Cosmological consequences of a rolling homogeneous scalar field

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The cosmological consequences of a pervasive, rolling, self-interacting, homogeneous scalar field are investigated. A number of models in which the energy density of the scalar field red-shifts in a specific manner are studied. In these models the current epoch is chosen to be scalar-field dominated to agree with dynamical estimates of the density parameter, $\Omega_{\rm dyn} \sim 0.2$, and zero spatial curvature. The required scalar-field potential is "nonlinear" and decreases in magnitude as the value of the scalar field increases. A special solution of the field equations which is an attractive, time-dependent, fixed point is presented. These models are consistent with the classical tests of gravitation theory. The Eötvös-Dicke measurements strongly constrain the coupling of the scalar field to light (nongravitational) fields. Nucleosynthesis proceeds as in the standard hot big-bang model. In linear perturbation theory the behavior of baryonic perturbations, in the baryon-dominated epoch, do not differ significantly from the canonical scenario, while the presence of a substantial amount of homogeneous scalar-field energy density at low red-shifts inhibits the growth of perturbations in the baryonic fluid. The energy density in the scalar field is not appreciably perturbed by nonrelativistic gravitational fields, either in the radiation-dominated, matter-dominated, or scalar-field-dominated epochs. On the basis of this effect, we argue that these models could reconcile the low dynamical estimates of the mean mass density with the negligibly small spatial curvature preferred by inflation.

I. INTRODUCTION AND SUMMARY

One consequence of observational astronomy over the last half-century has been the accumulation of fairly persuasive evidence that a substantial fraction of the gravitationally bound mass associated with observed structure in the Universe is nonluminous.[1] Dynamical estimates of the mass density on large (clusters of galaxies, etc.) scales (which assumes that galaxies trace mass) suggest $\Omega_{\rm dyn}=0.2\pm0.1$ (Ref. 2). [The density parameter Ω is the ratio of the mean mass density to the Einstein–de Sitter value, $\Omega(t)=8\pi G\rho/(3H^2)$, where G is Newton's gravitational constant, H is the Hubble constant, and ρ the relevant mass density.] The fact that the baryon mass density Ω_B needed for nucleosynthesis[3] falls within this range is sometimes taken as evidence that the nonluminous mass is purely baryonic and that the total mass density in all forms, Ω_0, is appreciably less than the Einstein–de Sitter value. If the equations of general relativity (i.e., Einstein's theory of gravity with baryons and radiation) govern cosmology, this would mean that spatial sections must have appreciable mean curvature. However, this conflicts with the inflation paradigm,[4,5] which offers the only known, reasonable, explanation for the remarkable homogeneity of the Universe within our horizon: the cosmological expansion during inflation, which is supposed to make the length scale over which the density varies much larger than our horizon, could be expected to do the same to the radius of curvature of spatial hypersurfaces (in the standard model, this would force Ω_0 to be unity to high accuracy[6]). To resolve this discrepancy between the dynamical estimates of the mean mass density and the negligibly small space curvature preferred by inflation one could assume that $(1-\Omega_{\rm dyn})$ is balanced by some kind of nonluminous, nearly homogeneous, energy density.

Many different kinds of nonluminous matter have been proposed for this purpose. The currently popular candidates are what are known (in Bond's classification) as cold nonluminous matter (hereafter, CDM, i.e., extremely weakly interacting matter with almost no primeval thermal velocity).[7] There are problems with this scenario [which assumes a scale-invariant spectrum for the adiabatic energy density fluctuations (see Ref. 8)]:[9] the lack of power on small scales means galaxies form late (this does not naturally agree with observational data); to reproduce the observed dynamical estimates of the clustered mass density, $\Omega_{\rm dyn}$, one must assume that galaxies cluster more strongly than mass (i.e., galaxies are "biased" with respect to the mass),[10] and, furthermore, this model predicts that aggregations of mass are anticorrelated on very large scales (which also does not seem to agree with the observations). Since CDM has very low pressure, it is again apparent that galaxies would have to form at relatively low red-shifts to prevent gravitational instability from removing the bias. Hot dark matter (HDM) is another type of nonluminous matter that has been extensively studied (a low-mass neutrino is an example of HDM). A drawback of this scenario is that the thermal energy of HDM erases inhomogeneities on small scales (galaxies, etc.), so galaxies can only form at low red-shifts from the fragmentation of bigger objects that condensed first.[11] At present the observations suggest that galaxies are old while clusters are young and still forming.[12]

The arguments against some form of CDM or HDM would be vitiated if there were evidence that the mass density estimated from dynamics increases with increasing length scale. While this may yet be observed, we are impressed with the contrary indication that Ω_{dyn} estimated on scales from ~ 30 kpc to ~ 10 Mpc seems to be nearly constant.[2] It might also be noted that the evidence for large-scale velocity fields[13] is not necessarily evidence for high clustered mass density: the large-scale velocity fields are at least roughly in line with what would be expected if clustered mass were distributed like galaxies with the density parameter of the clustered mass ~ 0.2 (Ref. 14).

The dynamical estimates of Ω_{dyn} would not be indicative of the value of Ω_0 if the mean mass density were dominated by relativistic matter. However, if the relativistic matter were primeval it would dominate the expansion rate at the epoch of light-element nucleosynthesis and so spoil the observational success of the standard calculations.[3] Relativistic dark matter produced by the decay of CDM (Ref. 15) remains a possibility that may be seen to be increasingly attractive if less speculative approaches continue to have problems.

A. Cosmological scalar field

As we have indicated, it might be desirable to have a nonluminous matter that resists gravitational collapse over a fairly large range of scales and that dominates the mass density only at low red-shifts. A cosmological constant Λ is exactly such a candidate.[16] Another possibility is a homogeneous scalar field that is very weakly coupled to ordinary matter (this might even be the same field that drove inflation).[17] If the potential of this scalar field Φ slowly decreased towards zero for large Φ, the mass density associated with it could act like a cosmological "constant" that decreases with time less rapidly than the mass densities of matter and radiation. In Ref. 17 we have analyzed the cosmological predictions of our preferred scalar-field models (the power-law potential models of Sec. V). We have found that these models (i) have a larger value of $H_0 t_0$ (at fixed Ω_B) as compared to the Einstein–de Sitter model (they, therefore, seem to agree better with the observational data), (ii) predict a value of the bolometric distance modulus, at red-shift $z = 1.5$ ($\sim$highest red-shift for which there is useful observational data), that is consistent with current observations and $\Omega_B \sim 0.2$, (iii) are not obviously inconsistent with the number counts of galaxies as a function of red-shift. (iv) do better than the Einstein–de Sitter case but not as well as the constant-Λ model in fitting the red-shift dependence of angular sizes of radio sources, (v) suppress the growth of linear density inhomogeneities, relative to the Einstein–de Sitter model, by a factor of $\sim 2-3$ (which does not seem very serious because of our limited understanding of how galaxies and clusters form), and (vi) reconcile the relatively large galaxy density fluctuations and small relative peculiar velocities observed on scales $\sim 1-10$ Mpc, about as well as the constant-Λ model does.

The discord between the estimates of the luminous mass density ($\Omega_{lum} \sim 0.01$) and the nucleosynthesis requirement that $\Omega_B \sim 0.1$ suggests that there must also be a significant amount of nonluminous baryonic matter in the Universe. Since baryons are pressureless, this nonluminous matter must be gravitationally clustered, that is, if galaxies formed before a red-shift of few. Dynamical estimates of the clustered mass density suggest $\Omega_{dyn} = 0.2 \pm 0.1$. We, therefore, believe that in the present class of models it is economical to assume that the only forms of nonluminous matter present are the nonluminous baryons (that have to be present), the usual massless neutrinos, and the scalar field. This is the assumption we adopt in the present paper.

The main purpose of this paper is to attempt to codify the constraints experiments and observations place on the potential of Φ, if it is to be a suitable candidate for the nonluminous matter. We construct and study some acceptable, albeit simplified, models to see whether there are any significant deviations from standard CDM scenarios. We are particularly interested in the behavior, in linear perturbation theory, of density perturbations in the scalar-field energy density and in the densities of baryons and radiation. We find that scalar-field energy-density fluctuations tend to decay inside the horizon, while in a baryon-dominated universe, the behavior of baryonic perturbations is effectively the same as in the canonical scenario. On the other hand, if the energy density in the homogeneous part of the scalar field is substantial, baryonic perturbations cannot grow. Since scalar-field perturbations do not grow inside the horizon, the scalar-field energy density will remain very much smoother than the baryonic distribution (in contradistinction to the canonical CDM scenario), so the peaks in the matter distribution will be almost entirely baryonic. This is, in effect, a naturally "biased" scenario for galaxy formation with $\Omega_{dyn} = \Omega_B$.

The mass of the scalar field fluctuation ($\sim$second derivative of the scalar potential) is related to the horizon size; the scalar-field fluctuation is exceedingly light. This is the reason why galaxy formation is dynamically "biased" in this scenario—the scalar-field fluctuations are much too hot (relativistic) to condense either through their mutual gravitational attraction or through the gravitational attraction of other matter. If the scalar field were to dominate early enough (like the canonical HDM candidate, the almost massless neutrino, does) it would suppress growth of baryonic structure on small scales. This is because the Universe expands faster than the perturbations can collapse.[18] (Note that the scalar field must be very weakly coupled to ordinary matter so that it does not drag the matter perturbations with it and thereby prevent them from collapsing, even before the scalar field comes to dominate the energy density of the Universe.)[19] This is one reason why we must assume that the Universe has only recently become dominated by the scalar-field energy density, if galaxies formed by gravitational instability. This effect motivates the choice of potentials $V(\Phi)$ discussed below.

We consider two, fairly distinct, classes of models for the potential of the scalar field. In the first set of models we assume that the energy density of the scalar field red-shifts in a certain way and then determine the potential of the scalar field that this requires (we refer to such models as fixed equation of state models). We find that the scalar-field potential is "nonlinear" (i.e., the scalar-field equation of motion is nonlinear); typically it tends to be made from exponentials of the scalar field. We have not succeeded in determining the general solution of the scalar-field equation of motion, but a special solution can be found (in which the scalar-field energy density redshifts in the requisite manner). Somewhat remarkably, we find that this solution dominates at large time and a study of phase space shows that it is an attractive, time-dependent, fixed point (in the cases of interest, it is the only attractive fixed point in phase space). This solution may, therefore, be chosen as a background solution for a study of the evolution of density inhomogeneities in linear perturbation theory. In the second set of models we choose a simple function (of Φ) for the scalar-field potential $V(\Phi)$ (we refer to such models as fixed potential models). In this paper we have studied potentials that are either exponentials or negative powers of the scalar field. One can conceive of other simple functional forms, for instance, a Gaussian potential; the main criteria that must be kept in mind are that the energy density of the scalar field should be significantly less than that of radiation near the nucleosynthesis epoch and that the Universe must have had a sufficiently long baryon-dominated epoch to allow galaxies to form (this requires that the scalar-field energy density must have come to dominate the Universe only fairly recently.) Again, in the fixed potential models, we find the attractive fixed-point solutions for the scalar field in both the radiation and baryon-dominated epochs.

The problem with the assumption of the coincidental similarity of the contributions of ordinary matter and the scalar field to the present expansion rate deserves further comment. As we have mentioned above, one reason for this assumption is the desire to have galaxies form (by gravitational instability) in our model. A second, related reason is that we want $\Omega_{\rm dyn} \sim 0.2$ now. This assumption must be made in most nonluminous matter models; although it renders these models rather unattractive, it certainly does not rule them out. For example, another (scalar-field) candidate for nonbaryonic matter (CDM) is a weakly coupled massive pseudoscalar field, the axion. When the coherent axion oscillations, in an approximately quadratic potential, are rapid enough (compared to the rate of expansion), pressure averages to zero and the axion fluid behaves like nonrelativistic matter.[20] The form the above assumption takes in this model is the requirement that the axion mass have a specified dependence on the present baryon density.

The models we have studied are meant to be classical and phenomenological—they are not meant to be fundamental theories. The coincidental similarity between the contributions of nonluminous and ordinary matter to the present expansion rate might arise through some, as yet not understood, microphysical relation between nonluminous and ordinary matter.[21] If the results of the classical cosmological tests, discussed in Ref. 17 for the power-law potential models, eventually converge to select a particular scalar-field model from this class of models, then this assumption deserves further scrutiny (if they do not converge to select one, the power-law potential models are ruled out).[22] The situation would then be somewhat reminiscent of the current status of the standard electroweak model. Although the electroweak model is a fairly accurate description of the physics of the weak interactions, the dynamics of spontaneous symmetry breaking in this model seems to require a conspiracy between various parameters in the theory[23] (the values of the numerical coincidences needed depend on the value chosen for the fundamental length scale at which the electroweak theory must be cut off). This numerical conspiracy certainly does not rule out the standard electroweak model, rather, it is best interpreted as reflecting the present lack of knowledge of an underlying, more fundamental, theory.

Another issue that deserves comment is the rather generic appearance of nonrenormalizable scalar-field potentials in our models. A significant part of theoretical cosmology consists of the art of determining a Lagrangian that is consistent with all available observational data. There is, by now, a fairly widespread belief that none of the current cosmological scenarios completely accomplish this. (This is a very subjective conclusion since a fair fraction of the observational data is quite tentative and the standard inflation modified hot big-bang scenario does have a large number of notable successes.) We believe it is instructive to focus on what seems to be a significant drawback of the standard scenario and to consider all possible resolutions of the conflict between the low dynamical estimates of the mean mass density and the negligibly small space curvature preferred by inflation. If the classical cosmological tests, discussed in Ref. 17, select one of these scalar-field models, it would not be the first time that a nonrenormalizable theory might be needed for the correct low-energy description of physics (the most celebrated example is Fermi's theory of the weak interactions which subsequently evolved into the more fundamental standard electroweak model). In any case, since the gravitational part of the theory is nonrenormalizable, there is no theoretical reason to prefer a theory where the scalar part is renormalizable. Eventually, one would hope to find a consistent quantum-mechanical theory from which such a nonrenormalizable theory might be extracted.

