Black-hole normal modes: A WKB approach. III. The Reissner-Nordström black hole

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(Received 9 January 1987)

Complex frequencies of the normal modes of the Reissner-Nordström black hole are computed by two independent methods. The first is a high-order WKB approach devised by Schutz and Will and extended by Iyer and Will for the Schwarzschild case. The second is direct numerical integration using a method developed by Chandrasekhar and Detweiler, thereby extending earlier results of Gunter. The WKB results agree with the numerical ones with an error less than 1% for the lowest-order modes. For somewhat higher orders, the numerical techniques fail but the WKB method continues to give eigenfrequencies that should be reasonably accurate.

I. INTRODUCTION

The Reissner-Nordström metric is a solution of the coupled Einstein-Maxwell equations and describes a spherically symmetric space-time appropriate for a black hole with mass M and charge Q. The perturbation theory for the exterior Reissner-Nordström geometry has been developed in the past decade by the work of Moncrief, Zerilli, Chandrasekhar, Xanthopoulos,[1-6] and other authors. A complete review can be found in Chap. 5 of Ref. 2.

The perturbation equations of the Reissner-Nordström geometry separate (Moncrief,[3-5] Zerilli[6]) into two pairs of Schrödinger-type equations describing the odd- and even-parity oscillations. They have the form

$$\frac{d^2}{dr_*^2}Z_j^{(\pm)}+(\omega^2-V_j^{(\pm)})Z_j^{(\pm)}=0\ , \tag{1}$$

where the $(+)$ corresponds to even and $(-)$ to odd-parity modes:

$$V_j^{(-)}=\frac{\Delta}{r^5}\left[l(l+1)r-q_i+\frac{4Q^2}{r}\right]\ , \tag{2}$$

$$V_j^{(+)}=V_j^{(-)}+2q_i\frac{d}{dr_*}\left[\frac{\Delta}{r^3[(l-1)(l+2)r+q_i]}\right]\ , \tag{3}$$

where

$$\Delta=r^2-2r+Q^2,\quad i,j=1,2\ (i\neq j)\ , \tag{4a}$$

$$q_1=3+[9+4(l-1)(l+2)Q^2]^{1/2}\ ,$$
$$q_2=3-[9+4(l-1)(l+2)Q^2]^{1/2}\ , \tag{4b}$$

and where Q is the black-hole charge, l is the angular harmonic index, and we have set $M=c=G=1$. The tortoise coordinate r_* is related to r by the equation

$$r_*=r+\frac{r_+^2}{r_+-r_-}\ln|r-r_+|-\frac{r_-^2}{r_+-r_-}\ln|r-r_-|\ . \tag{5}$$

r_+ and r_- are the radius of the outer and the inner horizon of the black hole and are the solutions of the equation $\Delta=0$. Equation (5) is the solution of the following convenient form:

$$\frac{dr_*}{dr}=\frac{r^2}{\Delta}\ . \tag{6}$$

Chandrasekhar[7] has shown that the solution $Z_j^{(-)}$ for the odd-parity oscillations can be deduced from the solution $Z_j^{(+)}$ for the even-parity oscillations, and thus (as in the Schwarzschild case) it is enough to find the solutions for one parity. We shall study the odd-parity modes.

The quasinormal modes represent the resonant nonradial deformations of black holes. They are solutions of the wave equation (1) which correspond to purely ingoing waves at the horizon and purely outgoing at infinity. The spectrum of their complex frequencies is discrete; the real part represents the frequency of the black-hole oscillation and the imaginary part the rate at which each mode is damped as a result of emission of gravitational and electromagnetic radiation. The complex frequencies are uniquely determined by the mass M, the charge Q, the angular harmonic index l, and the degree (overtone index n) of the mode.

There is a basic difference concerning the quasinormal modes of a Reissner-Nordström black hole compared with the Schwarzschild black hole; there is no quasinormal mode which is purely electromagnetic or purely gravitational. Instead any quasinormal mode of oscillation will be accompanied by the emission of both electromagnetic and gravitational radiation. Equation (1) in the special case of $Q=0$ corresponds to purely electromagnetic perturbations for Z_1 and purely gravitational perturbations for Z_2.

