Variational calculation of the spectrum of two-dimensional ϕ^4 theory in light-front field theory

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We demonstrate that a coherent state may be a valid vacuum in light-front field theory. Then by minimizing the sum of the expectation values of the light-front Hamiltonian *and* the momentum operators in a variational trial state, we evaluate the ground state (vacuum) of two-dimensional ϕ^4 field theory. The resulting expectation value in the coherent state is identical with the result of the effective-potential method in the equal-time formulation. Thus we demonstrate how to solve for the ground state of the strong-coupling $(\phi^4)_2$ problem on the light front. We also discuss the calculation of excited states.

I. INTRODUCTION

Variational approaches to quantum field theory have a long history. The Hartree approximation or the Gaussian effective potential (GEP) formalism has been particularly popular.[1] Recently there have been attempts to improve upon the Gaussian ansatz.[2] All these attempts at nonperturbative methods have been made in the conventional equal-time formulation of quantum field theory. Until recently the light-front formulation of field theory, originating from Dirac's work on different forms of dynamics, was employed solely for perturbative calculations.[3] This changed when discretized light-front quantization (DLFQ) [also called discretized light-cone quantization (DLCQ)] was developed for nonperturbative calculations.[4]

The DLFQ formulation is based on a Fock-space expansion and discretization in light-cone momentum variables. The greatest advantage of the light-front scheme over the equal-time formulation comes from the fact that the light-front momentum operator is a positive operator. If the states where the particles carry zero light-cone momentum are neglected, then the Fock-space vacuum is not dynamically related to the remainder of the states. However, in describing the vacuum structure itself, it is important to include the zero–light-cone–momentum ($k^+=0$) states. The importance of the point $k^+=0$ in light-front field theory has been stressed in the past by several authors.[5]

In strongly coupled field theories, there are well-known examples where the vacuum exhibits a nontrivial structure. The simplest example is the ϕ^4 field theory.[6] In our initial investigation of $(\phi^4)_2$ theory in the DLFQ scheme[7] we followed the approach of Ref. 4 and found that in the strong-coupling region the vacuum instability manifested itself through the appearance of negative eigenvalues of the invariant-mass-squared operator. We believe that within the DLFQ method this is due to the lack of a dynamical description for the ground state (vacuum) itself. In the present work we adapt the familiar GEP formalism to the ground state of $(\phi^4)_2$ theory in the light-front formulation. One of our major results is that, in addition to the light-front Hamiltonian operator, the light-front momentum operator also is affected by normal ordering. This is in sharp contrast to the equal-time formulation where the momentum operator is *not* affected by normal ordering, and variational calculations may be performed by simply minimizing the expectation value of the Hamiltonian operator. We show that a suitable procedure is to minimize the momentum plus Hamiltonian operator in a manner that yields a coherent-state representation for the dynamical vacuum in the strong-coupling domain.

We should note that this is not the only attempt to incorporate the nontrivial vacuum structure in the light-front formulation. By modifying the infrared-singular point of the light-front theory Glazek[8] has been able to incorporate ideas from QCD sum rules in the light-front formalism.

This paper is organized as follows. In Sec. II we describe the GEP formalism applied to the light-front formulation and show equivalence to the conventional equal-time result. In Sec. III we discuss the calculation of excited states in the DLFQ formalism. Section IV contains the summary and conclusions.

II. VARIATIONAL ESTIMATE OF THE GROUND STATE

Let us start from the light-front Hamiltonian operator for a general scalar field theory in 1 + 1 dimensions for illustrative purposes:

$$P^-=\lim_{L\to\infty}\int_{-L}^{+L}dx^-N_a\left[\tfrac{1}{2}m^2\phi_a^2+\frac{\lambda}{4}\phi_a^4+\cdots\right], \tag{1}$$

where the unspecified terms allow for higher-order nonderivative self-couplings. Here we have adopted the normal ordering with respect to the Fock-space vacuum $|\phi_a\rangle$ of quanta having mass m. The normal ordering with respect to a in the Hamiltonian operator can be interpreted as normal ordering with respect to the mass m. The field operator ϕ_a has the following expansion:

$$\phi_a(x^-)=\int_0^\infty \frac{dk^+}{2\pi 2k^+}[a(k^+)e^{-(i/2)k^+x^-}+a^\dagger(k^+)e^{(i/2)k^+x^-}] . \quad (2)$$

