Eigenmodes for fluctuations about the classical solutions in the generalized Liouville equation

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We investigate the eigenmodes for fluctuations about the instantonlike solutions of the generalized Liouville equation. We find that the scalar equation gives positive-definite eigenvalues whereas zero modes can be formed in one sector for the spinor case.

During the last 10 years, the ordinary, purely bosonic, Liouville theory[1] was reconsidered in the study of solitons and instantons,[2] and in the reformulations of the dual string model,[3] where the Liouville modes appear naturally for $D=26$ dimensions. Recently, it has been suggested[4] that two-dimensional gravitational dynamics should be governed by a Liouville-type dynamics, and an $N=1$ supersymmetric extension of the Liouville theory has also been investigated.[5] Two of us (G.K.A. and C.D.)[6] generalized the Liouville theory by adding a fermionic self-interaction term, and changing the sign of the bosonic potential. This model, although not supersymmetric, still maintains conformal symmetry,[7] possesses instantontype solutions for both the scalar and the spinor fields. In recent work,[8] it has been shown that the two-dimensional theory of gravity with dynamical metric and torsion, which was proposed in the context of strings, can be reduced to the model considered in Ref. 6.

Here, we consider the class of two-dimensional conformal-invariant theories described by the Lagrangian[6]

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+\frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi+\frac{\mu^2}{\beta^2}e^{\beta\phi}+\frac{\mu}{2\sqrt{2}}e^{\beta\phi/2}\,\bar{\psi}\psi+g(\bar{\psi}\psi)^2\ , \tag{1}$$

where the scalar field ϕ is dimensionless, the fermion field ψ has a scalar dimension $\frac{1}{2}$, the positive constants β,g are also dimensionless and the quantity μ^2 with the dimension of the square of mass is also taken positive. This Lagrangian (1) gives the following equations of motion:

$$\Box\phi-\frac{\mu^2}{\beta}e^{\beta\phi}-\frac{\mu\beta}{4\sqrt{2}}e^{\beta\phi/2}\,\bar{\psi}\psi=0\ , \tag{2a}$$

$$i\partial\!\!\!/\psi+\frac{\mu}{2\sqrt{2}}e^{\beta\phi/2}\psi+2g(\bar{\psi}\psi)\psi=0\ . \tag{2b}$$

One can find the following instantonlike solutions[6] of Eqs. (2) as a result of the conformal symmetry:

$$\psi_{\rm cl}=\frac{1}{1+x^2}(1+i\gamma\cdot\mathbf{x})C\ , \tag{3a}$$

$$\phi_{\rm cl}=\frac{1}{\beta}\ln\left[\frac{2}{\mu^2}\left[\frac{2A}{1+x^2}\right]^2\right] \tag{3b}$$

with

$$A_{1,2}=\frac{\beta^2\mp\sqrt{\Delta}}{\beta^2-32g} \tag{4a}$$

and C a constant spinor normalized by

$$(\bar{C}C)_{1,2}=-\frac{16}{\beta^2 A_{1,2}}(1+A^2)\ , \tag{4b}$$

where

$$\Delta=\beta^4+32g(\beta^2-32g)\ . \tag{4c}$$

For $g=0$, the solution for the scalar field is in the same form as the instanton solutions in the ordinary Liouville theory[9] and one can also show that solutions (3) are the solutions of the supersymmetric Liouville theory[10] with $A=-1$ and with an anticommuting constant spinor.

In this work, we study the quantum fluctuations about instantonlike classical solutions (3), and also investigate for which range of the coupling constants, g and β^2, the generalized Atiyah-Singer zero modes for the fermion equation appear.

One can find that the equations for quantum fluctuations are, for $\mu>0$,

$$\left[-\partial^2+\frac{4(A^2-1)}{(1+x^2)^2}\right]\varphi=\frac{\lambda^2\varphi}{(1+x^2)^2}\ , \tag{5a}$$

$$\left[i\partial\!\!\!/+\frac{A}{1+x^2}-\frac{64g(1+A^2)}{\beta^2A(1+x^2)}\right]\Psi=\frac{\lambda\Psi}{(1+x^2)}\ , \tag{5b}$$

where A is given in Eq. (4a). Here we are studying the eigenmodes on the unit sphere.

This equation can be solved for g/β^2 in terms of A. Two solutions exist for g/β^2:

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$$g/\beta^2=\tfrac{1}{32},\quad A=-\tfrac{1}{2}\ , \tag{6a}$$

and

$$g/\beta^2=\frac{1}{32}\left[1-\frac{A+1}{1+A^2}\right]\ , \tag{6b}$$

where A is a free parameter. For $A=-\frac{1}{2}$, the fluctuations about the classical solutions are not stable.

For the latter solution we get

$$\left[-\partial^2+\frac{4(A^2-1)}{(1+x^2)^2}\right]\varphi=\frac{\lambda^2\varphi}{(1+x^2)^2}\ , \tag{7a}$$

$$\left[i\partial\!\!\!/+\frac{2-A}{1+x^2}\right]\Psi=\frac{\lambda\Psi}{1+x^2}\ . \tag{7b}$$

For $A=1$, we get $g/\beta^2=0$. This is the case without the fermion-fermion interaction whose solutions were given in the ordinary Liouville theory.[9] From Eq. (6b) we see that A should be greater than 1, to get positive g/β^2 values.

