Critical dimensions for chiral bosons

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We give the Lagrangian formulation of a Bose model in 1+1 dimensions which describes a free chiral Lie-algebra-valued current. This model is a non-Abelian generalization of the chiral scalar model of Siegel. Both the Abelian and non-Abelian actions have a gauge invariance, which becomes anomalous when the models are quantized. The condition that this anomaly be canceled coincides with the string no-ghost condition.

Chiral bosons are one of the basic building blocks of string models. In this paper we shall argue that there are "critical dimensions" for which chiral bosons can be consistently quantized, which coincide with the critical dimensions of strings.

A classical covariant model describing a chiral scalar in 1+1 dimensions has been found by Siegel.[1] The action is

$$S=\int d^2x[-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\tfrac{1}{8}(\eta^\mu{}_\nu-\epsilon^\mu{}_\nu)(\eta^\rho{}_\sigma-\epsilon^\rho{}_\sigma)\times\lambda^{\nu\sigma}\partial_\mu\phi\partial_\rho\phi] \tag{1a}$$

$$=\int d^2x[\partial_+\phi\partial_-\phi+\tfrac{1}{2}\lambda^{--}(\partial_-\phi)^2] \tag{1b}$$

[light-cone coordinates are defined by $x^\pm=(1/\sqrt{2})(x^0\pm x^1)$; we use a Minkowski metric, with $\eta^{+-}=\eta^{-+}=-1$, $\epsilon^{+-}=-\epsilon^{-+}=-1$] where $\phi(x)$ is a real scalar field, and $\lambda^{\mu\nu}(x)$ is a real symmetric tensor field. Varying λ^{--} and ϕ yields the classical equations of motion

$$\tfrac{1}{2}(\partial_-\phi)^2=0\ , \tag{2a}$$

$$2(\partial_+\partial_-\phi)+\partial_-(\lambda^{--}\partial_-\phi)=0\ , \tag{2b}$$

respectively. The first equation implies $\partial_-\phi=0$; and so the second equation is automatically satisfied. Since λ^{--} drops out of the field equations, it must be a gauge degree of freedom. Indeed, the action (1) has the ("Siegel") invariance[1]

$$\delta\phi=\tfrac{1}{2}(\eta^{\mu\nu}-\epsilon^{\mu\nu})\xi_\nu\partial_\mu\phi=\xi^-\partial_-\phi\ , \tag{3a}$$

$$\delta\lambda^{--}=-2\partial_+\xi^--\lambda^{--}\overleftrightarrow{\partial}_-\xi^-\ , \tag{3b}$$

where $\xi^\mu(x)$ is an infinitesimal vector. Thus, classically this model describes one chiral scalar, with corresponding current $\partial_+\phi$.

It is natural to ask whether there exists a non-Abelian generalization of this model, which describes a free chiral Lie-algebra-valued current. In fact, there is. The action is given by

$$S=-\frac{1}{8\pi}\int d^2x\ \mathrm{tr}[\partial_+gg^{-1}\partial_-gg^{-1}+\tfrac{1}{2}\lambda^{--}(g^{-1}\partial_-g)^2]-\frac{1}{24\pi}\int_N \mathrm{tr}(g^{-1}dg)^3\ , \tag{4}$$

where $g(x)$ is a matrix field in some real, orthogonal representation of a compact, semisimple group G. Moreover, N is a three-dimensional manifold, whose boundary is spacetime. The action therefore consists of the Wess-Zumino-Witten (WZW) action,[2] with an additional coupling to the field $\lambda^{--}(x)$. To see that this model indeed describes only one (chiral) current, consider the λ^{--} equation of motion,

$$\tfrac{1}{2}\mathrm{tr}(g^{-1}\partial_-g)^2=0\ . \tag{5}$$

Since the group G is compact, this implies

$$g^{-1}\partial_-g=0\ . \tag{6}$$

Hence, only a single field λ^{--} is needed to set an entire Lie-algebra-valued current to vanish. The remaining current which the model describes is ∂_+gg^{-1}.

As in the Abelian case, the action (4) does not provide dynamics for the field λ^{--}, as it is a gauge degree of freedom. The gauge transformations under which the action is invariant are

$$\delta g=\xi^-\partial_-g\ , \tag{7a}$$

$$\delta\lambda^{--}=-2\partial_+\xi^--\lambda^{--}\overleftrightarrow{\partial}_-\xi^-\ . \tag{7b}$$

These are an immediate generalization of the Siegel transformation laws (3).