A recurring theme in attempts to generalize the classical theory of cosmology has been the belief that some kind of scalar field could have had important cosmological consequences. The motivations for such a belief have varied: they have ranged from attempts to develop a framework to rationalize Dirac's desire to dynamically explain what is now known as the hierarchy "problem" (e.g., the number $M_{\rm weak}/M_{\rm GUT} \sim 10^{-13}$) (Ref. 24) to attempts to elaborate on the steady-state cosmology model;[25] to Brans and Dicke's attempt to incorporate Mach's principle in general relativity.[26] None of these models made use of the type of scalar-field potential that leads to a time-varying Λ. The inexorable improvement of observational and experimental data has severally (almost fatally) constrained all of the above models. As we discuss in Sec. VI, the only constraint these experiments place on our model is that the scalar field can only be exceedingly weakly coupled to ordinary matter. It is interesting to note that the theory of superstrings, which is currently thought to hold the most promise for explaining the hierarchy "problem" (through the introduction of new symmetries, not dynamically) contains a Brans-Dicke scalar field (the dilaton). It is hoped that large quantum corrections (string-loop effects) will sufficiently modify classical superstring theory to make it consistent with current bounds on Brans-Dicke scalars.[27] It is much too premature to decide if the resulting scalar-field theory will bear any similarity to one of our models.[28] We should also note that there is a variety of other scalar fields in compactified superstring theories, including the scale factor of the compactified space. As with the case of the dilaton, a substantial amount of work will have to be done before one can decide whether or not any of these scalar fields would act like our Φ.

B. Time-variable cosmological "constant"

The idea of using a cosmological constant Λ to balance $(1-\Omega_{\rm dyn})$ to give a low dynamical value of the mean mass density has a fairly long history.[16,29] A possible problem with a constant Λ is that the presently required value defines an energy scale

$$\epsilon_0=\left[\frac{3(1-\Omega)H_0^2\hbar^3c^5}{8\pi G}\right]^{1/4}$$

$$\simeq 0.002(1-\Omega)^{1/4}h^{1/2}\ \text{eV} \qquad (1.1)$$

($H_0=100h$ km s^{-1}Mpc^{-1}). In the constant-Λ model the ratio of the potential energy of Λ (at reheating in the conventional inflationary scenario) to the scale set by the radiation temperature just after reheating ($\sim 10^{13}$ GeV) is of the order of 10^{-100}. We have argued, in Ref. 17, that this number could be as large as 10^{-25} in our models [i.e., the scale set by the scalar-field potential energy at reheating could be as high as $\sim 10^{6}$ GeV; the scalar field dynamics then reduces the energy scale of the present cosmological constant to near the value of Eq. (1.1)].[30] We therefore believe that our models have a distinct advantage over the model of Ref. 16, which requires a new energy scale $\sim 10^{-3}$ eV. As with any other small number in quantum field theory that is not "protected" by a symmetry,[31] there seems to be a problem with this small ratio (the quantum-mechanical cosmological-constant "problem"), at least when our phenomenological models are analyzed in quantum-mechanical perturbation theory—one must understand why quantum fluctuations do not drastically alter the classically desired value of the parameter, perhaps there is a symmetry that preserves the form of the potential.[32,33] It is unclear if this aspect of the fine-tuning problem can even be studied for these scalar field theories, in our phenomenological framework, since they cannot be consistently quantized (we hope to discuss this in more detail elsewhere).

Some of the recent discussions on the cosmological-constant "problem" and on cosmology with a time-variable cosmological "constant" are relevant to the issues we have studied in this paper. The main purpose of the models discussed in Refs. 34–38 was to present "classical" mechanisms that might account for the small present value of the cosmological constant (the resulting cosmological consequences of a, possibly, time-varying Λ was not examined). (We presume that these models were meant to be effective, phenomenological models, since most of them do not resolve the quantum-mechanical cosmological-constant "problem.") The models proposed in Ref. 34 are closest, in spirit, to the models we have considered here. These references discuss the classical Einstein–scalar-field system with various types of potentials for the scalar field and various forms of coupling between the scalar field and gravity. Banks[34] has argued that one can construct models in which an initially large effective cosmological constant is dynamically driven to smaller values (and eventually through zero to large negative values). He has pointed out that if, when the effective cosmological constant is zero, the scalar-field potential is chosen to be exceedingly flat for a time comparable to the age of the Universe, then the resulting cosmology might resemble the canonical scenario. [In our models $\Lambda(t)$ varies much faster than in Banks's model.] There are, however, many issues that need to be resolved before a viable classical theory of cosmology can be constructed from the models of Ref. 34.

The authors of the first two references in Ref. 35 have suggested that, in certain conditions, the inclusion of dilational symmetry in the standard model of particle physics (which results in a nontrivial potential for a scalar field) might lead to a small-enough Λ; however, the authors of the last reference in Ref. 35 argue that the resulting pseudo-Nambu-Goldstone boson of the spontaneously broken dilation invariance (particle physics is not dilation invariant) is in contradiction with experimental bounds on Brans-Dicke scalars. Brans-Dicke cosmological models with nontrivial potentials have also been studied in Ref. 36. Other intriguing suggestions include the possibility that "quantum" gravity screens a large cosmological constant (i.e., the observed value of Λ is small at low energies and large at high energies, a point of view somewhat similar to that which we advocate in this paper)[39] and that the dynamics of a third-rank antisymmetric tensor might play an important role in reducing Λ,[37] as well as various other ideas.[38] We believe that it is fair to conclude that, at least from the particle-physics point of view, the issue of whether or not a time variable Λ is physically significant is far from resolved.

Aside from the field theory models mentioned above, there has also been some discussions of the possible cosmological applications of a phenomenological time variable Λ (Refs. 40–43). Reference 40 has discussed the effects of a postulated time variable Λ on the age of the Universe. References 41 and 42 have proposed a model with a time-dependent Λ; in this model, even though Λ is assumed to be a function of time, the Lagrangian does not contain a term that depends on time derivatives of Λ. This model is, effectively, the $q\to 0$ limit of our model, i.e., the equation of state for the "vacuum" fluid has been taken to be $p_\Lambda=-\rho_\Lambda$ [where p and ρ are the pressure and energy density of the ideal fluid; in our models, the homogeneous scalar field does not interact with other nongravitational fields and obeys an equation of state $p_\Phi=\rho_\Phi(q-3)/3$]. In the absence of any interaction with matter or radiation this would force the cosmological constant to be constant, but, in the presence of interactions with matter or radiation, a solution of Einstein's equation and the assumed equation of covariant conservation of stress energy with a time variable Λ can be found. For these solutions, conservation of energy requires that any decrease in the energy density of the vacuum component be compensated for by a corresponding increase in the energy density of matter or radiation. The models of Refs. 41 and 42 are not complete since no explicit, microphysical, coupling between the vacuum fluid and matter or radiation has been presented (to mediate the interconversion of the vacuum component and the ordinary fluids and to, thereby, justify the assumed covariant conservation of stress energy). As noted in Ref. 42, there are severe observational constraints on the "spontaneous" production of matter or radiation that ensure that such a time variable Λ has no cosmologically significant effect during recent epochs. In our models, there is no "spontaneous" creation of matter or radiation; the decrease in the potential energy of the scalar field, as it rolls down the potential, is converted to kinetic energy of the scalar field.

In the time-variable Λ models of Ref. 43, the "vacuum" fluid has been furnished with an equation of state $p_\Lambda = -\rho_\Lambda$ as well as an equation of evolution which is assumed to relate the time derivative of ρ_Λ to a function of ρ_Λ, the energy density of other matter and radiation in the theory and Hubble's constant; ρ_Λ is also taken to obey the covariant conservation of stress energy. We are not aware of any obvious way of comparing this model to our models (it is also unclear if the assumed equations of motion of this model are a consequence of the stationarity of an action).

C. Inflation

Finally, we comment on the "patching" of our low-energy-density scalar-field cosmology to the usual inflationary scenario. We shall discuss this in the context of the simplest model of inflation[44] where a scalar field with a step-function potential (and a small linear term responsible for causing the field to slowly roll on top of the hill) drives the exponential expansion of the Universe.

One might generalize this potential to the following form: (i) an initial period when the potential is nearly flat (with a small linear driving term); (ii) an intermediate period when the potential has a suitably steep part so that some of the energy density of the scalar field may be converted to entropy; and (iii) a final period where the potential decreases much more slowly (i.e., is of the form discussed in one of the following sections).

Initially, the Universe must be dominated by the "constant" potential for a long enough time to ensure the desired resolution of the horizon problem.[5] (There are other potentials which also resolve the horizon problem;[45] we restrict our discussion to the relatively simple step-function potential of Ref. 44.) During the period when the scale factor is growing exponentially with time (i.e., when the potential is nearly flat), scalar-field quantum fluctuations continuously evolve out of the horizon. These fluctuations, which have substantial power at large wavelengths,[46] reenter the horizon in the radiation-, matter-, and scalar-field-dominated epochs. This part of the scenario is essentially the same as in the standard inflation modified, adiabatic, hot big-bang model. When these energy-density fluctuations reenter the horizon they are scale invariant.[44,47,8] The fluctuations at horizon crossing will determine the initial conditions for our analysis of fluctuations inside the horizon in Secs. VIII and IX.

In Sec. VI we have argued that the scalar field cannot couple to ordinary, light, matter if we are to avoid violating the equivalence principle. If the intermediate period of rapid roll down is to generate sufficient entropy (i.e., radiation), the scalar field must couple to heavy (mass $\sim M_{\rm GUT}$) particles which subsequently decay to light, ordinary, matter. We assume that the forces generated by such couplings will not violate any of the current experimental limits on the validity of the equivalence principle; however, the resolution of this issue would require a detailed analysis, which we plan to present elsewhere.[48,49]

D. Overview

In Sec. II, we introduce, and review some details of, the class of theories which we shall study. In Secs. III, IV, and V we study specific examples of these theories and present special, asymptotically dominant, solutions of the nonlinear Einstein–scalar-field equations (these are the attractive fixed points in the phase space of homogeneous and isotropic world models). In Sec. III we study a class of models in which the only contribution to the stress tensor is a spatially homogeneous scalar field whose energy density red-shifts as $\rho_\Phi \propto a^{-q}$, where a is the cosmological scale factor. The required scalar-field potential is an exponential. Exponential potentials have previously been studied, in the context of models of generalized inflation.[50,51] We analyze the four-dimensional phase-space structure of the spatially homogeneous Einstein–scalar-field equations and show that there is only one attractive critical point in the finite part of phase space (in this paper we use both the mathematics terminology "critical point" and the physics terminology "fixed point"). Some aspects of this phase space have also been examined in Ref. 51. These models are unrealistic since they contain neither baryons nor radiation. In Sec. IV we study a model with baryons and a pressureless scalar field (with $q=3$), which has a fixed, exponential, potential. We show that there is a critical point (in the Einstein-scalar-field phase space), but note that this solution is inconsistent with the assumptions of the canonical nucleosynthesis scenario, since it requires that the scalar-field energy density be significant during nucleosynthesis (if it is to be significant at any later epoch in the evolution of the Universe). We also present a solution of a slightly different model, which is consistent with the standard nucleosynthesis scenario, but is not much different from the constant Λ model. Our preferred models are presented in Sec. V. In these models the scalar-field potential is taken to be a negative power of the scalar field. Initially the Universe is assumed to be radiation dominated; in this epoch the feedback from the scalar field to the Einstein equations and the equation of motion for radiation is negligible, so it suffices to analyze the two-dimensional homogeneous scalar-field phase space. There is only one fixed-point solution, whose scalar-field energy density decreases, with time, less rapidly than the energy density of radiation. A similar analysis with similar conclusions is carried through for the matter-dominated epoch. Finally, in the scalar-field-dominated epoch, we are forced to resort to late-time asymptotic techniques to discuss the solutions of the coupled Einstein–scalar-field equations of motion (since we have not succeeded in unravelling the structure of the relevant four-dimensional phase space). In conclusion, aside from our preferred models, presented in Sec. V, most of the other models we have studied in Secs. III and IV seem to lead to unsatisfactory cosmologies (that are either inconsistent with observations or the canonical nucleosynthesis scenario) although our analysis of some of these models is far from complete (because of the complexity of some of these models we have had to make some very crude, simplifying, approximations; a more complete analysis might show that they also lead to realistic cosmologies).

In general, phase space can contain a variety of different kinds of phase trajectories. In Appendix A we show that the phase space of our preferred models has no limit cycles, we discuss the structure at ∞ in the two-dimensional phase plane and argue that the domain of attraction of the fixed point is exceedingly large. We have not studied these issues for any of the other models of Secs. III and IV.

Most of the rest of the paper is devoted to a more detailed investigation of our preferred models. Section VI discusses the compatibility of these Einstein–scalar-field theories with the classical tests of gravitation. We show that the Eötvös-Dicke measurements indicate that the scalar field is very weakly coupled to light matter. In Sec. VII we derive the relativistic linear-perturbation-theory equations that determine the evolution of spatial inhomogeneities. In Secs. VIII and IX we examine the growth of density inhomogeneities in the preferred models, an issue relevant to the formation of galaxies.

II. HOMOGENEOUS SCALAR FIELDS: GENERALITIES

We consider homogeneous, isotropic, spatially flat cosmologies described by the line element

$$ds^2=dt^2-a^2(t)(d\mathbf{x})^2 . \tag{2.1}$$

The Einstein–scalar-field action is

$$S=\frac{m_P^2}{16\pi}\int dt\, d^3x\sqrt{-g}\left[-R+\frac{g^{\mu\nu}}{2}\partial_\mu\Phi\partial_\nu\Phi-\frac{V(\Phi)}{2}\right] \tag{2.2}$$

(here $m_P=G^{-1/2}$ is the Planck mass); for a homogeneous scalar field Φ_0 the usual definition of the stress-energy tensor leads to

$$\begin{aligned}(\dot\Phi_0)^2&=16\pi m_P^{-2}(\rho_\Phi+p_\Phi) ,\\ V(\Phi_0)&=16\pi m_P^{-2}(\rho_\Phi-p_\Phi) ,\end{aligned} \tag{2.3}$$

where ρ_Φ and p_Φ are the energy density and pressure of the homogeneous scalar-field fluid and an overdot denotes a derivative with respect to time. If the scalar-field energy density red-shifts as

$$\rho_\Phi=\rho_\Phi^{(0)}\left[\frac{a_0}{a}\right]^q \tag{2.4}$$

(in most of what follows we assume $q\neq 0$), conservation of stress energy, neglecting interactions with fields other than gravity (in the models we shall consider, the homogeneous part of the scalar field does not couple to other fields), implies an equation of state

$$p_\Phi=\left[\frac{q-3}{3}\right]\rho_\Phi \tag{2.5}$$

and

$$\begin{aligned}(\dot\Phi_0)^2&=\frac{q}{3}16\pi m_P^{-2}\rho_\Phi ,\\ V(\Phi_0)&=\left[\frac{6-q}{3}\right]16\pi m_P^{-2}\rho_\Phi .\end{aligned} \tag{2.6}$$

A variant of the $q=6$ model has been studied by Dicke.[52]

We note that by using Eq. (2.3), conservation of stress energy,

$$\frac{dp_\Phi}{dt}a^3=\frac{d}{dt}[a^3(\rho_\Phi+p_\Phi)] , \tag{2.7}$$

may be rewritten as

$$\dot\Phi_0\left[\frac{d}{dt}(a^3\dot\Phi_0)+\frac{a^3}{2}\frac{\partial V}{\partial\Phi_0}(\Phi_0)\right]=0 , \tag{2.8}$$

the expression in large parentheses being the scalar-field equation of motion. So the scalar-field equation of motion implies the conservation of the scalar-field stress tensor, while a choice of an equation of state, Eq. (2.5), is equivalent to choosing how $(\dot\Phi_0)^2$ and V red-shift, Eq. (2.6).

