Feynman path integral for fermions

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With Lagrangian quantum field theory as a starting point, the general quantum transition amplitude is derived as a sum of action exponentials over all paths connecting the initial and final fermion states. For bookkeeping convenience in intermediate stages anticommuting parameters are introduced. All integrals are determined by completeness and unitarity. Grassmann calculus is not postulated.

I. INTRODUCTION

In 1948 Feynman[1] developed an idea of Dirac[2] to express the quantum-mechanical transition amplitude as an integral over all paths connecting the initial and final states, with each path weighted by a phase that is the classical action in units of Planck's constant:

$$\langle b \mid a \rangle \sim \sum_{\substack{\text{paths} \\ a \to b}} e^{iS_{ba}/\hbar} . \tag{1}$$

The beauty of this approach lies first in providing a simple quantum explanation of why classical systems follow paths along which the action is stationary and second in giving a rigorous method of calculating quantum results with only the familiar classical action as dynamical input. Application of the Feynman-path-integral (FPI) approach to ordinary quantum mechanics[3] is elegant and illuminating; its application to quantum field theory[4] in recent years has been very fruitful.

Problems in ordinary quantum mechanics usually have a classical analogue so there is little difficulty in choosing the correct classical action for the problem at hand. In quantum field theory the classical analogue is not so clear-cut but most agree that a satisfactory choice for the classical action is in the form of the quantum action with the quantum fields replaced by classical fields. This choice is immediately successful for boson fields[5] and can, in fact, be derived from quantum field theory for scalar fields[6] and photon fields.[7]

But the results of quantum field theory are not reproduced for fermion fields unless the classical fields are assumed to anticommute.[5] In modern field theory they are chosen to be elements of a Grassmann algebra.[4,8] Because of the *ad hoc* nature of this algebra and the identity of differentiation and integration therein, the connection of the field integrations to sums over paths is lost. Also lost is much of the intuitive beauty of the FPI approach.

If one uses fermion quantum field theory to calculate an arbitrary transition amplitude and then expresses it as an FPI what does the classical action look like and what does it tell us about the classical fermion field? This is the approach taken here. We begin by finding completeness relations for a set of fermion states and using these relations to derive general expressions for transition amplitudes in terms of sums over all possible occupancy amplitudes of each mode at each intermediate time. As a bookkeeping aid only, anticommuting parameters are introduced but the integrals of these parameters are derived from completeness rather than from Grassmann postulates.

II. FERMION MODES AND COMPLETENESS

Fermion and antifermion modes are described by the same four quantum numbers: three for momentum and one for spin. (We only consider spin one-half.) These modes may be put into a one-to-one correspondence with the positive integers α. For concision we define the operator a_α to destroy a fermion in the mode α when $\alpha > 0$ and to destroy an antifermion in the mode $-\alpha$ when $\alpha < 0$. Then we have

$$\{a_\alpha, a_\beta^\dagger\} = \delta_{\alpha\beta} , \quad \{a_\alpha, a_\beta\} = 0 \tag{2}$$

with both α and β running over the integers from $-\infty$ to $+\infty$ with zero excluded. The most general assembly of states that can be constructed with the creation operators $a_\alpha^\dagger$ is

$$| y \rangle \equiv \prod_\alpha (y_\alpha a_\alpha^\dagger + z_\alpha) | 0 \rangle . \tag{3}$$

The order of the factors in (3) will be discussed later. The corresponding bra vector is defined as

$$\langle y | \equiv \langle 0 | \prod_\alpha (y_\alpha^* a_\alpha + z_\alpha^*) . \tag{4}$$

When y_α is a complex variable, y_α^* is its complex conjugate; when y_α is a Grassmann variable, y_α^* is an independent Grassmann variable.