We have

$$a(k^+)\,|\,\phi_a\rangle=0 . \quad (3)$$

Recall that in the equal-time formulation the corresponding expansion is given by

$$\phi(x)=\int_{-\infty}^{+\infty}\frac{dk}{2\pi 2\omega_k}[a(k)e^{-ikx}+a^\dagger(k)e^{+ikx}] , \quad (4)$$

where

$$\omega_k=\sqrt{k^2+m^2} . \quad (5)$$

By comparing relations (2) and (4) we see that the light-front field carries no explicit dependence on the mass m as opposed to the equal-time formulation. One must, however, note that the light-front expansion is divergent at $k^+=0$. One possible way[9] to manage this divergence is to introduce the regularization

$$\phi_a(x^-)=\lim_{\Lambda\to\infty}\int_{m^2/\Lambda}^{\Lambda+m^2/\Lambda}\frac{dk^+}{2\pi 2k^+}[a(k^+)e^{-(i/2)k^+x^-}+a^\dagger(k^+)e^{(i/2)k^+x^-}] . \quad (6)$$

The operators a and $a^\dagger$ obey the commutation relations

$$[a(k^+),a^\dagger(k'^+)]=2\pi 2k^+\delta(k^+-k'^+) ,$$
$$[a(k^+),a(k'^+)]=0,\quad [a^\dagger(k^+),a^\dagger(k'^+)]=0 . \quad (7)$$

Let us introduce a linear transformation which preserves commutation relations:

$$b(k^+)=a(k^+)-f(k^+) , \quad (8)$$

where $f(k^+)$ is a c-number field. Let us also introduce the vacuum with respect to the b operator by

$$b(k^+)\,|\,\phi_b\rangle=0 . \quad (9)$$

Let us also introduce a mass parameter μ. The c-number field f and the mass μ will be treated as variational parameters in the following. Once f and μ have been determined variationally we will show how physical masses can be obtained through, for example, the DLFQ method.

To clarify the underlying physics one can construct the operator

$$U=\exp\left[\int\frac{dk^+}{2\pi 2k^+}f(k^+)[a^\dagger(k^+)-a(k^+)]\right] , \quad (10)$$

$$Ua(k^+)U^\dagger=a(k^+)-f(k^+) . \quad (11)$$

Defining

$$b(k^+)=Ua(k^+)U^\dagger , \quad (12)$$

since $a(k^+)\,|\,\phi_a\rangle=0$, we have $b(k^+)U\,|\,\phi_a\rangle=0$. Thus, the b-vacuum $|\,\phi_b\rangle$ is given by

$$|\,\phi_b\rangle=\exp\left[\int\frac{dk^+}{2\pi 2k^+}f(k^+)[a^\dagger(k^+)-a(k^+)]\right]|\,\phi_a\rangle . \quad (13)$$

That is, the new vacuum is a coherent state when described in terms of the original a quanta.

From Eq. (8) we have

$$\phi_a(x^-)=\phi_b(x^-)+\phi_c(x^-) , \quad (14)$$

where

$$\phi_c(x^-)=2\tilde{f}(x^-) . \quad (15)$$

Let us define

$$\frac{f(k^+)}{2k^+}=2\bar{f}(k^+) . \quad (16)$$

Now

$$\bar{f}(k^+)=\tfrac{1}{2}\int dx^-\tilde{f}(x^-)e^{(i/2)k^+x^-} . \quad (17)$$

If $\tilde{f}(x^-)$ is a constant independent of x^-, i.e., $\tilde{f}(x^-)=f_0$, then $\bar{f}(k^+)=f_0 2\pi\delta(k^+)$ and

$$|\,\phi_b\rangle=\exp\{f_0[a^\dagger(k^+=0)-a(k^+=0)]\}\,|\,\phi_a\rangle . \quad (18)$$

Thus the light-cone vacuum is a coherent state of zero-momentum bosons. Note that this condensate has arisen from very general considerations of a scalar field and does not depend on the Lagrangian in any way. The key issue is simply whether $|\,\phi_a\rangle$, the usual light-cone vacuum, or $|\,\phi_b\rangle$, the boson condensate, is lower in energy for a given field theory and given masses and coupling strengths [i.e., fixed m, λ in Eq. (1)].