In the next two sections we describe the techniques used to obtain the eigenfrequencies listed in Tables I and II. The final section provides interpretation and conclusions.

II. THE WKB METHOD

The WKB approximation scheme derived by Schutz and Will[8] has been improved recently by Iyer and Will.[9] They have shown it to be very efficient in determining the lowest overtones among the complex frequencies of an oscillating Schwarzschild black hole (Iyer[10]). The approximation gets better as the angular harmonic index l increases but it gets worse as the overtone index increases.

The method is based on the similarity of Eq. (1) with the one-dimensional Schrödinger equation for a potential barrier. We summarize it here. Consider the scattering of a wave incident on the barrier from $r_* = \infty$ with a given amplitude. In general, if the function $V(r_*)-\omega^2$ is positive over a reasonable range of r_*, then the reflected amplitude is comparable to the incident amplitude, while the transmitted amplitude is very small. But in our normal-mode problem the boundary conditions are different: at $r_* = \infty$ the waves are purely outgoing and for causality reasons at $r_* = -\infty$ (i.e., the horizon) the waves should be "outgoing" again. ("Outgoing" from the potential barrier for $r_* \to -\infty$ means the waves are going across the horizon into the hole.) Thus it is expected that in the normal-mode case the "reflected" and "transmitted" waves should have comparable amplitudes, with the incident wave's amplitude zero. In quantum mechanics this occurs when the energy (here the frequency squared) coincides with the peak of the potential $V(r_*)$. In this case the WKB method gives outgoing waves with equal amplitudes each of factor $\frac{1}{2}$ times the incident amplitude. This suggests that if normal modes exist for a given potential, they must exist "nearby" for complex frequencies such that $\max[V(r_*)] \simeq \omega^2$. In that case the classical turning points are very close and an exact analytic procedure for continuing the solution from one turning point to the other is available. Thus, outside the turning points one can take the normal WKB solutions to Eq. (1), but between the two turning points, $V(r_*)$ is replaced by the first terms of its Taylor expansion about its maximum. The solution between the two turning points then turns out to be given as a combination of parabolic cylinder functions.[8,9]

The matching conditions finally reduce the problem of finding the quasinormal-mode frequencies $\omega(n)$ of the solution of a very simple equation (Iyer and Will[9]):

$$\omega(n) = (V_0 + \Lambda) - i(n+\tfrac{1}{2})(-2V_0'')^{1/2}(1+O) , \tag{7}$$

where

$$\Lambda(n) = \frac{1}{8}\left[\frac{V_0^{(4)}}{V_0''}\right](a^2+\tfrac{1}{4}) - \frac{1}{288}\left[\frac{V_0'''}{V_0''}\right]^2(7+60a^2) , \tag{8a}$$

$$O(n) = \frac{1}{-2V_0''}\left[\frac{5}{6912}\left[\frac{V_0'''}{V_0''}\right]^4(77+188a^2) - \frac{1}{384}\left[\frac{V_0'''^2 V_0^{(4)}}{V_0''^3}\right](51+100a^2) + \frac{1}{2304}\left[\frac{V_0^{(4)}}{V_0''}\right]^2(65+68a^2) + \frac{1}{288}\left[\frac{V_0''' V_0^{(5)}}{V_0''^2}\right](19+28a^2) - \frac{1}{288}\left[\frac{V_0^{(6)}}{V_0''}\right](5+4a^2)\right] . \tag{8b}$$

Here $a = n+\frac{1}{2}$; the primes and the superscript (n) denote differentiation with respect to r_* of the corresponding potential $V_j^{(-)}$ given in Eq. (2); and the subscript 0 on a function denotes the value of the function at the point r_0 which corresponds to the peak of the potential. The derivatives of the potential have been calculated using the algebraic computing language MACSYMA (Ref. 11).

The listing of a FORTRAN subroutine which calculates the potential and its first six derivatives for any values of r and the parameters is available on request from the second author. Note, however that r_0 must be found numerically.

The WKB approximation method is expected to behave badly for large n, i.e., for $n \gg l$ because higher-order normal modes have larger imaginary parts which seem to increase as $\sim n/4$ for larger n, at least for the Schwarzschild black hole.[12] In this case the method used here is no longer adequate to handle the problem because the "turning points" in the complex r plane again become well separated, so the local Taylor expansion of $V(r_*)$ becomes a poorer approximation.