The scalar product of two such states is

$$\langle \bar{y} | y \rangle = \prod_\alpha (\bar{y}_\alpha^* y_\alpha + \bar{z}_\alpha^* z_\alpha) . \tag{5}$$

For unitarity we require that each mode be separately normalized,

$$y_\alpha^* y_\alpha + z_\alpha^* z_\alpha = 1 , \tag{6}$$

i.e., the total probability of any mode being occupied or unoccupied is 1. We look for a weight that will give us an overcompleteness relation

$$\int \mathcal{D}y \, |y\rangle\langle y| = 1 \, . \tag{7}$$

This requires

$$\int \mathcal{D}y_\alpha y_\alpha y_\alpha^* = \int \mathcal{D}y_\alpha z_\alpha z_\alpha^* = 1 \tag{8}$$

and

$$\int \mathcal{D}y_\alpha y_\alpha z_\alpha^* = \int \mathcal{D}y_\alpha z_\alpha y_\alpha^* = 0 \tag{9}$$

with

$$\mathcal{D}y \equiv \prod_\alpha \mathcal{D}y_\alpha \, . \tag{10}$$

Such a weight is not unique but as all integrals that can occur are those coming from (7), all weights are equivalent. When y is a complex number the simplest weight is

$$\mathcal{D}y_\alpha = \frac{2}{\pi} d^2 y_\alpha \tag{11}$$

with the range of integration being the interior of the unit circle in the complex y plane. It is, however, superfluous to specify the weight and (8) and (9) can be taken as defining all integrals. This is especially relevant when y is not a complex number.

III. THE BOOKKEEPING PROBLEM

Now we come to a crucial point. We can proceed in the well-known fashion[3,6,9] to develop transition amplitudes as sums over intermediate paths by using the completeness relation (7) at all intermediate times. Eventually we shall obtain the usual Feynman rules as a result. Because of the anticommutation rules (2), at intermediate stages of the calculation we find a large number of minus signs that must be kept track of—a bookkeeping problem, really. If our object were to obtain the Feynman rules we would either tolerate the bookkeeping or turn to more conventional and simpler derivations. But our object here is to find an elegant *intermediate stage*—the Feynman path integral—and it is precisely here that the minus signs spoil the elegance. Therefore as an artificial device to help us keep track of signs and maintain the FPI elegance we require that the coefficients of the creation operators in (3) be *anticommuting parameters:*

$$\{y_\alpha, y_\beta\} = \{y_\alpha, y_\beta^*\} = 0 \, . \tag{12}$$

These parameters occur only in the integrals (8) and (9) and are therein converted into ordinary numbers. With the y's anticommuting, the state $|y\rangle$ in (3) is independent of the order of the factors in its definition. The components of the vectors $|y\rangle$ and $\langle y|$ are Grassmann parameters. As the overall "phase" of a state is meaningless we are free to assume

$$z_\alpha^* = z_\alpha = (1 - y_\alpha^* y_\alpha)^{1/2} = 1 - \tfrac{1}{2} y_\alpha^* y_\alpha \, . \tag{13}$$

An immediate consequence of (13) and (12) is then

$$y_\alpha z_\alpha^* = y_\alpha, \quad y_\alpha^* z_\alpha = y_\alpha^* \, , \tag{14}$$

so that the fundamental integrals (8) and (9) reduce to

$$\int \mathcal{D}y_\alpha = 0 \, , \tag{15}$$

$$\int \mathcal{D}y_\alpha y_\alpha = \int \mathcal{D}y_\alpha y_\alpha^* = 0 \, , \tag{16}$$

$$\int \mathcal{D}y_\alpha y_\alpha y_\alpha^* = 1 \, . \tag{17}$$

Note that (16) and (17) are identical to the complex variable results. Equation (15) is different; it follows from $z^* z = 1 + y y^*$. [Compare Eq. (6).] These integrals are reminiscent of the Grassmann integrals

$$\int dy_\alpha^* y_\alpha^* = \int dy_\alpha y_\alpha = 1, \quad \int dy_\alpha = \int dy_\alpha^* = 0 \tag{18}$$

and, indeed, would follow from them if we set $\mathcal{D}y_\alpha = dy_\alpha^* dy_\alpha$. The main difference is that (15)–(17) are derived from completeness while the Grassmann integrals (18) are simply postulated with normalizations fixed separately. Also the Grassmann integrals involve more detail than is needed. Clearly all the results of Grassmann calculus, e.g., inverse Jacobians, invariance under shift of origin, etc., follow equally well from (15)–(17).

