Hypercylindrical black holes

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We show that the Schwarzschild black hole extended to five dimensions by the addition of an extra dimension is stable against linearized perturbations of the metric; thus the topology of the horizon can be thought of as an extra black-hole hair.

I. INTRODUCTION

Recent developments in fundamental theories of physics suggest that the Universe might have more than four dimensions.[1] It is believed that the extra dimensions played an important role in the early Universe;[2] if this is the case we must generalize familiar four-dimensional objects to their higher-dimensional counterparts. An interesting question is that of extending black holes, and considerable progress has been made in this area.[3,4] For example, in the case of five dimensions we find that we have at least two options: we could either have a hyperspherically symmetric black hole, or we could extend the four-dimensional black hole uniformly into the fifth dimension producing a cylindrical black hole. The stability of such objects has not been explored; their stability would imply that not only the mass, Yang-Mills charges, and the angular momenta were necessary to describe higher-dimensional black holes, but also the specification of the topology of the horizon.

Another recent area of interest and activity is cosmic strings. These have aroused much interest as they give a compelling description of the formation of structure in the Universe. There is evidence that cosmic strings in four dimensions are gravitationally stable;[5] however, the five-dimensional problem is slightly different. In four dimensions it is not possible for the string to have an event horizon (unless it is a small loop) and radial collapse of the string would result in a naked conical singularity. In five dimensions this is not the case; for example, the Schwarzschild metric extended into the fifth dimension is a solution of the Einstein equations which has a hypercylindrical event horizon; it may be possible for a cosmic string to shrink inside its event horizon, and it would be interesting to know if such a string were stable.

We consider the problem of perturbing a Schwarzschild spacetime extended by the addition of an extra dimension (henceforth denoted as $S_{\rm Sch}\times\mathbb{R}$). In Sec. II we describe our approach, which is similar to the analysis of Regge and Wheeler.[6] We choose to impose the transverse trace-free gauge on our perturbation, which leads us, via the Einstein equations, to the Lichnerowicz equation for the perturbation. Of critical importance are the boundary conditions. Clearly the perturbation has to be regular at large spatial distances from the string, but near the horizon we must be careful as our coordinate system is not regular. Thus we impose regularity of the perturbation with respect to an orthonormal tetrad. In Sec. III we show that it is possible to apply a Kaluza-Klein decomposition to the perturbation, regarding it as a combination of a scalar, vector, and tensor part with respect to the four-dimensional sections. In Sec. IV we discuss an unexpected static mode for the tensor perturbations and comment on the thermodynamics of black holes in five dimensions. Finally we summarize our results and conclude that the $S_{\rm Sch}\times\mathbb{R}$ string is classically stable.

II. THE FORMALISM

The first steps towards a proof of the stability of a black hole in four dimensions were established by Regge and Wheeler[6] and later refined and extended by various workers.[7,8] We intend to follow the basic outline of these methods.

The metric of the Schwarzschild black hole in five dimensions is given by

$$ds^2=+dz^2-\left[1-\frac{2m}{r}\right]dt^2+\left[1-\frac{2m}{r}\right]^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2) \quad (1)$$

(using units $c=G=1$). Clearly this metric is a solution of the five-dimensional vacuum Einstein equations:

$$G_{ab}(g_{cd})=0\ . \quad (2)$$

We perturb this metric by setting

$$g_{ab}\to g_{ab}+h_{ab}\ ,$$

where h_{ab} is small (in a sense to be made precise shortly). We assume that the perturbed metric also satisfies the Einstein equations

$$G_{ab}(g_{cd}+h_{cd})=0\ , \quad (3)$$

which give us the equations of motion for h_{ab}. The stability problem is then whether these equations allow oscillatory or undamped solutions for h_{ab}. In the former case we would say that $S_{\rm Sch}\times\mathbb{R}$ was stable, in the latter case unstable.

