Global gauge anomalies for simple Lie algebras

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We generalize the formula by Elitzur and Nair on the global-anomaly coefficients in even ($D=2n$)-dimensional space and analyze global anomalies for Sp(2N), SO(N), and SU(N) groups. In particular, we show that any irreducible representation of any Sp(N) and SU(2) group has no global anomalies in $D=8k$ dimensions. In $D=8k+4$ dimensions, SU(2) has Z_2-type global anomalies only if the spin J of an irreducible representation has the form $J=\frac{1}{2}(1+4l)=\frac{1}{2},\frac{5}{2},\frac{9}{2},\ldots$. For any SU(N) group in $D=2n$, the global-anomaly coefficients can be expressed in terms of so-called unstable James numbers of Stiefel manifold $SU(n+1)/SU(n-k)$ and generalized Dynkin indices $Q_{n+1}(\omega)$ for $SU(n+1)$.

I. INTRODUCTION AND SUMMARY OF RESULTS

It was noted by Witten[1] in 1982 that an SU(2) gauge theory in four dimensions with an odd number of left chiral doublets of fermions is mathematically inconsistent, because of the global (nonperturbative) anomaly in the theory. This is partly reflected by the fact that the homotopy group for SU(2) is nonvanishing, $\Pi_4(SU(2))=Z_2$. Subsequently, Witten[2] and others[3] derived general formulas for global anomalies, including gravitation. However, if we are interested only in the pure gauge anomalies, then another method due to Elitzur and Nair[4] is often more convenient for the computation of the pure global-gauge-anomaly coefficient $A(\omega)$ for the representation (rep) ω. This method has been utilized by many authors[4-6] for the study of $A(\omega)$ for various SU(N) groups in $D=2n$ dimensions. In particular, in Ref. 6 (which will be referred to as I), we generalized the formula for $A(\omega)$ by Elitzur and Nair and proved the following: Assuming that the rep ω under discussion possesses no local (perturbative) anomaly, both $SU(n-1)$ in $D=2n$ $(n\geq 3)$ and $SU(2k+1)$ in $D=4k+2$ have no global anomalies, but $SU(2k)$ in $D=4k$ may have a Z_2 global anomaly.

In this paper, we will extend the results of I. First of all, we show that the global anomaly coefficient $A(\omega)$ for a rep ω of $SU(n-k)$ $(0\leq k\leq n-2)$ in $D=2n$ is expressed as

$$A(\omega)=\exp\left[2\pi i\frac{1}{d_{n+1,k+1}}Q_{n+1}(\tilde{\omega})\right], \tag{1.1}$$

where

$$d_{n+1,k+1}=\frac{n!}{U(n+1,k+1)}=\text{integer}. \tag{1.2}$$

The integral number $U(n+1,k+1)$ is the James number[7] for the complex Stiefel manifold, $SU(n+1)/SU(n-k)$. The definition and discussion of the James numbers wil be given in Sec. V and the Appendix. Here $Q_{n+1}(\tilde{\omega})$ is the $(n+1)$th generalized Dynkin index[8] for the rep $\tilde{\omega}$ of the $SU(n+1)$ group. The rep $\tilde{\omega}$ of $SU(n+1)$ must satisfy the requirement that under the reduction of $SU(n+1)$ into $SU(n-k)$ the rep $\tilde{\omega}$ reduces to a direct sum of ω and singlets of $SU(n-k)$. As noted in I, such a $\tilde{\omega}$ can be always found for a given ω of $SU(n-k)$, provided that we allow negative multiplicities corresponding to fermions with opposite chirality.

As an application of Eqs. (1.1) and (1.2), we will show in Sec. IV the following.

(i) Any irreducible representation (irrep) ω of SU(2) has no global anomaly in $D=8k$.

(ii) Only spinor reps of SU(2) with spins $J=\frac{1}{2}(4l+1)=\frac{1}{2},\frac{5}{2},\frac{9}{2},\ldots$, have Z_2 global anomalies in $D=8k+4$. Neither reps with $J=\frac{1}{2}(4l+3)=\frac{3}{2},\frac{7}{2},\ldots$, nor reps with $J=$(any integer) have global anomalies.

(iii) SU(3) does not give rise to global gauge anomalies in $D=8k+6$ for a rep ω which is free of local anomaly. We emphasize the fact that the absence of the local anomaly in (i) and (ii) is automatically satisfied for any ω of SU(2) in both $D=8k$ and $8k+4$ dimensions.

As another generalization of the method given in I, we prove in Sec. III (without assuming the local anomaly-free condition) the following.

(i) Any rep of Sp(2N) (with rank N) has no global anomaly in $D=8k$. This is in conformity with the result stated on SU(2), since $SU(2)\simeq Sp(2)$.

(ii) Any tensor rep of $SO(2l+1)$ $(l\geq 3)$ has no global anomalies in $D=8k+4$.

(iii) Any tensor rep of $SO(2l)$ $(l\geq 4)$ in $D=8k+4$ possesses no global anomalies, provided that the Young tableau[9] corresponding to the irrep ω does not contain any column which has boxes containing the maximally allowable number l.

(iv) No global anomaly exists in $D=20$ for G_2, even though $\Pi_{20}(G_2)=Z_2$. Similarly, F_4 has no global anomaly in $D=16$ in spite of $\Pi_{16}(F_4)=Z_2\oplus Z_2$.

We organize our paper as follows. In Sec. II we prove various general propositions needed for the main results mentioned above. The cases for Sp($2N$) and SO(N) groups are discussed in Sec. III. We show that the present method is not applicable in general for spinor reps of SO(N) groups. In Sec. IV we prove the formula (1.1) with (1.2) for SU(N) and show the connection with the James numbers. Finally, we make a comparison of the present method with that of Ref. 3 in order to assess their respective merits and limits.

II. BASIC PROPOSITIONS

Before going into detail, let H be the gauge group under consideration, which is assumed to be a simple connected compact Lie group. A generic rep ω of H will be expressed as

$$\omega=\sum_j \oplus m_j\omega_j \tag{2.1}$$

in terms of a direct sum of irreps ω_j with multiplicity m_j which are non-negative. However, it is more convenient (and even necessary for some cases) for our purpose to allow formal uses of negative values for the multiplicity coefficients m_j's. Physically, the negative value of m_j implies that we are considering the underlying fermion field with negative (or right-handed) chirality in contrast with that of the positive (or left-handed) chirality. This convention is justified for the present problem by the following reason. We are only interested in the calculation of

$$\mathrm{Tr}^{(\omega)}X^{n+1}=\sum_j m_j \mathrm{Tr}^{(\omega_j)}X^{n+1}\ , \tag{2.2}$$

where X is a generic rep of a generic element of the Lie algebra of H. Such a convention is more convenient than dealing with a discussion of the group $H_L\otimes H_R$ instead of H, where L and R refer to the left-handed and right-handed chiral groups, respectively. We note that such a usage has been already employed in the analysis of global anomalies[4–6] as well as that of local anomalies.[10] A similar use of negative multiplicities has been also utilized[11] for studies of Kronecker products and branching rules of reps of classical groups.

