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Casimir force on a solid ball when $\epsilon(\omega)\mu(\omega)=1$

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The Casimir surface force on a solid ball is calculated, assuming the material to be dispersive and to be satisfying the condition $\epsilon(\omega)\mu(\omega)=1$, $\epsilon(\omega)$ being the spectral permittivity and $\mu(\omega)$ the spectral permeability. This particular condition simplifies the Casimir theory of dielectric media considerably. As a dispersion relation we choose the analogue of Sellmeir's form (with one absorption frequency), known from ordinary dispersion theory. We follow a combined numerical and analytic approach: the low values of the angular momentum variable are treated numerically, whereas the higher values are treated analytically by means of the Debye expansion. The dispersive effect is found to yield a strong, *attractive* contribution to the surface force. If the cutoff frequency ω_0 is large, the dispersion-induced surface force becomes proportional to ω_0.

I. INTRODUCTION

The study of the Casimir effect in spherical geometry is interesting from a formal point of view, since the situation is relatively tractable and leads to results that demonstrate in a striking way the effect of the geometrical constraint at $r=a$, a being the radius of the sphere. Moreover, the situation is of direct physical interest, as exemplified in the following two cases.

(1) Casimir's semiclassical model of the electron.[1] Casimir's original idea was that the force arising from vacuum oscillations should be able to stabilize the classical electron. Although the idea appeared to be contradicted by subsequent detailed calculations, by Boyer[2] and others (the force on a nondispersive perfectly conducting shell was calculated to be repulsive instead of attractive), we do not find it unreasonable that the Casimir idea will be revived again in some modified form and prove to be useful.

(2) The bag model in quantum chromodynamics. This case can be physically quite important. There have been many speculations about the possible role played by the zero-point oscillations in QCD. (As regards bag theory in general, the ready may consult reviews by Hasenfratz and Kuti[3] and DeTar and Donoghue.[4] The first bag-model calculation of zero-point energy was given by Bender and Hays.[5] Later, Milton[6] examined different aspects of the zero-point oscillations in the bag. We mention in this context also related papers by Olaussen and Ravndal.[7] Of particular interest as far as QCD is concerned is Johnson's idea[8] of picturing the vacuum as a space filled with bags, the boundaries of which confine color to asymptotically free regions.

The generic situation in all these cases is that of a perfectly conducting singular *shell*, situated in a vacuum. Among the modern mathematical methods, the multiple-scattering formalism of Balian and Duplantier[9] has proved to be a powerful and flexible tool in the handling of problems of this type. What we shall be concerned with in the following is however a different method, viz., the application of *Schwinger's source theory.*[10] This method, which is a point-splitting Green's-function method, is remarkably compact and effective. The source-theory treatment of Milton, De Raad, and Schwinger[11] on the singular-shell problem is well known. The electromagnetic Casimir surface force, as anticipated above, was in this calculation found to be repulsive.

One variant of the physical problem is obtained if we consider, instead of a shell, a *spherical ball* make up of a dielectric medium. Milton[12] has also considered this problem, assuming the medium to be nondispersive. The result of his calculation was that the medium gives rise to a finite, attractive, term in the force. Essentially similar results were later derived by one of the present authors,[13] dealing with electrostriction effects.

One particular subclass of the material-medium problem corresponds to letting the medium satisfy the condition

$$\epsilon\mu=1 , \tag{1.1}$$

ϵ being the permittivity and μ the permeability. We have earlier[14–16] considered various aspects of the Casimir theory (at zero temperature) for *nondispersive*, spherically symmetric, media satisfying (1.1). This condition, artificial as it might seem at first, is in fact appealing for several reasons. For one thing it eliminates the need for taking into account a *contact term* in the calculation of the Casimir force. Milton[12] has devised a procedure for subtracting off a contact term in the dielectric-medium case when the medium is an "ordinary" one, i.e., not satisfying (1.1). Although this method for constructing a contact term compensating the infinities in the formalism may be considered not to be strictly unambiguous, the method is nevertheless in our opinion physically reasonable and mathematically clear-cut. The significant advantage we encounter when dealing with media satisfying (1.1) is that the contact term simply becomes *equal to zero*.[15(a)] Another advantage of (1.1) is that it is of interest when trying to formulate a theory of the phenomenological vacuum in QCD, since the condition ensures the velocity of gluons to be equal to the velocity of light. The condition actually plays a significant role in Lee's model[17] for the vacuum exterior to the bag.

After all this it may be surprising to learn that the Casimir calculations referred to above are in fact *incomplete*. The reason is that they do not take *dispersive effects* into account. The inclusion of dispersion in the present problem is quite important since the dispersion-induced part of the Casimir force becomes strong. This force moreover is directed inwards, thus oppositely to the nondispersive part of the force.

One may ask at this point why it should be so important in the present case to take dispersion into account. Namely, in most problems in the electrodynamic theory of material media one manages well without paying attention to the dispersion. The reason is (we restrict ourselves to the source-theory approach) that an *infinite constant is lost* in the handling of the nondispersive formalism. When doing Casimir calculations one is confronted with a frequency integral of an infinite series over all angular momentum variables l. This series is usually calculated approximately by means of the Debye expansion, which is an asymptotic high-l expansion. The important point is that the sum and the integral are *interchanged*, and the frequency integration carried out first. In general it is known that when performing such an interchange operation for a series that is not uniformly convergent, one runs the risk of getting a wrong answer. Now the asymptotic Debye expansion is not uniformly convergent. So the loss of an infinite constant in the nondispersive calculations is actually what we might expect. [Actually we pointed out this possibility already in Sec. 3.4 in Ref. 15(b).]

Our recognition of the importance of the dispersive effect in the present problem was triggered by some computer calculations by Baacke (private communication): He put the general formulas for the interior and exterior Casimir energies for a dielectric sphere, as given by Eqs. (3.13) and (3.20) in Ref. 15(b), on a computer, without use of the Debye expansion. The results were reported to be in good agreement with our analytically computed results, Eq. (3.23) in Ref. 15(b), apart from a cutoff-dependent, divergent term. See also the papers by Baacke and co-workers.[18–20] These computer results clearly pointed toward the omission of a constant in the usual analytic, Debye expansion-based calculations.

The purpose of the present paper is to take dispersive effects into account from the beginning. We base our analysis on a simple, one-absorption-frequency, dispersion relation. Since all material susceptibilities tend towards zero at high frequencies, the aforementioned mathematical problems with the handling of the infinite series go away. *After* the calculation has been completed, it is straightforward to check the limit of nondispersive media by letting the absorption frequency ω_0 go to infinity. It is explicitly verified, in fact, that the expression for the surface force contains a term which, in the limit of high ω_0, is proportional to ω_0 [see Eq. (6.5) below]. This is simply a divergent term in the case of a nondispersive medium. We stress that this way of considering nondispersive media as a limit *at the end* of the calculation is superior to the previous method where the nondispersive property was assumed from the outset. In the dispersive case, we have control over the formalism and can check where the divergences actually come from.

To be more precise about the last-mentioned point: we do not state that all the nondispersive calculations[11–16] are simply incorrect. What is done in these works is to restrict oneself to finite quantities throughout, disregarding any dispersion-induced infinities, and that is just the natural way to proceed once the initial assumption about a nondispersive material has been adopted. The main issue is how realistic the underlying *model* is.

We mention that our results are essentially in agreement with Candelas,[21] who similarly stressed the need of including the strong, attractive, cutoff-dependent term in the force. Whereas his arguments are mainly of a general nature, we consider a specific model. We return to a comparison with some of his results in Sec. VI.