III. THE NUMERICAL METHOD

The numerical technique that we have used is based on that developed first by Chandrasekhar and Detweiler.[13] The boundary conditions as mentioned before are that the radiation is purely outgoing at infinity and purely ingoing on the horizon. Only complex frequencies can satisfy these boundary conditions. This makes the numerical integration of Eq. (1) very difficult, since the ratio of ingoing to outgoing solution in each asymptotic region falls off exponentially with r. Thus by direct numerical integration only frequencies with relatively small imaginary part can be found. Using this method Gunter[14] determined the fundamental frequencies ($n=0$) which have a very small imaginary part, less than 0.1. He also calculated the next overtone ($n=1$) modes, finding that their eigenfrequencies had similar real parts and imaginary part "approximately $0.3/M$." But he did not publish these. With our root-finding algorithms (described below), we have managed to determine these $n=1$ eigenfrequencies with reasonable accuracy, largely verifying Gunter's remarks about them. (See Tables I and II.)

TABLE I. The complex frequencies belonging to the eigenfunction Z_2 for the angular harmonic index. (a) $l=2$, (b) $l=3$, (c) $l=4$, and (d) $l=5$.

(a)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=0$				
0.00	0.373 16	0.089 22	0.373 67	0.088 96[a]
0.20	0.374 23	0.089 33	0.374 75	0.089 07[a]
0.40	0.377 92	0.089 64	0.378 44	0.089 40[a]
0.50	0.381 15	0.089 84	0.381 68	0.089 61[a]
0.70	0.391 90	0.090 03	0.392 50	0.089 90[a]
0.80	0.400 54	0.089 84	0.401 22	0.089 64[a]
0.99	0.428 30	0.085 32	0.429 30	0.084 27[a]
$n=1$				
0.00	0.346 02	0.274 91	0.346 71	0.273 91[b]
0.20	0.347 14	0.275 25	0.348 02	0.281 12[b]
0.40	0.350 98	0.276 09	0.349 42	0.275 45[b]
0.50	0.354 41	0.276 62	0.350 70	0.273 48[b]
0.70	0.365 80	0.276 25	0.368 95	0.275 02[b]
0.80	0.375 48	0.276 25	0.413 10	0.290 60[b]
0.99	0.403 38	0.264 34	0.404 93	0.265 02[b]
$n=2$				
0.00	0.302 93	0.471 06	0.301 05	0.478 277[b]
0.20	0.304 15	0.471 61		
0.40	0.308 18	0.472 92		
0.50	0.311 89	0.473 74		
0.70	0.324 01	0.473 75		
0.80	0.335 84	0.472 74		
0.99	0.367 18	0.456 76		
$n=3$				
0.00	0.247 46	0.672 90	0.251 50	0.705 15[b]
0.20	0.248 81	0.673 64		
0.40	0.253 10	0.675 44		
0.50	0.257 23	0.676 52		
0.70	0.270 29	0.676 34		
0.80	0.285 50	0.674 74		
0.99	0.326 00	0.655 79		
$n=4$				
0.00	0.178 75	0.878 67	0.207 52	0.946 87[b]
0.20	0.180 28	0.879 60		
0.40	0.184 91	0.881 85		
0.50	0.189 61	0.883 14		
0.70	0.203 94	0.882 74		
0.80	0.223 70	0.880 09		
0.99	0.278 67	0.857 16		

(b)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=0$				
0.00	0.599 27	0.092 73	0.599 44	0.092 70[a]
0.20	0.600 85	0.092 81	0.601 03	0.092 79[a]
0.40	0.606 88	0.093 09	0.607 05	0.093 06[a]
0.50	0.612 47	0.093 25	0.612 65	0.093 24[a]
0.70	0.631 68	0.093 47	0.631 87	0.093 12[a]
0.80	0.647 34	0.093 12	0.647 55	0.093 12[a]
0.99	0.699 80	0.087 13	0.700 10	0.086 98[a]
$n=1$				
0.00	0.582 35	0.281 41	0.582 64	0.281 60[b]
0.20	0.583 98	0.281 65	0.583 73	0.281 72[b]
0.40	0.590 22	0.282 46	0.590 22	0.282 40[b]

TABLE I. (*Continued*).