Let us now consider the light-front Hamiltonian for the ϕ^4 problem as an illustrative case:

$$P^-=\lim_{L\to\infty}\int_{-L}^{L}dx^-N_m\left[\tfrac{1}{2}m^2\phi_a^2+\frac{\lambda}{4}\phi_a^4\right] . \quad (19)$$

Now

$$\phi_a(x^-)=\phi_b(x^-)+\phi_c , \quad (20)$$

where $\phi_c=2f_0$. Thus,

$$P^-=\lim_{L\to\infty}\int_{-L}^{L}dx^-N_m\Bigg[\tfrac{1}{2}m^2\phi_b^2+\frac{\lambda}{4}\phi_b^4+\tfrac{1}{2}m^2\phi_c^2+\frac{\lambda}{4}\phi_c^4+m^2\phi_b\phi_c+\frac{6\lambda}{4}\phi_b^2\phi_c^2+\lambda\phi_b\phi_c^3+\lambda\phi_b^3\phi_c\Bigg] . \quad (21)$$

Now, by generalized Wick ordering[10] we have the following results:

$$N_m\phi^2 = N_\mu\phi^2 + \frac{1}{4\pi}\ln\frac{m^2}{\mu^2} ,$$

$$N_m\phi^3 = N_\mu\phi^3 + 3\left[\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right]N_\mu\phi , \tag{22}$$

$$N_m\phi^4 = N_\mu\phi^4 + 6\left[\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right]N_\mu\phi^2 + 3\left[\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right]^2 .$$

Sı̈bstituting back we arrive at

$$P^- = \lim_{L\to\infty}\int_{-L}^{L} dx^- N_\mu\left[\left[\tfrac{1}{2}m^2 + \frac{6\lambda}{4}\frac{1}{4\pi}\ln\frac{m^2}{\mu^2} + \frac{6\lambda}{4}\phi_c^2\right]\phi_b^2 + \lambda\phi_c\phi_b^3 + \frac{\lambda}{4}\phi_b^4 + \left[m^2\phi_c + \lambda\phi_c^3 + \lambda\phi_c 3\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right]\phi_b \right.$$
$$\left. + \tfrac{1}{2}m^2\frac{1}{4\pi}\ln\frac{m^2}{\mu^2} + 3\frac{\lambda}{4}\left[\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right]^2 + 6\frac{\lambda}{4}\phi_c^2\frac{1}{4\pi}\ln\frac{m^2}{\mu^2} + \tfrac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4}\phi_c^4\right] . \tag{23}$$

Thus the light-front Hamiltonian density is given by

$$\frac{\langle\phi_b\,|\,P^-\,|\,\phi_b\rangle}{2L} = \tfrac{1}{2}m^2\frac{1}{4\pi}\ln\frac{m^2}{\mu^2} + 3\frac{\lambda}{4}\left[\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right]^2 + \tfrac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4}\phi_c^4 + \frac{3}{2}\lambda\phi_c^2\frac{1}{4\pi}\ln\frac{m^2}{\mu^2} . \tag{24}$$

In the conventional formulation, the variational principle is applied to the vacuum expectation value of the Hamiltonian operator. In an analogous way we first apply the variational principle to the light-front Hamiltonian density. Minimizing this quantity with respect to μ^2 we arrive at

$$m^2 + 3\lambda\phi_c^2 + 3\lambda\frac{1}{4\pi}\ln\frac{m^2}{\mu^2} = 0 . \tag{25}$$

We are led to an inconsistency since Eq. (25) does not satisfy the requirement that $\mu = m$ when $\lambda = 0$. To uncover the source of this inconsistency we take a closer look at the difference between the momentum operator in the equal-time and the light-front formulation. In the equal-time formulation, the spectrum of the momentum operator runs from $-\infty$ to $+\infty$. Normal ordering does not induce new terms. In other words the momentum operator in the equal-time formulation carries no dependence on the mass of the quanta. However, the momentum operator in the light-front formulation has a quite different nature. We have

$$P^+ = \tfrac{1}{2}\lim_{L\to\infty}\int_{-L}^{+L} dx^- N_m(\partial^+\phi_a\partial^+\phi_a) , \tag{26}$$

where

$$\partial^+ = 2\frac{\partial}{\partial x^-} . \tag{27}$$

Again introducing the regularization we can write

$$N_m(\partial^+\phi_a\partial^+\phi_a) = N_\mu(\partial^+\phi_a\partial^+\phi_a) + \frac{1}{8\pi}(\mu^2 - m^2) . \tag{28}$$

Thus the light-front momentum density is given by

$$\frac{\langle\phi_b\,|\,P^+\,|\,\phi_b\rangle}{2L} = \frac{1}{8\pi}(\mu^2 - m^2) . \tag{29}$$