The plan of our paper is the following. Section II briefly generalizes Schwinger's source theory to the case of dispersive media. As dispersion relation, we choose the one-absorption-frequency Sellmeir form.[22] In Sec. III the general expression for the surface force is given, and methods for further handling of the expression are discussed. In Sec. IV we concentrate upon the Debye expansion, and sum the various series over l exactly, using complex function theory. The problem therewith becomes reduced to numerical evaluation of one-dimensional frequency integrals. Since the Debye expansion, however, is not expected to very accurate for low l, our main strategy is to rely upon separate numerical treatment for low l ($l \leq 10$), and the Debye expansion thereafter. The main results of our calculations are given in tabular form and they show in fact, as anticipated, that the dispersion-induced part of the force becomes attractive. We also calculate the Casimir *energy* for the sphere, and close our paper with some discussion on the relationship to previous work.

II. BASIC FORMALISM: DISPERSION RELATION

A. The fundamental equations

Assume that the material posseses a frequency-dependent permittivity $\epsilon(\omega)$ and a corresponding permeability $\mu(\omega)$ that satisfy the condition

$$\epsilon(\omega)\mu(\omega)=1 \tag{2.1}$$

for each ω. At high frequencies, (2.1) becomes satisfied automatically, since $\epsilon(\omega)\rightarrow 1$, $\mu(\omega)\rightarrow 1$ always when $\omega\rightarrow\infty$. We make use of Schwinger's source theory (Green's-function theory). To avoid repetition of known material, we refer to earlier works: the basic formalism applied to spherically symmetric systems in vacuum is contained in Ref. 11 (in turn, this followed from earlier work by Schwinger, DeRaad, and Milton[23], whereas the adaptation of the formalism to media satisfying (1.1) is contained in Ref. 15. We only supply here some remarks related to dispersion.

The main technique is to relate the electric field $\mathbf{E}(x)$ to the polarization $\mathbf{P}(x')$ through a dyad $\boldsymbol{\Gamma}(x,x')$. Its Fourier transform is $\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}',\omega)$. The constitutive relations in the present case are

$$\mathbf{D}(\omega)=\epsilon(\omega)\mathbf{E}(\omega), \quad \mathbf{B}(\omega)=\mu(\omega)\mathbf{H}(\omega) \; . \tag{2.2}$$

The Maxwell equations imply in Fourier space the following governing equation for the dyad $\boldsymbol{\Gamma}$:

$$-\nabla\times\nabla\times\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}',\omega)+\omega^2\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}',\omega)=-\mu(\omega)\omega^2\mathbf{1}\delta(\mathbf{r}-\mathbf{r}') \; . \tag{2.3}$$

This equation can be solved, as usual, in terms of vector spherical harmonics. The solution contains two scalar Green's functions F_l and G_l given in terms of spherical Bessel and Hankel functions. We write down the basic expression for the effective product of two electric field components:

$$iE_i(\mathbf{r})E_k(\mathbf{r}')|_{\text{eff}}=\Gamma_{ik}(\mathbf{r},\mathbf{r}',\omega) \; , \tag{2.4}$$

where reference to the frequency is suppressed on the left-hand side. From this we derive the following expressions for the effective products of the radial, and the orthogonal, electric field components:

$$iE_r(x)E_r(x')|_{\text{eff}}=\frac{1}{rr'}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\mu(\omega)\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}l(l+1)G_l(r,r') \; , \tag{2.5}$$

$$iE_\perp(x)E_\perp(x')|_{\text{eff}}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\mu(\omega)\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}\left[\omega^2F_l(r,r')+\frac{1}{rr'}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r'}r'G_l(r,r')\right]\right] \; . \tag{2.6}$$

These expressions refer to the same direction: $\Omega\to\Omega'$. The magnetic field products $iB_r(x)B_r(x')|_{\text{eff}}$ and $iB_\perp(x)B_\perp(x')|_{\text{eff}}$ are obtained from (2.5) and (2.6) if we substitute $G_l\leftrightarrow F_l$.

The electromagnetic boundary conditions at $r=a$ have to hold for each frequency separately. Let the permeabilities of the inside and the outside regions be, respectively, $\mu_1(\omega)$ and $\mu_2(\omega)$. Then we obtain the following expressions for the constants $A_{F,G}$ and $B_{F,G}$ appearing in F_l and G_l:

$$A_F=\frac{e_l(z)e_l'(z)}{s_l(z)s_l'(z)}B_F =[\mu_1(\omega)-\mu_2(\omega)]\frac{e_l(z)e_l'(z)}{D_l(z)} \; , \tag{2.7}$$

$$A_G=\frac{e_l(z)e_l'(z)}{s_l(z)s_l'(z)}B_G =[\mu_1(\omega)-\mu_2(\omega)]\frac{e_l(z)e_l'(z)}{\tilde{D}_l(z)} \; . \tag{2.8}$$

Here $z=ka$, $s_l=zj_l$, and $e_l=zh_l^{(1)}$ are the Riccati-Bessel functions, and we have defined

$$D_l(z)=\mu_1(\omega)s_l(z)e_l'(z)-\mu_2(\omega)s_l'(z)e_l(z) \; ,$$
$$\tilde{D}_l(z)=\mu_1(\omega)s_l'(z)e_l(z)-\mu_2(\omega)s_l(z)e_l'(z) \; . \tag{2.9}$$

B. The analog of Sellmeir's dispersion relation

We shall now make the dependence of ϵ and μ on ω explicit, by adopting a dispersion relation that is analytically simple and physically reasonable. In the usual theory of (nonmagnetic) dielectric dispersive media, several semiempirical formulas exist, containing one or more absorption frequencies. We choose in the following a formula for $\mu(\omega)$ which is analogous to Sellmeir's formula for $\epsilon(\omega)$ in ordinary electromagnetism.[22] Assuming only one single absorption frequency, at $\omega=\omega_0$, we can write the formula as

$$\mu(\omega)-1=\frac{\mu_0-1}{1-\omega^2/\omega_0^2} \; , \tag{2.10}$$

where μ_0, as well as ω_0, are input parameters (the constant μ_0 has no relationship to SI units). We shall allow the magnetic susceptibility (μ_0-1) to take positive as well as negative values.

Formula (2.10) is assumed to hold in frequency regions lying far away from ω_0. In these regions, where dispersive effects are weak, it is possible to define the (mean) electromagnetic energy density as a thermodynamic quantity (Ref. 24, Sec. 80):

$$w=\frac{1}{2}\left[\frac{d(\omega\epsilon)}{d\omega}E^2+\frac{d(\omega\mu)}{d\omega}H^2\right] \; . \tag{2.11}$$

However, near ω_0, Eq. (2.10) has to be supplemented with an imaginary term in order to take into account absorption. From general arguments (Ref. 24, Sec. 82) we know that $\epsilon(\omega)$ and $\mu(\omega)$, at least for nonmetals, have no singularities on the real axis. Mathematically, the relationship between the real and the imaginary parts are expressed through the Kramers-Kronig relations. Thermodynamically, one has to be most careful in the strongly absorbing region: even the expression for the electromagnetic energy ceases to have a thermodynamic meaning.

In spite of these features, we can nevertheless use the formula (2.10) without any difficulty in the present problem. The point is that $\mu(\omega)$, regarded as a function of a complex variable ($\omega=\omega'+i\omega''$) is *analytic* in the upper half-plane ($\omega''>0$). When calculating physical quantities, we will generally have to evaluate a frequency integral which, in view of symmetry about the origin, can be taken to run from $\omega=0$ to $\omega=\infty$. The path of integration is the usual one for Feynman propagators: it passes *above* the singularities on the real positive axis. The analyticity property of $\mu(\omega)$ thus permits us to make a *complex frequency rotation*,

$$\omega \rightarrow i\hat{\omega} , \tag{2.12}$$

so that without encountering any singularities we can transform the frequency integral into an integral along the positive imaginary axis. According to (2.10), $\mu(i\hat{\omega})$ becomes a real and positive quantity. The final expression for the surface force is free from singularities.