Before discussing this however, we should turn to the questions of gauge and boundary conditions. We choose a transverse trace-free gauge for h_{ab}, i.e.,

$$g^{ab}h_{ab}=0, \quad \nabla^a h_{ab}=0 \ . \tag{4}$$

We then see that the Einstein equations (3) for $g_{ab}+h_{ab}$ simplify to the Lichnerowicz equation for h_{ab}:

$$\nabla^e \nabla_e h_{ac} + 2R_{abcd}h^{bd}=0 \tag{5}$$

(where R_{abcd} is calculated from the background metric).

The boundary conditions we need to impose are that h_{ab} is regular (and small) compared with the background metric. Clearly there is no problem far from the black hole since our coordinates are regular and we simply require that h_{ab} be regular. However, near the horizon we must be careful, as the (t,r) coordinates become singular. Vishveshwara[7] overcame this problem by introducing Kruskal coordinates. We choose to approach it somewhat differently by imposing that all components of h_{ab} remain regular with respect to the orthonormal tetrad

$$\{e_a\}=\left\{\frac{\partial}{\partial z}, \left[1-\frac{2m}{r}\right]^{-1/2}\frac{\partial}{\partial t}, \left[1-\frac{2m}{r}\right]^{1/2}\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right\} . \tag{6}$$

We prefer this approach as it gives access to the boundary conditions more readily, and it corresponds with the intuitive notion of what we would mean by regularity of a perturbation.

Now we return to the problem of analyzing the perturbation equation (5). Since we have a high degree of symmetry of the background, it is possible to simplify these equations by decomposing the perturbation into its irreducible modes with respect to the group of symmetries of the background metric. In our case, the t and z translation symmetries mean that we can decompose the perturbation into frequency modes in the t and z variables, and spherical symmetry means that we can decompose h_{ab} into tensor spherical harmonics. In this paper we will focus attention on perturbations which are independent of θ and ϕ; thus, we can expand h^{ab} as

$$h^{ab}(z,t,r)=e^{i\mu z}e^{i\omega t}\begin{bmatrix} H^{zz}(r) & H^{zt}(r) & H^{zr}(r) & 0 & 0 \\ H^{zt}(r) & H^{tt}(r) & H^{tr}(r) & 0 & 0 \\ H^{zr}(r) & H^{tr}(r) & H^{rr}(r) & 0 & 0 \\ 0 & 0 & 0 & K(r) & 0 \\ 0 & 0 & 0 & 0 & \dfrac{K(r)}{\sin^2\theta} \end{bmatrix} . \tag{7}$$

Using this ansatz, Eq. (5) reduces to a set of coupled second-order ordinary differential equations in r with parameters μ and ω. By expanding the solutions in asymptotic series it is possible to find the leading-order behavior near the horizon ($r\to 2m$) and near infinity ($r\to\infty$). We then demonstrate that the regular solution corresponding to an unstable mode (imaginary ω) at the horizon cannot be matched to the regular solution corresponding to the same mode at infinity. We therefore deduce that the $S_{\rm Sch}\times\mathbb{R}$ is stable.

III. THE PERTURBATION EQUATIONS

We now examine the set of differential equations continued in Eq. (5). The interpretation of the metric perturbations in a Kaluza-Klein fashion renders the problem more transparent. This consists of reinterpreting the metric perturbation as scalar, vector, and tensor fields which propagate in a lower-dimensional spacetime, in our case four-dimensional Schwarzschild spacetime. In this framework the "frequency" of the perturbation in the z direction, denoted by μ in Eq. (7), corresponds to a mass parameter for the fields mentioned above.