IV. WAVE FUNCTIONS

The wave function of some arbitrary state, for example, the state

$$|a\rangle \equiv \prod_\alpha (a_\alpha^\dagger)^{k_\alpha} |0\rangle, \quad k_\alpha = 0 \text{ or } 1 \, , \tag{19}$$

is defined as

$$\Psi_a(y) \equiv \langle y | a \rangle \, . \tag{20}$$

For the example state this is

$$\Psi_a(y) = \prod_\alpha (y_\alpha^*)^{k_\alpha} (z_\alpha^*)^{1-k_\alpha} \, . \tag{21}$$

The order of the anticommuting parameters in (21) reflects the order of the creation operators in (19). These wave functions are normalized via

$$\int \mathcal{D}y \, \Psi_a^*(y) \Psi_a(y) = 1 \, , \tag{22}$$

provided that $|a\rangle$ is normalized. Finally note that the wave function of the vacuum is

$$\Psi_0(y) = \prod_\alpha z_\alpha^* \, . \tag{23}$$

V. TRANSITION AMPLITUDES

We can look upon $|y\rangle$ to be a $t=0$ interaction-picture state. The Heisenberg-picture state at time t is then

$$|yt\rangle \equiv e^{i\hat{H}t} |y\rangle \, . \tag{24}$$

Because this is a unitary relationship the Heisenberg states are also overcomplete at each time,

$$\int \mathcal{D}y \, |yt\rangle\langle yt| = 1 \, . \tag{25}$$

The transition amplitude from a state $|at_a\rangle$ to a state $|bt_b\rangle$ can then be written as

$$\langle bt_b \mid at_a \rangle = \int \mathcal{D}y^a \mathcal{D}y^b \langle bt_b \mid y^b t_b \rangle \langle y^b t_b \mid y^a t_a \rangle \times \langle y^a t_a \mid at_a \rangle = \int \mathcal{D}y^a \mathcal{D}y^b \Psi_b^*(y^b) \langle y^b t_b \mid y^a t_a \rangle \Psi_a(y^a) . \tag{26}$$

Thus we can concentrate on the transition kernel, $\langle y^b t_b \mid y^a t_a \rangle$.

We can express the kernel as a sum over all intermediate paths as follows. Break up the time interval, $T = t_b - t_a$, into $N+1$ equal parts of lengths ϵ and label the times by $t_j \equiv t_a + j\epsilon$, $j = 0, 1, \ldots, N+1$. Insert a complete set (25) at each intermediate time,

$$\langle y^b t_b \mid y^a t_a \rangle = \int \prod_1^N \mathcal{D}y^j \prod_0^N \langle y^{j+1} t_{j+1} \mid y^j t_j \rangle , \tag{27}$$

where $y^0 \equiv y^a$, $y^{N+1} \equiv y^b$. With $t_{j+1} = t_j + \epsilon$ and (24) we have

$$\langle y^b t_b \mid y^a t_a \rangle = \int \mathcal{D}y \prod_0^N \langle y^{j+1} \mid e^{-i\epsilon \hat{H}_{\text{ip}}(0)} \mid y^j \rangle \tag{28}$$

with now

$$\mathcal{D}y \equiv \prod_1^N \mathcal{D}y^j \equiv \prod_1^N \prod_\alpha \mathcal{D}y_\alpha^j \tag{29}$$

and $\hat{H}_{\text{ip}}(t)$ is the interaction-picture Hamiltonian with $\hat{H}_{\text{ip}}(0) = \hat{H}$. We note that occupancy amplitudes at different times are taken to anticommute:

$$\{y_\alpha^k, y_\beta^j\} = \{y_\alpha^k, y_\beta^{j*}\} = 0 . \tag{30}$$

A very useful lemma is the following exact relation:

$$\lim_{\epsilon \to 0} \prod_0^N (1 + \epsilon\alpha_j + \epsilon^2 \beta_j + \cdots) = \lim_{\epsilon \to 0} \prod_0^N e^{\epsilon \alpha_j} . \tag{31}$$