The fact that $\mu(\omega)$ reduces to a real function on the positive imaginary axis is actually in agreement with what can be derived from the causality principle for real dielectric materials (Ref. 24, Sec. 82).

The absorption frequency ω_0 plays in our theory the role of a soft, high-frequency cutoff. For definiteness, we shall choose $\omega_0=3\times10^{16}\ \text{sec}^{-1}$ to be typical value in our numerical work later on. This value simply is taken from analogy with the theory of ordinary dielectrics. Now, our medium satisfying (2.1) is after all no ordinary dielectric, and one might therefore find it more appropriate instead to choose a typical value of ω_0 of the order of $1/a$ (a is the radius), since this is what one should expect in the hadronic world. Anyway, we shall let the frequency input parameter cover two decades, the lowest alternative corresponding to $\omega_0=5/a$. In this way the hadronic case ought to be roughly covered also.

III. GENERAL EXPRESSION FOR THE SURFACE FORCE

We consider in the following a compact spherical ball of radius a, with a vacuum on the outside. Thus $\mu_2(\omega)=1$ in our previous notation, whereas the inside permeability will for simplicity be denoted by $\mu(\omega)$.

Assume that the two points $\mathbf{r}$ and $\mathbf{r}'$ lie in the same radial direction, close to each other. The spatial part of the electromagnetic energy-momentum tensor in nondispersive media is

$$S_{ik}=-E_iD_k-H_iB_k+\tfrac{1}{2}\delta_{ik}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}) . \tag{3.1}$$

In dispersive media we can in fact use just the same expression, replacing each bilinear term in (3.1) by its appropriate Fourier component. This is actually an important point in dispersion theory: the spatial part of the energy-momentum tensor (which is equal to minus the stress tensor) gets *no additional derivative terms* in the presence of dispersion. See, for instance, Ref. 24, Sec. 81. By contrast, as we have already seen in (2.11), the energy density gets additional derivative terms. Therefore, when dealing with dispersion theory, it is simpler to start considering *forces* rather than *energies*.

The Fourier component $F(\omega)$ of the radial surface force density is

$$F(\omega)=S_{rr}(\omega)\,|_{a-}-S_{rr}(\omega)\,|_{a+}$$
$$=\frac{\mu(\omega)-1}{2}\left[-E_r^2(\omega)-\frac{1}{\mu(\omega)}E_1^2(\omega)+\frac{1}{\mu(\omega)}H_r^2(\omega)+H_1^2(\omega)\right]_{a+} . \tag{3.2}$$

The surface force can thus be expressed in terms of the Green's-function dyad Γ_{ik}. We write the force as a Fourier integral over all frequencies:

$$F=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}F(\omega) \tag{3.3}$$

with $\tau=t-t'$. Some calculation results in the expression

$$F=\frac{i}{32\pi^2a^3}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega\tau}z\chi^2(\omega)\sum_{l=1}^{\infty}(2l+1)\frac{\lambda_l(z)\lambda_l'(z)}{D_l(z)\tilde{D}_l(z)} \tag{3.4a}$$

$$=\frac{i}{16\pi^2a^3}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega\tau}z\sum_{l=1}^{\infty}(2l+1)\frac{\frac{d}{dz}\ln[1+\lambda_l^2(z)]}{\left[1+\frac{i\chi^{-1}(\omega)}{s_l(z)e_l'(z)}\right]\left[1-\frac{i\chi^{-1}(\omega)}{s_l'(z)e_l(z)}\right]} , \tag{3.4b}$$

where $z=ka=|\omega|a$. We have introduced the frequency-dependent magnetic susceptibility

$$\chi(\omega)=\mu(\omega)-1 \tag{3.5}$$

and also the abbreviation

$$\lambda_l(z)=[s_l(z)e_l(z)]' \tag{3.6}$$

with a prime meaning differentiation with respect to z. To obtain (3.4), we have employed the fundamental differential equation for the Riccati-Bessel functions,[25] and also the Wronskian (in standard notation) $W\{s_l,e_l\}=i$.

Similarly as in the nondispersive case, we do not have to subtract any *contact term* from the force expression (3.4). The calculation proceeds just as previously,[15(a)] and will not be repeated here. The expression (3.4) accordingly gives the physical force directly.

Consider next the parameter τ in (3.4). We have kept this parameter so far in the formalism, to make the treatment parallel to that of nondispersive media. In the latter case, τ played an important role as a cutoff parameter. In the present case, however, τ can be omitted since the susceptibility $\chi(\omega)$ tends to zero at high frequencies and so suppresses the contribution to the force from those frequencies. Accordingly, we set henceforth $\tau=0$.

Assume now that the dispersion relation is such that the susceptibility $\chi(\omega)$ is symmetric about the origin: $\chi(\omega)=\chi(-\omega)$. Our model example (2.10) is of this kind. Then the integral over ω from minus infinity to infinity in (3.4) can be replaced by twice the integral from zero to infinity. We perform a complex frequency rotation, $\omega\rightarrow i\hat{\omega}$, as mentioned above. Since ω is now taken to be positive, we can relate the frequency rotation to the replacements

$$k\rightarrow i\hat{k}=i\hat{\omega},\quad z\equiv ka\rightarrow i\hat{k}a\equiv ix\ . \tag{3.7}$$

Accordingly, the susceptibility $\chi(\omega)$ in (3.4) becomes a function of the argument $i\hat{\omega}$, whereas the quantities s_l, e_l, D_l, $\tilde{D}_l$, λ_l become functions of ix. For simplicity, we henceforth regard the last-mentioned quantities to be functions of the real positive variable x, thus $s_l(ix)\rightarrow s_l(x)$, etc., or

$$s_l(x)=(\pi x/2)^{1/2}I_\nu(x)\ ,\qquad e_l(x)=(2x/\pi)^{1/2}K_\nu(x) \tag{3.8}$$

with $\nu=l+\frac{1}{2}$. This normalization corresponds to the Wronskian $W\{s_l,e_l\}=-1$. Similarly, we let, in the following, D_l and $\tilde{D}_l$ mean

$$D_l(x)=\mu(\omega)s_l(x)e_l'(x)-s_l'(x)e_l(x)\ ,\qquad \tilde{D}_l(x)=\mu(\omega)s_l'(x)e_l(x)-s_l(x)e_l'(x)\ , \tag{3.9}$$

where now a prime means differentiation with respect to x. Analogously, we define

$$\lambda_l(x)=[s_l(x)e_l(x)]',\quad \chi(x)=\frac{\chi_0}{1+x^2/x_0^2}\ , \tag{3.10}$$

with $\chi_0=\mu_0-1$, $x_0=\omega_0 a$ [as regards the latter equation, cf. (2.10)].

Expressions (3.4) for the surface force density can now be written as

$$F=-\frac{1}{8\pi^2a^4}\int_0^\infty dx\,x\chi^2(x)\sum_{l=1}^{\infty}\nu\frac{\lambda_l(x)\lambda_l'(x)}{D_l(x)\tilde{D}_l(x)} \tag{3.11a}$$

$$=-\frac{1}{4\pi^2a^4}\int_0^\infty dx\,x\sum_{l=1}^{\infty}\nu\frac{\frac{d}{dx}\ln[1-\lambda_l^2(x)]}{\left[1-\frac{\chi^{-1}(x)}{s_l(x)e_l'(x)}\right]\left[1+\frac{\chi^{-1}(x)}{s_l'(x)e_l(x)}\right]}\ , \tag{3.11b}$$

where $\chi(x)$ is to be inserted from (3.10). [Note that there is no contribution to the contour integral over ω from the part of the contour that forms a circular arc at $|\omega|\rightarrow\infty$. Namely, for $|\omega|\rightarrow\infty$ and fixed l we have[25] $s_l(z)\rightarrow\cos\phi$, $e_l(z)\rightarrow e^{i\phi}$, with $\phi=z-\frac{1}{2}\pi(l+1)$. Insertion into (3.4a) then shows that the contribution from the circular arc vanishes, since χ varies with ω as $1/\omega^2$ in this remote region.]