Hereafter, we restrict ourselves to the case of even dimensional spaces with $D=2n$ and assume that

$$\Pi_{2n}(H)\neq 0 \tag{2.3}$$

so that some rep ω of H may possess a global anomaly in $D=2n$. Now, the method of Ref. 4 (as well as that of I) is to reduce the problem of computing the global anomaly coefficient $A(\omega)$ of H to that of calculating the local anomaly of a group G in a rep $\tilde{\omega}$ as follows. For this purpose we must require that G and its rep $\tilde{\omega}$ must satisfy the conditions that (i) $G\supset H$ and $\Pi_{2n}(G)=0$ and (ii) $\tilde{\omega}$ reduces to a direct sum of ω and singlets of H when we restrict G to H. Note that the condition $\Pi_{2n}(G)=0$ implies that G has no global anomaly in $D=2n$. The second condition cannot be in general obeyed, unless we allow negative multiplicity coefficients $\tilde{m}_j$'s for $\tilde{\omega}$ of G as in Eq. (2.1):

$$\tilde{\omega}=\sum_j \oplus \tilde{m}_j\tilde{\omega}_j\ , \tag{2.4}$$

where $\tilde{\omega}_j$ are irreps of G with possibly negative multiplicity $\tilde{m}_j$. Since the theory will not be consistent if H has a local anomaly, we assume that the rep ω of H must obey

$$\mathrm{Tr}^{(\omega)}F^{n+1}\,|_H=0\ . \tag{2.5}$$

Here F are the Lie-algebra-valued field-strength differential two-forms.[12] As we have emphasized in I, Eq. (2.5) is stronger than that of the Green-Schwarz mechanism of anomaly cancellation,[13] and our results apply only to this stronger form of the local anomaly-free condition. However, we note that Eq. (2.5) is automatically satisfied for any self-contragredient rep ω of H in $D=4l\ (n=2l)$. In particular, if H is one of the groups

$$\begin{gathered}\mathrm{SU}(2),\ \mathrm{G}_2,\ \mathrm{F}_4,\ \mathrm{E}_7,\ \mathrm{E}_8,\ \mathrm{Sp}(2N)\ ,\\ \mathrm{SO}(2N+1)\ \ (N\geq 3),\ \ \mathrm{SO}(4N)\ \ (N\geq 2)\ ,\end{gathered} \tag{2.6}$$

then any rep ω of these groups automatically satisfies Eq. (2.5) in $D=4l$, since these groups allow[14] only real (orthogonal) or pseudoreal (symplectic) reps. As we noted elsewhere,[15] this reflects the fact that groups listed in Eq. (2.6) possess no odd-order Casimir invariants. In this paper we will not make any specific difference between SO(N) and Spin(N), since the distinction is rather immaterial for the present discussion of global anomalies.

Note that for the Lie groups H and G with $H\subset G$, G can be considered as a principal bundle over base space G/H with fiber H. More generally for a bundle $\beta=\{B,P,X,Y,\bar{G}\}$ with bundle space B, base space X, fiber Y, group $\bar{G}$, and projection P, let Y_0 be the fiber over $x_0\in X$ and $y_0\in Y_0$, and let $i:Y_0\to B$ and $j:B\to(B,Y_0)$ be the inclusion maps. Then we have the homotopy sequence[16] of (B,Y_0,y_0) given by

$$\to\Pi_N(Y_0)\xrightarrow{i_*}\Pi_N(B)\xrightarrow{j_*}\Pi_N(B,Y_0)\xrightarrow{\partial_*}\Pi_{N-1}(Y_0)\xrightarrow{i_*}\Pi_{N-1}(B)\to\ ,$$

where ∂ is the natural boundary operator and where i_*, j_*, and ∂_* are maps induced by i, j, and ∂, respectively. Let P_0 denote the restriction of P as a map $(B,Y_0,y_0)\to(X,x_0,x_0)$. Then P_0j is the projection $p:(B,y_0)\to(X,x_0)$. For $N\geq 2$ we have the isomorphism relation[16]

$$p_*:\Pi_N(B,Y_0)\sim\Pi_N(X,x_0)\ .$$

Defining $\Delta_*=\partial_*(P_{0*})^{-1}:\Pi_N(X,x_0)\to\Pi_{N-1}(Y_0,y_0)$, the exact homotopy sequence can be written as

$$\to\Pi_N(Y_0,y_0)\xrightarrow{i_*}\Pi_N(B)\xrightarrow{p_*}\Pi_N(X,x_0)\xrightarrow{\Delta_*}\Pi_{N-1}(Y_0,y_0)\xrightarrow{i_*}\Pi_{N-1}(B)\to\ .$$

Now, under our assumptions for H and G with $B=G$, $X=G/H$, and $Y=\bar{G}=H$ for all these Lie group spaces, the choice of the base points x_0 and y_0 are irrelevant in our discussion, since all the relevant homotopy groups based upon different points are isomorphic.

With the above preparations, we can now consider the following exact homotopy sequence:[16]

$$\Pi_{2n+1}(G)\xrightarrow{p_*}\Pi_{2n+1}(G/H)\xrightarrow{\Delta_*}\Pi_{2n}(H)\xrightarrow{i_*}\Pi_{2n}(G)=0\ . \tag{2.7}$$

Since H is assumed to be a simple compact connected Lie group, then $\Pi_{2n}(H)$ must consist of purely torsion groups, i.e., finite cyclic Abelian groups, proved by Serre.[17] Therefore, we may write

$$\Pi_{2n}(H)=\sum_{j=1}^{M}\oplus Z_{m_j}\quad(2\le m_j<\infty)\ . \tag{2.8}$$

Now, let m be the least common multiple of M integers, $m_1,m_2,\ldots,m_M$, appearing in Eq. (2.8). Then, in I, we have proved the following proposition.

Proposition 1. Suppose that we have $\Pi_{2n+1}(G)=Z$. Then the global anomaly coefficient $A(\omega)$ of H is given by

$$A(\omega)=\exp\left[2\pi i\frac{b}{m}Q_{n+1}(\tilde{\omega})\right]\ , \tag{2.9}$$

where b is an integer and $Q_{n+1}(\tilde{\omega})$ is the $(n+1)$th Dynkin index[8] of G in the rep $\tilde{\omega}$, which is always an integer.

We will prove the following propositions.

Proposition 2. Suppose that we have $\Pi_{2n+1}(G)=Z$ and $\Pi_{2n+1}(G/H)=Z\oplus T$, where T is a (finite) torsion group. Let $x\in\Pi_{2n+1}(G)$ and $y\in\Pi_{2n+1}(G/H)$ be generating elements of Z's. Suppose that they are related by

$$p_*(x)=dy+t\ , \tag{2.10}$$

where d is a nonzero integer and t is an element of T. Then, the global anomaly coefficient $A(\omega)$ of H is effectively given by

$$A(\omega)=\exp\left[2\pi i\frac{1}{d}Q_{n+1}(\tilde{\omega})\right] \tag{2.11}$$

in a sense that all other global anomalies are some integral powers of the expression $A(\omega)$ given by Eq. (2.11). There are two simple cases[18] where we can estimate the value of d immediately.

(i) If $\Pi_{2n+1}(G/H)=Z$ and $\Pi_{2n}(H)=Z_m$, then we have $d=m$.

(ii) If $\Pi_{2n+1}(G/H)=Z\oplus Z_l$ and $\Pi_{2n}(H)=Z_m$ with m, then l divides m. If m is an integral multiple of l^2, then[18] $d=m/l$. Moreover, if $l=m$, then $A(\omega)=1$ with $d=1$.

Three comments are in order.

(i) For a reason to be explained shortly, propositions 1 and 2 must be slightly modified when G is SO$(2n+2)$ with n being odd in $D=2n$, since then G possesses two independent $(n+1)$th-order indices $Q_{n+1}(\tilde{\omega})$ and $\hat{Q}_{n+1}(\tilde{\omega})$. This case will be discussed in Sec. III.

(ii) If G has no fundamental $(n+1)$th-order Casimir invariant,[8] then we may set $Q_{n+1}(\tilde{\omega})=0$, so that $A(\omega)=1$.

(iii) If $G=\mathrm{SU}(n+1)$ and $H=\mathrm{SU}(n-k)$ $(0\le k\le n-2)$, then the value of d is given by the formula Eq. (1.1). That is, d is inversely proportional to the James number.

Proposition 3. Suppose that $\Pi_{2n+1}(G)$ or $\Pi_{2n+1}(G/H)$ is zero or a finite group. Then we have $A(\omega)=1$ for H.