(b)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
0.50	0.595 96	0.282 83	0.594 88	0.283 99[b]
0.70	0.616 02	0.283 36	0.616 30	0.284 04[b]
0.80	0.632 34	0.282 12	0.629 08	0.281 12[b]
0.99	0.684 05	0.263 98	0.680 01	0.270 17[b]
$n=2$				
0.00	0.553 20	0.476 68	0.551 68	0.479 09[b]
0.20	0.554 88	0.477 06		
0.40	0.561 54	0.478 39		
0.50	0.567 37	0.478 78		
0.70	0.554 19	0.680 77		
0.80	0.606 24	0.476 97		
0.99	0.655 91	0.446 94		
$n=3$				
0.00	0.515 75	0.677 43	0.511 96	0.690 34[b]
0.20	0.517 49	0.677 93		
0.40	0.524 78	0.679 77		
0.50	0.530 55	0.680 11		
0.70	0.554 19	0.680 77		
0.80	0.572 67	0.676 96		
0.99	0.619 31	0.636 20		
$n=4$				
0.00	0.471 07	0.881654	0.470 19	0.915 65[b]
0.20	0.472 88	0.882 16		
0.40	0.481 03	0.884 48		
0.50	0.486 61	0.884 78		
0.70	0.513 00	0.885 37		
0.80	0.532 88	0.880 20		
0.99	0.576 40	0.830 06		

(c)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=0$				
0.00	0.809 10	0.094 17	0.809 18	0.094 16[a]
0.20	0.811 26	0.094 25	0.811634	0.094 25[a]
0.40	0.820 12	0.094 54	0.820 20	0.094 53[a]
0.50	0.828 49	0.094 74	0.828 58	0.094 74[a]
0.70	0.857 19	0.094 97	0.857 27	0.095 00[a]
0.80	0.880 47	0.094 65	0.880 57	0.094 67[a]
0.99	0.959 14	0.087 98	0.959 28	0.088 13[a]
$n=1$				
0.00	0.796 50	0.284637	0.796 63	0.284 33[b]
0.20	0.798 70	0.284 60	0.798 65	0.286 37[b]
0.40	0.807 73	0.285 45	0.808 42	0.285 28[b]
0.50	0.816 29	0.286 02	0.812 76	0.285 48[b]
0.70	0.845 61	0.286 49	0.850 44	0.267 13[b]
0.80	0.869 50	0.285 42	0.888 53	0.282 17[b]
0.99	0.947 35	0.264 46	1.036 64	0.267 68[b]
$n=2$				
0.00	0.773 64	0.478 97	0.772 71	0.479 91[b]
0.20	0.775 88	0.479 34		
0.40	0.785 25	0.480 71		
0.50	0.794 12	0.481 60		
0.70	0.824 39	0.481 83		
0.80	0.849 34	0.479 81		
0.99	0.924 10	0.442 46		

TABLE I. (*Continued*).

(c)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=3$				
0.00	0.743 31	0.678 30		
0.20	0.745 61	0.678 78		
0.40	0.755 45	0.680 65		
0.50	0.764 72	0.681 80		
0.70	0.795 92	0.681 42		
0.80	0.822 26	0.678 27		
0.99	0.889 94	0.622 73		
$n=4$				
0.00	0.707 21	0.881 27		
0.20	0.709 56	0.881 85		
0.40	0.720 02	0.884 22		
0.50	0.729 78	0.885 60		
0.70	0.761 78	0.884 43		
0.80	0.789 81	0.880 09		
0.99	0.845 45	0.805 78		

