If there are only two terms in the potential, then as $R \to \infty$ one again has flat free motion in α and a fast time in that limit. If there are more than two terms, then the situation is as before and a slow time results.

The choice of different power-law time gauges to describe the same system leads to a time evolution of the logarithmic dependent variable which looks very different. Similarly, changing from a logarithmic-dependent variable to a power dependent variable also makes a radical change in the equivalent one-dimensional scattering problem which describes the solutions. This can easily be seen from the energy constraint (5.9) for the two-potential-term case. If the time is adapted to the term associated with the smaller of the two exponents, then the difference q is positive and one has an increasing exponential potential in the logarithmic variable β^0 (ignoring the sign of the coefficient), while in the other case, adapting the time to the term associated with the larger exponent leads to a negative difference q and a decreasing exponential potential. On the other hand, for each of these times gauges one can transform to the natural power dependent variable leading to parabolic potentials, but the two different power variables are reciprocally related so that the vertex of one parabolic potential corresponds to infinite values of the other and vice versa. The asymptotic values of the potential at infinite values of β^0 corresponding to zero or infinite values of the scale factor are sometimes finite and sometimes infinite depending on which of the four possibilities one chooses. These changes are due to the rescaling of the potential which occurs in the choice of either independent or dependent variables.

Of course in higher-dimensional theories it is precisely the anisotropic case which is interesting. Suppose one considers the simple model of a spatial geometry which is the product of two (intrinsically) isotropic subspaces. With the decomposition (2.6)–(2.8), let ${}^{(1)}\beta^0$ and ${}^{(2)}\beta^0$ be the average scale factors associated with the individual subspaces, with $\beta^0 = d_1{}^{(1)}\beta^0 + d_2{}^{(2)}\beta^0$. One has the Hamiltonian

$$H = x(\tfrac{1}{2}\eta_{AB}\dot{\beta}^A\dot{\beta}^B) + 2D(D-1)x^{-1}e^{2D\beta^0}[d_1(d_1-1)k_1 e^{-2{}^{(1)}\beta^0} + d_2(d_2-1)k_1 e^{-2{}^{(2)}\beta^0}] = 0 . \quad (5.12)$$

If allowed to be nontrivial as in the case of a flat subspace, the individual anisotropies within the subspaces undergo free motion, leaving behind an effective potential. In terms of the orthogonal variables adapted to the first subspace, one has

$$H = \tfrac{1}{2}x(-\dot{\bar{\beta}}^{0\,2} + \dot{\bar{\beta}}^{+2}) + 2D(D-1)x^{-1}e^{2d_1\zeta\bar{\beta}^0}\{[4D(D-1)]^{-1}\Sigma^2_{\text{partial}}e^{-2d_1\zeta\bar{\beta}^0} + d_1(d_1-1)k_1 e^{-2\zeta\bar{\beta}^0} + d_2(d_2-1)k_2 e^{2(D-1)\chi\bar{\beta}^+}\} = 0 . \quad (5.13)$$

If $k_2 = 0$ then $\bar{\beta}^+$ is also free and the only effect of the flat free motion of the extra dimensions is to add to the shear constant associated with the free variables in the first subspace, if it is already nonzero. In the Taub time gauge this shear term in the Hamiltonian is constant and the Hamiltonian evolution equations for the Weyl rescaled metric associated with the remaining dimensions are identical with the equations in which the additional dimensions are not present. The effect of the additional dimensions simply makes the conserved Hamiltonian for the Weyl rescaled partial metric alone equal to a nonzero constant. This fact alone explains many higher-dimensional exact solutions that have appeared in the literature. Note that it is the orthogonal variables $\bar{\beta}^0$ and $\bar{\beta}^+$ which permit a decoupling, not the original variables $^{(1)}\beta^0$ and $^{(2)}\beta^0$. However, as long as $k_1k_2\neq 0$, one does not get a decoupling of the two nonfree modes $\bar{\beta}^0$ and $\bar{\beta}^+$. One may use the super-Hamiltonian constraint to transfer the coupling from the potential terms to the kinetic terms but one cannot eliminate it.

The Weyl choice of power-law-lapse time gauge adapted to a splitting of this kind is equivalent to allowing x to depend only on the Weyl rescaled partial metric, in which case the terms in the Hamiltonian involving only those variables alone have the "standard Einstein form,"[39] i.e., without multiplicative factors involving the additional metric or scalar variables. The Hamiltonian approach is well suited to the task of picking out useful time gauges, as shown in many applications by Maeda and Halliwell.