It should be noted that the Wess-Zumino term in the action (4) is separately invariant under the transformations (7). The coefficient of the Wess-Zumino term is determined by demanding conformal invariance;[2] or, equivalently, by requiring that the model should couple correctly to a background non-Abelian gauge field.

Having described the classical Lagrangian formulation of chiral bosons, let us now turn to quantization. We begin with the simpler case of a chiral scalar (1). Since one must eventually perform a functional integral over the gauge field λ^{--}, it is necessary to fix the Siegel invariance. Choosing the quantum gauge[3] $\lambda_q^{--}=0$ leads to the ghost Lagrangian[4]

$$\mathcal{L}_{\mathrm{ghost}}=b^{++}(-2\partial_+c^--\lambda^{--}\overleftrightarrow{\partial}_-c^-)\ . \tag{8}$$

Consider the generating functional $W[\lambda^{--}]$, defined by functional integration over ϕ and the ghosts:

$$\exp(iW[\lambda^{--}])=\int[d\phi][d(\text{ghosts})] \times\exp\{i(S[\phi,\lambda^{--}]+S_{\text{ghost}})\} . \quad (9)$$

We normalize the measure so that the functional integral is unity for $\lambda^{--}=0$. To lowest order in λ^{--}, the generating functional is given by

$$W[\lambda^{--}]=\frac{1}{8}\int d^2x\int d^2y\, i\langle T^*U_{--}(x)U_{--}(y)\rangle \times\lambda^{--}(x)\lambda^{--}(y) , \quad (10)$$

where

$$U_{--}(x)=2\frac{\delta S}{\delta\lambda^{--}(x)} =(\partial_-\phi)^2-2[2b^{++}\partial_-c^-+(\partial_-b^{++})c^-] . \quad (11)$$

By straightforward computation, one finds

$$i\langle T^*U_{--}(x)U_{--}(y)\rangle=\frac{1}{24\pi}(1-26)\frac{\partial_-^3}{\partial_+}\delta(x-y) , \quad (12)$$

where the contribution -26 comes from the ghosts. Hence, under a Siegel transformation,

$$\delta_\xi W[\lambda^{--}]=\frac{25}{48\pi}\int d^2x\,\lambda^{--}\partial_-^3\xi^- . \quad (13)$$

That is, the Siegel symmetry is anomalous.[4] If the model is to describe only a chiral scalar, this anomaly must be canceled.

We know of two ways of canceling the Siegel anomaly. The first, proposed in Ref. 4, is to modify the Siegel action (1) by adding a new term which is linear in ϕ:

$$S=\int d^2x[\partial_+\phi\partial_-\phi+\tfrac{1}{2}\lambda^{--}(\partial_-\phi)^2-\alpha\lambda^{--}\partial_-^2\phi] . \quad (14)$$

This action has a tree-level Siegel anomaly proportional to α^2, which can be made to cancel against the one-loop anomaly (13), by choosing $\alpha^2=25/48\pi$. However, there is a difficulty in coupling this model to background (two-dimensional) gravity $g_{\mu\nu}$. Setting $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, and working to first order in the fields λ^{--} and $h^{\mu\nu}$, we find that the correct linearized gravitational couplings are given by the action

$$S=\int d^2x[\partial_+\phi\partial_-\phi+\tfrac{1}{2}h^{++}(\partial_+\phi)^2+\tfrac{1}{2}h^{--}(\partial_-\phi)^2 +\tfrac{1}{2}\lambda^{--}(\partial_-\phi)^2-\alpha\lambda^{--}\partial_-^2\phi-\alpha h^{--}\partial_-^2\phi] . \quad (15)$$

Notice the last term in (15). It is needed to cancel a curved-space Siegel anomaly. However, because of the presence of this term, it is not clear how the couplings to gravity can be generalized to the full nonlinear level. (For further details, see Ref. 5.)

There is a second way to cancel the Siegel anomaly, which avoids such difficulties with coupling to gravity. Instead of a single chiral scalar, we consider a set of d such scalars ϕ^α, $\alpha=1,\ldots,d$ with action

$$S=\int d^2x(\partial_+\phi^\alpha\partial_-\phi^\alpha+\tfrac{1}{2}\lambda^{--}\partial_-\phi^\alpha\partial_-\phi^\alpha) . \quad (16)$$

From (12), we see that the Siegel anomaly is proportional to $d-26$; hence, for $d=26$, the anomaly is absent. It is straightforward to couple this model to gauge and gravitational backgrounds.