(d)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=0$				
0.00	1.012 25	0.094 87	1.012 29	0.094 87[a]
0.20	1.015 06	0.094 96	1.015 10	0.094 96[a]
0.40	1.027 10	0.095 27	1.027 15	0.095 27[a]
0.50	1.038 48	0.095 49	1.038 53	0.095 49[a]
0.70	1.077 12	0.095 81	1.077 16	0.095 80[a]
0.80	1.108 27	0.095 49	1.108 31	0.095 49[a]
0.99	1.213 73	0.088 44	1.213 81	0.088 70[a]
$n=1$				
0.00	1.002 15	9.285 83	1.002 22	0.285 82[b]
0.20	1.004 99	0.286 08	1.001 41	0.280 80[b]
0.40	1.017 18	0.286 99	1.001 54	0.289 26[b]
0.50	1.028 72	0.287 64	1.001 54	0.289 90[b]
0.70	1.067 97	0.288 54	1.005 21	0.289 27[b]
0.80	1.099 62	0.287 46	1.000 10	0.280 34[b]
0.99	1.204 34	0.264 99	1.204 22	0.272 23[b]
$n=2$				
0.00	0.983 26	0.479 90	0.982 70	0.480 33[b]
0.20	0.986 16	0.480 31		
0.40	0.998 64	0.481 78		
0.50	1.010 45	0.482 79		
0.70	1.050 87	0.484 15		
0.80	1.083 35	0.482 03		
0.99	1.185 02	0.440 76		
$n=3$				
0.00	0.957 48	0.677 80	0.956 85	0.682 18[b]
0.20	0.960 46	0.678 36		
0.40	0.973 31	0.680 34		
0.50	0.985 49	0.681 66		
0.70	1.027 53	0.683 36		
0.80	1.061 01	0.679 91		
0.99	1.154 90	0.615 71		

TABLE I. (*Continued*).

(d)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=4$				
0.00	0.926 36	0.879 19		
0.20	0.929 44	0.879 91		
0.40	0.942 75	0.882 37		
0.50	0.955 35	0.883 96		
0.70	0.999 47	0.885 94		
0.80	1.033 97	0.880 95		
0.99	1.113 08	0.790 36		

[a]Numerical results coming from Gunter.
[b]New numerical results that we have found.

TABLE II. The complex frequencies belonging to the eigenfunction Z_1 for the angular harmonic index. (a) $l=2$, (b) $l=3$, (c) $l=4$, and (d) $l=5$.

(a)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=0$				
0.00	0.457 13	0.095 06	0.457 60	0.095 00[a]
0.20	0.462 50	0.095 42	0.462 96	0.095 37[a]
0.40	0.479 48	0.096 48	0.479 93	0.096 44[a]
0.50	0.493 25	0.096 22	0.493 68	0.097 19[a]
0.70	0.536 14	0.098 85	0.536 51	0.098 77[a]
0.80	0.569 76	0.098 99	0.570 13	0.099 07[a]
0.99	0.692 50	0.089 17	0.692 75	0.088 64[a]
$n=1$				
0.00	0.435 83	0.290 97	0.436 54	0.290 71[b]
0.20	0.441 43	0.291 93	0.441 55	0.294 24[b]
0.40	0.459 28	0.294 84	0.467 79	0.291 84[b]
0.50	0.473 77	0.296 88	0.473 66	0.294 24[b]
0.70	0.519 13	0.301 25	0.526 08	0.300 08[b]
0.80	0.554 06	0.300 68	0.539 65	0.293 28[b]
0.99	0.678 94	0.271 15	0.653 82	0.273 34[b]
$n=2$				
0.00	0.402 32	0.495 86	0.401 21	0.501 58[b]
0.20	0.408 22	0.497 31		
0.40	0.427 40	0.501 84		
0.50	0.443 01	0.504 96		
0.70	0.492 51	0.511 55		
0.80	0.528 40	0.509 04		
0.99	0.657 03	0.461 08		
$n=3$				
0.00	0.360 50	0.705 64		
0.20	0.366 76	0.707 59		
0.40	0.387 79	0.713 66		
0.50	0.404 95	0.717 82		
0.70	0.460 35	0.726 52		
0.80	0.496 40	0.721 86		
0.99	0.632 15	0.658 04		
$n=4$				
0.00	0.309 80	0.917 90		
0.20	0.316 54	0.920 33		
0.40	0.340 01	0.926 85		
0.50	0.359 23	0.932 96		
0.70	0.422 56	0.943 50		
0.80	0.458 37	0.936 90		
0.99	0.606 40	0.858 95		

TABLE II. (*Continued*).