(i) The scalar perturbation. The component h^{zz} can be interpreted as a scalar field on a Schwarzschild background. Its equation of motion decouples from all other components of the perturbation and reduces to [using the ansatz (7)]

$$\frac{(r-2m)^2}{r^2}\frac{d^2}{dr^2}H^{zz}+\frac{2(r-m)(r-2m)}{r^3}\frac{d}{dr}H^{zz} - \left[\Omega^2+\frac{\mu^2(r-2m)}{r}\right]H^{zz}=0 , \tag{8}$$

where $\Omega=i\omega$. The regular solutions at the horizon and infinity are

$$H^{zz}\propto (r-2m)^{2m\Omega} \quad \text{as } r\to 2m , \tag{9a}$$

$$H^{zz}\propto e^{-r\sqrt{\Omega^2+\mu^2}} \quad \text{as } r\to\infty \tag{9b}$$

(using the convention $\Omega>0$). Without loss of generality we can choose the constant of proportionality in (9a) to be positive, then $H^{zz}\to 0$ as $r\to 2m$ and $dH^{zz}/dr>0$; but then in order to have a regular solution at infinity (i.e., $H^{zz}\to 0$) we must have a maximum at some value of r. This is impossible by inspection of Eq. (8), since at such a point $dH^{zz}/dr=0$ and $d^2H^{zz}/dr^2<0$. Therefore, the scalar perturbation cannot lead to an instability of the $S_{\rm Sch}\times\mathbb{R}$ spacetime.

This result was expected because, as noted earlier, this perturbation is similar to a massive scalar field and four-dimensional black holes are stable to such perturbations. Thus we must take this part of the perturbation to be zero in the search of instability from other types of perturbation.

(ii) Vector perturbations. In this case we look at the az $(a\neq z)$ components. The equations for h^{zt} and h^{zr} are coupled, but can be disentangled using the transverse gauge condition. We obtain, for the h^{zr} perturbation,

$$\frac{d^2}{dr^2}H^{zr}+\frac{2(r-m)}{r(r-2m)}\frac{d}{dr}H^{zr} -\left[\frac{2(r-4m)}{r^2(r-2m)}+\frac{\Omega^2r^2}{(r-2m)^2}+\frac{\mu^2 r}{r-2m}\right]H^{zr}=0 . \tag{10}$$

The regular solutions are of the form

$$H^{zr}\propto(r-2m)^{2m\Omega} \quad \text{as } r\to 2m , \tag{11a}$$

$$H^{zr}\propto e^{-r\sqrt{\Omega^2+\mu^2}} \quad \text{as } r\to\infty . \tag{11b}$$

However, the boundary conditions imply that $H^{zr}(1-2m/r)^{-1/2}$ must be regular; this imposes the restriction that $\Omega>1/4m$ from Eq. (11a). We can then see that the coefficient multiplying H^{zr} in Eq. (10) above is strictly positive outside the horizon and the argument of the previous section applies. Hence, the component H^{zr} must be taken to be zero for any unstable perturbation. The solution for H^{zt} can be obtained by integrating the gauge condition using the previous result and no regular unstable solution exists. Therefore, we conclude that the $S_{\mathrm{Sch}}\times\mathbb{R}$ spacetime is stable against the vector perturbations.

(iii) Tensor perturbations. Finally we study the tensor perturbations. These behave like tensor fields in a four-dimensional Schwarzschild background. Using the Lichnerowicz equation, the gauge conditions, and the previous results we obtain, after a lengthy calculation,

$$\left[\frac{m}{2r^3}-\frac{\Omega^2 r}{2m}-\frac{\mu^2(r-2m)}{2m}\right]\frac{d^2}{dr^2}H^{tr}+\left[\frac{3m(r-m)}{r^4(r-2m)}-\frac{\Omega^2(r+m)}{m(r-2m)}-\frac{\mu^2}{m}\right]\frac{d}{dr}H^{tr} -\left[\frac{3mr^2-6m^2r+2m^3}{r^5(r-2m)^2}+\frac{\Omega^2(2r^2-8mr+3m^2)}{2mr(r-2m)^2} +\frac{\mu^2(2r^2-10mr+13m^2)}{2mr^2(r-2m)}+\frac{r}{2m}\left[\mu^2+\frac{\Omega^2 r}{r-2m}\right]^2\right]H^{tr}=0 , \tag{12}$$

and the regular solutions behave like

$$H^{tr}\propto(r-2m)^{-1+2m\Omega} \quad \text{as } r\to 2m , \tag{13a}$$

$$H^{tr}\propto e^{-r\sqrt{\Omega^2+\mu^2}} \quad \text{as } r\to\infty . \tag{13b}$$

In this case we must have $\Omega>1/2m$ in order to have a regular perturbation at the horizon. By a similar argument to the previous case, we see that the two solutions cannot match.

