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Effective scalar field theory of p-adic string

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Using the effective scalar field theory of the p-adic string, we show the equivalence of two previously derived sets of classical Feynman rules: one by the present authors, the other by Brekke *et al.* We use this equivalence to demonstrate a symmetry of the p-adic tree amplitude $A^m(p)$ under $p \to 1/p$. We then present a scalar field theory with a continuous parameter ξ that reduces to the two previously discussed examples as special cases.

String and superstring theories are very promising as candidates for a unified theory which includes gravity with the other elementary-particle forces. Strings are also relevant to other areas of physics, e.g., cosmology and superconductivity. It is becoming clear that some new technique is necessary to make real progress toward confronting strings with empirical data.

In the last year, there have been several discussions of the p-adic approach to string theory.[1,2] In particular, it was shown in Ref. 1 how to find the open-string N-point tree amplitude for a p-adic string where the world-sheet coordinates are p-adic valued.

For a fixed prime number p, the amplitude is derivable from Feynman rules which correspond to a scalar field theory[1] with a nonlocal propagator and nonpolynomial interaction. The Lagrangian of this field theory is

$$L_0=\frac{1}{2}\phi\left[\frac{1-p^{1+\Box/2}}{1-p^{-1}}\right]\phi-\frac{1}{(1+p^{-1})p^{-1}(p^{-1}-1)}\left[(1+\phi)^{1+p^{-1}}-1-(1+p^{-1})\phi-\frac{(1+p^{-1})p^{-1}}{2}\phi^2\right] . \tag{1}$$

This is written in Euclidean spacetime, although it can be continued directly to Minkowski spacetime. The Feynman rules which reproduce the S-matrix N-point tree amplitudes of Ref. 1 are obtained directly from Eq. (1) as

$$\Pi_0=(1-p^{-1})(1-p^{\alpha})^{-1} , \tag{2a}$$

$$V_0^m=(-1)^{m+1}\prod_{x=2}^{m-2}(x-p^{-1}) \quad (m\geq 4) , \tag{2b}$$

$$V_0^3=1 ,$$

with $\alpha=1-\frac{1}{2}k^2$ in Euclidean spacetime.

Independently of Ref. 1, Brekke, Freund, Olson, and Witten[3] have recently presented a set of Feynman rules which are similar to our Eq. (2), taken from Ref. 1, but which differ in an interesting manner. The Feynman rules of Ref. 3 are

$$\Pi_1=(1-p^{-1})p^{\alpha}(1-p^{\alpha})^{-1} , \tag{3a}$$

$$V_1^m=\prod_{x=2}^{m-2}(1-xp^{-1}) \quad (m\geq 4) , \quad V_1^3=1 . \tag{3b}$$

In the present article we shall explain the equivalence of the two sets of Feynman rules (and corresponding scalar field theories) at the classical level and show that it implies a symmetry of the p-adic tree amplitudes. We shall then introduce a more general class of Feynman rules characterized by a continuous parameter (ξ).

The equation of motion from the Lagrangian of Eq. (1) is

$$(1-p^{1+\Box/2})\phi=-p(1+\phi)^{p^{-1}}+p+\phi . \tag{4}$$

This has two candidate vacua: first, $\phi=0$ (the tachyonic vacuum), and second, $\phi=-1$. This is easily seen by rewriting Eq. (4) as

$$p^{\Box/2}(1+\phi)=(1+\phi)^{1/p} , \tag{5}$$

so that we may perturb around $\phi=0$ or $\eta\equiv\phi+1=0$.