Using conservation of stress energy, the Einstein equations may be reduced to

$$\left[\frac{\dot a}{a}\right]^2=\frac{8\pi}{3m_P^2}\rho-\frac{\kappa^2}{a^2} \tag{2.9}$$

[here κ is the inverse of the coordinate radius of curvature of the spatial hypersurfaces, $\kappa^2>0$ (<0) for closed (open) models and $\kappa=0$ for spatially flat models] or, alternatively,

$$\frac{\ddot a}{a}=-\frac{4\pi}{3m_P^2}(\rho+3p) , \tag{2.10}$$

which does not explicitly depend on κ^2.

III. SCALAR-FIELD-DOMINATED COSMOLOGY

In this section we shall, mostly, consider models in which the only contribution to the stress tensor is a homogeneous scalar field. These models do not lead to a satisfactory cosmology, since they contain neither baryons nor radiation (in the second part of this section we comment on some models that contain baryons and radiation), rather they are simple enough for us to use them to illustrate the techniques that we shall need to use to examine more realistic models.

A. Scalar-field-dominated models

Assuming that the scalar-field energy density red-shifts with index q, we have, from Eqs. (2.6) and (2.9) (here we have set $\kappa^2=0$),

$$\frac{\partial\Phi_0}{\partial a}=\frac{\dot{\Phi}_0}{\dot{a}}=\frac{(2q)^{1/2}}{a} \tag{3.1}$$

or

$$\frac{a}{a_0}=\exp\left[\frac{1}{\sqrt{2q}}(\Phi_0-\Phi_0^{(0)})\right] \tag{3.2}$$

(where $\Phi_0^{(0)}$ is the value of Φ_0 at $a=a_0$) which, from Eq. (2.6), implies

$$V(\Phi_0)=\left[\frac{6-q}{3}\right]16\pi m_P^{-2}\rho_\Phi^{(0)}\times\exp\left[-\left[\frac{q}{2}\right]^{1/2}(\Phi_0-\Phi_0^{(0)})\right] . \tag{3.3}$$

The Einstein–scalar-field equations (with this potential) have a special solution [Eq. (3.5) below] for which $a(t)\propto t^{2/q}$, $\Phi_0(t)\propto \ln(t)$, and $\rho_\Phi\propto t^{-2}$. In this section we study spatially homogeneous perturbations (not necessarily small) about the special solution, in order to see if Φ_0 might approach the special solution from a wide range of possible initial conditions (possibly specified at the end of reheating). We shall have to study the structure of the four-dimensional, spatially homogeneous, phase space $(\Phi_0,\dot{\Phi}_0,a,\dot{a})$ because the perturbations in the scalar field generate perturbations of similar magnitude in the gravitational field. To do this we need to consider the scalar-field equation of motion and Einstein's equation, which are given by

$$\begin{aligned}&\ddot{y}+3\frac{\dot{a}}{a}\dot{y}-\frac{q}{4}H^2e^{-y}=0 ,\\ &12\frac{\ddot{a}}{a}+\frac{4}{q}(\dot{y})^2-H^2e^{-y}=0 ,\end{aligned} \tag{3.4}$$

where we have used the potential given in Eq. (3.3), $y=\sqrt{q/2}(\Phi_0-\Phi_0^{(0)})$ and

$$H^2=\left[\frac{6-q}{3}\right]16\pi m_P^{-2}\rho_\Phi^{(0)} ;$$

H is not Hubble's constant. A special solution of these equations is

$$\begin{aligned}&a_e(t)=a_0[1+M(t-t_0)]^{2/q} ,\\ &y_e(t)=2\ln[1+M(t-t_0)] ,\end{aligned} \tag{3.5}$$

where $M=qH/[2\sqrt{2(6-q)}]$ (for this solution $\rho_\Phi\propto a^{-q}$); to study the structure of the phase space of these equations it is convenient to make the change of variables

$$\begin{aligned}&y(t)=y_e(t)+u(t) ,\\ &a(t)=a_e(t)v(t) ,\\ &t=t_0-M^{-1}+e^x .\end{aligned} \tag{3.6}$$

[We emphasize that this is only a change of variables and that all the dynamical information in the equations of motion for $a(t)$ and $\Phi_0(t)$ is now encoded in the equations of motion for $u(x)$ and $v(x)$.] The equations that govern the evolution of $u(x)$ and $v(x)$ in phase space are then given by

$$\begin{aligned}&u'=p ,\\ &p'=\left[\frac{q-6}{q}\right]p-3\frac{rp}{v}-6\frac{r}{v}+\frac{q}{4}\frac{H^2}{M^2}(e^{-u}-1) ,\\ &v'=r ,\\ &r'=\left[\frac{q-4}{q}\right]r-\frac{4}{3q}vp-\frac{1}{3q}p^2v+\frac{H^2}{12M^2}v(e^{-u}-1) ,\end{aligned} \tag{3.7}$$

where primes denote derivatives with respect to x. The critical points of this system are those points at which the "velocities" vanish. The only critical point in the finite part of phase space is $(u_0,p_0,v_0,r_0)=(0,0,\bar{v},0)$, where $\bar{v}$ is an arbitrary constant corresponding to the freedom in rescaling a_0 (we have assumed that $q\neq 6$).

It suffices to use linear analysis to study the stability of the nonlinear problem near a critical point (except for the case in which the critical point is a center).[53] Perturbing about the critical point $(u,p,v,r)=(0+u_1,0+p_1,\bar{v}+v_1,0+r_1)$ we have

$$\begin{aligned}&u_1'=p_1 ,\\ &p_1'=\left[\frac{q-6}{q}\right]p_1-6\frac{r_1}{\bar{v}}-\frac{q}{4}\frac{H^2}{M^2}u_1 ,\\ &v_1'=r_1 ,\\ &r_1'=\left[\frac{q-4}{q}\right]r_1-\frac{4\bar{v}}{3q}p_1-\frac{H^2}{12M^2}\bar{v}u_1 .\end{aligned} \tag{3.8}$$

The eigenvalues of small oscillations are $\lambda_1=0$, $\lambda_2=-1$, $\lambda_3=(q-6)/q$, and $\lambda_4=(2q-4)/q$. [Defining ν through $p_\Phi=\nu\rho_\Phi$, we find $\lambda_4=(2+6\nu)/(3+3\nu)$. It is pleasing to note that λ_2 and λ_4 agree with the solutions, Eq. (86.12), given in Sec. 86 of Ref. 54 for the temporal behavior of density inhomogeneities on scales much larger than the horizon. As expected, on these large scales, gravity does not distinguish between different microphysical theories.] For $q<2$ these eigenvalues are negative real numbers so the point $(0,0,\bar{v},0)$ is a stable fixed point and all trajectories approach it as asymptotically straight lines as $t \to \infty$. (Here, and in the next two sections, by a stable critical point, we mean a critical point that is stable under spatially homogeneous perturbations. Spatially inhomogeneous perturbations will be discussed in Secs. VII–IX.)

For some exceptional values of finite, nonzero, q ($q = 6, 3, 2, \frac{4}{3}, -2$), two of the eigenvalues are degenerate. We shall treat these exceptional values of q separately. The corresponding eigenvectors (for nonexceptional values of q) are given by

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ \bar{v}/q \\ -\bar{v}/q \end{bmatrix}, \quad \begin{bmatrix} 1 \\ (q-6)/q \\ \bar{v}/3 \\ (q-6)\bar{v}/(3q) \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ (2q-4)/q \\ \bar{v}(2-3q)/[3q(q-2)] \\ 2\bar{v}(2-3q)/(3q^2) \end{bmatrix}.$$

From these expressions we see that the zero eigenvalue λ_1 corresponds to the arbitrariness of rescaling a_0. Using the method described in Sec. 10 of Ref. 54, to generate the solution corresponding to time translation invariance, it can be verified that it takes the form of the eigenvector corresponding to the eigenvalue λ_2. The solutions corresponding to the eigenvalues λ_3 and λ_4 also describe decaying perturbations (for $q < 2$). From the formulas presented above, it can be verified that the fractional perturbations $\delta\Phi_0/\Phi_0$ and $\delta a/a$ corresponding to the eigenvalues λ_2, λ_3, and λ_4 vary as powers (λ_i) of time (up to a logarithmic factor for the scalar-field fractional perturbation). We may use Eq. (2.9) as an operational definition of the curvature of the spatial hypersurfaces (κ^2) induced by these perturbations. Working to first order in the perturbations we find that κ^2 vanishes for the first three modes while, for λ_4,

$$\kappa^2 \propto \frac{2}{q^3}(q+2)(q-\tfrac{4}{3})a_0^2\bar{v}^{\,2}M^{4/q} .$$

So, in summary, the λ_1 mode corresponds to the freedom in rescaling a_0, the λ_2 mode corresponds to the freedom of shifting the zero of time, the λ_3 mode describes decaying isocurvature fluctuations, and the λ_4 mode corresponds to adiabatic fluctuations that grow if $q > 2$ and decay if $q < 2$.

For the exceptional value $q = 2$, the fourth eigenvector, presented above, is not linearly independent of the other three eigenvectors. The linearly independent solution may be determined by choosing a basis, for the small fluctuations, in which the evolution matrix reduces to Jordan form. The linearly independent solution is then given by

$$\begin{bmatrix} -3/(2\bar{v}) \\ 0 \\ x \\ 1 \end{bmatrix},$$

and we find that the fractional perturbation in the scale factor only grows logarithmically with time while $\kappa^2 \propto -a_0^2\bar{v}^{\,2}[M(q=2)]^2$. For the only other exceptional value, $q = \frac{4}{3}$, of interest to us, the fourth eigenvector above must be replaced by the linearly independent solution

$$\begin{bmatrix} x \\ 1-x \\ \bar{v}(1+3x/4) \\ -\bar{v}(1+3x)/4 \end{bmatrix} e^{-x},$$

and we find that the fractional perturbation in the scale factor decays as $t^{-1}\ln(t)$ while

$$\kappa^2 \propto 5a_0^2\bar{v}^{\,2}[M(q=\tfrac{4}{3})]^{1/3}/4 .$$

In the case of a cosmological model dominated by ordinary matter or radiation, the perturbation mode with $\kappa^2 \neq 0$, which is called the adiabatic mode, has a density contrast, $\delta\rho/\rho$, that grows with time, with amplitude proportional to κ^2. The condition that the value of $\delta\rho/\rho$, at the present epoch, be acceptably small, is equivalent to a limit on the space curvature fluctuation represented by κ^2. In the present model, with $q > 2$, the condition that Φ remain close to homogeneous is again a limit on space curvature fluctuations. It is interesting that if $q < 2$, Φ approaches Φ_0 as $t \to \infty$ even in the presence of nonzero space curvature fluctuations.

If the system asymptotically approaches a fixed point the volume of phase space must decrease. We have, to lowest order in δx,

$$\begin{aligned} u(x+\delta x) &= u(x) + \delta x\, u'(x) , \\ p(x+\delta x) &= p(x) + \delta x\, p'(x) , \\ v(x+\delta x) &= v(x) + \delta x\, v'(x) , \\ r(x+\delta x) &= r(x) + \delta x\, r'(x) , \end{aligned} \tag{3.9}$$

under which the volume $V(x)$ [not to be confused with the scalar-field potential $V(\Phi_0)$] in phase space transforms as

$$\begin{aligned} V(x+\delta x) &= \frac{\partial(u(x+\delta x), p(x+\delta x), v(x+\delta x), r(x+\delta x))}{\partial(u(x), p(x), v(x), r(x))} V(x) \\ &= \left[1 + \delta x \left[\frac{\partial u'}{\partial u} + \frac{\partial p'}{\partial p} + \frac{\partial v'}{\partial v} + \frac{\partial r'}{\partial r} \right] \right] V(x) , \end{aligned} \tag{3.10}$$

where the off-diagonal elements of the Jacobian are higher order in δx and have been neglected. This implies

$$\frac{V(x+\delta x)-V(x)}{V(x)\delta x}=\frac{\partial u'}{\partial u}+\frac{\partial p'}{\partial p}+\frac{\partial v'}{\partial v}+\frac{\partial r'}{\partial r}$$

or

$$\frac{d}{dx}\ln V(x)=2\frac{q-5}{q}-3\frac{v'}{v} \quad (3.11)$$

(the first term on the right-hand side is just the trace of the matrix which governs the evolution of the linearized system), which gives

$$V\left[\frac{a}{a_0}\right]\propto\left[v\left[\frac{a}{a_0}\right]\right]^{-3}\left[\frac{a_0}{a}\right]^{5-q} . \quad (3.12)$$

Near the critical point the first factor on the right-hand side is constant and the volume in phase space decreases for $q<5$; however, only for $q<2$ does the system approach an attractive fixed point (the decrease of the volume of phase space is necessary but not sufficient to establish that the system approaches a fixed point). Equation (3.12) is not of much use for studying stability away from the critical point.

In conclusion, we have shown that our assumed solution, Eq. (3.5), is the only stable fixed-point solution (in the finite part of phase space) of the Einstein–scalar-field equations for the potential given by Eq. (3.3) (if $q<2$). If $q>2$, the solution is unstable and inhomogeneities grow in the standard manner. These models do not lead to satisfactory cosmologies since they do not contain baryons or radiations. We shall consider more realistic models in Secs. IV and V.