This completes the proof of stability for the $S_{\mathrm{Sch}}\times\mathbb{R}$ spacetime in the case of (θ,ϕ)-independent perturbations. In the case of (θ,ϕ)-dependent perturbations the only modification brought about by the fifth dimension is an additional μ^2-dependent term in the H^{ab} coefficient; however, this only strengthens the four-dimensional stability argument, as we have shown.

IV. DISCUSSION AND CONCLUSION

In the previous section we studied the perturbations with $\Omega\neq 0$, i.e., nonstatic perturbations. However, there do exist nontrivial solutions of the perturbation equations for $\Omega=0$, e.g., a static vector perturbation of a four-dimensional black hole corresponds to charging up the black hole. In analyzing the static modes of (5) we found an unexpected perturbation of the form

$$h^{ab}=e^{i\mu z}\,\mathrm{diag}[0,H^{tt}(r),H^{rr}(r),K(r),K(r)/\sin^2\theta] . \tag{14}$$

Again, using the Lichnerowicz equation and the gauge conditions we obtain

$$\frac{r-2m}{r}\frac{d^2}{dr^2}H^{rr}+\frac{2(2r-3m)(r-4m)}{r^2(r-3m)}\frac{d}{dr}H^{rr} -\frac{8m}{r^2(r-3m)}H^{rr}=-\mu^2H^{rr} , \tag{15}$$

which is an eigenvalue equation for H^{rr}. A similar equation has already been studied in the context of the negative mode of the Schwarzschild instanton.[9,10] Its solutions are not known analytically, but it can be solved using numerical techniques which show the existence of a regular solution only for the value $\mu^2\simeq 0.22/m^2$ (Ref. 11). This indicates the existence of a (z,r)-dependent solution in five dimensions with parameter μ^2, which scales the usual four-dimensional geometry as we proceed along the z direction. If we were to interpret this perturbation in a Kaluza-Klein spirit, we would conclude that this static solution represented a black-hole hair from a massive spin-2 field (with mass parameter μ).

Usually in Kaluza-Klein theories, the extra coordinate is identified periodically giving the extra dimension the topology of a circle. This would restrict the parameter μ to take values of the form $(2\pi/L)q$ for integer q, L being the proper length of the extra dimension. The previous argument given for stability remains unchanged; however, the static solution discussed above exists only when the circumference of the extra dimension is a multiple of $m/\sqrt{0.22}$.

Like its four-dimensional counterpart, the extended Schwarzschild black hole will emit Hawking radiation.

It is easy to calculate its temperature by asking for the regularity of the Euclidean section. The result is similar to the four-dimensional case: $T=8\pi km$, where k is the Boltzmann constant, m is the mass (in five dimensions) per unit length for the hypercylindrical black hole and we assume $c=\hbar=1$ and the five-dimensional Newton's constant, $G_5=1$. The entropy of such a black hole will of course be unbounded if the length of the fifth dimension is infinite; however, it is possible to calculate the entropy per unit length, which is given by $S/L=4\pi m^2$. The entropy S for a length L of the black hole is $S=4\pi M^2/L$, where M is the total mass of that length. It is interesting to compare this to the entropy for a hyperspherical black hole $S=(\pi 2^7/3^3)^{1/2}M^{3/2}$. In this case the entropy of the hypercylindrical black hole increases linearly with M, and that of the hyperspherical black hole increases as $M^{3/2}$. This suggests that the former would be favored for low masses, and the latter for high masses. It is tempting to suggest that this could provide a mechanism to trigger dimensional reduction; this is presently under investigation.

To summarize, we have shown that the extension of a four-dimensional black hole into an extra dimension is stable against linearized perturbations of the metric. This implies that the topology of the horizon is necessary to classify black holes in higher dimensions and thus can be thought of as an extra black-hole hair.

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