If $\Pi_{2n+1}(G)=0$, then $\Pi_{2n+1}(G/H)$ is isomorphic to $\Pi_{2n}(H)$ by the exactness of the homotopic sequence.

Proposition 4. If $\tilde{\omega}$ of G satisfies the local anomaly-free condition

$$\mathrm{Tr}^{(\tilde{\omega})}F^{n+1}\,|_G=0\ , \tag{2.12}$$

then we have $A(\omega)=1$.

Note that the validity of Eq. (2.12) immediately implies that of Eq. (2.5). Moreover, if G is one of the groups listed in Eq. (2.6), then the condition Eq. (2.12) is automatically satisfied in $D=4l$ (i.e., $n=2l=$even integer).

Proposition 5. Suppose that $\Pi_{2n+1}(G/H)$ contains a torsion part T such that its image $\Delta_*(T)$ coincides with $\Pi_{2n}(H)$. Then we have $A(\omega)=1$.

Moreover, if we have $\Pi_{2n+1}(G)=Z$ and $\Pi_{2n+1}(G/H)=Z\oplus T$ as in proposition 2, then proposition 5 is equivalent to proposition 2 for the case where $d=1$ and T is out of necessity isomorphic to $\Pi_{2n}(H)$.

Now, we will proceed for proofs[19] of these propositions. First, note that the map Δ_* is onto, since the exactness of the homotopy sequence Eq. (2.7) requires $\mathrm{Im}\Delta_*=\mathrm{Ker}i_*$. Thus, for any $h\in\Pi_{2n}(H)$, we can always find an element $\tilde{g}\in\Pi_{2n+1}(G/H)$, satisfying

$$\tilde{g}\xrightarrow{\Delta_*}h\xrightarrow{i_*}0\ .$$

However, since m is the smallest common multiple of $\{m_1,m_2,\ldots,m_M\}$ in Eq. (2.8), h^m is a trivial element of $\Pi_{2n}(H)$ so that $\Delta_*(\tilde{g}^{\,m})=0$. Therefore, the exactness condition, $\mathrm{Im}p_*=\mathrm{Ker}\Delta_*$, implies the existence of $g_0\in\Pi_{2n+1}(G)$, satisfying

$$g_0\xrightarrow{p_*}\tilde{g}^{\,m}\xrightarrow{\Delta_*}0\ . \tag{2.13}$$

Now, as in Ref. 4, we may identify h as a representative of the element $h(x):D^{2n}\to H$ for $x\in D^{2n}$. Compactifying D^{2n} into S^{2n}, then $h\in\Pi_{2n}(H)$. Following Ref. 4, we construct $\tilde{g}(y)\in G$, where $y\in D^{2n+1}$ such that $\tilde{g}(y)$ coincides with $h(x)$ on the boundary $\partial(D^{2n+1})=S^{2n}$. This construction is due to the following observation: $\tilde{g}(y):D^{2n+1}\to G$ is classified by the *relative* homotopy group, $\Pi_{2n+1}(G,H)$, which is isomorphic to $\Pi_{2n+1}(G/H)$ (Ref. 16). Moreover, the local anomaly-free condition, Eq. (2.5), guarantees the fact that the anomaly coefficient $A(\omega)$ does not depend upon the detail of $\tilde{g}(y)$, but is a functional of $\Pi_{2n+1}(G/H)$. For a given $h\in\Pi_{2n}(H)$, it may happen that we can find more than one $\tilde{g}\in\Pi_{2n+1}(G/H)$, which will be mapped to h. In such a case, we can choose any one of them for our purpose. This remark will be relevant for proposition 5.

Now, under these preparations, the global anomaly coefficient $A(\omega)$ can be computed[4] as a local anomaly of G by

$$A(\omega)=\exp\left[i\int_{D^{2n+1}}\gamma(\tilde{g},A,F)\right] \tag{2.14}$$

in the notation of Ref. 4. The integral in Eq. (2.14) is a linear functional of $\Pi_{2n+1}(G/H)$. Then, proposition 4 is almost self-evident, since there is no local anomaly of G under the assumption of proposition 4. Although a more formal proof based upon Eq. (2.14) can be given along the line of Ref. 4, we will not go into detail.

Next, we prove the first part of proposition 5. We rewrite Eq. (2.14) further as

$$A(\omega)=\exp\left[\frac{i}{N}\int_{D^{2n+1}}\gamma(\tilde{g}^N,A,F)\right] \tag{2.15}$$

when N is an arbitrary finite integer. If the assumption of proposition 5 is valid, then we can choose $\tilde{g}\in T$ for any $h\in\Pi_{2n}(H)$. However, since T is a finite group, there exists a finite positive integer N that $(\tilde{g})^N$ corresponds to a trivial element. Therefore, Eq. (2.15) implies $A(\omega)=1$. The second half of proposition 5 will be proved at the end.

We turn to proposition 3. Since N is arbitrary, we may choose $N=m$ in Eq. (2.15) and use the homotopy sequence Eq. (2.13) to rewrite Eq. (2.15) as

$$\begin{aligned}A(\omega)&=\exp\left[\frac{i}{m}\int_{D^{2n+1}}\gamma(\tilde{g}^m,A,F)\right]\\&=\exp\left[\frac{i}{m}\int_{S^{2n+1}}\gamma(g_0,A,F)\right]\end{aligned} \tag{2.16}$$

in the same notation as in Ref. 4. For an arbitrary finite nonzero integer N, we can rewrite Eq. (2.16) as

$$\begin{aligned}A(\omega)&=\exp\left[\frac{i}{mN}\int_{D^{2n+1}}\gamma(\tilde{g}^{mN},A,F)\right]\\&=\exp\left[\frac{i}{mN}\int_{S^{2n+1}}\gamma(g_0^N,A,F)\right].\end{aligned} \tag{2.17}$$

Suppose that $\Pi_{2n+1}(G)$ is a finite group. Then, there exists a finite positive integer N such that g_0^N is the trivial element of $\Pi_{2n+1}(G)$. Thus Eq. (2.17) leads to $A(\omega)=1$.

In order to prove proposition 1, let us suppose $\Pi_{2n+1}(G)=Z$ and let x be its generator. Then, the element $g_0\in\Pi_{2n+1}(G)$ can be written as

$$g_0=x^b \tag{2.18}$$

for an integer b. Therefore, Eq. (2.16) leads to

$$\begin{aligned}A(\omega)&=\exp\left[\frac{i}{m}\int_{S^{2n+1}}\gamma(x^b,A,F)\right]\\&=\exp\left[i\frac{b}{m}\int_{S^{2n+1}}\gamma(x,A,F)\right]\\&=\exp\left[2\pi i\frac{b}{m}Q_{n+1}(\tilde{\omega})\right]\end{aligned} \tag{2.19}$$

as in I, provided that we have only one $(n+1)$th-order Dynkin index $Q_{n+1}(\tilde{\omega})$. This proves proposition 1. Note that the uniqueness condition for $Q_{n+1}(\tilde{\omega})$ excludes the case of $G=\mathrm{SO}(2n+2)$, with n being odd. We will come back to this exceptional case in Sec. III.

Now, we proceed to prove proposition 2. Since $\Pi_{2n+1}(G/H)=Z\oplus T$ and y is the generator of the Z part of $\Pi_{2n+1}(G/H)$, we can write

$$\tilde{g}=y^k+t_0 \tag{2.20}$$

for some integer k and some $t_0\in T$. Hence, the sequence (2.13) may be written as

$$x^b\xrightarrow{p_*}y^{km}+t_0^m\xrightarrow{\Delta_*}0\ . \tag{2.21}$$

On the other hand, we have, by assumption,

$$x\xrightarrow{p_*}y^d+t\xrightarrow{\Delta_*}0 \tag{2.22}$$

for some $t\in T$. Consequently, we get

$$b=\frac{m}{d}k\ . \tag{2.23}$$

Actually, we can choose $k=1$ by the following reason. Since Δ_* is onto, there exists an element $h_0\in\Pi_{2n}(H)$, satisfying $y\xrightarrow{\Delta_*}h_0$. This implies that $A(\omega)$ for $k=1$ is the possible global anomaly generating element. Any other element $\tilde{g}\in\Pi_{2n+1}(G/H)$ can give an anomaly coefficient of the form $[A(\omega)|_{k=1}]^k$, since the torsion parts T do not give any contribution of $A(\omega)$ as can be seen in the proof of proposition 5. Thus, for the basic global anomaly, we can set $k=1$ in Eq. (2.21) and have $b/m=1/d$. For the special case of $\Pi_{2n}(H)=Z_m$ with $\Pi_{2n+1}(G/H)=Z$, we have $d=m$ as can be seen from the proof given in I.