B. Other models

The models presented in Secs. IV and V are simple enough for us to be able to analyze them with the techniques developed in the first part of this section. We have also considered spatially flat models in which the scalar field obeys the same equation of state in the radiation-dominated matter-dominated, and current, scalar-field-dominated epochs. These models are more complicated than the other models studied in this paper and our analysis is not complete. One of these models is an example of a possibly satisfactory, although complicated cosmology. These models can be classified by two new real numbers, $\rho_\Phi^{(0)}$ (the energy density of the scalar field now) and q (the index that determines how this energy density red-shifts), as well as the usual numbers that characterize baryons and radiation, and will be called E_q models. For the simplest model E_3 (in which the energy density of the scalar field red-shifts like a pressureless fluid) we find

$$V(\Phi_0)=S(e^{A(\Phi_0-\Phi_0^{(0)})}-\epsilon e^{-A(\Phi_0-\Phi_0^{(0)})})^{-6} , \quad (3.13)$$

where $\Phi_0^{(0)}$ is the present value of the scalar field, $\tilde{\epsilon}=\rho_R^{(0)}(\rho_B^{(0)}+\rho_\Phi^{(0)})^{-1}$, $B=1+(1+\tilde{\epsilon})^{1/2}$, $\epsilon=\tilde{\epsilon}B^{-2}$,

$$A=\left[\frac{\rho_B^{(0)}+\rho_\Phi^{(0)}}{24\rho_\Phi^{(0)}}\right]^{1/2} ,$$

and $S=4^5\pi m_P^{-2}\rho_\Phi^{(0)}B^{-6}$. (Here $\rho_B^{(0)}$ and $\rho_R^{(0)}$ are the present energy densities in baryons and radiation.)

A special solution of the scalar-field equation of motion is

$$\Phi_0-\Phi_0^{(0)}=A^{-1}\ln\left\{B^{-1}\left[\left[\frac{a}{a_0}\right]^{1/2}+\left[\tilde{\epsilon}+\frac{a}{a_0}\right]^{1/2}\right]\right\} .$$

Using this solution the phase-space equations for the scalar field can be reduced to first-order form; however, the coefficients in these equations depend on the independent variable. The equation that results upon setting the "velocities" to zero has seven roots. The relevant critical point is stable for a certain range of parameters and for all time. The complexity of the algebraic equation that determines the position of the other critical points has prevented us from completing our analytical study of this model. In any case, this is not a realistic model since the scalar-field energy density and the baryon energy density red-shift in a similar manner.

We have also studied a slightly more interesting spatially flat model: E_2. We find

$$V(\Phi_0)=\frac{64\pi}{3m_P^2}\rho_\Phi^{(0)}(A_1e^{(\Phi_0-\Phi_0^{(0)})/2}+A_2+A_3e^{-(\Phi_0-\Phi_0^{(0)})/2})^{-2} , \quad (3.14)$$

where

$$A_1=\frac{1}{2}\left[\left[\frac{\rho_R^{(0)}}{\rho_\Phi^{(0)}}+\frac{\rho_B^{(0)}}{\rho_\Phi^{(0)}}+1\right]^{1/2}+1+\frac{1}{2}\frac{\rho_B^{(0)}}{\rho_\Phi^{(0)}}\right] ,$$

$$A_2=-\frac{1}{2}\frac{\rho_B^{(0)}}{\rho_\Phi^{(0)}} ,$$

$$A_3=\frac{1}{2}\left[\left[\frac{1}{2}\frac{\rho_B^{(0)}}{\rho_\Phi^{(0)}}\right]^2-\frac{\rho_R^{(0)}}{\rho_\Phi^{(0)}}\right]\times\left[\left[\frac{\rho_R^{(0)}}{\rho_\Phi^{(0)}}+\frac{\rho_B^{(0)}}{\rho_\Phi^{(0)}}+1\right]^{1/2}+1+\frac{1}{2}\frac{\rho_B^{(0)}}{\rho_\Phi^{(0)}}\right]^{-1} ,$$

and the special solution of the scalar-field equation of motion is given by

$$\Phi_0=\Phi_0^{(0)}=2\ln\left[\frac{1}{2A_1}\left\{\frac{a}{a_0}\left[\frac{\rho_R^{(0)}}{\rho_\Phi^{(0)}}\left[\frac{a_0}{a}\right]^2+\frac{\rho_B^{(0)}}{\rho_\Phi^{(0)}}\left[\frac{a_0}{a}\right]+1\right]^{1/2}+\frac{a}{a_0}+\frac{1}{2}\frac{\rho_B^{(0)}}{\rho_\Phi^{(0)}}\right\}\right] .$$

Our analysis of this model has been as incomplete as our analysis of E_3. In particular, we have not studied the complete homogeneous phase space but have restricted our attention to the two-dimensional homogeneous scalar-field phase plane. We, however, have no reason to believe that the cosmology following from this model (or for that matter from any $E_{q<3}$ model) cannot be realistic, although it requires a rather special form for the potential. This issue can only be settled after a more detailed analysis is performed (we have no plans to do this).

IV. FIXED EXPONENTIAL POTENTIAL MODELS

In the previous section we have studied models in which the functional form of the scalar-field potential was determined by holding fixed the scalar-field equation of state (the functional form of the potential, therefore, depended on the dominant contribution to the stress tensor). In this section, and in the next section, we shall consider models in which the scalar-field fluid has a specified equation of state at some epoch. This equation of state determines the functional form of the scalar-field potential (as a function of the scalar field) at that epoch. We then use the scalar-field equation of motion (with this potential) to determine the scalar-field equation of state at all other times. These models are again classified by the same two new real numbers as the models of Sec. III and will be referred to as V_q models.

We shall first study $V_3^{\exp}$. Following the analysis of Sec. III we have in the baryon–scalar-field-dominated epoch (for a spatially flat cosmology)

$$\frac{\partial \Phi_0}{\partial a}=\frac{1}{Aa} \tag{4.1}$$

or

$$\frac{a}{a_0}=\exp[A(\Phi_0-\Phi_0^{(0)})] , \tag{4.2}$$

where

$$A=\left[\frac{\rho_\Phi^{(0)}+\rho_B^{(0)}}{6\rho_\Phi^{(0)}}\right]^{1/2} ,$$

and $\Phi_0^{(0)}$ is the present value of the scalar field. From Eq. (2.6) we find that this results in

$$V(\Phi_0)=16\pi m_P^{-2}\rho_\Phi^{(0)}\exp[-A(\Phi_0-\Phi_0^{(0)})] . \tag{4.3}$$

In the baryon–scalar-field-dominated epoch, the scalar-field equation of motion, Einstein's equation and the equation of covariant conservation of the baryon stress tensor are given by

$$\begin{aligned} &\ddot{y}+3\frac{\dot a}{a}\dot y-2H^2e^{-y}=0 ,\\ &3\frac{\ddot a}{a}+4\pi m_P^{-2}\left[\rho_\Phi^{(0)}\left[\frac{(\dot y)^2}{2H^2}-e^{-y}\right]+\rho_B\right]=0 , \\ &\dot\rho_B+3\frac{\dot a}{a}\rho_B=0 , \end{aligned} \tag{4.4}$$

where $y=3A(\Phi_0-\Phi_0^{(0)})$ and $H^2=6\pi m_P^{-2}(\rho_\Phi^{(0)}+\rho_B^{(0)})$ (again, H is not Hubble's constant). A special solution of these equations is

$$\begin{aligned} &a_e(t)=a_0[1+H(t-t_0)]^{2/3} ,\\ &y_e(t)=2\ln[1+H(t-t_0)] ,\\ &\rho_{B,e}(t)=\frac{\rho_B^{(0)}}{[1+H(t-t_0)]^2}\propto\rho_\Phi(t) . \end{aligned} \tag{4.5}$$

To study the stability of this solution, it is convenient to make the change of variables

$$\begin{aligned} &y(t)=y_e(t)+u(t) ,\\ &a(t)=a_e(t)v(t) ,\\ &\rho_B(t)=\rho_{B,e}(t)[1+w(t)] ,\\ &t=t_0-H^{-1}+e^x . \end{aligned} \tag{4.6}$$

The equations that govern the evolution of $u(x)$, $v(x)$, and $w(x)$ in phase space are then given by

$$\begin{aligned} &u'=p ,\\ &p'=-\left[1+\frac{3r}{v}\right]p-\frac{6r}{v}-2(1-e^{-u}) ,\\ &v'=r ,\\ &r'=-\frac{r}{3}-\frac{4\pi}{3m_P^2H^2}v\left[\rho_\Phi^{(0)}\left[1-e^{-u}+2p+\frac{p^2}{2}\right]+\rho_B^{(0)}w\right] ,\\ &w'=-3\frac{r}{v}(1+\omega) . \end{aligned} \tag{4.7}$$

The only critical point in the finite part of phase space is $(u_0,p_0,v_0,r_0,w_0)=(0,0,\bar v,0,0)$, where $\bar v$ is an arbitrary constant corresponding to the freedom of rescaling a_0. Perturbing about this critical point, one finds that the eigenvalues of small oscillations are $\lambda_1=0$, $\lambda_2=-1$, $\lambda_3=\frac{2}{3}$, and

$$\lambda_{4,5}=\frac{1}{2}\left[-1\pm\left[1-8\frac{\rho_B^{(0)}}{\rho_\Phi^{(0)}+\rho_B^{(0)}}\right]^{1/2}\right] .$$

This system is unstable in the λ_3 direction and is a stable "spiral" point in the $\lambda_{4,5}$ directions for any nonzero value of $\rho_B^{(0)}$. The λ_1 mode corresponds to the arbitrariness in rescaling a_0, the λ_2 mode corresponds to the freedom of shifting the origin of time, the λ_3 mode is the usual growing perturbation and the $\lambda_{4,5}$ modes describe decaying perturbations. In the limit $\rho_B^{(0)}\to 0$ these eigenvalues reduce to the eigenvalues of the $q=3$ model of Sec. III A. In this limit we have a new zero eigenvalue λ_5, which corresponds to the freedom of shifting ρ_B (when ρ_B is negligible).

Although this is not a realistic model, since ρ_Φ/ρ_B is time independent, it is instructive to examine the behavior of ρ_Φ during the radiation-dominated epoch. Replacing the independent variable t by $x=a/a_0$ in the scalar-field equation of motion, and assuming that ρ_Φ is not significant during the radiation-dominated epoch, we find

$$y''+\frac{2}{x}y'+Sx^2e^{-y}=0 \ , \tag{4.8}$$

where $S=-9(\rho_\Phi^{(0)}+\rho_B^{(0)})/(2\rho_R^{(0)})$. The critical-point solution is $y(x)=\ln(-Sx^4/4)$. Using Eqs. (4.2) and (2.6) we find, for the homogeneous scalar-field fluid, in the radiation-dominated epoch,

$$\rho_\Phi=\frac{4}{3}\left[\frac{\rho_\Phi^{(0)}}{\rho_B^{(0)}+\rho_\Phi^{(0)}}\right]\rho_R \ . \tag{4.9}$$

This solution for a pure exponential potential requires that the energy density in the scalar field red-shifts in exactly the same way as the energy density of the dominant component of the stress tensor. This means that the Universe could have been initially radiation dominated only in the limit $\rho_\Phi^{(0)}\to 0$. This solution is phenomenologically untenable since the scalar-field energy density would have to contribute a significant fraction of the energy density at nucleosynthesis, if it does so at the present epoch.

We have also examined $V_{q<3}^{\exp}$ models. It may be verified that when the Universe is baryon dominated, these models are very similar to the models of the next section. When the Universe is dominated by the scalar-field energy density they reduce to the models of Sec. III.

It is instructive to consider a different solution of the model corresponding to the potential

$$V(\Phi_0)=\kappa e^{-\alpha\Phi_0} \ , \tag{4.10}$$

where α and κ are real parameters. In the approximate solution, valid for $t \ll t_1$ and $t \gg t_1$ (with different values for the constant C),

$$a(t)\propto t^n, \quad \Phi_0(t)=\frac{1}{\alpha}\ln[C(t^2+t_1^2)] \ , \tag{4.11}$$

the scalar-field equation of motion reduces to

$$3n+1-\frac{2t^2}{t^2+t_1^2}=\frac{\alpha^2}{4}\frac{\kappa}{C} \ ; \tag{4.12}$$

hence, for $t \ll t_1$,

$$C=\frac{\alpha^2\kappa}{4(3n+1)} \tag{4.13}$$

and, for $t \gg t_1$,

$$C=\frac{\alpha^2\kappa}{4(3n-1)} \ . \tag{4.14}$$

The scalar-field energy density is given by

$$\rho_\Phi(t)=\frac{m_P^2}{8\pi\alpha^2}\frac{1}{(t^2+t_1^2)^2}[3nt^2+(3n+1)t_1^2] \ , \tag{4.15}$$

which is constant for $t \ll t_1$ and varies as t^{-2} for $t \gg t_1$. Since $\rho_B \propto a^{-3}\propto t^{-3n}$, for $n>\frac{2}{3}$ the Universe is dominated by the scalar-field energy density for $t \gg t_1$. During this epoch Einstein's equation becomes

$$n^2=n/\alpha^2 \ ; \tag{4.16}$$

so once n is specified we may determine α. For $t \ll t_1$, the energy density in the scalar field is time independent and the Universe is baryon dominated. To satisfy Einstein's equation during this epoch we must have the usual $n=\frac{2}{3}$. Clearly at even earlier epochs, when radiation is present, the energy density in radiation will be substantially larger than that in the scalar field. We have not analyzed the phase-space structure of this solution, although we have studied it numerically and find it to be fairly stable. This solution for a pure exponential potential will not perturb the canonical nucleosynthesis scenario; however, it does require a very substantial drop in the scalar-field energy density at reheating since the scalar-field energy density is initially time independent. This model is very similar to the constant Λ model, the only significant difference is during the scalar-field-dominated epoch, when $\rho_\Phi \propto t^{-2}$. Models that require less fine-tuning will be discussed in the next section.