This is easily established by comparing terms on either side and using the fact that all terms like $\sum \epsilon^2 \alpha_j^2$ and $\sum \epsilon^2 \beta_j$, etc., vanish in the limit $\epsilon \to 0$ with $N\epsilon$ finite.

Using the lemma we can write the transition kernel in a useful alternative form:

$$\langle y^b t_b \mid y^a t_a \rangle = \int \mathcal{D}y \prod_0^N \langle y^{j+1} \mid (1 - i\epsilon \hat{H}_{\text{ip}}(0)) \mid y^j \rangle . \tag{32}$$

(The limit $\epsilon \to 0$, $N\epsilon$ finite, is understood in all equations.)

VI. FREE-PARTICLE KERNEL

The free Hamiltonian in fermion quantum field theory has the Hermitian form

$$\hat{H}_0 = \int d^3x : \hat{\bar{\psi}}(\mathbf{x}) (\tfrac{1}{2} \boldsymbol{\gamma} \cdot \overleftrightarrow{\nabla} + m) \hat{\psi}(\mathbf{x}) : , \tag{33}$$

where

$$\hat{\psi}(\mathbf{x}) = \sum_\alpha \psi_\alpha(\mathbf{x}) [\theta(\alpha) a_\alpha + \theta(-\alpha) a_\alpha^\dagger] \tag{34}$$

is the fermion quantum field in the interaction picture. The Dirac wave functions satisfy

$$\int d^3x \, \bar{\psi}_\alpha(\mathbf{x}) (\tfrac{1}{2} \boldsymbol{\gamma} \cdot \overleftrightarrow{\nabla} + m) \psi_\beta(\mathbf{x}) = \omega_\alpha \delta_{\alpha\beta} \epsilon(\alpha) . \tag{35}$$

Note that ψ_α is a positive- or negative-energy wave function as α is positive or negative, respectively. Here $\omega_\alpha > 0$ is the energy of the corresponding mode ($\omega_{-\alpha} \equiv \omega_\alpha$). One easily gets

$$\hat{H}_0 = \sum_\alpha \omega_\alpha a_\alpha^\dagger a_\alpha . \tag{36}$$

From the anticommutation rules (2), one has

$$e^{-i\hat{H}_0 T} = \prod_\alpha [1 + a_\alpha^\dagger a_\alpha (e^{-i\omega_\alpha T} - 1)] \tag{37}$$

and immediately the kernel is seen to be

$$\langle y^b t_b \mid y^a t_a \rangle^0 = \langle y^b \mid e^{-i\hat{H}_0 T} \mid y^a \rangle = \prod_\alpha (y_\alpha^{b*} y_\alpha^a e^{-i\omega_\alpha T} + z_\alpha^{b*} z_\alpha^a) . \tag{38}$$

It is very useful to express the kernel as a path integral. To this end we use (32) to write

$$\begin{aligned} \langle y^b t_b \mid y^a t_a \rangle^0 &= \int \mathcal{D}y \prod_0^N \langle y^{j+1} \mid (1 - i\epsilon \hat{H}_0) \mid y^j \rangle \\ &= \int \mathcal{D}y \prod_0^N \left[\prod_\alpha (y_\alpha^{j+1*} y_\alpha^j + z_\alpha^{j+1*} z_\alpha^j) - i\epsilon \sum_\beta \omega_\beta y_\beta^{j+1*} y_\beta^j \prod_{\alpha \neq \beta} (y_\alpha^{j+1*} y_\alpha^j + z_\alpha^{j+1*} z_\alpha^j) \right] \\ &= \int \mathcal{D}y \prod_0^N \prod_\alpha (\gamma_\alpha y_\alpha^{j+1*} y_\alpha^j + z_\alpha^{j+1*} z_\alpha^j) \end{aligned} \tag{39}$$