Equation (3.11) is our final general result. Each of the expressions (3.11a) and (3.11b) may, depending on circumstances, be convenient starting points for further evaluation.

There are essentially two different ways of approach now to be chosen if we want to make the calculation of the force complete. One option is that we may attack the problem numerically from the outset, making use of the recurrence equations for the Riccati-Bessel functions and their derivatives,

$$s_{l+1}(x)=s_{l-1}(x)-\frac{2l+1}{x}s_l(x)\ , \tag{3.12}$$

$$e_{l+1}(x)=e_{l-1}(x)+\frac{2l+1}{x}e_l(x)\ , \tag{3.13}$$

$$s_l'(x)=s_{l-1}(x)-\frac{l}{x}s_l(x)\ , \tag{3.14}$$

$$e_l'(x)=-e_{l-1}(x)-\frac{l}{x}e_l(x)\ , \tag{3.15}$$

which in principle permit the evaluation of the functions for all values of l and x if the lowest-order analytic functions are given:

$$s_0(x)=\sinh x,\quad s_1(x)=-\frac{1}{x}\sinh x+\cosh x\ ,\qquad e_0(x)=e^{-x},\quad e_1(x)=\left[1+\frac{1}{x}\right]e^{-x}\ . \tag{3.16}$$

The other option is that we may make use of the analytic Debye expansion, whereby it will be possible to reduce the fundamental force expression to a one-dimensional frequency integral.

In the following we shall actually use a combination of these two methods, leaning on the numerical method for the low values of l and the Debye method thereafter. In the next section we shall consider the latter method, and sum the series over *all* values of l.

IV. USE OF THE DEBYE EXPANSION

A. Construction of the basic expressions

The Debye expansion is a uniform asymptotic expansion, being most accurate for high values of l. As already anticipated, it will be convenient for our purpose to sum the series over all l in this section. The corrections from the low values of l will be taken into account later.

We proceed as follows.

(1) The starting point is the expression (3.11b), in which the Riccati-Bessel functions are expanded to $O(\nu^{-2})$. In particular, the numerator in (3.11b), i.e., the derivative of the logarithm, which is first expanded in λ_l as a smallness parameter, is effectively expanded in ν^{-1} because the leading term in λ_l is of order ν^{-1}.

(2) All terms are summed exactly, from $l=1$ to $l=\infty$.

(3) We end up with a one-dimensional frequency integral, from zero to infinity, which is evaluated by moderate numerical effort.

The basic Debye expansions for s_l, e_l and their derivatives have been given previously[11,14(b),15,16(b)] and will not be repeated here. We need the expansion coefficients u_k, v_k up to $k=2$; these are given, for instance, in Ref. 11. In accordance with common notation[25] we henceforth let the symbol z mean x/ν. Further, we define

$$t(z)=(1+z^2)^{-1/2}\ . \tag{4.1}$$

It is convenient in our context to introduce a new symbol $\alpha=\alpha(z)$, defined as

$$\alpha(z)=\left[\frac{\mu_0+1}{2}\right]^{1/2}\frac{x_0}{z}\ . \tag{4.2}$$

It is also useful to note that the permeability $\mu(x)=\mu(\nu z)$, as defined by (3.10) in our model, satisfies the relation

$$\frac{\mu(\nu z)-1}{\mu(\nu z)+1}=\frac{\mu_0-1}{\mu_0+1}\frac{\alpha^2}{\nu^2+\alpha^2}\ . \tag{4.3}$$

We develop the formalism in analogy to our earlier treatment on nondispersive media.[15] The inverse denominator in (3.11b) is expanded as

$$\frac{1}{\left[1-\dfrac{\chi^{-1}(x)}{s_l e_l'}\right]\left[1+\dfrac{\chi^{-1}(x)}{s_l' e_l}\right]}=\left[\frac{\mu_0-1}{\mu_0+1}\right]^2\left[\frac{\alpha^2}{\nu^2+\alpha^2}\right]^2\left[1-\frac{t^6}{4}\frac{\left[\nu^2+\dfrac{2\alpha^2}{\mu_0+1}\right]\left[\nu^2+\dfrac{2\alpha^2\mu_0}{\mu_0+1}\right]}{\nu^2(\nu^2+\alpha^2)^2}+O(\nu^{-4})\right]\ . \tag{4.4}$$

In the derivation of this expression we have made use of the expressions for u_1, v_1, and u_2, v_2. The coefficients in the next approximation, u_3 and v_3, are not needed to obtain the result (4.4); the $O(\nu^{-3})$ terms drop out automatically. Using (4.4), we can now express (3.11b) as

$$F=-\frac{1}{8\pi a^4}\left[\frac{\mu_0-1}{\mu_0+1}\right]^2\sum_{l=1}^{\infty}T_l(\mu_0,x_0)\ , \tag{4.5a}$$

$$T_l(\mu_0,x_0)=J_l(\mu_0,x_0)-\frac{1}{\nu^2}\frac{\mu_0}{(\mu_0+1)^2}I_l(\mu_0,x_0)+O(\nu^{-4}) \tag{4.5b}$$

with

$$J_l(\mu_0,x_0)=\frac{2\nu^2}{\pi}\int_0^{\infty}dz\, z\left[\frac{\alpha^2}{\nu^2+\alpha^2}\right]^2\frac{d}{dz}\ln(1-\lambda_l^2)\ , \tag{4.6}$$

$$I_l(\mu_0,x_0)=\frac{2\nu^2}{\pi}\frac{(\mu_0+1)^2}{4\mu_0}\int_0^{\infty}dz\, z\alpha^4 t^6\frac{\left[\nu^2+\dfrac{2\alpha^2}{\mu_0+1}\right]\left[\nu^2+\dfrac{2\alpha^2\mu_0}{\mu_0+1}\right]}{(\nu^2+\alpha^2)^4}\frac{d}{dz}\ln(1-\lambda_l^2)\ . \tag{4.7}$$

The special case of a nondispersive medium corresponds to $\omega_0=\infty$, i.e., $x_0=\infty$ or $\alpha=\infty$. Then J_l and I_l reduce to their respective counterparts as given by Eqs. (2.51) and (2.52) in Ref. 15(a) (for time separation $\tau=0$).

Now expand the logarithmic factor in (4.6) and (4.7) in λ_l as a smallness parameter. We first calculate

$$\lambda_l(x)=\frac{t^3}{2\nu}\left[1-\frac{1}{8\nu^2}(2-27t^2+60t^4-35t^6)+O(\nu^{-4})\right]\ , \tag{4.8}$$

which shows that λ_l is of order ν^{-1}, as mentioned above. Inserting this expression into

$$\ln(1-\lambda_l^2)=-\lambda_l^2-\tfrac{1}{2}\lambda_l^4+O(\lambda_l^6) \tag{4.9}$$

and differentiating with respect to z we obtain, taking into account that $dt/dz=-zt^3$,

$$\nu^2\frac{d}{dz}\ln(1-\lambda_l^2)=\tfrac{3}{2}zt^8\left[1-\frac{1}{4\nu^2}(2-36t^2+100t^4-71t^6)+O(\nu^{-4})\right]\ . \tag{4.10}$$

We have multiplied by a factor ν^2 to make the leading term of order unity.