If we have $\Pi_{2n}(H)=Z_m$ and $\Pi_{2n+1}(G/H)=Z\oplus Z_l$, then we can prove that l divides m when we assume $d\neq 0$. It is sufficient to prove that Δ_* restricted to Z_l is a one-to-one map to Z_m. Suppose that we have $\Delta_*(t_0)=0$ for some $t_0\in Z_l$. Then $t_0\in\mathrm{Ker}\Delta_*=\mathrm{Im}p_*$, so that there exists an element $x_0\in\Pi_{2n+1}(G)$, satisfying $p_*(x_0)=t_0$. Since we can express $x_0=x^k$ for some integer k in terms of the generating element $x\in Z$, then, $t_0=p_*(x_0)=kp_*(x)=k(dy+t)$. Since $t_0\in Z_l$, $k=0$, and $t_0=0$, l must divide m. Moreover, if $l=m$, then $\Delta_*(Z_l)=Z_m$. In this case, proposition 5 will lead to $A(\omega)=1$ and $d=1$, as we will see shortly. If m is an integral multiple of l^2, then the proof given in I together with the inspection of Eq. (2.10) gives $d=m/l$.

Now we study the relationship between propositions 2 and 5. We first note that proposition 2 implies $A(\omega)=1$ if $d=\pm1$, since $Q_{n+1}(\tilde{\omega})$ is an integer. Second, we remark that proposition 5 is valid, even though $\Pi_{2n+1}(G/H)$ and $\Pi_{2n+1}(G)$ have structures more complicated than those specified in proposition 2. However, if we assume that $\Pi_{2n+1}(G)=Z$ and $\Pi_{2n+1}(G/H)=Z\oplus T$, then proposition 5 is equivalent to the case of $d=\pm1$ in proposition 2 by the following reason. Suppose that $\Delta_*(T)=\Pi_{2n}(H)$ as in proposition 5, then for the generating element $y \in Z$ of $\Pi_{2n+1}(G/H)$, we set $\Delta_* y = t_0 \in \Pi_{2n}(H)$. However, by the assumption, there exists an element $\tilde{t}_0 \in T$ such that

$$\Delta_* \tilde{t}_0 = -t_0 \ .$$

Therefore, we find that $\Delta_*(y+\tilde{t}_0)=0$. Then, since $\mathrm{Ker}\Delta_* = \mathrm{Im}p_*$, there exists an element $g_2 \in \Pi_{2n+1}(G)$, satisfying

$$p_*(g_2) = y + \tilde{t}_0 \ . \tag{2.24}$$

Let $g_2 = x^N$ for some integer N. Then

$$p_*(g_2) = Np_*(x) = N(dy+t)$$

by Eq. (2.10). Comparing this with Eq. (2.24), we must have

$$Nd = 1 \ .$$

However, since both N and d are integers, we have $N = d \pm 1$. Conversely, suppose that we have $d = \pm 1$ in proposition 2. If $d=-1$, then we use $-y$ instead of y so that we can set $d=1$ without loss of generality for our purpose. Then we prove $\Delta_*(T) = \Pi_{2n}(H)$, which satisfy the condition of proposition 5.

The proof is as follows. Let us note that

$$\mathrm{Im}p_* = \bigcup_{k=-\infty}^{\infty} (ky+kt) \ ,$$

if we assume that $d=1$. However, since $\mathrm{Im}p_* = \mathrm{Ker}\Delta_*$, this implies that we have

$$k\Delta_* y + k\Delta_* t = 0 \tag{2.25}$$

for any integer k. Now, let h be a generic element of $\Pi_{2n}(H)$. Since Δ_* is onto, there exists an integer l and $\tilde{t} \in T$ such that

$$h = \Delta_*(y^l + \tilde{t}) = l\Delta_* y + \Delta_* \tilde{t} \ .$$

Using Eq. (2.25) with $k=l$, we have

$$h = \Delta_*(\tilde{t} - lt).$$

Since $\tilde{t} - lt \in T$, this proves that $\Delta_*(T) = \Pi_{2n}(H)$. Since $d=1$ in proposition 2 implies that $A(\omega)=1$, proposition 2 with $d=1$ and proposition 5 are consistent with each other, as it should be. Also, as we may see from the proof given above, we can in general find more than one element $\tilde{g}$ in $\Pi_{2n+1}(G/H)$ which corresponds to the same element h of $\Pi_{2n}(H)$ under the condition of proposition 5. Also, since Δ_* restricted to T is a one-to-one map as we have proved in connection with proposition 2, we see that if $d=1$, then T and $\Pi_{2n}(H)$ are isomorphic. We simply mention here that these facts will be an immediate consequence of a more general result $d = \mathrm{ord}[\Pi_{2n}(H)]/\mathrm{ord}(T)$ when $\Pi_{2n+1}(G) = Z$ and $\Pi_{2n+1}(G/H) = Z \oplus T$ which will be reported elsewhere.[18]

III. CHOICE OF $\tilde{\omega}$ AND Sp(2N) AND SO(N) GROUPS

In order to use our formulas, we have to find G and $\tilde{\omega}$ of G for a given ω of H, satisfying the condition that $\tilde{\omega}$ reduces to a direct sum of ω and singlets of H for the reduction of G into H. As we have emphasized in I, this is not in general possible, unless we allow negative multiplicity coefficients $\tilde{m}_j$ in Eq. (2.4), corresponding to the opposite chirality of the underlying fermions.

A. Choice of $\tilde{\omega}$ and G

$H = SU(N)$. First, we consider the case of $H = \mathrm{SU}(N)$. It is clearly sufficient for us to consider the case of ω being an irrep of H. Then, let Γ be the Young tableau[9] associated with ω. For any $\tilde{N} \geq N+1$, we can always find $\tilde{\omega}$ of $G = \mathrm{SU}(\tilde{N})$. We will show this by induction as in I. If ω is the fundamental rep of H with one box in its Young tableau, then we choose $\tilde{\omega}$ to be the corresponding fundamental rep of $G = \mathrm{SU}(\tilde{N})$ with one box. Clearly, this $\tilde{\omega}$ reduces to ω and singlets for the reduction of G to H. Now, we proceed to prove the general case by induction. Suppose that the Young tableau Γ associated with ω of SU(N) contains k boxes. Suppose that our assertion is correct for any ω of SU(N) whose Young tableau contains boxes equal to or less than k. Let ω be the rep with $k+1$ boxes in its Young tableau Γ. Let $\tilde{\omega}'$ be the irrep of SU($\tilde{N}$) with the same Young tableau Γ with $k+1$ boxes. Clearly $\tilde{\omega}'$ will reduce to a sum of ω and other reps whose Young tableau contain boxes less than or equal to k for the reduction of SU($\tilde{N}$) to SU(N). Therefore, by the induction hypothesis, we can always find $\tilde{\omega}$ of SU(N) such that it reduces to a direct sum of ω and singlets of SU(N), if we allow negative coefficients $\tilde{m}_j$ in Eq. (2.4). Thus, our statement will hold generally.

$H = Sp(2N)$. The same induction proof will clearly hold for any rep ω of $H = \mathrm{Sp}(2N)$ (N = rank of H) by choosing $G = \mathrm{Sp}(2\tilde{N})$ for any $\tilde{N} \geq N+1$.