For the case of Eq. (3) the Lagrangian is

$$L_1=\tfrac{1}{2}\psi\left(\frac{p^{-1-\Box/2}-1}{1-p^{-1}}\right)\psi-\frac{p^3}{(p+1)p(p-1)}\left[\left(1+\frac{\psi}{p}\right)^{p+1}-1-\frac{p+1}{p}\psi-\frac{(p+1)p}{2p^2}\psi^2\right] . \tag{6}$$

The equation of motion from variation of Eq. (6) can be written

$$p^{-\Box/2}\left(1+\frac{\psi}{p}\right)=\left(1+\frac{\psi}{p}\right)^p , \tag{7}$$

with two vacuum solutions $\psi=0$ and $\psi=-p$. If we define $\chi=1+\psi/p$ then χ is related to the η of the earlier formulation by

$$\eta=\chi^p . \tag{8}$$

This explains how the on-shell tree amplitudes are the same. Since the relationship is

$$\phi=\psi+\text{higher order} , \tag{9}$$

the redefinition is within the class once studied by Borchers.[4]

It is important to note that the two theories are classically equivalent not only in the perturbative sense of the tree S matrix for the tachyonic vacuum but also nonperturbatively for the soliton solution[3] in the χ (or η) vacuum. One may readily check that the classical soliton energy is the same.

Inspection of the two sets of Feynman rules shows that

$$\Pi_1(1/p)=p\Pi_0(p) , \tag{10a}$$

$$V_1^m(1/p)=p^{m-3}V_0^m(p) . \tag{10b}$$

Now for a general tree diagram with v propagators and n_i vertices of type V^i one therefore finds, for the m-point amplitude,

$$A_m(1/p)=p^kA_m(p) , \tag{11}$$

with

$$k=v+\sum_{i=3}^{\infty}n_i(i-3)=(m-3) , \tag{12}$$

using the topological identities

$$\sum in_i=2v+m , \tag{13a}$$

$$\sum n_i=v+1 . \tag{13b}$$

In reaching the new result [Eq.(11)], we have used the previously derived fact that the two sets of Feynman rules give the same $A_m(p)$.

We may introduce a new parameter ξ which interpolates between the two sets of Feynman rules as follows. We define a propagator

$$\Pi_\xi=(1-p^{-1})[(1-p^{\alpha})^{-1}-\xi] . \tag{14}$$

This coincides with Eq. (2a) for $\xi=0$ and with Eq. (3a) for $\xi=1$, and explains the subscript in those equations. To define the vertex V_ξ^m we need to introduce the combinatoric quantities $C^m_{(v,\{n_i\})}$ defined as the number of inequivalent m-point tree amplitudes containing v propagators and n_i vertices of type V^i $(i\geq 3)$. The requirement that the tree amplitude A_m be independent of ξ then dictates that V_ξ^m is given by either of the (equivalent) expressions

$$V_\xi^m=\sum C^m_{(v,\{n_i\})}[\xi(1-p^{-1})]^v\prod_{i=3}^{m}V_0^{n_i} \tag{15a}$$

$$=\sum C^m_{(v,\{n_i\})}[(\xi-1)(1-p^{-1})]^v\prod_{i=3}^{m}V_1^{n_i} . \tag{15b}$$

These expressions are readily derived by writing Π_ξ in Eq. (14) as a sum of Π_0, or Π_1, respectively, with a contact term and equating the resultant tree amplitudes, after rearranging the contact term from the propagator to the vertex.

Under p into $1/p$ the propagator and vertex satisfy

$$\Pi_\xi(1/p)=p\Pi_{1-\xi}(p) , \tag{16a}$$

$$V_\xi^m(1/p)=p^{m-3}V_{1-\xi}^m(p) , \tag{16b}$$

as a generalization of Eq. (10) which implies Eq. (11).

The coefficients $C^m_{(v,\{n_i\})}$ can be evaluated explicitly in simple cases. For simplest (and only even) prime number $p=2$ it is sufficient to know that

$$C^m_{(v=m-3,n_3=m-2;n_i=0,i\geq 4)}=(2m-5)!! . \tag{17}$$

The Lagrangian for $p=2$ and general ξ is

$$L=\phi\left(\frac{1-2^{1+\Box/2}}{(1-\xi)+\xi 2^{1+\Box/2}}\right)\phi-\sum_{m=3}^{\infty}(2m-5)!!\left(\frac{\xi-1}{2}\right)^{m-3}\frac{\phi^m}{m!} . \tag{18}$$