Equation (7a), on the sphere, is written as[11]

$$-4(1-y)y\frac{d^2}{dy^2}\varphi_m+[2y-6(1-y)]\frac{d}{dy}\varphi_m +2(2\,|\,m\,|+1)\frac{d}{dy}\varphi_m=[\lambda^2-4(A^2-1)]\varphi_m\ , \tag{8}$$

where

$$\varphi=\varphi_m(y)e^{im\theta}r^{|m|}\ , \tag{9a}$$

$$y=\frac{1}{1+r^2}\ , \tag{9b}$$

$$\varphi_m=y^{\alpha}\sum_{n=0}^{N-\alpha}a_n y^n\ , \tag{9c}$$

where $\alpha=0$ and $|\,m\,|$.

The series is stopped as a polynomial of degree N and

$$\lambda^2-4(A^2-1)=4(N^2+N)\ . \tag{10}$$

We see that for $A>1$, $0<\lambda^2$ for $N\geq 0$

Starting with the scalar Liouville theory D'Hoker *et al.*[12] found a similar result, that λ^2 is positive definite upon fluctuations about the static solution, contrary to the claims of Barbasov *et al.*[13]

If we take $\Psi=\binom{u}{v}$, Eq. (7b) is reduced to the form

$$-4(1-y)y\frac{d^2}{dy^2}u_m+[2y-6(1-y)]\frac{d}{dy}u_m +2(2\,|\,m\,|+1)\frac{d}{dy}u_m+4(1-y)\frac{d}{dy}u_m -(2\,|\,m\,|-2m)u_m=(\lambda-2+A)^2u_m\ , \tag{11}$$

where

$$u_m(y)e^{+im\theta}r^{|m|}=u\ , \tag{12a}$$

$$y=\frac{1}{1+r^2}\ . \tag{12b}$$

The equation for v is the same equation with m replaced by $(-m)$. This equation has solutions

$$u_m^{(1)}=\sum_{n=0}^{N}b_n y^n \tag{13a}$$

and

$$u_m^{(2)}=y^{(m+1)}\sum_{n=0}^{N-m-1}c_n y^n \tag{13b}$$

for $m>0$. For $m<0$, we get

$$u_m^{(3)}=\sum_{n=1}^{N}d_n y^n \tag{14a}$$

and

$$u_m^{(4)}=y^{|m|}\sum_{n=0}^{N-|m|}e_n y^n\ . \tag{14b}$$

The eigenvalues are given by

$$N=\pm\left[\frac{\lambda}{2}+1-\frac{A}{2}\right]\ . \tag{15}$$

For $A=2$, we get the free case for the fermion, which is the same as the vacuum. The equation for the scalar, in particular, is stable for this value of A, with positive-definite eigenvalues. For $A=4$ we get the first zero modes for the fermion equation, with normalizable eigenfunctions

$$u_0=\frac{1}{1+r^2} \tag{16a}$$

and

$$u_1=\frac{re^{-i\theta}}{1+r^2} \tag{16b}$$

over the sphere. Thus, the topological signature of a true instanton is seen in the fermion equation when $g/\beta^2=\frac{3}{136}$. Note that to get $A=4$ we have to set the ratio of the two coupling constants to this definite value, which fixes a definite sector. For $A>4$ we find that normalizable solutions exist for λ less than zero. This is clearly seen if one writes Eq. (7b) as

$$\partial_{\bar z}u=\frac{(\lambda+A-2)v}{(1+\bar z z)}\ , \tag{17a}$$

$$-\partial_z v=\frac{(\lambda+A-2)u}{1+\bar z z}\ , \tag{17b}$$

where $z=re^{i\theta}$. If $\lambda+A-2=N$, these equations have solutions

$$\Psi_1=\frac{\bar z^{N-1}}{1+\bar z z}\begin{bmatrix}\bar z\\ 1\end{bmatrix} \tag{18a}$$

and

$$\Psi_2=\frac{Z^{N-1}}{(1+\bar z z)^N}\begin{bmatrix}1\\ z\end{bmatrix}\ . \tag{18b}$$

For $N<0$, these solutions are not normalizable with the measure $d\mu=d\bar z\,dz/1+\bar z z$. If $A>4$, we can take $N>0$

for $\lambda < 0$. Such solutions are not found among the fluctuation modes of the models studied before.[11,14]

In this work, we found that the fluctuations around the instanton solutions (3) of the model (1) are stable. Although, we have a free parameter A, which can be interpreted as the instanton number in this model, to get the standard results one has to fix this constant A to 4 which fixes the ratio of the coupling constants in the theory. For this value of the ratio of the coupling constants we get the generalized Atiyah-Singer mode, our $\not{D}$ operator has a nonderivative diagonal part in it, in the fermion equation. The scalar equation has only positive-definite eigenmodes, similar to the static solution case.[12] One can also investigate similar solutions in the bosonic string models, such as the model with torsion on the string world sheet,[8] since the Lagrangian of both models look like different versions of a single unifying model.

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