$H = SO(N)$. For the case of $H = \mathrm{SO}(N)$, the situation is different. First, consider the case of $H = \mathrm{SO}(2l-1)$, and ω being its tensor rep. In this case, the induction method works for any $G = \mathrm{SO}(\tilde{N})$ with $\tilde{N} \geq 2l$. However, for the spinor rep of $H = \mathrm{SO}(2l-1)$, the Young tableau method is not applicable. We know that the group SO($2l-1$) contains the unique fundamental spinor rep of dimension 2^{l-1} and that the group SO($2l$) possesses two fundamental spinor reps of the same dimension. When we reduce SO($2l$) to SO($2l-1$), any one of these two fundamental spinors of SO($2l$) reduces to the unique spinor rep of SO($2l-1$). However, the spinor rep of SO($2l+1$) will reduce to the direct sum of two fundamental spinor reps of SO($2l$) under the reduction. This implies that we can choose only $G = \mathrm{SO}(2l)$ for the spinor rep of $H = \mathrm{SO}(2l-1)$, but we cannot in general find any G and $\tilde{\omega}$ for the spinor rep as well as self-dual reps of $H = \mathrm{SO}(2l)$. A more careful analysis of the branching rule[9,11] indicates the following.

(i) For any tensor rep of $H = \mathrm{SO}(2l-1)$, we can choose any SO($\tilde{N}$), satisfying $\tilde{N} \geq 2l$.

(ii) For any rep (including the spinor rep) of $H = \mathrm{SO}(2l-1)$, we can choose only $G = \mathrm{SO}(2l)$.

(iii) Let ω be any tensor rep of $H = \mathrm{SO}(2l)$ whose Young tableau Γ does not contain any column with maximally allowable l boxes in it. Note that this condition excludes self-dual tensor reps. For such a case, we can use any $G=\mathrm{SO}(\tilde{N})$ as long as $\tilde{N}\geq 2l+1$.

(iv) For spinor reps and self-dual tensor reps of $H=\mathrm{SO}(2l)$ $(l\geq 4)$, it appears in general that we cannot find any $\tilde{\omega}$ and G satisfying the desired condition. In other words, for such cases, our method seems powerless.

B. Sp(2N)

For $D=2n$, we must choose G so that $\Pi_{2n}(G)=0$, in addition to the condition for $\tilde{\omega}$. For symplectic groups, the Bott periodicity theorem[20] implies that

$$\Pi_k(\mathrm{Sp}(2\tilde{N}))=\begin{cases} Z, & k\equiv 3,7 \pmod 8 ,\\ Z_2, & k\equiv 4,5 \pmod 8 ,\\ 0, & k\equiv 0,1,2,6 \pmod 8 ,\end{cases} \tag{3.1}$$

for $4\tilde{N}\geq k-1$. Therefore, for $H=\mathrm{Sp}(2N)$, we can choose $G=\mathrm{Sp}(2\tilde{N})$ for any $\tilde{N}$, as long as $\tilde{N}\geq N+1$ and $4\tilde{N}\geq 2n-1$ for $D\equiv 0,2,6$ (mod 8). In particular, we have $\Pi_{2n+1}(G)=0$ for $D=8p$. Consequently, we find that *Sp(2N) has no global anomalies in $D=8p$* by proposition 3. We can use proposition 4 to show the same, since in $D=8p$ the local anomaly-free condition Eq. (2.12) is automatically satisfied. Using the isomorphism of $\mathrm{Sp}(2)\simeq \mathrm{SU}(2)$, we see that *SU(2) has no global anomalies in $D=8p$.* We will prove the same in the next section by choosing $G=\mathrm{SU}(\tilde{N})$ $(\tilde{N}\geq 3)$. For the case of $D\equiv 2,6$ (mod 8), we find that $\Pi_{2n+1}(G)=Z$, so that we can use only propositions 1 or 2. In this case, we use the trace identities given in I and prove that *both H=Sp(2n) and Sp(2n-2) have no global anomalies in $D=4l+2$.* However, in Sec. V we show more generally that *any group H listed in Eq. (2.6) has no global anomalies in $D=4l+2$,* if the local anomaly-free condition Eq. (2.5) is satisfied.

C. SO(N)

The Bott periodicity theorem[20] states that

$$\Pi_k(\mathrm{SO}(\tilde{N}))=\begin{cases} Z, & k\equiv 3,7 \pmod 8 ,\\ Z_2, & k\equiv 0,1 \pmod 8 ,\\ 0, & k\equiv 2,4,5,6 \pmod 8 ,\end{cases} \tag{3.2}$$

for $\tilde{N}\geq k+2$. For $H=\mathrm{SO}(N)$, we may choose $G=\mathrm{SO}(\tilde{N})$ with $\tilde{N}\geq N+1$ and $\tilde{N}\geq D+2$ in $D\equiv 2,4,6$ (mod 8), in order to satisfy $\Pi_{2n}(G)=0$. If $D=8p+4$ $(n=4p+2)$, then Eq. (3.2) implies that $\Pi_{2n+1}(G)=0$ also. Consequently, in $D=8p+4$ *no global anomalies exist for any tensor rep of H=SO(2l-1) as well as any tensor rep of H=SO(2l) where Young tableau contains columns with only boxes less than l.* Note that tensor reps of $H=\mathrm{SO}(2l)$ satisfying the condition are automatically self-contragredient reps which satisfy Eq. (2.5) in $D=8p+4$. For a general rep ω of $H=\mathrm{SO}(2l-1)$, we can choose only $G=\mathrm{SO}(2l)$. Hence, let $n=4p+2$ with $D=8p+4$. Moreover, the choice of l is further restricted by the requirements of $\Pi_{2n}(G)=0$, but $\Pi_{2n}(H)\neq 0$. For $l=n+1$ with $n\geq 5$, we have

$$\begin{aligned} &\Pi_{2n}(\mathrm{SO}(2n+2))=0, \quad n\not\equiv 0 \pmod 4 ,\\ &\Pi_{2n}(\mathrm{SO}(2n+1))=Z_2 ,\\ &\Pi_{2n+1}(\mathrm{SO}(2n+2)/\mathrm{SO}(2n+1))=Z ,\\ &\Pi_{2n+1}(\mathrm{SO}(2n+2))=\begin{cases} Z\oplus Z, & n\equiv 1 \text{ or } 3 \pmod 4 ,\\ Z\oplus Z, & n\equiv 0 \pmod 4 ,\\ Z, & n\equiv 2 \pmod 4 .\end{cases}\end{aligned} \tag{3.3}$$

Actually the first one is valid for $n\geq 1$ by Bott's periodicity theorem. The second one is valid for $n\geq 5$ while $\Pi_4(\mathrm{SO}(5))=Z_2$, $\Pi_6(\mathrm{SO}(7))=0$, $\Pi_8(\mathrm{SO}(9))=Z_2\oplus Z_2$ for $n=2,3,4$. The third one is valid for $n\geq 1$ by the fact that $\Pi_m(\mathrm{SO}(m+1)/\mathrm{SO}(m))=\Pi_m(S^m)$. The last one is valid for $n\geq 2$.