The last expansion makes it natural to divide J_l into two parts:

$$J_l(\mu_0,x_0)=J_l^{(0)}+\frac{1}{\nu^2}J_l^{(2)} \tag{4.11}$$

with

$$J_l^{(0)}=\frac{3}{\pi}\int_0^\infty dz\, z^2 t^8 \left[\frac{\alpha^2}{\nu^2+\alpha^2}\right]^2 , \tag{4.12}$$

$$J_l^{(2)}=-\frac{3}{4\pi}\int_0^\infty dz\, z^2 t^8 \left[\frac{\alpha^2}{\nu^2+\alpha^2}\right]^2 \times(2-36t^2+100t^4-71t^6) . \tag{4.13}$$

When judging the order of ν^{-1} in these expressions, we have to face the complications arising from the dispersion factor $[\alpha^2/(\nu^2+\alpha^2)]^2$. Actually one might imagine to expand this factor also, in powers of α/ν. In view of (4.2), this would be equivalent to an expansion in x_0/ν as smallness parameter. We wish, however, to maintain x_0 as a freely adjustable input parameter—as we have seen the case of a nondispersive medium corresponds to $x_0=\infty$—and we therefore choose to judge the order of the expansion without paying regard to the dispersive effect. Consequently, both $J_l^{(0)}$ and $J_l^{(2)}$ are reckoned as being of $O(\nu^0)$.

In addition to these two terms, there is a remainder of $O(\nu^{-4})$ in J_l, arising from the last term in (4.10); this remainder is omitted from (4.12) and (4.13), and is assumed to be absorbed in the common remainder in (4.5b).

The expression for I_l, defined in (4.7), is treated similarly. From (4.5b) it is apparent that we need to retain only the first term to the right in the expansion (4.10). That means, I_l needs only to be expanded to $O(\nu^0)$:

$$I_l=\frac{3}{\pi}\frac{(\mu_0+1)^2}{4\mu_0}\int_0^\infty dz\, z^2\alpha^4 t^{14} \times\frac{\left[\nu^2+\frac{2\alpha^2}{\mu_0+1}\right]\left[\nu^2+\frac{2\alpha^2\mu_0}{\mu_0+1}\right]}{(\nu^2+\alpha^2)^4} . \tag{4.14}$$

For, the correction term which we have omitted here is of $O(\nu^{-2})$; when multiplied by ν^{-2} as required by (4.5b) it becomes of $O(\nu^{-4})$ and can as such be absorbed in this equation's last term.

B. Summation of the series

Our next task is to sum the terms $J_l^{(0)}$, $J_l^{(2)}$, and I_l over l. This can be done exactly. Each of the series has the form $\sum_1^\infty f_l$, where the summand f_l satisfies

$$\sum_{l=1}^\infty f_l=-f_0+\tfrac{1}{2}\sum_{l=-\infty}^\infty f_l \tag{4.15}$$

in view of its quadratic dependence on ν. The first term to the right corresponds to $l=0$.

We replace l by a complex variable ζ, and consider $f(\zeta)$ as an analytic function of ζ. This function will have poles at $\zeta=\zeta_r$, $r=1,2,\ldots$. Thus taking the integral of the function $\pi f(\zeta)\cot\pi\zeta$ around a closed contour C enclosing the poles at ζ_r as well as the poles at all integer values of ζ from minus infinity to infinity we obtain, since $f(\zeta)$ is such that

$$|\zeta f(\zeta)|\to 0 \quad\text{when}\quad |\zeta|\to\infty ,$$

the following formula:[26]

$$\sum_{l=-\infty}^\infty f(l)=-\sum_r \mathrm{Res}[\pi f(\zeta)\cot\pi\zeta] =-\sum_r\frac{1}{(n-1)!}\left[\frac{d^{n-1}}{d\zeta^{n-1}}[(\zeta-\zeta_r)^n\pi f(\zeta)\times\cot\pi\zeta]\right]_{\zeta=\zeta_r} . \tag{4.16}$$

The sum is taken over the residues. n is the order of the pole at $\zeta=\zeta_r$.

By means of this method we can calculate the useful formula

$$\sum_{l=1}^\infty\frac{1}{\nu^2+\alpha^2}=-\frac{4}{1+4\alpha^2}+\frac{\pi}{2\alpha}\tanh\pi\alpha , \tag{4.17}$$

whose derivative with respect to α yields

$$\sum_{l=1}^\infty\frac{1}{(\nu^2+\alpha^2)^2}=-\frac{16}{(1+4\alpha^2)^2}+\frac{\pi}{4\alpha^3}\tanh\pi\alpha -\frac{\pi^2}{4\alpha^2}\frac{1}{\cosh^2\pi\alpha} . \tag{4.18}$$

Thus, $J_l^{(0)}$, as defined in (4.12), can be summed as

$$\sum_{l=1}^\infty J_l^{(0)}=\frac{3}{\pi}\int_0^\infty dz\, z^2 t^8\left[-\left[\frac{4\alpha^2}{1+4\alpha^2}\right]^2+\frac{\pi^2\alpha^2}{4\cosh\pi\alpha}\left[\frac{\sinh\pi\alpha}{\pi\alpha}-\frac{1}{\cosh\pi\alpha}\right]\right] . \tag{4.19}$$

To calculate the sum of $\nu^{-2}J_l^{(2)}$ we might use (4.16) directly. It is however more convenient to resolve into partial fractions the expression

$$\frac{\alpha^4}{\nu^2(\nu^2+\alpha^2)^2}=\frac{1}{\nu^2}-\frac{\alpha^2}{(\nu^2+\alpha^2)^2}-\frac{1}{\nu^2+\alpha^2} , \tag{4.20}$$

and thereafter use (4.17) and (4.18), together with the formula

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}=\tfrac{1}{2}\pi^2-4\ , \tag{4.21}$$

to obtain the desired expression

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}J_l^{(2)}=\frac{3}{\pi}\int_0^{\infty}dz\,z^2t^8\left[\left[\frac{4\alpha^2}{1+4\alpha^2}\right]^2+\frac{\pi^2}{16\cosh\pi\alpha}\left[\frac{3\sinh\pi\alpha}{\pi\alpha}-\frac{1}{\cosh\pi\alpha}\right]-\frac{\pi^2}{8}\right]$$
$$\times(2-36t^2+100t^4-71t^6)\ . \tag{4.22}$$

The summation of $\nu^{-2}I_l$, with I_l given in (4.14), can be done similarly. We first resolve into partial fractions the quantity

$$\frac{\left[\nu^2+\dfrac{2\alpha^2}{\mu_0+1}\right]\left[\nu^2+\dfrac{2\alpha^2\mu_0}{\mu_0+1}\right]}{\nu^2(\nu^2+\alpha^2)^4}=\frac{4\mu_0}{(\mu_0+1)^2}\left[\frac{1}{\alpha^4\nu^2}-\frac{1}{\alpha^2(\nu^2+\alpha^2)^2}-\frac{1}{\alpha^4(\nu^2+\alpha^2)}\right]$$
$$+\left[\frac{\mu_0-1}{\mu_0+1}\right]^2\left[\frac{\alpha^2}{(\nu^2+\alpha^2)^4}+\frac{1}{(\nu^2+\alpha^2)^3}\right]\ , \tag{4.23}$$

and thereafter sum the individual terms by using (4.21), (4.18), and (4.17), as well as the formulas

$$\sum_{l=1}^{\infty}\frac{1}{(\nu^2+\alpha^2)^3}=-\frac{64}{(1+4\alpha^2)^3}+\frac{3\pi}{16\alpha^5}\tanh\pi\alpha-\frac{3\pi^2}{16\alpha^4}\frac{1}{\cosh^2\pi\alpha}-\frac{\pi^3}{8\alpha^3}\frac{\tanh\pi\alpha}{\cosh^2\pi\alpha}\ , \tag{4.24}$$