with

$$\gamma_\alpha \equiv 1 - i\epsilon\omega_\alpha \ . \tag{40}$$

The integrals in (39) can be evaluated using (8) and (9). With

$$\lim_{N\to\infty} (\gamma_\alpha)^{N+1} = e^{-i\omega_\alpha T} \tag{41}$$

the result (38) is reproduced.

VII. THE FREE ACTION

The free action is defined by

$$\langle y^b t_b \mid y^a t_a \rangle^0 = \int \mathcal{D}y \, e^{iS^0_{ba}} \ , \tag{42}$$

which means that

$$e^{iS^0_{ba}} = \prod_{k=0}^{N} \prod_\alpha (\gamma_\alpha y_\alpha^{k+1*} y_\alpha^k + z_\alpha^{k+1*} z_\alpha^k) \ , \tag{43}$$

so that

$$\begin{aligned} S^0_{ba} &= -i \sum_{k=0}^{N} \sum_\alpha \ln(\gamma_\alpha y_\alpha^{k+1*} y_\alpha^k + z_\alpha^{k+1*} z_\alpha^k) \\ &= -i \sum_{k=0}^{N} \sum_\alpha \ln[1 - i\epsilon\omega_\alpha y_\alpha^{k+1*} y_\alpha^k + (y_\alpha^{k+1*} - y_\alpha^{k*}) y_\alpha^k \\ &\qquad + (z_\alpha^{k+1*} - z_\alpha^{k*}) z_\alpha^k] \ . \end{aligned} \tag{44}$$

Here we used (6). We can expand (44) in the usual logarithm series; all terms higher than the second order vanish because $y^2 = 0$. The result is

$$\begin{aligned} S^0_{ba} &= \frac{i}{2} \sum_\alpha (y_\alpha^{b*} y_\alpha^b - y_\alpha^{a*} y_\alpha^a) \\ &\quad - i \sum_{k\alpha} (\gamma_\alpha y_\alpha^{k+1*} y_\alpha^k - y_\alpha^{k*} y_\alpha^k) \ . \end{aligned} \tag{45}$$

Note that this exact result has been obtained without the usual assumption that $y^{k+1} - y^k = O(\epsilon)$. This is just as well since that assumption is rather meaningless (i.e., formal) for anticommuting parameters.

VIII. FREE ACTION IN DIRAC FORM

Although (42) and (45) give the Feynman path integral for fermions it is still desirable to write the action in a familiar classical form—in terms of classical fermion fields. In order to find the form of the classical field we calculate matrix elements of the possible quadratic operators

$$\langle y \mid a_\alpha^\dagger a_\beta \mid y \rangle = y_\alpha^* y_\beta \ , \tag{46}$$

$$\langle y \mid a_\alpha a_\beta \mid y \rangle = -y_\alpha y_\beta \ , \tag{47}$$

$$\langle y \mid a_\alpha^\dagger a_\beta^\dagger \mid y \rangle = -y_\alpha^* y_\beta^* \ . \tag{48}$$

Here use was made of (14).

For operators that do not change the charge of the state, (47) and (48) can only occur when one operator refers to a particle, the other an antiparticle, while in (46) both operators refer to a particle or to an antiparticle. In conventional notation with $b_\alpha, b_\alpha^\dagger$ referring to antiparticles we have

$$\langle y \mid f(a_\alpha, a_\beta^\dagger, b_\gamma, b_\delta^\dagger) \mid y \rangle = f(y_\alpha, y_\beta^*, -y_\gamma, -y_\delta^*) \ , \tag{49}$$

where f involves an even number of operators. Thus we expect the proper classical Dirac field will be

$$\psi(\mathbf{x}, t_k) = \sum_\alpha \psi_\alpha(\mathbf{x})[\theta(\alpha) y_\alpha - \theta(-\alpha) y_\alpha^*] \ . \tag{50}$$