V. NEGATIVE-PRESSURE SCALAR FIELD WITH A POWER-LAW POTENTIAL

In this section we construct another class of spatially flat modes: V_q^{power}. Consider a scalar-field whose energy density red-shifts with exponent q (<3) in a universe dominated by radiation. Using the same arguments as in the previous section we find that the scalar-field equation of motion is

$$y''+\frac{2}{x}y'+Ax^2y^{-(4+q)/(4-q)}=0 \ . \tag{5.1}$$

We have assumed $\rho_\Phi \ll \rho_R$ so $a(t)$ is not perturbed. Here $y=\Phi_0-\Phi_0^{(0)}$ ($\Phi_0^{(0)}$ is the value of Φ_0 at $a=0$), $x=a/a_0$, primes denote derivatives with respect to x, and

$$A=-2q\left[\frac{6-q}{4-q}\right]\frac{\rho_\Phi^{(0)}}{\rho_R^{(0)}}\left[2q\left[\frac{2}{4-q}\right]^2\frac{\rho_\Phi^{(0)}}{\rho_R^{(0)}}\right]^{q/(4-q)}$$

The special solution of Eq. (5.1) is

$$y_e(x)=\left[2q\left[\frac{2}{4-q}\right]^2\frac{\rho_\Phi^{(0)}}{\rho_R^{(0)}}\right]^{1/2}x^{(4-q)/2} \ . \tag{5.2}$$

As in the previous two sections, we wish to examine the behavior of spatially homogeneous solutions of Eq. (5.1), not necessarily close to this special solution, in which $\Phi_0(t)\propto t^{(4-q)/4}$ and $\rho_\Phi(t)\propto t^{-q/2}$. To study the structure of the spatially homogeneous phase space of Eq. (5.1) it is convenient to make the change of variables, $(y,x)\to(u,\tau)$,

$$y(x)=y_e(x)u(x), \quad x=e^\tau \ . \tag{5.3}$$

The phase-space equations governing the evolution of $u(\tau)$ are then given by

$$\begin{aligned}\dot u&=p \ ,\\ \dot p&=-(5-q)p\\ &\quad-\left[\frac{6-q}{2}\right]\left[\frac{4-q}{2}\right](u-u^{-(4+q)/(4-q)}) \ ,\end{aligned} \tag{5.4}$$

where an overdot denotes a derivative with respect to τ. The only relevant critical point is at $(u_0,p_0)=(1,0)$.

(In general, there are a number of critical points at $p=0$ with u a $[(4-q)8^{-1}]$th root of unity; since u must be real, we may discard the complex fixed points. For some isolated values of q, Eq. (5.1) is invariant under $y \to -y$; in this case there are two real critical points; since we are interested in solutions for which $y'>0$ we discard one of the critical points.) On linearizing about this critical point we find that the eigenvalues of small fluctuations are given by

$$\lambda_{1,2}=\tfrac{1}{2}[q-5\pm i(23+2q-q^2)^{1/2}] \ . \qquad (5.5)$$

These eigenvalues show that the critical point is in fact a spiral fixed point, for the values of q $(0<q<3)$ that we are interested in. The volume in phase space decreases with time as $V(a/a_0)=(a_0/a)^{(5-q)}$. Clearly, if $q<4$, ρ_Φ/ρ_R is an increasing function of time. A more detailed discussion of the phase space of these equations is presented in the Appendix. The eigenvalues $\lambda_{1,2}$ are derived in another way in Sec. IX A.

Let us now consider the behavior of the scalar field in the same potential (as a function of the scalar field) during the baryon-dominated epoch $(\rho_\Phi \ll \rho_B)$. The scalar-field equation of motion is

$$x^2y''+\tfrac{5}{2}xy'+A\frac{\rho_R^{(0)}}{\rho_B^{(0)}}x^3y^{-(4+q)/(4-q)}=0 \ . \qquad (5.6)$$

A special solution of this equation is

$$y_e(x)=Bx^{3(4-q)/8} \ , \qquad (5.7)$$

where

$$B=\left[\frac{128}{9}\frac{q(6-q)}{(4-q)^2(8-q)}\frac{\rho_\Phi^{(0)}}{\rho_B^{(0)}}\right]^{(4-q)/8} \times\left[2q\left[\frac{2}{4-q}\right]^2\frac{\rho_\Phi^{(0)}}{\rho_R^{(0)}}\right]^{q/8} .$$

Making the change of variables

$$y(x)=y_e(x)u(x), \quad x=e^\tau \qquad (5.8)$$

[we have used the same symbols u,τ to denote the new variables introduced to rewrite Eq. (5.6) as we did to rewrite Eq. (5.1); we hope this does not lead to undue confusion], the equations governing the evolution of trajectories in phase space become

$$\dot u=p \ , \qquad \dot p=-\tfrac{3}{4}(6-q)p-\tfrac{9}{64}(4-q)(8-q)(u-u^{-(4+q)/(4-q)}) \ . \qquad (5.9)$$

The only relevant critical point is $(u_0,p_0)=(1,0)$ and the eigenvalues of small fluctuations are given by

$$\lambda_{1,2}=\tfrac{3}{8}[q-6\pm i(28+4q-q^2)^{1/2}] \ , \qquad (5.10)$$

which show that it is a spiral fixed point for the range of q of interest. The volume in phase space decreases with time as $V(a/a_0)=(a_0/a)^{3(6-q)/4}$. These eigenvalues are derived in a different way in Sec. IX B. A more detailed discussion of the phase space of these equations is presented in the Appendix.

Since there is only one fixed point for the scalar-field equation of motion in both the radiation- and baryon-dominated epochs, it seems reasonable to assume (in our simplified model) that there is only one fixed point during the transition of the Universe from radiation to baryon dominance.

From the special solution, Eq. (5.7), we see $(\dot\Phi_0)^2$, $V(\Phi_0)\propto(a_0/a)^{3q/4}$ so during the baryon-dominated epoch the scalar stress tensor red-shifts slower than in the radiation-dominated epoch (if $q>0$), while ρ_Φ/ρ_B is an increasing function of time (if $q<4$). We note that in both the radiation- and matter-dominated epochs, the fractional perturbations in the scalar field, $\delta\Phi_0/\Phi_0$, are given by u_1 (where $u=u_0+u_1$).

In the scalar-field-dominated epoch, the scalar-field equation of motion and Einsten's equation are coupled because we can no longer neglect the scalar-field contribution to the stress tensor. These equations are given by

$$\ddot y+3\frac{\dot a}{a}\dot y+Cy^{-(4+q)/(4-q)}=0 \ , \qquad \left[\frac{\dot a}{a}\right]^2=\frac{1}{12}(\dot y)^2-\frac{C}{12}\left[\frac{4-q}{q}\right]y^{-2q/(4-q)} \ , \qquad (5.11)$$

where

$$C=-\frac{q(6-q)}{3(4-q)}16\pi m_P^{-2}\rho_\Phi^{(0)} \times\left[2q\left[\frac{2}{4-q}\right]^2\frac{\rho_\Phi^{(0)}}{\rho_R^{(0)}}\right]^{q/(4-q)} .$$

In this case, we may safely work with the version of Einstein's equation given by Eq. (2.8) since we are only interested in determining the spatially homogeneous solution for $\kappa^2=0$. We find, at large times, the scalar-field stress tensor is dominated by the potential term and the asymptotic form of the solution is

$$\Phi_0-\Phi_0^{(0)}=Dt^{(4-q)/(8-q)} \ , \qquad \frac{a}{a_0}=\exp(Wt^{(8-2q)/(8-q)}) \ , \qquad (5.12)$$

where

$$D=\left[\frac{2q}{3}\frac{(8-q)(6-q)^{1/2}}{(4-q)^2}(16\pi m_P^{-2}\rho_\Phi^{(0)})^{1/2}\right]^{(4-q)/(8-q)} \times\left[2q\left[\frac{2}{4-q}\right]^2\frac{\rho_\Phi^{(0)}}{\rho_R^{(0)}}\right]^{q/[2(8-q)]} ,$$
$$W=\left[\frac{8-q}{8-2q}\right]\left[\frac{-C(4-q)}{12q}\right]^{1/2}D^{-q/(4-q)} \ , \qquad (5.13)$$

and, at late times,

$$\rho_\Phi(t)=Et^{-2q/(8-q)} \ , \qquad (5.14)$$

where

$$E=\frac{6-q}{6}\rho_\Phi^{(0)}\left[\frac{\pi}{9m_P^2}(8-q)^2(6-q)\frac{(\rho_R^{(0)})^2}{\rho_\Phi^{(0)}}\right]^{-q/(8-q)}$$

As expected, at late times, the Universe expands slower than if it were dominated by a constant cosmological constant.

The evolution of spatially inhomogeneous perturbations, in these models, is analyzed in Secs. VII–IX.

VI. THE CLASSICAL TESTS OF GRAVITY THEORIES

We summarize here the constraints on the scalar term in the action from the classical tests of gravity theories. Our analysis of the cosmological tests for the power-law potential model is described in Ref. 17. We note, in particular, that the energy density of the scalar field has been chosen to red-shift slower than radiation (i.e., in one of the models of Secs. III–V), so the scalar energy density would not greatly modify the expansion rate at nucleosynthesis. (The scalar energy density at nucleosynthesis is determined by its energy density now. Typically we desire $\rho_\Phi^{(0)}/\rho_B^{(0)} \sim 5$.)

Expanding the scalar field about the homogeneous background field Φ_0, we find that the equation of motion for the fluctuation ϕ is given by

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}-\frac{1}{a^2}\nabla^2\phi+m_\phi^2\phi+\cdots=0 , \tag{6.1}$$

where $m_\phi^2=\frac{1}{2}V''(\Phi_0)$. In model E_3 (Sec. III), $\epsilon\sim 10^{-4}$, so for red-shifts $1+z\lesssim 10^4$, Eq. (3.13) reduces to

$$V=Se^{-6A(\Phi_0-\Phi_0^{(0)})} , \tag{6.2}$$

so

$$V''=36A^2V=24\pi m_P^{-2}(\rho_B^{(0)}+\rho_\Phi^{(0)})\left[\frac{a_0}{a}\right]^3 , \tag{6.3}$$

which gives

$$m_\phi \simeq 5\times 10^{-28}\Omega_0^{1/2}h\left[\frac{a_0}{a}\right]^{3/2} \text{cm}^{-1}$$
$$\simeq 10^{-32}\Omega_0^{1/2}h\left[\frac{a_0}{a}\right]^{3/2} \text{eV} , \tag{6.4}$$

where we have taken $\rho_B^{(0)}+\rho_\Phi^{(0)}=5\times 1.9\times 10^{-29}\Omega_0 h^2$ g cm^{-3}. For the power-law potentials considered in Sec. V, in the baryon-dominated epoch, we find

$$m_\phi^2=\frac{3\pi}{8m_P^2}\rho_B^{(0)}(4+q)(8-q)\left[\frac{a_0}{a}\right]^3 . \tag{6.5}$$

This results in numbers comparable to those of Eq. (6.4). As one would expect by dimensional analysis, m_ϕ^{-1} is on the order of the distance to the horizon when $\Omega_0\sim 1$.

Coherent scalar-boson exchange between light ($\ll M_{\text{GUT}}$) fermions will result in an effective Newton's constant:

$$G_{N,\text{eff}}=G_N+\frac{\langle g_y\rangle^2}{\langle m_f\rangle^2}e^{-m_\phi r} , \tag{6.6}$$

where $\langle g_y\rangle$ is a dimensionless number representing the weighted average (over the experimental objects) strength of the Yukawa interaction, $\langle m_f\rangle$ is the weighted average fermion mass and r is the proper size of the experiment. Since m_ϕ is exceedingly small, for any experiment $e^{-m_\phi r}\sim 1$.

By far the strongest constraint on $\langle g_y\rangle$ is the Eötvös-Dicke experiment[55] which indicates that the acceleration towards the Sun is independent of the material to an accuracy of better than a part in 10^{10}. Thus the difference between the accelerations of a neutron (a_n) and a proton (a_p), on the Earth, due to the Sun (which we may crudely approximate as being composed of N neutrons of mass m_n and P protons of mass m_p), is

$$\frac{a_n-a_p}{g}=\frac{m_P^2}{Nm_n+Pm_p}\left[g_n^2\frac{N}{m_n}-g_p^2\frac{P}{m_p}\right] , \tag{6.7}$$

where g is the acceleration due to the Sun at the Earth and g_n and g_p are the Yukawa couplings of the scalar field to neutrons and protons. From the Eötvös-Dicke experiment, we have $|(a_n-a_p)/g|\lesssim 10^{-10}$. As a rough approximation, we take $m_n\sim m_p\sim 1$ GeV. Eq. (6.7) then reduces to

$$(N+P)^{-1}(g_n^2N-g_p^2P)\lesssim 10^{-48} , \tag{6.8}$$

where $(N+P)^{-1}N$ and $(N+P)^{-1}P$ are factors of order unity. To avoid having to require an unreasonable coincidental similarity between different Yukawa couplings, we must require that the scalar field couple only exceedingly weakly to light matter, i.e.,

$$g_n, g_p \leq 10^{-24} , \tag{6.9}$$

This constraint on Yukawa couplings might lead to a serious problem for two, not totally unrelated, reasons. Even if we ensure that the scalar does not explicitly couple to ordinary, light matter, radiative corrections might generate effective Yukawa couplings whose values differ by more than is allowed by Eq. (6.8). (It is unclear how one would estimate the magnitude of radiatively generated Yukawa couplings in such a nonrenormalizable field theory. One might, however, be able to ensure the absence of such couplings through the use of an appropriate symmetry.) The second problem arises if we wish to use the decay of the scalar field to heavy (GUT mass) particles (which subsequently decay to light particles) to create matter and radiation. We have not estimated if this leads to a contradiction with the bound of Eq. (6.8).

One might have thought that the gravitational field of the Sun would perturb the local value of Φ. If so, the local perturbation of ρ_Φ could affect the classical tests of gravity theory in the Solar System. Wagoner[56] showed that this is not so if one ignores the general expansion of the Universe. (The class of theories we are considering is Wagoner's case ψ=const.) The effect of the expansion of the Universe can be taken into account as follows. Ignoring ρ_Φ, we can choose a spherically symmetric static line element in the neighborhood of the Sun. In these coordinates the equation for the scalar field is

$$g^{00}\frac{\partial^2\Phi}{\partial t^2}+\frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}\left[\sqrt{-g}\,g^{rr}\frac{\partial\Phi}{\partial r}\right]+\tfrac{1}{2}V'(\Phi)=0\ , \tag{6.10}$$

where we have assumed that $\Phi=\Phi(t,r)$. In the limit $r\to 0$, g^{00} and g^{rr} approach constant values and $\sqrt{-g}\to r^2$, so if Φ varied as the power-law solution r^{-1} at small r, it would have to be singular at $r\to 0$, which is not allowed. Rather, the r dependence is

$$\Phi\simeq\Phi\left[t-\frac{Hr^2}{2}\right]\ , \tag{6.11}$$

where the second term in the argument comes from the first-order Doppler shift in the transformation from expanding coordinates, and we have ignored gravitational red-shift terms of order Gm/r. This means that the local mass density in the scalar field would be very nearly equal to the large-scale mean value, which, as we know from discussions of the cosmological constant, would have negligible effect on the Solar System.