First, consider the case of $n=4p+2$ $(p\geq 0)$ where $\Pi_{2n}(G)=0$ but $\Pi_{2n}(H)\neq 0$. Then, $G=\mathrm{SO}(2n+2)$ has the unique $(n+1)$th-order Dynkin index, $Q_{n+1}(\omega)$. Note that it is the unique odd-order Dynkin index for $\mathrm{SO}(2n+2)$. In this case, $Q_{n+1}(\omega)$ will vanish identically for all ordinary (non-self-dual) tensor reps whose Young tableaux do not contain any $(n+1)$th row. We now have to normalize $Q_{n+1}(\omega)$ to be 1 and -1 for two fundamental spinor reps of $\mathrm{SO}(2n+2)$, respectively. Note that $Q_{n+1}(\omega)$ here corresponds to $\hat{Q}_{n+1}(\omega)$ of Ref. 8. Then, the anomaly coefficient is given by

$$A(\tilde{\omega})=\exp[i\pi Q_{n+1}(\tilde{\omega})] ,$$

since we have $d=m=2$ in proposition 2. Since the local anomaly-free condition Eq. (2.5) is automatically satisfied for $H=\mathrm{SO}(2n+1)$ for $n=$even, we see that $A(\tilde{\omega})=-1$ for the fundamental spinor rep. Thus, we conclude that $\mathrm{SO}(2n+1)$ in $D=2n=8p+4=4,12,20,\ldots$, can have a Z_2 global anomaly for spinor reps (but not for tensor reps, including self-dual ones).

Next, consider the case of $n=2k+1=$odd for $H=\mathrm{SO}(2n+1)$ in $D=2n$. The general result to be proven in Sec. V by another method implies that any locally anomaly-free self-contragradient representation ω of any H has no global anomaly in $D=4k+2$. Therefore, any locally anomaly-free rep ω of $H=\mathrm{SO}(2n+1)$ $(n=\text{odd}\geq 3)$ has no global anomaly in $D=4k+2=2n$. In particular, SO(7) in $D=6$ has no global anomaly in conformity with $\Pi_6(\mathrm{SO}(7))=0$. However, this fact is rather difficult to be shown by the present method. The reasons are as follows. First, for $G=\mathrm{SO}(2n+2)$ with $n=2k+1=$odd, we have

$$\Pi_{2n+1}(G)=\Pi_{2n+1}(\mathrm{SO}(2n+2))=Z\oplus Z \quad \text{for } n\geq 3 ,$$

in contrast with all the previous cases where we had $\Pi_{2n+1}(G)=Z$. This exceptional behavior is quite likely related to the fact that only $G=\mathrm{SO}(2n+2)$ with n being odd alone has two independent $(n+1)$th-order Dynkin indices $Q_{n+1}(\omega)$ and $\hat{Q}_{n+1}(\omega)$ (Ref. 8). We must normalize $Q_{n+1}(\tilde{\omega})$ to be 1 for the fundamental vector rep, but $\hat{Q}_{n+1}(\tilde{\omega})$ must be 1 and -1 for two fundamental spinor reps, since the latter vanish for the vector rep. Thus, for odd n, Eq. (2.11) of proposition 2 must be slightly modified as

$$A(\tilde{\omega})=\exp\{i\pi[b_1 Q_{n+1}(\tilde{\omega})+b_2\tilde{Q}_{n+1}(\tilde{\omega})]\}\ , \tag{3.4}$$

where b_j are integers. Any global anomaly is generated by two possible independent fundamental anomalies, corresponding to $(b_1=1,\ b_2=0)$ and $(b_1=0,\ b_2=1)$. Although $\hat{Q}_{n+1}(\omega)$ vanish for any non-self-dual tensor reps, $Q_{n+1}(\omega)$ is nonvanishing for spinor reps as well as for self-dual tensor reps. However, the numerical values of $Q_{n+1}(\omega)$ for spinors and self-dual tensor reps are always even for $n\geq 5$. Therefore, we may say that $Q_{n+1}(\tilde{\omega})$ controls global anomalies of all non-self-dual tensor reps, while $\hat{Q}_{n+1}(\tilde{\omega})$ does the same for both spinor reps and self-dual tensor reps of SO($2n+2$). The explicit formulas for both $Q_{n+1}(\omega)$ and $\hat{Q}_{n+1}(\omega)$ are found in Ref. 8. Here, we give a formula for $Q_p(\omega_s)$, where p is any even positive integer $(2\leq p\leq 2n)$ and ω_s is any of two fundamental spinor reps of SO($2n+2$):

$$Q_p(\omega_s)=\frac{1}{p}2^{n-1}(2^p-1)B_p\ , \tag{3.5}$$

where B_p is the Bernoulli number, i.e., $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, etc. From Eq. (3.5), we can prove that $Q_{n+1}(\tilde{\omega}_s)$ is an even integer for $n\geq 5$ as we have stated earlier. The derivation of Eq. (3.5) will be given elsewhere. The fact that $A(\tilde{\omega})$ given by Eq. (3.4) cannot assume the value -1 for any locally anomaly-free ω is not self-evident by this method since we have to take into account the local anomaly-free condition Eq. (2.5).

So far we have used G from the stable region where the Bott periodicity holds. We may use G from the unstable region. For example, $\Pi_{2n}(\mathrm{SO}(2n-5))=0$ for $n=6$. For $D=8P$ $(n=4P=4,8,12,\ldots)$, we have $\Pi_{2n}(\mathrm{SO}(2n+2))=Z_2\neq 0$, and hence we cannot use $G=\mathrm{SO}(2n+2)$. However, we can use $G=\mathrm{SU}(\tilde{N})$ for sufficiently large $\tilde{N}$ as we see from Eqs. (2.2) and (2.9). Such a case will be discussed elsewhere.

D. Exceptional Lie groups

Concluding this section, let us investigate the cases where H is one of the exceptional Lie groups G_2, F_4, E_6, E_7, and E_8.

For $H=\mathrm{G}_2$, we can always choose $G=\mathrm{SO}(N)$ or SU(N) for any N larger than 6 and satisfy the requirements specified in Sec. II. We note that two fundamental reps ω_1 and ω_2 of G_2 have dimensions 7 and 14, respectively. For both of them, we can find[21] formal tensor reps $\tilde{\omega}_1=7$ and $\tilde{\omega}_2=21\ominus 7$ of $G=\mathrm{SO}(7)$, where 7 and 21 are tensor reps of SO(7) with their dimensions 7 and 21, respectively. Since any irrep ω of G_2 can be constructed from products of two fundamental irreps ω_1 and ω_2, we can readily prove by induction that we can always find a general tensor rep $\tilde{\omega}$ of SO(7), such that $\tilde{\omega}$ reduces to sum of ω and singlets of G_2. Since $\tilde{\omega}$ is a tensor rep of SO(7), it is not difficult to find a tensor rep $\tilde{\omega}$ of $G=\mathrm{SO}(\tilde{N})$ for any $\tilde{N}\geq 7$. Also, by embedding SO(7) into SU(7), we can do the similar thing for $G=\mathrm{SU}(\tilde{N})$ $(\tilde{N}\geq 7)$. For the case of $G=\mathrm{SU}(7)$, this fact has been used in I to prove that any local anomaly-free rep ω of G_2 in $D=6$ has no global anomaly. In $D=8p+4$ for any positive integer p, we can show that G_2 has no global anomaly by choosing $G=\mathrm{SO}(\tilde{N})$ for a sufficiently large odd $\tilde{N}$ and using either proposition 3 or 4. However, the cases of $p=0$ and $p=1$ are trivial, since $\Pi_4(\mathrm{G}_2)=\Pi_{12}(\mathrm{G}_2)=0$ (Ref. 22). The first nontrivial case occurs at $p=2$ $(D=20)$ where $\Pi_{20}(\mathrm{G}_2)=Z_2$.

For $H=\mathrm{F}_4$, we can use $G=\mathrm{E}_6$ to do the similar thing. Using Table I (Ref. 22) we find that

$$\Pi_8(\mathrm{F}_4)=Z_2,\ \ \Pi_8(\mathrm{E}_6)=0,\ \ \Pi_9(\mathrm{E}_6)=Z\ ,$$

$$\Pi_{14}(\mathrm{F}_4)=Z_2,\ \ \Pi_{14}(\mathrm{E}_6)=0,\ \ \Pi_{15}(\mathrm{E}_6)=Z\ ,$$

$$\Pi_{16}(\mathrm{F}_4)=Z_2\oplus Z_2,\ \ \Pi_{16}(\mathrm{E}_6)=0\ ,$$

$$\Pi_{17}(\mathrm{E}_6)=Z\oplus Z_2\ ,$$

and $\Pi_D(\mathrm{F}_4)=0$ for $D=4,6,10,12$. Because E_6 possesses no fundamental Casimir invariants of order 15 and 17, proposition 1 implies that F_4 in $D=14$ and 16 has no global anomalies. In particular, in $D=16$, the local anomaly-free condition is automatically satisfied for F_4.