Using (35) one gets

$$\int d^3x \, \bar{\psi}(\mathbf{x}, t_k)(\tfrac{1}{2}\boldsymbol{\gamma}\cdot\overleftrightarrow{\nabla} + m)\psi(\mathbf{x}, t_k) = \sum_\alpha \omega_\alpha y_\alpha^{k*} y_\alpha^k \tag{51}$$

so that

$$\int d^4x \, \bar{\psi}(x)(\tfrac{1}{2}\boldsymbol{\gamma}\cdot\overleftrightarrow{\nabla} + m)\psi(x) = \sum_{k=0}^{N} \sum_\alpha \epsilon\omega_\alpha y_\alpha^{k*} y_\alpha^k \ . \tag{52}$$

Next with

$$\int d^3x \, \psi_\alpha^\dagger(\mathbf{x})\psi_\beta(\mathbf{x}) = \delta_{\alpha\beta} \tag{53}$$

one finds

$$\begin{aligned} &\epsilon \int d^3x \, \bar{\psi}(\mathbf{x}, t_k) \tfrac{1}{2}\gamma^0 \overleftrightarrow{\partial_0} \psi(\mathbf{x}, t_k) \\ &\qquad = \frac{i}{2} \sum_\alpha (y_\alpha^{k*} y_\alpha^{k+1} - y_\alpha^{k+1*} y_\alpha^k) \end{aligned} \tag{54}$$

so that

$$\begin{aligned} \int d^4x \, \bar{\psi}(x)(\tfrac{1}{2}\gamma^0 \overleftrightarrow{\partial_0})\psi(x) &= \frac{i}{2} \sum_{k=0}^{N} \sum_\alpha (y_\alpha^{k*} y_\alpha^{k+1} \\ &\qquad - y_\alpha^{k+1*} y_\alpha^k) \ . \end{aligned} \tag{55}$$

Here we have defined the time derivative of y_α^k by

$$\epsilon \dot{y}_\alpha^k = y_\alpha^{k+1} - y_\alpha^k \ . \tag{56}$$

To identify S^0_{ba} in (45) with the familiar classical expressions (52) and (55), one must make some *formal* manipulations. First one writes

$$\begin{aligned} (y_\alpha^{k+1*} - y_\alpha^{k*}) y_\alpha^k &= \frac{\epsilon}{2} \frac{d}{dt} (y_\alpha^{k*} y_\alpha^k) \\ &\quad + \tfrac{1}{2}(y_\alpha^{k+1*} y_\alpha^k - y_\alpha^{k*} y_\alpha^{k+1}) \ . \end{aligned} \tag{57}$$

When the sum over k (=integration over t) is performed, the time derivative here yields surface terms that exactly cancel those in (45). Thus we have, to lowest order

$$\begin{aligned} S^0_{ba} &= -i \sum_{k=0}^{N} \sum_\alpha [-i\epsilon\omega_\alpha y_\alpha^{k*} y_\alpha^k \\ &\quad + \tfrac{1}{2}(y_\alpha^{k+1*} y_\alpha^k - y_\alpha^{k*} y_\alpha^{k+1})] \ . \end{aligned} \tag{58}$$

Thus with (52) and (55) we have

$$S^0_{ba} = -\int_{t_a}^{t_b} d^4x\ \bar{\psi}(x)(\tfrac{1}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu + m)\psi(x)\ , \tag{59}$$

which gives the free action in Dirac form.