Next, let us consider the quantization of the non-Abelian model (4). As in the Abelian case, there is an anomaly in the Siegel symmetry (7), which must be canceled. However, the approach of Ref. 4 of adding another term to the action cannot be generalized to the non-Abelian case. Indeed, the term one would add is

$$-\alpha\lambda^{--}\text{tr}\partial_-(g^{-1}\partial_-g) ,$$

which is identically zero, since $g^{-1}\partial_-g$ is Lie-algebra valued. Hence, in order to cancel the Siegel anomaly, we must follow the second approach; namely, we must add to the model additional chiral fields.

As an example, consider the model describing a chiral Lie-algebra-valued current ∂_+gg^{-1} and a set of d chiral Abelian currents $\partial_+\phi^\alpha$, $\alpha=1,\ldots,d$ with corresponding actions (4) and (16), respectively. Again we fix the Siegel symmetry with the quantum gauge choice $\lambda_q^{--}=0$, and we consider the generating function $W[\lambda^{--}]$ defined by

$$\exp(iW[\lambda^{--}])=\int[d\phi^\alpha][dg][d(\text{ghosts})] \times\exp\{i(S[\phi^\alpha,\lambda^{--}]+S[g,\lambda^{--}] +S_{\text{ghost}})\} . \quad (17)$$

The generating functional can again be expanded as in Eq. (10). For the case that the group G is simple, the two-point function of $U_{--}\equiv 2\delta S/\delta\lambda^{--}$ is given by

$$i\langle T^*U_{--}(x)U_{--}(y)\rangle=\frac{1}{24\pi}(d+c_G-26)\frac{\partial_-^3}{\partial_+}\delta(x-y) , \quad (18a)$$

where

$$c_G=d_G/(1+C_A/\kappa) . \quad (18b)$$

Here, $d_G=\dim G$, and κ and C_A are defined in terms of the generators T_a of G as

$$[T_a,T_b]=if_{abc}T_c,\quad T_a^\dagger=T_a , \qquad \text{tr}(T_aT_b)=\kappa\delta_{ab},\quad f_{acd}f_{bcd}=C_A\delta_{ab} . \quad (19)$$

(See Ref. 6.) For a level-one representation of a simply laced group, the quantity c_G is equal to the rank of the group. If G is a direct product of simple groups, c_G is given by a sum of terms (18b), one such term for each simple factor.

From (18), we learn that the Siegel anomaly is absent, provided that the condition

$$d+c_G-26=0 \quad (20)$$

is satisfied. Remarkably, this is precisely the no-ghost condition for a string on a group manifold.[6] Moreover, we observe that for a chiral Lie-algebra-valued current, there is a restriction

$$c_G\le 26 . \quad (21)$$

In contrast, for the nonchiral case,[2] there is no such restriction.

We have seen that there exists a classical Lagrangian description of a free chiral Lie-algebra-valued current. In particular, the σ model (4) can accommodate chiral $E_8 \times E_8$ gauge currents. This may help provide a bosonic formulation of the $E_8 \times E_8$ heterotic string[7] in which all symmetries are realized linearly. A supersymmetric extension of the action (4) is also available.[8] In fact, the coupling of (supersymmetric) chiral bosons to arbitrary metric, antisymmetric, and dilaton backgrounds has also been determined.[8]

Moreover, we have seen that the condition that chiral bosons can be consistently quantized coincides with the string no-ghost condition. One can think of Siegel symmetry (3) as the gauging of "half" of the group of two-dimensional reparametrizations (diffeomorphisms), $\delta x^- = \xi^-(x^\mu)$, $\delta x^+ = 0$ (Refs. 1 and 4). Hence, for a chiral boson, the critical dimension is determined by the requirement that the quantum theory be invariant under half of the group of diffeomorphisms. This may be related to the quantum $\mathrm{Diff}S^1/S^1$ invariance considered in Ref. 9, and also to the holomorphic factorization of the string partition function in 26 (Euclidean) dimensions.[10] On the other hand, the classical string action is invariant under the full group of two-dimensional diffeomorphisms. The quantum theory must also respect this full invariance. It is intriguing that only half of this invariance is needed to fix the critical dimension.

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