$$\sum_{l=1}^{\infty}\frac{1}{(\nu^2+\alpha^2)^4}=-\frac{256}{(1+4\alpha^2)^4}+\frac{5\pi}{32\alpha^7}\tanh\pi\alpha-\frac{5\pi^2}{32\alpha^6}\frac{1}{\cosh^2\pi\alpha}-\frac{\pi^3}{8\alpha^5}\frac{\tanh\pi\alpha}{\cosh^2\pi\alpha}-\frac{\pi^4}{48\alpha^4}\frac{2\sinh^2\pi\alpha-1}{\cosh^4\pi\alpha}\ , \tag{4.25}$$

which are obtained by successive differentiations of (4.18). In this way we find

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}I_l=\frac{3}{\pi}\int_0^{\infty}dz\,z^2t^{14}\left[\tfrac{1}{2}\pi^2-\frac{64\alpha^4}{(1+4\alpha^2)^2}-\frac{3\pi}{4\alpha}\tanh\pi\alpha+\frac{\pi^2}{4}\frac{1}{\cosh^2\pi\alpha}\right.$$
$$+\frac{(\mu_0-1)^2}{4\mu_0}\left[-\frac{64\alpha^4}{(1+4\alpha^2)^2}+\frac{1024\alpha^8}{(1+4\alpha^2)^4}+\frac{11\pi}{32\alpha}\tanh\pi\alpha-\frac{11\pi^2}{32}\frac{1}{\cosh^2\pi\alpha}\right.$$
$$\left.\left.-\frac{\pi^3\alpha}{4}\frac{\tanh\pi\alpha}{\cosh^2\pi\alpha}-\frac{\pi^4\alpha^2}{48}\frac{2\sinh^2\pi\alpha-1}{\cosh^4\pi\alpha}\right]\right]\ . \tag{4.26}$$

We have thus finally obtained the following expression for F, with use of the Debye expansion for all l:

$$F=-\frac{1}{8\pi a^4}\left[\frac{\mu_0-1}{\mu_0+1}\right]^2\sum_{l=1}^{\infty}\left[J_l^{(0)}+\frac{1}{\nu^2}J_l^{(2)}-\frac{1}{\nu^2}\frac{\mu_0}{(\mu_0+1)^2}I_l+O(\nu^{-4})\right]\ , \tag{4.27}$$

in which the order of the correction term is shown explicitly. Recall that α, as defined in (4.2), depends on z, and also that $J_l^{(0)}$, $J_l^{(2)}$, and I_l are all of $O(\nu^0)$. The various terms in (4.27), which have now been reduced to single integrals, can readily be calculated on a computer, with μ_0 and x_0 as input parameters.

For a nondispersive medium, $x_0=\infty$, (4.27) is the analogue of Eq. (2.56) in Ref. 15(a).

V. CALCULATIONAL METHOD AND RESULTS

Because of the limited accuracy of the Debye expansion for low l we have found it to be most appropriate, as mentioned earlier, to choose the numerical method of approach for l ranging from $l=1$ to $l=10$. The general expression (3.11a) is the starting point, with $\mathcal{X}(x)$ given by (3.10). The basic recurrence equations for the Riccati-

Bessel functions are (3.12)–(3.15); they correspond to the following recurrence equations for the various factors in the integrand:

$$\lambda_l(x)=-s_l(x)e_{l-1}(x)+s_{l-1}(x)e_l(x)-\frac{2l}{x}s_l(x)e_l(x)\ , \tag{5.1}$$

$$\lambda_l'(x)=2\left[\frac{l}{x}[s_l(x)e_{l-1}(x)-s_{l-1}(x)e_l(x)]-s_{l-1}(x)e_{l-1}(x)+\left[1+\frac{(2l+1)l}{x^2}\right]s_l(x)e_l(x)\right]\ , \tag{5.2}$$

$$D_l(x)=-\chi(x)s_l(x)\left[e_{l-1}(x)+\frac{l}{x}e_l(x)\right]-1\ , \tag{5.3}$$

$$\tilde{D}_l(x)=\chi(x)\left[s_{l-1}(x)-\frac{l}{x}s_l(x)\right]e_l(x)+1\ . \tag{5.4}$$

By using these equations, together with (3.12)–(3.16), it is in principle possible to calculate the contribution to the force from the mentioned low values of l. We can interchange the sum with the integral without any problems.

For the lowest values of l, $l=1,2,3,4$, the recurrence equations were found to work well, except for the low values of x where the calculated values of $s_l(x)$ were numerically unstable. (It ought to be emphasized that these problems were purely *numerical*.) Therefore the recurrence equations were not used in this region; instead, a series expansion was worked out for the integrand for small values of x, $x\in[0,0.3]$, and thereafter integrated analytically (see the Appendix). For the higher values of x, i.e., in the interval $x\in[0.3,\infty\rangle$, numerical integration was carried out using Simpson's method. As a check on the numerical integration we compared Simpson's method with Romberg's method and found good agreement; in fact, the differences were so small that they do not turn up in any digit in our tables given below.

For the next values of l, $l=5-10$, the numerical instabilities in $s_l(x)$ were found to be more pronounced. First, for $x\in[0,0.3]$, the same series expansion as mentioned above was used. Thereafter, in the region $x\in[0.3,3.1]$, a satisfactory approach was found to be to use the recurrence equation for $s_l(x)$ in the *backward* direction, starting from tabular data for $s_9(x)$ and $s_{10}(x)$. Such data are given in steps of 0.1 in Ref. 25. The numerical integration could then be carried out using Simpson's method. For the greater values of x, $x\in[3.1,\infty\rangle$ (and still $l=5-10$), the calculations were easily done using the recurrence equations and Simpson's method in an ordinary way.

Finally, for $l=11-\infty$, we made use of the Debye expansion as spelled out in Sec. IV, subtracting off from (4.27) the contribution to the force from $l=1-10$. The various integrals over z were conveniently calculated using Simpson's method.

Proceeding in this way, we expect the analysis to be quite accurate. We recall that in many treatments where the Debye approximation is involved, one accepts the expansion for *all* values of l, down to $l=1$, and expects a reasonable accuracy nevertheless. In more accurate work, such as in the one of Milton, DeRaad, and Schwinger,[11] one applies a numerical treatment for the lowest values of l ($l=1-4$) and the Debye expansion thereafter. In the present case, where we apply the numerics considerably further, up to $l=10$, we would expect beforehand that the agreement between the exact method and the Debye method should be quite good for l lying around $l=10$. Actually, we have made an explicit check of this, for $l=10$, and have found very good agreement. The differences do not have any influence upon our tabular data (Table I).

It is convenient to present the results in nondimensional form, by dividing F by the finite result calculated earlier[11] for a singular shell:

$$F_0=0.092\,35/8\pi a^4\ . \tag{5.5}$$

As noted above, we take $\omega_0=3\times10^{16}\ \text{sec}^{-1}$ to be a typical value for the absorption (or cutoff) frequency. As regards the dimensions of the sphere, we take $2a=10^{-4}$ cm to be a typical diameter (if effects from sphericity are to be present, the diameter must necessarily be small). In dimensional terms, x_0 is defined as $x_0=\omega_0 a/c$. We accordingly obtain $x_0=50$ as a typical value for the cutoff. Table II shows calculated values of the ratio F/F_0 with this value for x_0, and with various chosen values for the static permeability μ_0. To show the dependence of the results upon x_0, we give in Tables I and III the corresponding results for $x_0=5$ and 500. As we noted above, the hadronic case is expected roughly to correspond to x_0 of the order of unity, so that Table I is pertinent in this case.