(b)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=0$				
0.00	0.656 73	0.095 63	0.656 90	0.095 62[a]
0.20	0.664 20	0.095 99	0.664 37	0.095 97[a]
0.40	0.687 12	0.096 98	0.687 28	0.096 97[a]
0.50	0.705 25	0.097 65	0.705 40	0.097 65[a]
0.70	0.760 36	0.098 98	0.760 50	0.098 98[a]
0.80	0.802 71	0.099 12	0.802 84	0.099 11[a]
0.99	0.951 89	0.088 91	0.952 06	0.089 32[a]
$n=1$				
0.00	0.641 47	0.289 80	0.641 74	0.289 73[b]
0.20	0.649 14	0.290 93	0.649 14	0.288 60[b]
0.40	0.672 65	0.293 67	0.674 24	0.294 41[b]
0.50	0.691 25	0.295 57	0.693 36	0.295 57[b]
0.70	0.747 92	0.299 23	0.748 68	0.294 75[b]
0.80	0.791 51	0.299 40	0.793 52	0.300 11[b]
0.99	0.940 34	0.266 27	0.943 88	0.270 68[b]
$n=2$				
0.00	0.615 11	0.490 06	0.613 85	0.492 05[b]
0.20	0.623 10	0.491 70		
0.40	0.647 57	0.496 16		
0.50	0.666 97	0.499 17		
0.70	0.726 22	0.504 53		
0.80	0.771 87	0.504 27		
0.99	0.916 28	0.442 64		
$n=3$				
0.00	0.581 41	0.695 55		
0.20	0.589 83	0.697 77		
0.40	0.615 54	0.703 75		
0.50	0.635 93	0.707 66		
0.70	0.698 50	0.714 54		
0.80	0.746 77	0.713 60		
0.99	0.878 35	0.618 40		
$n=4$				
0.00	0.541 60	0.904 26		
0.20	0.550 57	0.907 05		
0.40	0.577 83	0.914 52		
0.50	0.599 47	0.919 36		
0.70	0.666 18	0.926 61		
0.80	0.717 70	0.925 93		
0.99	0.825 50	0.794 84		

(c)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=0$				
0.00	0.853 02	0.095 86	0.853 10	0.095 86[a]
0.20	0.862 52	0.096 21	0.862 60	0.096 21[a]
0.40	0.890 92	0.097 16	0.891 00	0.097 16[a]
0.50	0.913 03	0.097 80	0.913 10	0.097 79[a]
0.70	0.979 59	0.098 99	0.979 66	0.098 98[a]
0.80	1.030 34	0.099 05	1.030 39	0.099 03[a]
0.99	1.206 56	0.089 37	1.206 65	0.089 79[a]
$n=1$				
0.00	0.841 14	0.289 34	0.841 27	0.289 31[b]
0.20	0.850 79	0.290 37	0.846 85	0.292 55[b]
0.40	0.879 64	0.293 13	0.910 95	0.279 62[b]

TABLE II. (*Continued*).

(c)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
0.50	0.902 13	0.295 00	0.912 91	0.295 30[b]
0.70	0.969 86	0.298 39	0.970 60	0.299 92[b]
0.80	1.021 53	0.298 45	1.025 24	0.298 49[b]
0.99	1.197 60	0.267 75	1.204 68	0.266 75[b]
$n=2$				
0.00	0.819 56	0.487 01	0.818 73	0.487 84[b]
0.20	0.829 47	0.488 66		
0.40	0.859 11	0.493 08		
0.50	0.882 30	0.496 12		
0.70	0.952 08	0.501 32		
0.80	1.005 42	0.501 09		
0.99	1.179 12	0.445 23		
$n=3$				
0.00	0.790 94	0.689 23		
0.20	0.801 21	0.691 48		
0.40	0.831 86	0.697 43		
0.50	0.856 04	0.701 58		
0.70	0.928 51	0.708 28		
0.80	0.984 06	0.707 53		
0.99	1.150 20	0.621 76		
$n=4$				
0.00	0.756 97	0.895 02		
0.20	0.767 67	0.897 85		
0.40	0.799 55	0.905 28		
0.50	0.825 02	0.910 53		
0.70	0.900 77	0.918 58		
0.80	0.959 03	0.917 22		
0.99	1.109 91	0.797 79		

(d)