VII. LINEAR PERTURBATION THEORY FOR THE EVOLUTION OF INHOMOGENEITIES: GENERALITIES

In this section we derive the equations of relativistic linear perturbation theory that determine the evolution of inhomogeneities, generalizing the equations of Chap. V of Ref. 54 to include the effects of the scalar field.

The equation we need to linearize are (i) Einstein's equations

$$R_{\mu\nu}=8\pi m_P^{-2}(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T)\ , \tag{7.1}$$

which determine how gravitational perturbations evolve, (ii) the scalar-field equation of motion

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\Phi)+\frac{1}{2}\frac{\partial V}{\partial\Phi}(\Phi)=0\ , \tag{7.2}$$

which determines how scalar-field perturbations evolve, and (iii) the equation of covariant conservation of stress energy

$$T_\mu{}^\nu{}_{;\nu}=0\ , \tag{7.3}$$

which determines how perfect-fluid perturbations evolve.

We work in synchronous gauge and linearize the metric about a spatially flat Friedmann-Robertson-Walker background, so the line element is given by

$$ds^2=dt^2-a^2(t)(\delta_{ij}-h_{ij})dx^i dx^j\ ,$$

the scalar field about a homogeneous background,

$$\Phi(\mathbf{x},t)=\Phi_0(t)+\phi(\mathbf{x},t)\ , \tag{7.4}$$

and the perfect fluid about a homogeneous background.[54]

We find that the homogeneous part of the scalar field obeys

$$\ddot\Phi_0+3\frac{\dot a}{a}\dot\Phi_0+\tfrac{1}{2}V'(\Phi_0)=0\ , \tag{7.5}$$

which is just Eq. (2.8), while the first-order equation of motion is given by

$$\ddot\phi+3\frac{\dot a}{a}\dot\phi-\frac{1}{a^2}\nabla^2\phi+\tfrac{1}{2}V''(\Phi_0)\phi-\tfrac{1}{2}\dot h\dot\Phi_0=0\ . \tag{7.6}$$

We shall need the expansion,to quadratic order in the perturbations, of the scalar-field stress tensor:

$$T_{00}=\frac{m_P^2}{32\pi}[(\dot\Phi_0)^2+V(\Phi_0)]+\frac{m_P^2}{16\pi}[\dot\Phi_0\dot\phi+\tfrac{1}{2}V'(\Phi_0)\phi]+\frac{m_P^2}{32\pi}\left[(\dot\phi)^2+\frac{1}{a^2}(\nabla\phi)^2+\tfrac{1}{2}V''(\Phi_0)\phi^2\right]\ , \tag{7.7}$$

$$T_{0i}=\frac{m_P^2}{16\pi}(\dot\Phi_0\partial_i\phi)+\frac{m_P^2}{16\pi}(\dot\phi\partial_i\phi)\ , \tag{7.8}$$

$$\begin{aligned}T_{ij}=&\frac{m_P^2a^2}{32\pi}\delta_{ij}[(\dot\Phi_0)^2-V(\Phi_0)]+\frac{m_P^2a^2}{32\pi}\{\delta_{ij}[2\dot\Phi_0\dot\phi-V'(\Phi_0)\phi]-h_{ij}[(\dot\Phi_0)^2-V(\Phi_0)]\}\\&+\frac{m_P^2a^2}{32\pi}\left[\delta_{ij}\left[(\dot\phi)^2-\frac{1}{a^2}(\nabla\phi)^2-\tfrac{1}{2}V''(\Phi_0)\phi^2\right]-h_{ij}[2\dot\Phi_0\dot\phi-V'(\Phi_0)\phi]+\frac{2}{a^2}\partial_i\phi\partial_j\phi\right]\ .\end{aligned} \tag{7.9}$$

We have previously noted that covariant conservation of the scalar-field stress energy is equivalent to Eqs. (7.5) and (7.6). Comparing the zeroth- and first-order terms of the perturbed scalar-field stress tensor to those of a perturbed perfect-fluid stress tensor, we may make the identifications

$$\rho_\Phi\delta=\frac{m_P^2}{16\pi}[\dot\Phi_0\dot\phi+\tfrac{1}{2}V'(\Phi_0)\phi]\ , \tag{7.10}$$

$$-a^2(\rho_\Phi+p_\Phi)u^i=\frac{m_P^2}{16\pi}(\dot\Phi_0\partial_i\phi)\ , \tag{7.11}$$

$$c_s^2\rho_\Phi\delta=\frac{m_P^2}{16\pi}[\dot\Phi_0\dot\phi-\tfrac{1}{2}V'(\Phi_0)\phi]\ . \tag{7.12}$$

Here δ is the fractional perturbation in the energy density of the scalar-field fluid, u^i is the coordinate peculiar velocity, and c_s the speed of sound in the scalar-field fluid. Using the results of Sec. II, we see that Eqs. (7.10) and (7.11) give operational definitions of the scalar-field energy-density perturbation and peculiar velocity; however, Eq. (7.12) shows that the speed of propagation of acoustic waves in a scalar-field background is a spacetime-dependent quantity. This means that the perturbed scalar field does not behave like a perturbed ideal fluid. This is exceedingly fortunate, since inhomogeneities in a negative-pressure perfect fluid (excluding the case $p=-\rho$) collapse on very small scales. (This would strongly affect the classical tests of gravity theories in the Solar System.)

For a linear perfect fluid (which we shall use to describe baryons and radiation), with the definitions

$$\rho(\mathbf{x},t)=\rho_b(t)[1+\delta(\mathbf{x},t)] ,$$

$$p(\mathbf{x},t)=p_b(t)+\nu\rho_b(t)\delta(\mathbf{x},t) ,$$

$$T^{\mu\nu}=(\rho+p)u^\mu u^\nu-g^{\mu\nu}p$$

(where ρ_b and p_b are the homogeneous background energy density and pressure of the baryon or radiation fluid), and the assumption

$$\nu=\frac{dp}{d\rho}=\frac{p}{\rho} ,$$

where ν is a constant, we find

$$\left[\frac{\dot a}{a}\right]^2=\frac{8\pi}{3m_P^2}(\rho_b+\rho_\Phi) , \tag{7.13}$$

$$\frac{d}{dt}(a^3\rho_b)+3a^2\dot a p_b=0 , \tag{7.14}$$

$$\dot\delta+(1+\nu)\left[\theta-\frac{\dot h}{2}\right]=0 , \tag{7.15}$$

$$\dot\theta+(2-3\nu)\frac{\dot a}{a}\theta+\frac{\nu}{1+\nu}\frac{\nabla^2\delta}{a^2}=0 , \tag{7.16}$$

$$\frac{1}{2}\ddot h+\frac{\dot a}{a}\dot h=4\pi m_P^{-2}\rho_b\delta(1+3\nu)+\dot\Phi_0\dot\phi-\tfrac{1}{4}V'(\Phi_0)\phi , \tag{7.17}$$

$$\dot h_{,i}-\dot h_{ij,j}=-16\pi m_P^{-2}a\rho_b(1+\nu)v^i+\dot\Phi_0\partial_i\phi , \tag{7.18}$$

$$\frac{1}{a^2}(h_{ij,kk}+h_{,ij}-h_{ik,jk}-h_{jk,ik})-\frac{3\dot a}{a}\dot h_{ij}-\frac{\dot a}{a}\dot h\delta_{ij}-\ddot h_{ij}=8\pi m_P^{-2}\delta_{ij}\rho_b\delta(1-\nu)+\frac{\delta_{ij}}{2}V'(\Phi_0)\phi \tag{7.19}$$

(here $v^i=au^i$ is the proper fluid velocity in synchronous gauge and $\theta=v^i{}_{,i}/a$). Compared to Chap. V of Ref. 54, we have two more equations [(7.5) and (7.6)] and Eqs. (7.17)–(7.19) have new source terms.

We find that the wave equations for the fluctuations ϕ and δ are given by (after a spatial Fourier transform, where k is the spatial coordinate momentum)

$$\ddot\phi+3\frac{\dot a}{a}\dot\phi+\frac{k^2}{a^2}\phi+\tfrac{1}{2}V''(\Phi_0)\phi=\tfrac{1}{2}\dot h\dot\Phi_0=\frac{\dot\Phi_0}{1+\nu}\dot\delta+\dot\Phi_0\theta , \tag{7.20}$$

where we have used Eq. (7.15) to rewrite (7.6), and

$$\ddot\delta+2\frac{\dot a}{a}\dot\delta+\frac{\nu k^2}{a^2}\delta-\frac{4\pi}{m_P^2}(1+\nu)(1+3\nu)\rho_b\delta=(1+\nu)\dot\Phi_0\dot\phi-\frac{1+\nu}{4}V'(\Phi_0)\phi-3\nu(1+\nu)\frac{\dot a}{a}\theta , \tag{7.21}$$

where we have used Eqs. (7.15) and (7.16) to rewrite Eq. (7.17).

For some purposes, it proves useful to have the equation

$$\ddot\delta+(2-3\nu)\frac{\dot a}{a}\dot\delta+\frac{\nu k^2}{a^2}\delta=\frac{1+\nu}{2}\left[\ddot h+(2-3\nu)\frac{\dot a}{a}\dot h\right] , \tag{7.22}$$

which can be derived from Eqs. (7.15) and (7.16).

We shall analyze solutions of these equations, for the power-law potential models of Sec. V, in the following two sections. Our analysis outside the horizon (for long-wavelength fluctuations) shall be restricted to the radiation- and baryon-dominated epochs. This is because our lowest-order solutions, in the scalar-field-dominated epoch, are only valid at asymptotically large times. However, since the scalar field could have come to dominate the Universe only very recently, the behavior of fluctuations outside the horizon, in the scalar-field-dominated epoch, are not of much current observational significance. Results of a numerical analysis of short-wavelength fluctuations in the baryon distribution (in the baryon- and scalar-field-dominated epochs) are described in Ref. 17.

VIII. EVOLUTION OF SHORT-WAVELENGTH INHOMOGENEITIES IN THE POWER-LAW POTENTIAL MODEL

In this section we study the evolution of short-wavelength inhomogeneities (i.e., fluctuations inside the horizon) in the scalar field, in the radiation fluid and in the baryon fluid during the three distinct epochs of the power-law scalar-field potential model of Sec. V.

Let us first consider fluctuations in the scalar field, ϕ, and in a relativistic fluid (i.e., $\nu\neq 0$), δ. For short-wavelength fluctuations (i.e., fluctuations for which $k^2\gg a^2$) we see, from Eqs. (7.20) and (7.22), that ϕ and δ oscillate rapidly. This regime is best studied in a WKB approximation (Sec. 16 of Ref. 54). We take the fluctuations to be of the form

$$\phi(t)=F(t)e^{i[\psi(t)+C_\phi]},\quad \delta(t)=G(t)e^{i\sqrt{\nu}\psi(t)} , \tag{8.1}$$

where $\psi(t)$ is a rapidly varying function of time, C_ϕ is an arbitrary constant, and $F(t)$ and $G(t)$ vary slowly with time. Since there are no $\nabla^2 h$ terms in Eq. (7.17) we see that h cannot oscillate rapidly, in fact, h will not be able to react, in lowest order, to the rapidly oscillating source terms in Eq. (7.17). From Eq. (7.15) or (7.16) we see that θ will oscillate coherently with δ. However, δ does not oscillate coherently with ϕ, Eq. (8.1). Hence the dominant terms in the adiabatic expansions of the first of Eqs. (7.20) and Eq. (7.22) are the first three terms on the left-hand sides. Working to first order in the adiabatic approximation, we find

$$\phi(t) \propto \frac{1}{a(t)} \exp\left[ik \int^t \frac{dt'}{a(t')}\right] ,$$
$$\delta(t) \propto [a(t)]^{(3\nu-1)/2} \exp\left[i\sqrt{\nu}k \int^t \frac{dt'}{a(t')}\right] . \tag{8.2}$$

Perturbations in the scalar field and in a relativistic fluid always oscillate inside the horizon. It is pleasing to note that the leading behavior $\dot{\phi} \propto ka^{-1}\phi$ implies that the quadratic small fluctuation term $\langle \delta T_{00}^{(\phi)} \rangle$ in Eq. (7.7) red-shifts like a^{-4}; i.e., inside the horizon, scalar field fluctuations behave like relativistic matter.

For a zero-pressure fluid, it proves convenient to work with Eq. (7.21). When $\nu=0$ this equation reduces to

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}-\frac{4\pi}{m_P^2}\rho_b\delta=\dot{\Phi}_0\dot{\phi}-\tfrac{1}{4}V'(\Phi_0)\phi . \tag{8.3}$$

To lowest order, the source terms average to zero and the equation reduces to the standard form discussed in Sec. 11 of Ref. 54 and analyzed for the power-law potential models in Ref. 17.

We next consider special cases of the above results for the power-law potential models of Sec. V. We shall present the general form of the wave equations, in the radiation- and baryon-dominated epochs, for the power-law potential models, since they are also required for our analysis of fluctuations outside the horizon in Sec. IX.

A. Radiation-dominated epoch

In this case the wave equations that govern the evolution of small fluctuations ϕ and δ (the fluctuation in the radiation fluid) derived in Sec. VII, become

$$\ddot{\phi}+\frac{3}{2t}\dot{\phi}\left[\frac{k^2}{2Rt}+\frac{E_R}{2t^2}\right]\phi-\frac{3B_R}{4t^{q/4}}\dot{\delta}-\frac{B_R}{t^{q/4}}\theta=0 \tag{8.4}$$

and

$$\ddot{\delta}+\frac{1}{t}\dot{\delta}+\left[\frac{k^2}{6Rt}-\frac{1}{t^2}\right]\delta-\frac{4B_R}{3t^{q/4}}\dot{\phi}+\frac{D_R}{3t^{(4+q)/4}}\phi+\frac{2}{3t}\theta=0 , \tag{8.5}$$

where we have taken $\nu=\frac{1}{3}$. The time-independent coefficients B_R, D_R, E_R, and R are given by

$$B_R=\left[\frac{q}{2}\frac{\rho_\Phi^{(0)}}{\rho_R^{(0)}}\right]^{1/2}\left[\frac{32\pi}{3m_P^2}\rho_R^{(0)}\right]^{(4-q)/8} , \tag{8.6}$$

$$D_R=-(6-q)\left[\frac{4\pi}{3m_P^2}q\rho_\Phi^{(0)}\right]^{1/2}\left[\frac{32\pi}{3m_P^2}\rho_R^{(0)}\right]^{-q/8} , \tag{8.7}$$

$$E_R=\frac{(6-q)(4+q)}{8} , \tag{8.8}$$

$$R=\left[\frac{8\pi}{3m_P^2}\rho_R^{(0)}a_0^4\right]^{1/2} . \tag{8.9}$$

From Eq. (8.2) we find

$$\psi(t)=\left[\frac{2}{R}\right]^{1/2}kt^{1/2}+C_\psi \tag{8.10}$$

(where C_ψ is a constant of integration) and

$$F(t)=\frac{C_F}{t^{1/2}},\quad G(t)=C_G \tag{8.11}$$

(here C_F and C_G are constants of integration).