VI. OTHER EXAMPLES

An interesting application of power-law-lapse time gauges occurs in Moncrief's study of generalized Taub-NUT (Newman-Unit-Tambourino) spacetimes,[40] whose metrics are fiber-invariant solutions of the vacuum Kaluza-Klein theory on the nontrivial S^1 fiber bundle $R\times S^3$ over $R\times S^2$, and in the closely related spacetimes where S^3 and S^2 are replaced by T^3 and T^2, respectively. These metrics are nondiagonal and inhomogeneous, but the time gauge Moncrief found useful to discuss the field equations for them is a Weyl choice of power-law-lapse time gauge with respect to the $2+1$ split of the spatial tangent space. Beginning in zero-shift spatial gauge with an explicit Weyl rescaling of the metric variables and an explicit factor of t^2 adapted to the Killing horizon at $t=0$:

$$^{(4)}ds^2=e^{-2\gamma}(-N^2dt^2+g_{ab}\omega^a\omega^b)+t^2e^{2\gamma}(k\omega^3+\beta_a\omega^a)^2 , \qquad (6.1)$$

where k is a constant and $(N,g_{ab},\gamma,\beta_a)$ are functions only of the base manifold coordinates (t,x^a), $a=1,2$ he imposes the Taub condition $N^2=N^2_{(0)}{}^{(2)}g$ on the three-dimensional Weyl rescaled metric (where $N_{(0)}$ is independent of t). Using the notation

$$^{(4)}ds^2=-\bar{N}^2dT^2+\bar{g}_{ij}\omega^i\omega^j , \qquad (6.2)$$

where $T=\ln t$ is a reparametrization of the time, shows that $\bar{N}^2=k^2N^2_{(0)}{}^{(3)}\bar{g}$, i.e., the t slicing is just the Taub time gauge apart from a reparametrization which brings the Killing horizon at $T=-\infty$ to the finite time $t=0$.

Another inhomogeneous four-dimensional example worth examining is the case of the Szekeres[5] dust solutions with nonzero cosmological constant studied by Barrow and Stein-Schabes,[41] where the Einstein equations reduce to the Friedmann equation in which the usual constant parameters and the scale factor depend on a spatial coordinate. With $D=3$, $\gamma=1$, and $\Sigma=0$, (5.1) becomes

$$H=-\tfrac{1}{2}x\dot{\beta}^{02}+12x^{-1}(2\kappa\rho_{(0)}e^{3\beta^0}-6ke^{4\beta^0}+2\Lambda e^{6\beta^0})=0 . \qquad (6.3)$$

This corresponds to the case in which the exponents of (5.2) are $(q_1,q_2,q_3)=(3,4,6)$.

When only two of the three parameters $(\rho_{(0)},k,\Lambda)$ are nonzero, this may be converted to a linear or quadratic potential scattering problem in a number of different power-law-lapse time gauges,[4] but when all three are nonzero one is forced to a third degree polynomial potential which results in elliptic functions. The choice of power-law lapse $N=e^{\beta^0/2}$, $x=12e^{5\beta^0/2}$ which is the conformal choice (5.5) adapted to the missing potential term with exponent $q=5$, leads to such a potential in terms of the power variable which is the scale factor itself $R=e^{\beta^0}$. Defining $M=\kappa\rho_{(0)}/3$ leads to

$$H=6e^{\beta^0/2}(-\dot{R}^2+M-kR+\tfrac{1}{3}\Lambda R^3)=0 . \qquad (6.4)$$

This may be integrated to obtain the time as a function of R; assuming $\Lambda>0$,

$$t-t_0=-\int_R^\infty(M-k\tilde{R}+\Lambda\tilde{R}^3/3)^{-1/2}d\tilde{R}$$
$$=-(12/\Lambda)^{1/2}\mathcal{P}^{-1}(R) . \qquad (6.5)$$

Inversion of this result leads to the scale factor

$$R=\mathcal{P}[-(\Lambda/12)^{1/2}(t-t_0)] , \qquad (6.6)$$

where $\mathcal{P}$ is the Weierstras elliptic function[42] with invariants $(g_2,g_3)=(12/\Lambda)(k,-M)$ and discriminant $\Delta=27(12/\Lambda)^2(4k^3/9\Lambda-M^2)$. The potential always has a negative root (denoted by R_0 by Barrow and Stein-Schabes[41] and expressible in terms of hyperbolic functions[43]). This is the only real root when $k<0$ but for large enough $k>0$ such that the discriminant is positive, all three roots are real and at least one is positive, changing the character of the solutions.

In the first case, one has either expanding or contracting solutions, while in the second, one has bounce and recollapse solutions, with asymptotic bounce and asymptotic recollapse solutions and a static solution at the critical case in which the discriminant vanishes. Note that $R\to\infty$ is reached from any positive value of R in a finite amount of coordinate time in this time gauge, which is also true of $R\to 0$ (when allowed by the solution in the case $k>0$). This is a "slow-time" time gauge both at $R\to\infty$ and $R\to 0$.