The results are generally seen to be negative, corre-

TABLE I. Relative surface force density F/F_0 when $x_0=5$ for some given values of μ_0. The contributions from the lowest l are given separately. The bottom line gives the sum over all l.

l	$\mu_0=0.1$	$\mu_0=0.4$	$\mu_0=0.8$	$\mu_0=1.2$	$\mu_0=2$	$\mu_0=5$	$\mu_0=10$	$\mu_0=50$
1	−0.5429	−0.1523	−0.0106	−0.0072	−0.1009	−0.4290	−0.6677	−0.9530
2	−0.4102	−0.1208	−0.0087	−0.0061	−0.0878	−0.3916	−0.6257	−0.9204
3	−0.3133	−0.0955	−0.0071	−0.0051	−0.0755	−0.3559	−0.5884	−0.9044
4	−0.2407	−0.0754	−0.0057	−0.0042	−0.0642	−0.3195	−0.5476	−0.8853
5−10	−0.6932	−0.2286	−0.0184	−0.0141	−0.2288	−1.3046	−2.4831	−4.8023
11−∞	−0.5029	−0.1815	−0.0161	−0.0134	−0.2488	−1.9548	−5.0361	−22.1671
Sum	−2.7033	−0.8541	−0.0666	−0.0501	−0.8060	−4.7554	−9.9487	−30.6325

TABLE II. Same as Table I, but with $x_0=50$.

l	$\mu_0=0.1$	$\mu_0=0.4$	$\mu_0=0.8$	$\mu_0=1.2$	$\mu_0=2$	$\mu_0=5$	$\mu_0=10$	$\mu_0=50$
1	−0.6871	−0.1853	−0.0124	−0.0083	−0.1120	−0.4536	−0.6892	−0.9598
2	−0.6720	−0.1836	−0.0123	−0.0083	−0.1114	−0.4481	−0.6775	−0.9381
3	−0.6671	−0.1831	−0.0123	−0.0083	−0.1115	−0.4485	−0.6776	−0.9373
4	−0.6604	−0.1819	−0.0123	−0.0082	−0.1113	−0.4480	−0.6771	−0.9368
5−10	−3.7551	−1.0460	−0.0712	−0.0481	−0.6541	−2.6597	−4.0378	−5.6115
11−∞	−26.4813	−8.4282	−0.6590	−0.4967	−7.9927	−47.1841	−98.8146	−305.180
Sum	−32.9230	−10.2080	−0.7796	−0.5779	−9.0929	−51.6420	−105.574	−314.563

sponding to an inward force. Moreover, the strength of the force increases as the cutoff x_0 increases. The relative importance of each l is seen to diminish with increasing l.

VI. CONCLUSIONS AND FINAL REMARKS

We arrive at the following conclusions.

(1) The most striking property of our calculated results is that, with the input values for x_0 and μ_0 chosen, they predict the surface force to be *attractive*. The attractiveness in fact persists for each value of l separately. For great values of x_0 the force becomes very strong.

(2) The reason for this circumstance lies entirely in the inclusion of the dispersive effect. Having developed the formalism, it is now easy to demonstrate in detail the properties that we explained qualitatively in Sec. I. For our purpose it is sufficient to consider the limiting case of large x_0. The delicate part of the problem is the evaluation of the zeroth-order part $J_l^{(0)}$ of the integral J_l, cf. Eqs. (4.11) and (4.12). From (4.19) we have, when x_0 is large,

$$\sum_{l=1}^{\infty} J_l^{(0)} \rightarrow \frac{3}{\pi}\int_0^{\infty} dz\, z^2 t^8(-1+\tfrac{1}{4}\pi\alpha)$$

$$= -\frac{3}{32}+\frac{x_0}{8}\left[\frac{\mu_0+1}{2}\right]^{1/2} . \quad (6.1)$$

It is clear that the second term, proportional to x_0, is due to the dispersive effect. Let us compare this with what is obtained in the conventional nondispersive theory, i.e., when the medium is assumed nondispersive from the outset. In the latter case we know from earlier works[11,14(b)] that it is necessary to introduce a cutoff parameter δ to avoid divergences when summing over l. [The nondispersive expression for $J_l^{(0)}$ can be obtained from (4.12) by putting $\alpha=\infty$ and introducing a factor $\cos(\delta\nu z)$.] We shall not go into detail about this, but write down the result:

$$\lim_{\delta\to 0}\sum_{l=1}^{\infty} J_l^{(0)}(\delta) = -\tfrac{3}{32} \quad \text{(nondispersive theory)} . \quad (6.2)$$

The point is that we get the same expression as in (6.1), *apart from the cutoff term*. Thus, if we start with the general dispersive formalism and let $x_0\to\infty$, which is physically the most realistic way of obtaining the case of nondispersive media, we are left with an infinite term in the force, a term which has been left out in the conventional nondispersive theory.

It ought to be stressed that the dispersive term describes a *physical* contribution to the force; it is not merely a spurious contact term in the formalism. This follows from the fact that the dispersive force depends on the finite dimensions of the sphere, and is moreover finite for any finite values of x_0.

(3) Let us calculate also the remaining terms in the general force expression (4.27) in the case of great x_0: from (4.22) we have

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}J_l^{(2)} \rightarrow (\tfrac{1}{2}\pi^2-4)\frac{9}{2^{13}} , \quad (6.3)$$

and correspondingly, from (4.26),

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}I_l \rightarrow (\tfrac{1}{2}\pi^2-4)\frac{63}{2^{11}} . \quad (6.4)$$

Then by using (4.27) and (5.5) we obtain for the force ratio

$$\frac{F}{F_0}\rightarrow 1.004\left[\frac{\mu_0-1}{\mu_0+1}\right]^2\left[1+0.310\frac{\mu_0}{(\mu_0+1)^2} - 1.348x_0\left[\frac{\mu_0+1}{2}\right]^{1/2}\right] , \quad (6.5)$$

TABLE III. Same as Table I, but with $x_0=500$.

l	$\mu_0=0.1$	$\mu_0=0.4$	$\mu_0=0.8$	$\mu_0=1.2$	$\mu_0=2$	$\mu_0=5$	$\mu_0=10$	$\mu_0=50$
1	−0.6894	−0.1858	−0.0124	−0.0083	−0.1121	−0.4539	−0.6894	−0.9599
2	−0.6781	−0.1849	−0.0124	−0.0083	−0.1117	−0.4489	−0.6781	−0.9382
3	−0.6787	−0.1856	−0.0125	−0.0083	−0.1122	−0.4500	−0.6788	−0.9376
4	−0.6789	−0.1859	−0.0125	−0.0084	−0.1124	−0.4504	−0.6791	−0.9374
5−10	−4.0724	−1.1168	−0.0751	−0.0503	−0.6758	−2.7050	−4.0759	−5.6231
11−∞	−328.516	−101.950	−7.7886	−5.7739	−90.8560	−516.016	−1055.01	−3144.44
Sum	−335.314	−103.809	−7.9134	−5.8574	−91.9804	−520.524	−1061.81	−3153.84

where we have omitted the correction term. Here the negative contribution to the force from the dispersive effect is shown clearly. [The reason for the factor 1.004 in front is that our accuracy in the evaluation of J_l goes only to $O(\nu^{-2})$; the nondispersive result (5.5) for F_0 obtained by Milton, DeRaad, and Schwinger[11] was found in a more accurate calculation.]

The high-x_0 formula (6.5) is surprisingly accurate, even for low values of x_0. Let us choose $\mu_0=10$ for definiteness; then inserting $x_0=500$ in (6.5) we obtain a value which deviates only about 0.01% from the value in Table III. If we insert $x_0=50$ or $x_0=5$, the deviations are somewhat greater, as expected, viz., about 0.05% and 0.15%, respectively. Equation (6.5) predicts the force to be slightly weaker than the tabular data do. Recall that our numerical data are expected to be roughly applicable in the hadronic case (x_0 of the order of unity) also.