A convenient measure of the temporal behavior of energy-density fluctuations in the scalar field ϕ is $\langle \delta T_{00}^{(\phi)} \rangle / T_{00}^{(\Phi_0)}$, where $T_{00}^{(\Phi_0)}$ and $\delta T_{00}^{(\phi)}$ are defined through Eq. (7.7). We find, for the slowest decaying component,

$$\frac{\langle \delta T_{00}^{(\phi)} \rangle}{T_{00}^{(\Phi_0)}} \propto \frac{k^2}{t^{(4-q)/2}} , \tag{8.12}$$

which decays for $q<4$. The scalar-field source terms in the graviton equations of motion, Eqs. (7.17)–(7.19), behave in much the same way as the radiation source terms. Hence the analysis of the graviton equations in our model will not differ significantly from the analysis in the standard radiation-dominated model. So, inside the horizon, in the radiation-dominated epoch, the inhomogeneities in the scalar field and in the radiation fluid do not grow. We have ignored the fluctuations in the baryonic fluid since they behave in much the same way as in the standard radiation-dominated universe.[18]

B. Baryon-dominated epoch

Rather than explicitly solving the time-averaged form of Eq. (8.3) (i.e., averaged over times long compared to the oscillation period of the scalar field but short compared to the time scale set by the expansion), we choose to investigate Eqs. (7.20) and (7.21), the wave equations for the fluctuations ϕ and δ (the fluctuation in the baryon energy density). In the baryon-dominated epoch we have $\nu=0$. Using Eq. (7.16) to replace $\theta(t)$ in Eq. (7.20) by $C_\theta t^{-4/3}$, we find that these equations become

$$\ddot{\phi}+\frac{2}{t}\dot{\phi}+\left[k^2\left[\frac{2}{3M}\right]^{4/3}\frac{1}{t^{4/3}}+\frac{E_B}{2t^2}\right]\phi-\frac{B_B}{t^{q/4}}\dot{\delta}-\frac{B_B}{t^{q/4}}\frac{C_\theta}{t^{4/3}}=0 \tag{8.13}$$

and

$$\ddot{\delta}+\frac{4}{3t}\dot{\delta}-\frac{2}{3t^2}\delta-\frac{B_B}{t^{q/4}}\dot{\phi}+\frac{D_B}{4t^{(q+4)/4}}\phi=0 . \tag{8.14}$$

Since $\nu=0$, $\theta(t)$ is not a source for $\delta(t)$ during the baryon-dominated epoch. The time-independent coefficients B_B, D_B, E_B, and M are given by

$$B_B=\left[\left[\frac{6-q}{8-q}\right]\frac{32\pi}{3m_P^2}\rho_R^{(0)}\right]^{(4-q)/8}\left[\frac{q}{2}\frac{\rho_\Phi^{(0)}}{\rho_R^{(0)}}\right]^{1/2}, \tag{8.15}$$

$$D_B=-\left[q(6-q)(8-q)\frac{4\pi}{3m_P^2}\rho_\Phi^{(0)}\right]^{1/2}\times\left[\left[\frac{6-q}{8-q}\right]\frac{32\pi}{3m_P^2}\rho_R^{(0)}\right]^{-q/8}, \tag{8.16}$$

$$E_B=\frac{(4+q)(8-q)}{8}, \tag{8.17}$$

$$M=\left[\frac{8\pi}{3m_P^2}\rho_B^{(0)}a_0^3\right]^{1/2}. \tag{8.18}$$

Equation (8.14) does not include the effect of radiation drag on the baryonic perturbations. This may be accounted for in exactly the same manner as in the standard baryon-dominated model.[19]

For short-wavelength fluctuations $[k^2 \gg (3Mt/2)^{4/3}]$, we see that ϕ will oscillate. Using the WKB approximation for $\phi(t)$, (8.1), we find, to leading order,

$$\psi(t)=k\left[\frac{12t}{M^2}\right]^{1/3}+C_\psi, \tag{8.19}$$

where C_ψ is a constant of integration. To the next order in the adiabatic expansion of Eqs. (8.13) and (8.14) we have

$$\dot F(t)+\frac{2}{3}\frac{F(t)}{t}=0, \tag{8.20}$$

$$\ddot\delta(t)+\frac{4}{3}\frac{\dot\delta(t)}{t}-\frac{2}{3}\frac{\delta(t)}{t^2}=H_\delta(t), \tag{8.21}$$

where the relevant term, to this order, in $H_\delta(t)$ is given by

$$H_\delta(t)=ik\left[\frac{2}{3M}\right]^{2/3}\frac{B_B}{t^{2/3+q/4}}F(t)e^{i\psi(t)}.$$

Equations (8.20) and (8.21) may be integrated to give

$$F(t)=\frac{C_F}{t^{2/3}}, \tag{8.22}$$

$$\delta(t)=C_1t^{2/3}+\frac{C_2}{t}+\frac{1}{t}\int^t(t')^{2/3}\left[\int^{t'}(t'')^{1/3}H_\delta(t'')dt''\right]dt' \tag{8.23}$$

(here C_1, C_2, and C_F are constants of integration). We see, from Eq. (8.23), that baryonic inhomogeneities, in our model, behave in exactly the same way as in the standard baryon-dominated model (Sec. 11 of Ref. 54): the oscillatory term in Eq. (8.23) is suppressed by one factor of k^{-1} ($H_\delta \sim k$ and integrating twice by parts brings down two powers of k^{-1}). The presence of the scalar field does not affect the growth of baryonic inhomogeneities.

Using Eq. (8.22) we find, for the slowest decaying term,

$$\frac{\langle\delta T_{00}^{(\phi)}\rangle}{T_{00}^{(\Phi_0)}}\propto\frac{k^2}{t^{(16-3q)/6}}; \tag{8.24}$$

this means that scalar-field inhomogeneities do not grow (for the relevant range of q) during the baryon-dominated epoch of our model. The analysis of the graviton equations of motion, Eqs. (7.17)–(7.19), in our model, is not significantly different from the analysis in the standard baryon-dominated model, since the new scalar source terms oscillate much too rapidly for the gravitons to be be able to react.

C. Scalar-field-dominated epoch

In the scalar-dominated epoch, scalar-field perturbations do not grow to lowest order, Eq. (8.2), as is true in general inside the horizon. The rapidly oscillating source terms in the wave equation for the baryonic fluctuations will only contribute terms of order k^{-1}; ignoring them, the baryonic fluctuations obey the usual equation

$$\ddot\delta+2\frac{\dot a}{a}\dot\delta-4\pi m_P^{-2}\rho_b\delta=0. \tag{8.25}$$

The complicated time dependence of the scale factor, Eq. (5.12), has prevented us from analytically integrating this equation, even for asymptotically large times. A numerical integration of Eq. (8.25), discussed in Ref. 17, shows that the presence of a substantial amount of scalar-field energy density inhibits the growth of perturbations in the baryonic fluid (a very similar effect occurs in a radiation-dominated universe[18]). This is one of the reasons why the scalar field could have come to dominate the Universe only very recently if galaxies formed by gravitational instability.

IX. EVOLUTION OF LONG-WAVELENGTH INHOMOGENEITIES IN THE POWER-LAW POTENTIAL MODEL

In this section we use the equations of Sec. VII to study the evolution of long-wavelength inhomogeneities, outside the horizon, in the scalar field and the radiation fluid in a universe dominated by radiation and in the scalar field and the baryonic fluid in a universe dominated by baryons. The scalar-field potential is taken to be of the power-law form derived in Sec. V

A. Radiation-dominated epoch

In the radiation-dominated epoch, for long-wavelength fluctuations ($k^2 \to 0$), we find, from Eq. (7.16),

$$\theta(t)=\frac{C_\theta}{t^{1/2}}, \tag{9.1}$$

where C_θ is a constant of integration. For long-wavelength fluctuations, Eqs. (8.4) and (8.5) reduce to

$$\ddot\phi+\frac{3}{2t}\dot\phi+\frac{E_R}{2t^2}\phi-\frac{B_RC_\theta}{t^{(2+q)/4}}=\frac{3B_R}{4t^{q/4}}\dot\delta, \tag{9.2}$$

$$\ddot{\delta}+\frac{1}{t}\dot{\delta}-\frac{1}{t^2}\delta+\frac{2C_\theta}{3t^{3/2}}=\frac{4B_R}{3t^{q/4}}\dot{\phi}-\frac{D_R}{3t^{(4+q)/4}}\phi \ . \tag{9.3}$$

Assuming $\dot{\phi}\sim\phi/t$, $\dot{\delta}\sim\delta/t$, it may be verified that the perturbations of δ induced by the scalar-field source terms in Eq. (9.3) induce a perturbation of the scalar field, via Eq. (9.2), $\Delta\phi$ that obeys $\Delta\phi/\phi\sim\rho_\Phi/\rho_R$ and similarly $\Delta\delta/\delta\sim\rho_\Phi/\rho_R$. So, in the limit $\rho_\Phi \ll \rho_R$ (i.e., when the Universe is radiation dominated) we may neglect the δ and ϕ source terms on the right-hand sides of Eqs. (9.2) and (9.3) and solve the equations for ϕ and δ. If we wish to, we may then define the sources in terms of these solutions and then solve the modified equations to determine order (ρ_Φ/ρ_R) corrections. Explicitly, $\Delta\phi/\phi$ and $\Delta\delta/\delta$ are of order $B_R^2 t^{(4-q)/2}$ or of order $B_R D_R t^{(4-q)/2}$ which are of the order

$$(\rho_\Phi^{(0)}/\rho_R^{(0)})(m_p^{-2}\rho_R^{(0)}t^2)^{(4-q)/4}\sim\rho_\Phi/\rho_R \ .$$

Equivalently, the energy density in the ϕ perturbation induced by the $\delta\rho_R$ source in Eq. (9.2) obeys $\Delta\rho_\phi\sim(\rho_\Phi/\rho_R)\delta\rho_R$. We have the limit $\delta\rho_R/\rho_R<10^{-4}$ on the scales probed by the microwave background anisotropy and, on smaller scales, $\delta\rho_R/\rho_R \ll 1$ from the condition that density fluctuations not over produce black-holes. Thus the perturbations to ϕ produced by the baryon (and radiation) energy-density fluctuations wanted to produce galaxies are always small: $\Delta\rho_\phi/\rho_\Phi \ll 1$. Once the scalar-field fluctuations enter the horizon, they will begin to oscillate away, Sec. VIII.

The solution of Eq. (9.2), with the source terms neglected, is

$$\phi(t)=C_+t^{n_+}+C_-t^{n_-}+\frac{4}{3}\frac{B_RC_\theta}{(6-q)}t^{(6-q)/4} \ , \tag{9.4}$$

where $C_\pm$ are time-independent functions of **k** and

$$n_\pm=\tfrac{1}{4}[-1\pm i(23+2q-q^2)^{1/2}] \ . \tag{9.5}$$

The solution of Eq. (9.3) with the source terms neglected is

$$\delta(t)=C_{\delta 1}t+C_{\delta 2}t^{-1}+\tfrac{8}{9}C_\theta t^{1/2} \ , \tag{9.6}$$

where $C_{\delta 1,\delta 2}$ are time-independent functions of **k**. The third term in Eq. (9.4) is proportional to

$$C_\theta\left[\frac{\rho_\Phi}{\rho_R}\right]^{1/2}t^{1/2} \ ; \tag{9.7}$$

it is a growing perturbation. There are other growing perturbations, some of which grow faster than Eq. (9.7). To determine these modes, we may use Eqs. (9.4) and (9.6) to define the source terms in Eqs. (9.2) and (9.3) and then integrate the resulting equation to determine all corrections to order (ρ_Φ/ρ_R). For instance, retaining the fastest growing mode in δ, $C_{\delta 1}$, we find that the source in Eq. (9.2) becomes

$$\tfrac{3}{4}B_RC_{\delta 1}\frac{1}{t^{q/4}} \ .$$

This induces a new mode in ϕ:

$$\phi(t)=\frac{3}{2}\frac{C_{\delta 1}}{13-2q}B_Rt^{(8-q)/4}\propto\left[\frac{\rho_\Phi}{\rho_R}\right]^{1/2}t \ . \tag{9.8}$$

The fastest growing mode in ϕ, $\propto C_\theta$, induces a term

$$\Delta\delta(t)\sim\frac{\rho_\Phi}{\rho_R}\delta_{1/2} \ , \tag{9.9}$$

where $\delta_{1/2}$ is the mode $\propto C_\theta$ in Eq. (9.6). Clearly such terms, $\Delta\delta$, are subdominant and are of significance only at low red-shift, when $\rho_\Phi/\rho_R\sim 1$.

The fractional perturbations in the scalar field $\delta\Phi_0/\Phi_0=\phi/\Phi_0$ corresponding to the modes $C_\pm$, vary as powers of time, $\phi/\Phi_0\propto t^\lambda$, with

$$\lambda_{1,2}=\tfrac{1}{4}[q-5\pm i(23+2q-q^2)^{1/2}] \ . \tag{9.10}$$

This agrees with Eq. (5.5) (where $x\sim t^{1/2}$).

Furthermore, as is easily verified, in this limit the scalar source terms in the graviton equations of motion are negligible compared to the radiation source terms [they lead to corrections of order (ρ_Φ/ρ_R)]. So, outside the horizon, the inhomogeneities in the scalar field essentially decouple from the inhomogeneities in the radiation fluid and in the gravitational field (to the lowest order). The analysis of the radiation-graviton system is presented in Sec. 86 of Ref. 54.