Note that the second term in Eq. (18) can be summed to give

$$-\frac{1}{(\xi-1)^3}\tfrac{8}{3}\{[1+\phi(1-\xi)]^{3/2}-1-\tfrac{3}{2}\phi(1-\xi)-\tfrac{3}{8}\phi^2(1-\xi)^2\} . \tag{19}$$

The equation of motion obtained by taking the stationary value of Eq. (18) is

$$\frac{1}{1+\xi}\left(\frac{1-2^{\Box/2}}{1-\xi+\xi 2^{1+\Box/2}}\right)\phi=\frac{1}{(1-\xi)^2}\left(-[1+(1-\xi)\phi]^{1/2}+1+\frac{1-\xi}{1+\xi}\phi\right) . \tag{20}$$

By defining now

$$\eta=\left(\frac{[1+(1-\xi)\phi]^{1/2}-\xi}{1-\xi}\right)^2 , \tag{21}$$

one finds after some algebra that Eq. (20) can be rewritten

$$2^{\Box/2}\eta = \eta^{1/2} \tag{22}$$

which agrees with Eq. (5) and Eq. (7) for special ξ values, and hence the classical theory for $p=2$ is independent of ξ. Again we stress this holds both perturbatively (S-matrix tree amplitudes) and nonperturbatively (classical soliton).

For general p, the appropriate transformation law is obtained from the equation

$$\xi p \eta^{1/p} + (1-\xi)\eta = \phi + 1 + \xi(p-1) \ . \tag{23}$$

This is a pth-order equation for η and hence for $p \geq 5$ involves solution of a quintic or higher-order equation; such an equation is not in general soluble by radicals (Galois).

For $p=3$, the cubic equation can be solved explicitly and one finds

$$3^{\Box/2}\eta = \eta^{1/3} \ , \tag{24}$$

with

$$\eta = \frac{1}{2(1-\xi)}[(1+2\xi+\phi+\Phi)^{1/3} + (1+2\xi+\phi-\Phi)^{1/3}]^3 \ , \tag{25a}$$

$$\Phi = \left[\phi^2 + 2(1+2\xi)\phi + \frac{1+3\xi}{1-\xi}\right]^{1/2} \ . \tag{25b}$$

This could be derived also from the $p=3$ Lagrangian using Eq. (15b) for $V_\xi^m(p=3)$ and noting that[5]

$$C_\xi^m(v=m-3-n_4, n_3=m-2-2n_4; n_4; n_i=0, i\geq 5) = \frac{1}{2^{m-2}}\left(\frac{2}{3}\right)^{n_4}\frac{[2m-(n_4+4)]!}{[m-(2n_4+2)]!}\frac{1}{n_4!} \ . \tag{26}$$

Introduction of the ξ parameter thus allows one to generalize the classical Feynman rules of Refs. 1 and 3. The classical physics including the N-point tree amplitudes in the tachyonic vacuum and the soliton solution in the shifted vacuum (which has no particles perturbatively) are invariant under ξ.

Loop amplitudes for the fixed-p scalar field theory do depend dramatically on ξ because the propagator Π_ξ of Eq. (14) is exponentially convergent for $k^2 \rightarrow \infty$ when $\xi=1$ but goes to a constant for all $\xi \neq 1$ when $k^2 \rightarrow \infty$. Thus, loop diagrams diverge for all $\xi \neq 1$. We believe it is, however, not meaningful to add loops for a fixed p. What seems more likely is that one should form the infinite product over p separately for the tree diagrams, the 1-loop diagrams, and so on, to regain the corresponding string amplitudes. If this is correct, then the parameter ξ may be useful in studying the infinite-product properties of the loop amplitudes.

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[5] We have checked that substitution of Eq. (26) into Eq. (15b) leads to an equation of motion coinciding with Eqs. (24) and (25), at least for several nontrivial terms as a power series in ϕ.