In discussing exact solutions, it is useful to distinguish two classes of exact cosmological solutions, according to their dependence on the time variable. An explicit exact solution will be a solution of some field equations in which the metric components can be expressed as explicit functions of the time variable using either elementary functions or known special functions, while an implicit exact solution will be one in which the time variable is instead expressed as an explicit function of the metric variables. Using a simple change of variable not explicitly stated, Barrow and Stein-Schabes reduce the integral (6.5) to a nonstandard form of an elliptic integral of the first kind,[42] which yields the time as an explicit function of an elliptic integral composed with still another change of dependent variable which reduces the integral to standard form. This is an implicit exact solution in contrast with the explicit exact solution associated with (6.6) involving an elliptic function (elliptic functions result from inverting certain elliptic integrals). Of course, if one is going to resort to such complicated functions, one might as well choose R itself as the time variable[5] and express the associated lapse function as a function of R by solving (6.4) for dt/dR, leading to a time gauge which is not characterized by a power-law lapse.

Until quite recently, most of the work in exact spatially homogeneous cosmological solutions has been done in four spacetime dimensions. Diagonal exact solutions in this dimension have at most two variables which do not undergo free motion, and almost every case where explicit exact solutions occur relies on the decoupling properties of power-law-lapse time gauges. In many cases this was not realized by the discoverers nor even obvious from the way in which the solutions were presented.

A good example[4] is the family of diagonal Einstein-Maxwell solutions with zero cosmological constant which are either locally rotationally symmetric or intrinsically locally rotationally symmetric (diagonal models of Bianchi types I and II). For this family, the Taub time gauge makes the free shear term in the Hamiltonian constant when present. The Misner time gauge, as already explained, is a Weyl gauge choice of the power-law lapse, and makes constant the term in the Hamiltonian due to the preferred submanifolds of constant curvature associated with the local rotational symmetry. The Brill[22] time gauge does the same thing to the term in the Hamiltonian contributed by the locally rotationally symmetric electromagnetic field and is a scale-invariant time gauge. Each of these are "absolute choices" of power-law-lapse time gauge adapted to the respective potential terms and each leads to a complete decoupling of the nontrivial variables. Other power-law-time gauges accomplish this in special cases or lead to partial decouplings.[4]

The case of the product manifold three geometries of this class (Kantowski-Sachs and its zero and negative curvature analogs) is summarized in a work by Vajk and Eltgroth[44] which attempts to collect all known results and derive them systematically. They use a scale-invariant power-law-time gauge of Siklos type adapted to the preferred two-manifolds, which in fact leads to a complete decoupling of two power variables, both of which are characterized by parabolic potential scattering problems. In their article only one decoupling was realized, and the second metric variable was found by integrating a coupled equation using a table of integrals. Many exact solution articles share this nearsighted habit of just doing the minimum calculation, often by trial and error, to produce explicit integrations.

In practice the power-law-lapse time gauges which allow decoupling and the actual variables which decouple are easier to identify by looking directly at the Ricci form of the evolution equations, although hindsight shows that these choices are simply related to the behavior of certain potential terms in the Hamiltonian. Through the kinetic terms the geometry of the DeWitt metric plays a key role in this problem, which clearly merits more careful scrutiny from the Hamiltonian point of view.

VII. CONCLUSIONS

Power-law lapses are presently playing a crucial role in the discussion of solutions of higher-dimensional gravitational theories, both in the cosmological context involving a spacelike slicing of spacetime and in the noncosmological context involving a timelike slicing. With very few exceptions, exact solutions or partially integrated field equations of many gravitational theories when the field equations are reduced to ordinary differential equations owe their existence to such power-law coordinate gauges.

Clearly exact solutions do have a certain importance in the grand scheme of things, but too often, especially in the cosmological context, either producing them seems to become an end in itself or an unsophisticated analysis has missed some useful information about the system it studies. Thinking a bit more carefully about the structure of the equations which produce them is certainly needed if the exact solutions are to contribute more toward a better understanding of the dynamics (or statics) of gravitational field equations. This also applies to a certain extent to qualitative analyses.

The ideas presented in this article are clearly insufficient. One needs to apply them in individual studies and carry them further, hopefully making links between such studies to draw more general conclusions. In the cosmological context, one needs to understand the properties of the time variables which may be preferred and their relationship to the proper time near singularities and at timelike infinities when they exist, a question which involves the notions of "fast" and "slow" times. It is important to emphasize how different a picture of the classical dynamics unfolds depending on which choice of independent and dependent variables one uses to describe it. This occurs in the somewhat broader context (i.e., independent of hypersurface symmetries) of Weyl rescalings in higher-dimensional theories where certain scalar potentials have radically different asymptotic properties in different conformal gauges and as a consequence lead to different predictions in some semiclassical calculations.[45–47] This problem is already modeled by the curvature potential for the gravitational field alone in different time gauges in spatially homogeneous cosmologies. Even in quantum calculations, because they often involve manipulations of classical equations, one can afford to think about these questions. If more individuals do think in this direction, then the goal of this article will have been met.

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