B. Baryon-dominated epoch

In the baryon-dominated epoch, for long-wavelength fluctuations ($k^2\to 0$), in the limit $\rho_\Phi \ll \rho_B$, it may be verified, from an analysis similar to that of Sec. IX A, that the only significant new feature (compared to the analysis in Sec. 86 of Ref. 54) is the equation of motion of scalar inhomogeneities [Eq. (8.13)] which reduces to

$$\ddot{\phi}+\frac{2}{t}\dot{\phi}+\frac{E_B}{2t^2}\phi=0 \ , \tag{9.11}$$

where the constant E_B has been defined in Sec. VIII B. The solution of this equation is given by

$$\phi(t)=C_+t^{n_+}+C_-t^{n_-} \ , \tag{9.12}$$

where $C_\pm$ are time-independent functions of **k** and

$$n_\pm=\tfrac{1}{4}[-2\pm i(28+4q-q^2)^{1/2}] \ . \tag{9.13}$$

The fractional perturbations $\delta\Phi_0/\Phi_0=\phi/\Phi_0$ vary as powers of time t^λ with

$$\lambda_{1,2}=\tfrac{1}{4}[q-6\pm i(28+4q-q^2)^{1/2}] \ . \tag{9.14}$$

This agrees with Eq. (5.10) (where $x\sim t^{2/3}$).

X. DISCUSSION

In the past three decades there has been much discussion of cosmological models with scalar fields. The postulated scalar fields have been used for a variety of purposes—to elaborate on the steady-state cosmology model, to dynamically explain the hierarchy problem, to incorporate Mach's principle in gravitation theory, to dynamically suppress the classical cosmological constant, as a candidate for CDM (the axion), and to drive inflation. In this paper we have suggested yet another use for a cosmological scalar field. (It is worth keeping in mind, however, that there is not yet any experimental or observational indication that such scalar fields exist.) Our models require a rather definite form of scalar-field potential. It is not inconceivable that a potential of the kind we require might arise in one of the currently popular theories of particle physics. Scalar fields are present in the theory of superstrings as well as in a large number of Kaluza-Klein and supergravity models.

Our models are meant to provide a possible way to resolve the discrepancy between the low dynamical estimates of the mean mass density and the negligibly small space curvature preferred by the inflationary scenario. Two appealing features of our models are that they can simultaneously accommodate the nucleosynthesis requirement of low baryon density and the wanted large total mass density of the Universe[57] and that they require that the distribution of galaxies (light) be clumpier than the distribution of mass (which is dominated by the scalar-field energy density). There are other models which also preserve the observational consequences of the standard nucleosynthesis scenario while allowing a substantially larger baryon mass density. These models are either based on the assumption that the QCD quark-hadron transition would strongly perturb the standard nucleosynthesis scenario (without affecting, too strongly, the calculated abundances of most of the light elements)[58] or on the assumed existence of a late-decaying particle (for instance, the gravitino), which could be responsible for a low-energy nucleosynthesis epoch (again, the abundance of most light elements would not greatly disagree with the observed abundances).[59] There is also the standard CDM scenario, which assumes the existence of a large amount of nonbaryonic matter with low present pressure.[7] However, further analysis is needed to decide whether or not any of these other models "bias" galaxy formation in an astrophysically consistent way.

The scalar-field solutions that we have presented are remarkably stable (for the appropriate range of a parameter), so, given the equations of motion, no fine-tuning is needed to make the time evolution of the scalar-field energy density approach the wanted form. Because of this time variation we can argue that the classical cosmological constant is small (compared to the scale set by the radiation temperature at reheating) now because the Universe is old (compared to the age of the Universe at the time of reheating). This is somewhat reminiscent of the mechanism proposed to dynamically reduce the gravitational constant.[26,24] Numerically, using Eq. (14) of Ref. 17, a crude estimate of the energy scale set by the cosmological constant at reheating, $m_V \sim (V_R)^{1/4}$ (where V_R is the value of the potential of the scalar field at reheating), gives $m_V \sim 5.6\times 10^6$ GeV (for $q \sim 3$ in the preferred models of Sec. V; this is about as high as this scale can be in these simple power-law potential models). This should be compared to the scale in the constant-Λ model, Eq. (1.1), which is $\sim 10^{-3}$ eV. Although a cursory examination might seem to suggest that the reduction in the fine-tuning of the cosmological constant, in these models, has been achieved at the cost of introducing an, equally undesirable, exceedingly low-mass scalar-field fluctuation (i.e., an exceedingly flat potential), this is, in fact, not true. A rough estimate of the mass of the scalar-field fluctuation, Eq. (6.5), at reheating, gives $m_\phi \sim 1.2$ MeV. Subsequently, the cosmological expansion reduces this to $\sim 10^{-32}$ eV at the present epoch. (We note that these models have not been proposed to, and do not, resolve the quantum-mechanical cosmological constant "problem.")

Our models offer no new insight into why the net mean mass density should approach zero as the cosmological expansion parameter $a(t)$ approaches infinity, but the models do suggest a possible connection between the inflaton field that drove inflation and the cosmological constant of the present epoch, as outlined in Sec. I C. In our models, the scalar field interacts at exceedingly weakly with light matter (if it interacts at all). Hence the scalar-field stress tensor is covariantly conserved. A major unsolved issue here is whether, in the unified scalar-field picture, the coupling of Φ to matter can be arranged so as to produce entropy during reheating without violating the Eötvös-Dicke experiment. Also, quantum mechanically, the present models presumably suffer from the same fine-tuning problem that renormalizable scalar-field theories have.[23] These models need a mechanism that prevents the potential from being drastically altered by quantum-mechanical effects.

Perhaps the most pressing unresolved issue is the problem of galaxy formation. We suspect that the present model is not viable in the usual adiabatic perturbations picture because without nonbaryonic matter it is difficult to reconcile the limits on the small-scale anisotropy of the microwave background radiation with the wanted present amplitude of mass-density fluctuations.[60] The other, currently popular, models for galaxy formation are isocurvature fluctuations,[61] the explosion scenario,[62] and cosmic strings.[63] We suspect that any of these mechanisms can be incorporated in our models. We defer a detailed discussion of galaxy formation to later work.[64]

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APPENDIX: GLOBAL STRUCTURE OF PHASE SPACE OF EQS. (5.4) AND (5.9)

In Sec. V we have analyzed the power-law potential models in the radiation- and matter-dominated epochs. We found a fixed point in both epochs (in the finite part of the phase plane), studied small fluctuations about these fixed points, and showed that the fixed points were stable (for some range of q). This analysis was local; it cannot be used to show whether there are any Poincaré limit cycles in the phase plane or to study critical points at ∞. It is also of interest to determine the domain of attraction of the fixed points found in Sec. V and to analyze the possibility of chaotic behavior of phase trajectories. Fortunately, the rather extensive theory of the two-dimensional phase plane, Ref. 53, allows us to clarify some of these issues.

1. Poincaré limit cycles

Equations (5.4) and (5.9) are of the form

$$\dot{x}=f_1(x,y), \quad \dot{y}=f_2(x,y) . \tag{A1}$$

If there is a simply connected closed orbit C in the x,y plane that encloses an area A, then

$$0=\oint_C[f_1(x,y)dy-f_2(x,y)dx]$$
$$=\int_A\left[\frac{\partial f_1(x,y)}{\partial x}+\frac{\partial f_2(x,y)}{\partial y}\right]dx\,dy , \tag{A2}$$

where we have used the Stokes-Green's theorem and assumed that $f_1(x,y)$ and $f_2(x,y)$ are sufficiently smooth. From Eq. (A2) we see that there can be no closed phase trajectories in the phase plane unless $\nabla\cdot\mathbf{f}$ vanishes or changes sign. This is known as Bendixson's theorem. Evaluating $\nabla\cdot\mathbf{f}$ for Eqs. (5.4) and (5.9) we see that there are no limit cycles in either case (for the relevant range of q).

2. The structure of phase space at ∞

A convenient strategy to adopt for such an analysis is to map ∞ to a finite part of the phase plane and then study the critical point structure of the finite part of the phase space of the transformed equations. We shall first consider the mapping

$$v=\frac{u}{p}, \quad r=\frac{1}{p} , \tag{A3}$$

which maps all points at infinity to the finite part of phase space except the line $p=0$; we shall then consider the transformation

$$v=\frac{p}{u}, \quad r=\frac{1}{u} , \tag{A4}$$

which maps all points at infinity to the finite part of phase space except the line $u=0$.

Using the transformation (A3) one finds, for the system described by Eq. (5.4), that the fixed points in the finite part of the (v,r) phase plane are at

$$(v,r)=\left[-\frac{2}{4-q},0\right] \tag{A5}$$

and

$$(v,r)=\left[-\frac{2}{6-q},0\right] \tag{A6}$$

the linearized equations at these fixed points are given by

$$\dot{v}_1=-v_1, \quad \dot{r}_1=\frac{4-q}{2}r_1$$

and

$$\dot{v}_1=v_1, \quad \dot{r}_1=\frac{6-q}{2}r_1 .$$

The first fixed point is a saddle point for $q<4$ and the second fixed point is an unstable node for $q<6$ (the critical point in the finite part of the original phase plane has been mapped to ∞). Under the mapping (A4) our original fixed point remains in the finite part of phase space and the new fixed points have exactly the same behavior as (A5) and (A6). There are methods for deciding which of these four fixed points at infinity are distinct, however, for our purposes it suffices to know that none of the fixed points at infinity are stable. We have thus established a global picture of the original phase space of Eq. (5.4): there is a single attractive fixed point in the finite part of phase space and all phase trajectories that start in the finite part of the phase plane must flow into it; the phase trajectories that flow into the saddle points at infinity must flow out of the unstable fixed points at infinity. Very similar conclusions hold for the phase-space structure of Eq. (5.9).

3. Estimate of the domain of attraction of the fixed points of Sec. V

Equations (5.4) and (5.9) can be rewritten in the second-order form

$$\ddot{u}+F_1\dot{u}+F_2(u)=0 , \tag{A7}$$

where F_1 does not depend on u. It is convenient to define a "potential energy"

$$V(u)=\int^u F_2(u')du' \tag{A8}$$

and an "energy"

$$E(t)=\tfrac{1}{2}(\dot{u})^2+V(u) . \tag{A9}$$

From Eq. (A7) it is easily established that

$$\frac{dE}{dt}=-F_1(\dot{u})^2 \tag{A10}$$

or

$$E(t_1)-E(t_0)=-F_1\int_{t_0}^{t_1}(\dot{u})^2dt . \tag{A11}$$

Now $F_1>0$ for both Eqs. (5.4) and (5.9), hence the "energy" decreases along any phase flow. The contours of constant "energy," for the radiation-dominated model, are given by

$$\frac{p^2}{2}+\left[\frac{6-q}{2}\right]\left[\frac{4-q}{2}\right]\times\left[\frac{u^2}{2}+\left[\frac{4-q}{2q}\right]u^{-2q/(4-q)}\right]=C_E , \tag{A12}$$

where C_E is a constant. These contours are centered (and minimized) at $(u,p)=(1,0)$ and extend over the whole of the finite part of the phase plane. Phase trajectories flow to minimize the "energy"; hence the domain of attraction of our fixed point is essentially the whole phase plane.

Although there is a value of q (in some of our models) which acts as a dividing point between stable and unstable behavior at the critical point in the finite part of the phase plane, we expect the (possibly bifurcating) solution corresponding to this value of q to, probably, have no cosmological significance since the value of q is not in the range $0<q<3$. The models of Sec. V evolve in the phase plane governed by differential equations with "time"-independent coefficients (autonomous differential equations). As far as we are aware there are no examples of such dynamical systems that exhibit chaotic behavior; however, we are also unaware of any theorem which guarantees that this is universally true. We believe that the arguments presented in this appendix, almost certainly, rule out the existence of chaotic phase-space trajectories in these models.

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[21]One would hope that an explanation of this coincidental similarity can be achieved consistent with the requirement that the scalar field and the baryonic fluid interact, if at all, only exceedingly weakly.

[22]In Ref. 17 we presented contours of three cosmological parameters in the two-dimensional (Ω_B,α) parameter space for the power-law potential models (α is the power-law index of the scalar-field potential). If the cosmological measurements converge to select a point in the (Ω_B,α) space, this will be strong circumstantial evidence for a power-law potential model. Other scalar-field potential models may be compared to the observational data in a similar manner. At this stage we can make no further observational predictions. However, once a detailed model for entropy production and galaxy formation is constructed, these models should have further observational consequences.

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[27]See, for example, W. Fischler and L. Susskind, Phys. Lett. B **171**, 363 (1986); **173**, 262 (1986). In the standard superstring scenario, it is believed that once supersymmetry is broken string-loop corrections will generate a potential for the dilaton. It is hoped that the form of this potential will ensure that the resulting time variation of the dilaton "vacuum" expectation value (and hence that of coupling constants such as the fine-structure constant which depend on the dilaton "vacuum" expectation value) does not conflict with observational bounds on the time variation of fundamental constants.

[28]A simple-minded analysis (which ignores the need to compactify the higher-dimensional superstring theory) suggests that it might be difficult to reconcile a slowly rolling dilaton "vacuum" expectation value with the observational bounds on the time variation of fundamental constants.

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[32]In our models there are really two, somewhat different, facets of the cosmological-constant "problem." There is the issue of preserving the shape of the potential as well as the issue of ensuring that the energy at the global minimum of the potential is exceedingly small, compared to the Planck scale.

[33]In any current, satisfactory, theory of particle physics, the cosmological constant at low energies is, essentially, a free pa-

rameter. This even seems to be true in superstring theories (at least at the present stage of their development), since the symmetry (supersymmetry) which might ensure that Λ vanishes at the Planck scale must be broken below $\gtrsim 1$ TeV. In the absence of a theory which determines the value of the cosmological constant and in the absence of conclusive observational evidence that $\Lambda=0$ (in fact the observational data is not inconsistent with the existence of a Λ energy density an order of magnitude larger than the present energy density in baryons), the only option seems to be to study the observational consequences of models with a nonvanishing cosmological "constant."

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[49]The cosmological consequences of our models do not depend on the parity of the scalar field. In certain cases a pseudoscalar field might be more desirable than a scalar field (although this does restrict the form of the potential). The forces mediated by a pseudoscalar coupled to ordinary matter are usually more difficult to detect than the scalar mediated force; see, for example, J. E. Moody and F. Wilczek, Phys. Rev. D **30**, 130 (1984). It is, at present, unclear whether sufficient entropy can be generated in such pseudoscalar models.

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