Unfortunately, for $H=\mathrm{E}_6$, E_7, and E_8, we cannot find in general $\tilde{\omega}$ and G, satisfying the requirement that $\tilde{\omega}$ reduces to ω and singlets of H. Thus, the present method is not applicable for the general study of global anomalies for these groups. However, for special cases, we can say something. Note that we have $\Pi_{20}(\mathrm{F}_4)=\Pi_{20}(\mathrm{E}_8)=0$, as well as the fact that both F_4 and E_8 are groups listed in

TABLE I. Homotopy groups $\Pi_k(G)$ for compact exceptional groups (courtesy of H. Toda).

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
G_2	0	0	∞	0	0	3	0	2	6	0	$\infty+2$	0	0	$168+2$	2	$6+2^2$	$8+2$
F_4	0	0	∞	0	0	0	0	2	2	0	$\infty+2$	0	0	2	∞	2^2	2
E_6	0	0	∞	0	0	0	0	0	∞	0	∞	12	0	0	∞	0	$\infty+2$
E_7	0	0	∞	0	0	0	0	0	0	0	∞	2	2	0	∞	2	2
E_8	0	0	∞	0	0	0	0	0	0	0	0	0	0	0	∞	2	2

k	18	19	20	21	22	23	24	25	26	27	28	29	30
G_2	240	6	2	0	$1386+8$								
F_4	$720+3$	2	0	3^2	27								
E_6	$720+6$	3	1512	3^2	$27+3$								
E_7	12	$\infty+2$	2	6	108	$\infty+2^2$	2^3	2^2					
E_8	24	0	0	2	0	$\infty+2$	2^2	6	0	∞	3		

Eq. (2.6). Thus, by proposition 4, any subgroup H of $G=\mathrm{F}_4$ or E_8 has no global anomaly in $D=20$, if the rep ω of H can be derived from $\tilde{\omega}$ of F_4 or E_8. Thus, those reps of G_2 and E_6 which can be derived from some reps of F_4 and E_8 have no global anomalies in $D=20$. For $H=\mathrm{SO}(9)$, the rep **36**$\oplus$ **16** is obtained from **52** of F_4 (Ref. 21). Since any non-self-dual tensor rep of SO(N) in $D=8p+4$ has no global anomaly, this implies that the 32-dimensional spinor of SO(9) has no global anomaly in $D=20$. Similarly, for $H=\mathrm{SO}(16)$, the rep **120**$\oplus$ **128** is obtained from **248** of E_8 (Ref. 21). Therefore, the fundamental spinor rep of SO(16) with its dimension 128 has no global anomaly in $D=20$, since the adjoint rep (**120**) has no global anomaly in $D=20$.

IV. SU(N) AND JAMES NUMBER

Let $G\supset H'\supset H$ and consider the homomorphisms between two homotopy sequences:[16]

$$\begin{array}{ccccccc} \Pi_{2n+1}(G) & \xrightarrow{p_*} & \Pi_{2n+1}(G/H) & \xrightarrow{\Delta_*} & \Pi_{2n}(H) & \xrightarrow{i_*} & \Pi_{2n}(G) \\ \downarrow i & & \downarrow q_* & & \downarrow \tilde{q}_* & & \downarrow i \\ \Pi_{2n+1}(G) & \xrightarrow{\tilde{p}_*} & \Pi_{2n+1}(G/H') & \xrightarrow{\tilde{\Delta}_*} & \Pi_{2n}(H') & \xrightarrow{\tilde{i}_*} & \Pi_{2n}(G)\ . \end{array} \tag{4.1}$$

We are interested in the case of

$$G=\mathrm{SU}(n+1),\quad H'=\mathrm{SU}(n)\ ,$$

$$H=\mathrm{SU}(n-k)\quad (1\le k\le n-2)\ .$$

Then the quotient group G/H is the complex Stiefel manifold $W_{n+1,k+1}=\mathrm{SU}(n+1)/\mathrm{SU}(n-k)$. Using the fact that in general

$$\Pi_{2n+1}(G/H)=\Pi_{2n+1}(W_{n+1,k+1})=Z\oplus T\ , \tag{4.2}$$

where T denotes a torsion group, we have

$$\begin{array}{ccccccc} Z & \xrightarrow{p_*} & Z\oplus T & \xrightarrow{\Delta_*} & \Pi_{2n}(H) & \xrightarrow{i_*} & 0 \\ \downarrow i & & \downarrow q_* & & \downarrow \tilde{q}_* & & \downarrow i \\ Z & \xrightarrow{\tilde{p}_*} & Z & \xrightarrow{\tilde{\Delta}_*} & Z_{n!} & \xrightarrow{\tilde{i}_*} & 0\ . \end{array} \tag{4.3}$$

The commutativity of the diagram implies

$$q_* p_* = \tilde{p}_* i\ . \tag{4.4}$$

Let x and z be the generators of $Z=\Pi_{2n+1}(\mathrm{SU}(n+1))$ and $Z=\Pi_{2n+1}(\mathrm{SU}(n+1)/\mathrm{SU}(n))$, respectively, and y be the generator of Z, contained in $\Pi_{2n+1}(W_{n+1,k+1})$. Then, the integer $U(n+1,k+1)$ defined by

$$q_*(y)=U(n+1,k+1)z \tag{4.5}$$

is called the (unstable) James number.[7,23] We remark that the James numbers are important in mathematics, since it will give information on existence or absence of a global section in the Stiefel manifold. On the other hand, it is easy to see

$$\tilde{p}_* i(x)=n!z \tag{4.6}$$

from the exact homotopy sequence in the second row of Eq. (4.3). Thus, the commutativity relation (4.4) gives[19]

$$p_*(x)=d_{n+1,k+1}y+t \tag{4.7}$$

for some $t\in T$, where $d_{n+1,k+1}$ is given by

$$d_{n+1,k+1}=\frac{n!}{U(n+1,k+1)}\ . \tag{4.8}$$

Comparing Eq. (4.7) with proposition 2, we find that $d=d_{n+1,k+1}$, which leads to Eqs. (1.1) and (1.2). The fact that $d_{n+1,k+1}$ is an integer is a trivial fact, since the map p_* is a homomorphism from Z to $Z\oplus T$. Unfortunately, the numerical values of the James number $U(n+1,k+1)$ are known only for few cases.[24,25] In particular, for $k=n-2$, corresponding to $H=\mathrm{SU}(2)$, we have[25]

$$d_{n+1,n-1}=\begin{cases} 1 & \text{for } n\equiv 0 \pmod 4\ , \\ 2 & \text{for } n\equiv 2 \pmod 4\ , \end{cases} \tag{4.9}$$

for even n, while for odd n

$$d_{n+1,n-1}=\begin{cases} d_{n+1,n-2}=\mathrm{denom}B_{n-1} & \text{for } n\equiv 3 \pmod 4\ , \\ \tfrac{1}{2}d_{n+1,n-2}=\tfrac{1}{2}\mathrm{denom}B_{n-1} & \\ \text{for } n\equiv 1 \pmod 4\ , & \end{cases} \tag{4.10}$$

where B_k is the Bernoulli number, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, etc., and denomB_k denotes the denominator of B_k, i.e., a product of primes p, which satisfies the condition that $p-1$ divides k.