(4) It is worth recalling that in the nondispersive calculations[14(b),15(a)] the force was found to be invariant under the substitution $\mu\rightarrow 1/\mu$. When the medium is dispersive, this kind of symmetry (for μ_0) does not hold. See the tables, or formula (6.5).

(5) Whereas it is simple to obtain an approximative expression for the force in the limit of large x_0, a corresponding approximation at the opposite extreme, for x_0 much less than unity, would be more complicated. Consider, for instance, the first term in (4.27), as given in (4.19). The quantity α runs from zero to infinity, regardless of the value of x_0, and there is a significant contribution to the integral from low z. A simple high-z expansion for the hyperbolic functions cannot be used.

(6) Formula (6.5) for the force in the limit of large x_0 makes it possible to derive a corresponding formula for the Casimir energy E. Let

$$E_0=0.092\,35/2a \tag{6.6}$$

be the Casimir energy corresponding to the force (5.5). When the material in the sphere is homogeneous, there are no electromagnetic forces in the interior, and thus the only forces occurring are the surface forces. The formula

$$-\partial E/\partial a=4\pi a^2 F \tag{6.7}$$

can thus be used to calculate the energy ratio E/E_0 that corresponds to (6.5):

$$\frac{E}{E_0}\rightarrow 1.004\left[\frac{\mu_0-1}{\mu_0+1}\right]^2\left[1+0.310\frac{\mu_0}{(\mu_0+1)^2}+1.348x_0\left[\frac{\mu_0+1}{2}\right]^{1/2}\ln a\right]+\text{const}\ . \tag{6.8}$$

(7) It is interesting finally to make a brief comparison with the discussion of Candelas for the case of a spherical shell [Ref. 21(a), Sec. 7.2]. As we mentioned in Sec. I, Candelas's arguments are of a general nature. Assuming there to be an angular-momentum cutoff $l=a\Lambda$ for the imposition of perfect conductor boundary conditions with Λ determined by the microstructure of the shell, he arrives at a cutoff-dependent term in the energy that is linearly dependent on Λ but *independent* of the radius. The surface force corresponding to an energy term of this kind is *zero*, cf. Eq. (6.7). Thus, if we focus attention only on how the dispersive terms vary with cutoff and with radius, we have, according to Candelas,

$$E_{\text{disp}}=-\tfrac{1}{2}\Lambda,\quad F_{\text{disp}}=0\ , \tag{6.9}$$

whereas our model calculation yields (for the case of large x_0)

$$E_{\text{disp}}\propto\omega_0\ln a,\quad F_{\text{disp}}\propto-\omega_0/a^3\ . \tag{6.10}$$

Since Candelas's cutoff Λ plays qualitatively speaking the same role as our ω_0, it is seen that our results are mainly in agreement, as far as the cutoff is concerned. There is some difference, however, in the predicted variation with radius. Instead of obtaining an energy completely independent of radius, we find a weak, logarithmic dependence. After all, it is according to our opinion physically reasonable that the radius ought to turn up in the expression for the energy, all the time that this energy is just a consequence of the presence of the sphere.

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APPENDIX: THE CONTRIBUTION FROM LOW FREQUENCIES

As mentioned above, we performed for technical reasons a series expansion of the integrand in the general force expression (3.11) for small values of x. Needless to say, the general expression is not very transparent to physical interpretation, so that it may be desirable to write down some of the results for small x explicitly, thereby making the formalism more easy to inspect in this particular frequency region.

Since $x=\hat{\omega}a/c$ in dimensional terms, the condition $x\ll 1$ implies that the electromagnetic wavelengths corresponding to these frequencies are much greater than the dimensions of the sphere.

We start from the small argument approximation for $I_\nu(x)$, as given in Ref. 25, p. 375. $K_\nu(x)$ can then be calculated by means of the general formula

$$K_\nu(x)=\frac{\pi}{2\sin\pi\nu}[I_{-\nu}(x)-I_\nu(x)]\ . \tag{A1}$$

In this way we derive the following expressions for the Riccati-Bessel functions:

$$s_l(x)=\frac{\sqrt{\pi}}{\Gamma(\nu+1)}\left[\frac{x}{2}\right]^{\nu+1/2}\times\left[1+\frac{\frac{1}{4}x^2}{\nu+1}+\frac{(\frac{1}{4}x^2)^2}{2(\nu+1)(\nu+2)}+O(x^6)\right]\ , \tag{A2}$$

$$e_l(x)=\frac{\Gamma(\nu)}{\sqrt{\pi}}\left[\frac{x}{2}\right]^{-\nu+1/2}\times\left[1-\frac{\frac{1}{4}x^2}{\nu-1}+\frac{(\frac{1}{4}x^2)^2}{2(\nu-1)(\nu-2)}+O(x^6)\right] \tag{A3}$$

(strictly speaking we have assumed that $l>2$ when giving the order of the remainder here). Differentiation of the product s_le_l gives the function λ_l:

$$\lambda_l(x)=\frac{1}{2\nu}\left[1-\frac{\frac{3}{2}x^2}{\nu^2-1}+\frac{\frac{15}{8}x^4}{(\nu^2-1)(\nu^2-4)}+O(x^6)\right], \tag{A4}$$

which upon differentiation once more yields

$$\lambda_l'(x)=-\frac{3x}{2\nu(\nu^2-1)}\left[1-\frac{\frac{5}{2}x^2}{\nu^2-4}+O(x^4)\right]. \tag{A5}$$

We work in the following to an accuracy of $O(x^4)$, as given in this formula.

We need also D_l and $\tilde{D}_l$. Writing the definition equations (3.9) as

$$\begin{aligned}D_l(x)&=\chi(x)s_l(x)e_l'(x)-1\ ,\\ \tilde{D}_l(x)&=\chi(x)s_l'(x)e_l(x)+1\ ,\end{aligned} \tag{A6}$$

and inserting the expansions

$$\begin{aligned}s_l(x)e_l'(x)&=-\frac{\nu-\frac{1}{2}}{2\nu}\left[1+\frac{\frac{3}{4}x^2}{(\nu^2-1)(\nu-\frac{1}{2})}+O(x^4)\right],\\ s_l'(x)e_l(x)&=\frac{\nu+\frac{1}{2}}{2\nu}\left[1-\frac{\frac{3}{4}x^2}{(\nu^2-1)(\nu+\frac{1}{2})}+O(x^4)\right],\end{aligned} \tag{A7}$$

we can work out the contribution to the force from the frequency region $x\in[0,\Delta]$, taking into account also that

$$\chi(x)=\chi_0\left[1-\frac{x^2}{x_0^2}+O(x^4)\right]. \tag{A8}$$

The result becomes

$$F(x\le\Delta)=-\frac{\Delta^3}{2\pi^2a^4}\sum_{l=1}^{\infty}\frac{\nu}{(\nu^2-1)\left[4\nu^2\left[\frac{\mu_0+1}{\mu_0-1}\right]^2-1\right]}\left[1-\frac{\frac{9}{5}\Delta^2}{\nu^2-1}\frac{1}{4\nu^2\left[\frac{\mu_0+1}{\mu_0-1}\right]^2-1}-\frac{\frac{3}{10}\Delta^2}{\nu^2-1}\frac{8\nu^2-17}{\nu^2-4}\right.$$
$$\left.-\frac{\frac{48}{5}\Delta^2}{x_0^2}\frac{\mu_0+1}{(\mu_0-1)^2}\frac{\nu^2}{4\nu^2\left[\frac{\mu_0+1}{\mu_0-1}\right]^2-1}+O(\Delta^4)\right]. \tag{A9}$$

The final sum over l is made numerically. In our calculation we used $\Delta=0.3$ as the upper limit. It is apparent that the contribution to the force from low frequencies is negative and thus of the same sign as the full force.

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