Q	Approximate-WKB Re(ω)	Approximate-WKB Im(ω)	Numerical Re(ω)	Numerical Im(ω)
$n=0$				
0.00	1.047 87	0.095 98	1.045 91	0.095 98[a]
0.20	1.059 34	0.096 33	1.059 38	0.096 33[a]
0.40	1.092 87	0.097 24	1.092 91	0.097 42[a]
0.50	1.118 75	0.097 83	1.118 79	0.097 83[a]
0.70	1.196637	0.098 94	1.196640	0.098 93[a]
0.80	1.255 38	0.098 93	1.255 41	0.098 47[a]
0.99	1.458 84	0.089 61	1.458 89	0.089 79[a]
$n=1$				
0.00	1.038 15	0.289 11	1.038 22	0.289 10[b]
0.20	1.049 73	0.290 13	1.076 59	0.285 86[b]
0.40	1.083 63	0.292 82	1.077 76	0.290 03[b]
0.50	1.109 79	0.294 56	1.075 59	0.291 40[b]
0.70	1.188 35	0.297 78	1.200 00	0.295 00[b]
0.80	1.248 07	0.297 62	1.250 77	0.289 44[b]
0.99	1.451 49	0.268 53	1.441 24	0.271 57[b]
$n=2$				
0.00	1.019 97	0.485 25	1.019 44	0.485 65[b]
0.20	1.031 78	0.486 90		
0.40	1.066 33	0.491 25		
0.50	1.093 00	0.494 03		
0.70	1.173 30	0.499 15		
0.80	1.234 27	0.498 49		
0.99	1.436 43	0.446 68		

TABLE II. (*Continued*).

	(d)			
	Approximate-WKB		Numerical	
Q	Re(ω)	Im(ω)	Re(ω)	Im(ω)
$n=3$				
0.00	0.995 13	0.685 11		
0.20	1.007 24	0.687 36		
0.40	1.042 67	0.693 27		
0.50	1.070 02	0.696 98		
0.70	1.152 71	0.703 77		
0.80	1.215 23	0.702 23		
0.99	1.413 01	0.623 85		
$n=4$				
0.00	0.965 18	0.888 40		
0.20	0.977 65	0.891 24		
0.40	1.014 14	0.898 64		
0.50	1.042 29	0.903 22		
0.70	1.126 96	0.911 54		
0.80	1.192 16	0.908 82		
0.99	1.380 51	0.800 15		

[a]Numerical results coming from Gunter.
[b]New numerical results that we have found.

The Chandrasekhar-Detweiler method is to use the transformation

$$Z_j=\exp\left[i\int\phi_j dr_*\right] \tag{9}$$

in order to transform Eq. (1) into a Riccati-type differential equation of the form

$$i\frac{d\phi}{dr_*}+\omega^2-\phi_j^2-V_j^{(-)}=0\ . \tag{10}$$

The appropriate boundary conditions for ϕ_j are

$$\phi_j\to-\omega \quad \text{as } r_*\to\infty \tag{11a}$$

and

$$\phi_j\to\omega \quad \text{as } r_*\to-\infty\ . \tag{11b}$$

The frequencies can be found by searching the complex ω plane in order to locate values of ω for which both boundary conditions are satisfied. In this search we are helped a lot by the fact that we have already approximately located the appropriate ω by the WKB method and thus the numerical search is much less time consuming. We integrate Eq. (10) inwards from $r_*=23$ where the initial values of integration have been found using asymptotic expansions of the form

$$Z_i^{(-)}=e^{-i\omega r_*}\sum_{n=1}^{\infty}A_n^{(i)}r^{1-n}\ , \tag{12}$$

where the coefficients $A_n^{(i)}$ are given by the recurrence relation[12]

$$\begin{aligned}-2i\omega(n-1)A_n^{(i)}=&[(n-1)(n-2)-l(l+1)]A_{n-1}^{(i)}\\&+[q_j-2(n-1)(n-3)]A_{n-2}^{(i)}\\&+[n(n-5)Q^2]A_{n-3}^{(i)}\ ,\end{aligned} \tag{13}$$

where $i,j=1,2$ ($i\neq j$), $n=0,1,\ldots,$ $A_0^{(i)}=1$, and $A_n^{(i)}=0$ for $n<0$. The inward integration stops near the peak of the potential and the value of ϕ_j is compared with the one found by outward integration of Eq. (10) starting at $r_*=-18$. Again, this integration begins from the asymptotic expansion