The dynamical dependence of the light-front momentum operator also implies that the application of the variational principle must be examined. Since both the vacuum expectation values of the Hamiltonian and of the momentum operator are bounded from below in the light-front formulation, we may in principle apply the variational principle to a linear combination. It can easily be shown that the only consistent result that $\mu = m$ when $\lambda = 0$ emerges when these two expectation values are combined with equal weight. Thus we consider the sum of the light-front Hamiltonian density and the momentum density given by

$$F = \frac{1}{8\pi}(\mu^2 - m^2) + \tfrac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4}\phi_c^4 + \tfrac{1}{2}(m^2 + 3\lambda\phi_c^2)\frac{1}{4\pi}\ln\frac{m^2}{\mu^2} + \frac{3\lambda}{4}\left[\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right]^2 . \tag{30}$$

This expression coincides with the expression for the Hamiltonian density obtained by Chang[6] in the equal-time formulation. To determine the appropriate mass μ, we evaluate

$$\frac{\partial F}{\partial\mu^2} = 0 . \tag{31}$$

Thus, we end up with

$$m^2 + 3\lambda\phi_c^2 + 3\lambda\frac{1}{4\pi}\ln\frac{m^2}{\mu^2} = \mu^2 . \tag{32}$$

For given values of λ, ϕ_c, and m we solve this equation to determine μ and calculate F. For convenience we set $m^2 = 1.0$. Plots of F vs ϕ_c for different values of λ are shown in Fig. 1. The minimum of F moves from $\phi_c = 0$ to $\phi_c \neq 0$ as λ increases and eventually passes through its critical value.

Minimizing F with respect to ϕ_c we arrive at

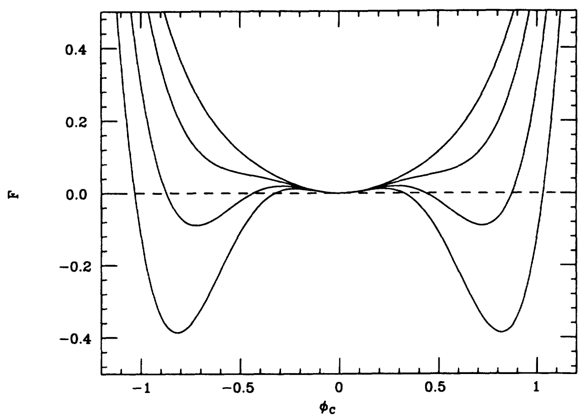

FIG. 1. The function F (for definition see the text) plotted vs ϕ_c for values of λ ranging from 4.0 to 16.0 in steps of 4.0. The smaller the value of λ the more the curve resembles a simple parabola.

$$\phi_c\left[m^2+\lambda\phi_c^2+3\lambda\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right]=0\ . \tag{33}$$

Combining the two minimization conditions we arrive at the known duality condition:

$$\frac{m^2}{\lambda}+\frac{\mu^2}{\lambda}=3\frac{1}{4\pi}\ln\frac{2\mu^2}{m^2}\ . \tag{34}$$

III. CALCULATION OF EXCITED STATES

Once we have achieved a dynamical description for the vacuum in the strong-coupling region we can build the spectrum of excited states. We first add c numbers to the operators P^+ and P^- so that

$$\langle\phi_b\,|\,P^+\,|\,\phi_b\rangle=0\ ,$$

$$\langle\phi_b\,|\,P^-\,|\,\phi_b\rangle=0\ .$$

Then introducing discretization following the procedure of Ref. 4, one can diagonalize the mass operator and readily find the eigenvalues and corresponding eigenvectors.

A word of caution is however in order. The Gaussian-effective-potential formalism seems to work fairly well away from the critical region. Near the critical region there are problems.[6] For the description of the critical region more sophisticated techniques seem necessary.[11]

IV. SUMMARY AND CONCLUSIONS

By introducing a coherent state of light-cone zero-momentum bosons, we have succeeded in building a dynamical model for the vacuum in the light-front quantization scheme. A key ingredient behind this success is the identification of the special role played by the light-front momentum operator. For the ϕ^4 example in $1+1$ dimensions, we have obtained a result identical to the GEP in the equal-time formulation by minimizing the vacuum expectation value of the sum of the Hamiltonian and momentum operators in the light-front formulation. We have also indicated how one can calculate the excited states built on the new vacuum.

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