IX. INTERACTING FERMIONS—QUANTUM ELECTRODYNAMICS

Consider quantum electrodynamics as an example of interacting fermions. We assume the photon matrix elements have already been calculated[7] so we are now interested in the matrix elements of the operator

$$\hat{\mathcal{H}}^k_1(\mathbf{x}) \equiv ie :\hat{\bar{\psi}}(\mathbf{x})\gamma\cdot a^k(\mathbf{x})\hat{\psi}(\mathbf{x}): \tag{60}$$

between fermion states. Here $a^k_\mu(\mathbf{x})$ is the classical photon field at time t_k. Using (49) one gets immediately

$$\begin{aligned}\mathcal{H}_1(\mathbf{x},t_k) &\equiv \langle y^k \mid \hat{\mathcal{H}}^k_1(\mathbf{x}) \mid y^k\rangle \\ &= ie\bar{\psi}(\mathbf{x},t_k)\gamma\cdot a^k\psi(\mathbf{x},t_k)\ .\end{aligned} \tag{61}$$

Now we show that

$$\begin{aligned}H_1 &\equiv \int_{t_a}^{t_b} d^4x\ \mathcal{H}_1(\mathbf{x},t) \\ &\equiv ie\int_{t_a}^{t_b} d^4x\ \bar{\psi}(x)\gamma\cdot a(x)\psi(x)\end{aligned} \tag{62}$$

adds to S^0_{ba} in the expected fashion:

$$S_{ba} = S^0_{ba} - H_1\ . \tag{63}$$

We start with the kernel in the form

$$\langle y^b t_b \mid y^a t_a\rangle = \int \mathcal{D}y \prod_0^N \langle y^{k+1} \mid (1 - i\epsilon\hat{H}) \mid y^k\rangle\ , \tag{64}$$

with

$$\hat{H} = \hat{H}_0 + \hat{H}_1\ , \tag{65}$$

and write out the product in detail:

$$\begin{aligned}\langle y^b t_b \mid y^a t_a\rangle = \int \mathcal{D}y \Bigg[&\prod_0^N \langle y^{k+1} \mid (1 - i\epsilon\hat{H}_0) \mid y^k\rangle - i\epsilon \sum_{j=0}^N \langle y^{j+1} \mid \hat{H}_1 \mid y^j\rangle \prod_{k\neq j} \langle y^{k+1} \mid (1 - i\epsilon\hat{H}_0) \mid y^k\rangle \\ &+ (-i\epsilon)^2 \sum_{j<l} \langle y^{l+1} \mid \hat{H}_1 \mid y^l\rangle\langle y^{j+1} \mid \hat{H}_1 \mid y^j\rangle \prod_{k\neq j,l} \langle y^{k+1} \mid (1 - i\epsilon\hat{H}_0) \mid y^k\rangle + \cdots \Bigg]\ .\end{aligned} \tag{66}$$

In a typical term the coefficient of the matrix elements of $\hat{H}_1$ is of the form

$$\prod_{k\neq j,l,\ldots} \langle y^{k+1} \mid (1 - i\epsilon\hat{H}_0) \mid y^k\rangle\ . \tag{67}$$

Note that each of the factors missing from this product is almost unity:

$$\langle y^{j+1} \mid (1 - i\epsilon\hat{H}_0) \mid y^j\rangle = 1 + O(\epsilon)\ . \tag{68}$$

Since the number of such factors does not increase with N, we can insert the missing factors without error in the limit $\epsilon \to 0$. Thus with

$$\prod_0^N \langle y^{k+1} \mid (1 - i\epsilon\hat{H}_0) \mid y^k\rangle \equiv e^{iS^0_{ba}} \tag{69}$$

we have

$$\begin{aligned}\langle y^b t_b \mid y^a t_a\rangle &= \int \mathcal{D}y\ e^{iS^0_{ba}} \left[1 - i\epsilon \sum_{j=0}^N \langle y^{j+1} \mid \hat{H}_1 \mid y^j\rangle + (-i\epsilon)^2 \sum_{j<l} \langle y^{j+1} \mid \hat{H}_1 \mid y^j\rangle\langle y^{l+1} \mid \hat{H}_1 \mid y^l\rangle + \cdots\right] \\ &= \int \mathcal{D}y\ e^{iS^0_{ba}} \exp\left[-i\epsilon \sum_{j=0}^N \langle y^{j+1} \mid \hat{H}_1 \mid y^j\rangle\right] \qquad (70) \\ &= \int \mathcal{D}y\ e^{i(S^0_{ba} - H_1)} \qquad (71) \\ &= \int \mathcal{D}y\ e^{iS_{ba}}\ . \qquad (72)\end{aligned}$$