$$Z_i^{(-)}=e^{i\omega r_*}\sum_n B_n^{(i)}(r-r_+)^{n-1}\ , \tag{14}$$

where the coefficients $B_n^{(i)}$ have been determined from the recurrence relation

$$\begin{aligned}&[2i\omega r_+^4+2r_+^2-2Q^2r_++(n-1)(r_+-r_-)r_+^2]nB_{n+1}^{(i)}\\&\quad=[l(l+1)r_+^2-q_jr_++4Q^2-(n-1)(n-2)(3r_+-2r_-)r_+-(n-1)(8i\omega r_+^3+4r_+-2Q^2)]B_n^{(i)}\\&\qquad+[2l(l+1)r_+-q_j-(n-2)(n-3)(3r_+-r_-)-(n-2)(12i\omega r_+^2+2)]B_{n-1}^{(i)}\\&\qquad+[l(l+1)-(n-3)(n-4+8i\omega r_+)]B_{n-2}^{(i)}+[(4-n)(2i\omega)]B_{n-3}^{(i)}\ ,\end{aligned} \tag{15}$$

where $i,j=1,2$ $(i\neq j)$, $n=0,1,\ldots$, $B_0^{(i)}=1$, and $B_n^{(i)}=0$ for $n<0$. The quasinormal mode frequency is the value of ω for which the two integrations inward and outward give the same value for the function ϕ_j at the matching point.

The failure of this numerical method to determine higher-order quasinormal modes has also been pointed out by Chandrasekhar and Detweiler[13] as well as by Detweiler[15] and Gunter.[14] Recently Leaver[12] found a much better method to determine the frequencies of the Schwarzschild and Kerr black hole with high accuracy, even for frequencies with very large imaginary part. Unfortunately it seems that this method does not apply to the study of the quasinormal modes of the Reissner-Nordström black hole. This makes the WKB method all the more important.

For the numerical integration of Eq. (1) we used the NAG library[16] routine DO2EAF which applies Gear's integration method;[17] for the search for quasinormal mode frequencies on the complex ω plane, Powell's hybrid algorithm for root search in nonlinear equations was used via the NAG library routine CO5NCF (Refs. 18 and 19).

The only check that we have on our results is by comparing our $n=1$ and $Q=0$ with the corresponding results of Leaver's method. The error seems to be usually 1% and we believe that for $Q\neq0$ also it will be of the same order.

IV. CONCLUSIONS

Tables I and II contain the results of our two methods plus earlier results by Gunter.[14] What is apparent from a first glance is that as the charge of the black hole increases the real part of the frequency also increases; in the extreme case ($Q=M=1$) it is about 50% greater than the corresponding frequency of the uncharged black hole.

The real part of the frequency seems to follow the same pattern as for the uncharged black hole: as the overtone number n increases, $\mathrm{Re}(\omega)$ decreases towards zero. For the Schwarzschild case, Leaver[12] showed numerically that the mode for $n=8$ has purely imaginary frequency. Chandrasekhar[20] showed that this mode, which is an algebraically special perturbation for which an exact solution of the perturbation equations exists, also exists for the Reissner-Nordström case. Our results cannot establish its existence, but the decrease we see in $\mathrm{Re}(\omega)$ is indicative.

The imaginary part of the frequency, which represents the damping of the oscillations, changes much less than the real part and shows a maximum for $Q=0.7$ or 0.8. In general it increases as the order n of the mode increases and that is the reason why for larger n both the numerical method and the approximate WKB method fail to determine the higher modes.

A well-known property[10,12,21] of the quasinormal mode frequencies of the Schwarzschild black hole, that the complex frequencies are symmetrically distributed about the imaginary axis, is also present in the Reissner-Nordström case.

Comparing the approximate method we used with the one by Ferrari and Mashhoon[21] we find that for modes we can calculate accurately numerically, our WKB method gives errors roughly an order of magnitude smaller than theirs. This is not surprising, as the approximate potential they used in order to get semianalytic results does not fit the true potential beyond lowest order in l. See Schutz and Will[8] and Iyer and Will[9] for further discussion of this point.

ACKNOWLEDGMENT

K.D.K. acknowledges the financial support of the University of Thessaloniki.

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