In going from (70) to (71) we have replaced $\langle y^{j+1}|\hat{H}_1|y^j\rangle$ by $\langle y^j|\hat{H}_1|y^j\rangle+O(\epsilon)$.

This is as far as we need to go for our present purpose: to establish the fermion FPI from Lagrangian field theory. We remark that the anticommuting nature of the occupancy amplitudes was introduced as an extremely useful bookkeeping device. All such anticommuting parameters occur in conjunction with an integration weight Dy, and ordinary numbers result immediately. It was not necessary to make the Grassmann postulates; all results follow from completeness.

In summary, the principal advances contained in this paper are (1) the integrations of anticommuting parameters are derived from completeness, (2) the field integrations are properly defined by expressing the classical fields in terms of the anticommuting mode parameters, (3) the Feynman path integral for fermions is derived from canonical Lagrangian quantum field theory, and (4) the Feynman rules follow from the Feynman path integral. (In the usual treatment the $-i\epsilon$ in the fermion propagator must be put in "by hand"; here it follows naturally. See the Appendix.)

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APPENDIX: THE FEYNMAN PROPAGATOR

Some discussion of the derivation of the Feynman rules is necessary for two reasons. The first is that the usual prescription[4,5] of replacing a time-ordered product of quantum spinor fields by a path integral of an ordinary product of classical spinor fields is not valid as such; it only works for expressions that are bilinear in the fields at a given time. This is not a problem since the interaction is always bilinear and local. The second reason is that the formal treatment of S keeps only terms of lowest possible order in ϵ while proper care for the boundary conditions requires that terms of order ϵ be kept. This is in close analogy to the usual prescription for Green's functions that requires the poles be infinitesimally removed from the real axis during integration.

If we write

$$iS^0_{ba}=-y^\dagger My \tag{A1}$$

and evaluate M from (45),

$$M=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -\gamma & 1 & 0 & 0 & \\ 0 & -\gamma & 1 & 0 & \\ 0 & 0 & -\gamma & 1 & \\ \vdots & & & & \end{bmatrix}, \tag{A2}$$

one immediately gets the Feynman propagator with the correct causal property (see Ref. 6 for details). However if we use the formal expression (59), we would get

$$M=\begin{bmatrix} i\epsilon\omega & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ -\frac{1}{2} & i\epsilon\omega & \frac{1}{2} & 0 & 0 & \\ 0 & -\frac{1}{2} & i\epsilon\omega & \frac{1}{2} & 0 & \\ 0 & 0 & -\frac{1}{2} & i\epsilon\omega & \frac{1}{2} & \\ \vdots & & & & & \end{bmatrix}, \tag{A3}$$

which gives equal weight to the forward and backward directions of time and thus gives an incorrect causal property to the propagator. To correct this oversimplification we must modify the formal expression for S into a causal product

$$S^0_{ba}\equiv-\int_{t_a}^{t_b}d^4x\,[\bar{\psi}^{(+)}(\mathbf{x},t+\epsilon)(\tfrac{1}{2}\gamma^\mu\overleftrightarrow{\partial}_\mu+m)\psi^{(+)}(\mathbf{x},t) +\bar{\psi}^{(-)}(\mathbf{x},t)(\tfrac{1}{2}\gamma^\mu\overleftrightarrow{\partial}_\mu+m)\psi^{(-)}(\mathbf{x},t+\epsilon)]\,, \tag{A4}$$

which is equivalent to (59) at the formal level. Now the correct propagator results.

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