A. $H=\mathrm{SU}(2)$

For $D=8k$ ($n=4k$), we see that $d_{n+1,n-1}=1$ so that the formula (1.1) implies that

$$A(\omega)=\exp[2\pi i Q_{n+1}(\tilde{\omega})]=1\ .$$

Therefore, *SU(2) has no global anomalies in $D=8k$ dimensions.* This conclusion of course agrees with that of the previous section based on the isomorphism between SU(2) and Sp(2).

For $D=8k+4$ ($n=4k+2$), $d_{n+1,n-1}=2$ so that

$$A(\omega)=\exp[\pi i Q_{n+1}(\tilde{\omega})]\ .$$

Now, recall that $\tilde{\omega}$ is a rep of SU($n+1$) which reduces to a direct sum of ω and singlets of SU(2). Then, as in Ref. 26, we can prove by induction that

$$Q_{n+1}(\tilde{\omega})\,|_{\mathrm{SU}(n+1)}=Q_2(\tilde{\omega})\,|_{\mathrm{SU}(n+1)} \pmod 2\quad (n\ge 2)\ . \tag{4.11}$$

Moreover, by branching index sum rule,[8] we can show the validity of

$$Q_2(\tilde{\omega})\,|_{\mathrm{SU}(n+1)}=Q_2(\omega)\,|_{\mathrm{SU}(2)}\equiv Q_2(\omega)$$

as in Ref. 26. Consequently, we conclude that

$$A(\omega)=\exp[\pi i Q_2(\omega)] \tag{4.12}$$

for $H=\mathrm{SU}(2)$ in $D=8k+4$ dimensions. Since the local anomaly-free condition for SU(2) in $D=8k+4$ is automatically satisfied, ω could be chosen as any rep of

SU(2). We consider ω to be an irrep, corresponding to a $(2J+1)$-dimensional rep with $J=0,\frac{1}{2},1,\frac{3}{2},2,\ldots$. We can then easily show that

$$Q_2(\omega)=\tfrac{2}{3}J(J+1)(2J+1)\ ,$$

which is even for all integers J as well as half-integer values of J of the form $J=\frac{1}{2}(4l+3)=\frac{3}{2},\frac{7}{2},\frac{11}{2},\ldots$. On the other hand, if J is a half-integer of the form $J=\frac{1}{2}(4l+1)=\frac{1}{2},\frac{5}{2},\frac{9}{2},\ldots$, then $Q_2(\omega)$ is an odd integer. Thus, we have proved that *among irreps of SU(2) only the half-integer spin irreps of the form $J=\frac{1}{2}(4l+1)$ have Z_2 global anomaly in $D=8k+4$.* In particular, this reproduces the result of Refs. 5 and 26 for $D=4$ ($k=0$).

For $D=8k+2$ or $8k+6$, the present method is not useful, since first we do not know the precise values of the James number and second we have to utilize the local anomaly-free condition Eq. (2.5) for ω. However, we will see in the next section that SU(2) in $D=4l+2$ has no global anomalies, provided that the local anomaly-free condition is satisfied.

B. SU(3) in $D=4l+2$

Using the fact that SU(2) has no global anomalies in $D=4l+2$, we can prove that any local anomaly-free rep ω of SU(3) will have no global anomalies in $D=8k+6$.

First note that

$$A(\omega)\,|_{SU(3)}=\exp\left[2\pi i\frac{1}{d_{n+1,n-2}}Q_{n+1}(\tilde{\omega})\right] . \tag{4.13}$$

Now, let us consider the branching of SU(3) into SU(2), where ω will reduce to a rep ω_0 of SU(2). Then $\tilde{\omega}$ of SU$(n+1)$ will also reduce to a direct sum of ω_0 and singlets of SU(2). Moreover, the local anomaly-free condition for SU(2) is automatically satisfied by our condition for ω being local anomaly-free in SU(3). Therefore, the global anomaly coefficient of ω_0 of SU(2) is given by

$$A(\omega_0)\,|_{SU(2)}=\exp\left[2\pi i\frac{1}{d_{n+1,n-1}}Q_{n+1}(\tilde{\omega})\right]$$

for the same $\tilde{\omega}$. However, if $D=8k+6$, we know from Eq. (4.10) that $d_{n+1,n-2}=d_{n+1,n-1}$ and hence we find

$$A(\omega)\,|_{SU(3)}=A(\omega_0)\,|_{SU(2)}\ .$$

Using the fact that $A(\omega_0)\,|_{SU(2)}=1$ in $D=4l+2$, we conclude that *SU(3) does not have global anomalies in $D=8k+6$ for local anomaly-free reps.* In particular, SU(3) has no global anomaly in $D=6$, which agrees with the result of I. For $D=8k+2$, we have $d_{n+1,n-2}=2d_{n+1,n-1}$ so that SU(3) may have a Z_2 global anomaly in $D=8k+2$. However, we can show by an explicit calculation that SU(3) in $D=10$ has no global anomaly.

C. $H=$SU$(n-k)$ with $k=0,1,2,3$

In connection with I, we study these cases, using James numbers.

For $H=$SU(n) in $D=2n$, the James number is given by

$$U(n+1,1)=1\ . \tag{4.14}$$

Therefore, we find

$$A(\omega)=\exp\left[2\pi i\frac{1}{n!}Q_{n+1}(\tilde{\omega})\right] ,$$

which is the same as in Ref. 4 and I. As we have explained in I, the quantity $(1/n!)Q_{n+1}(\tilde{\omega})$ is an integer if $n=$odd, but a half-integer in general if $n=$even, because of the local anomaly-free condition, Eq. (2.5).

For $H=$SU$(n-1)$, the corresponding James numbers are

$$U(n+1,2)=\begin{cases}2 & \text{for } n=\text{even}\geq 4\ ,\\ 1 & \text{for } n=\text{odd}\geq 3\ ,\end{cases} \tag{4.15}$$

which lead to

$$A(\omega)=\begin{cases}\exp\left[4\pi i\frac{1}{n!}Q_{n+1}(\tilde{\omega})\right] & \text{for } n=\text{even}\geq 4\ ,\\ \exp\left[2\pi i\frac{1}{n!}Q_{n+1}(\tilde{\omega})\right] & \text{for } n=\text{odd}\geq 3\ ,\end{cases}$$

in agreement with the calculation of I for the minimum value of $b=2$ for $n=3$ and $n=$even≥ 4 and $b=1$ for $n=$odd≥ 5. As we have shown in I, we have always $A(\omega)=1$ for these cases, because of the local anomaly-free condition, Eq. (2.5).

For $H=$SU$(n-2)$ and SU$(n-3)$, the situations become more complicated. The James numbers are given by

$$U(n+1,3)=U(n+1,4)=\frac{24(n-2,2)}{(24,n-2)(n+1,8)}\ , \tag{4.16}$$

where (p,q) is the greatest common divisor of p and q. The complication is of group-theoretical origin. The anomaly coefficient in these cases is given by

$$A(\omega)=\exp\left[i\frac{2\pi}{n!}\frac{24(n-2,2)}{(24,n-2)(n+1,8)}Q_{n+1}(\tilde{\omega})\right] .$$

The problem of calculating $Q_{n+1}(\tilde{\omega})$ becomes horrendous in these cases, since we have to reduce the trace identity of I for $G=$SU$(n+1)$ to $H=$SU$(n-2)$ or SU$(n-3)$. As a result, we are unable to give any general statement in these cases. However, assuming the local anomaly-free condition, we have verified that no global anomalies exist for the following special cases: SU(4) in $D=12$, SU(6) in $D=16$, SU(8) in $D=20$, and SU(10) in $D=24$ for SU$(n-2)$ cases with $n=6,8,10,12$. Similarly, no global anomalies exist for SU(3) in both $D=14$ and 18 with the local anomaly-free condition. Since these calculations involve a considerable amount of algebras, we will not discuss them. Note that in the Appendix, we give a method of computing the James numbers $U(n+1,k)$ for $k=5$ or 6 where we utilize the notion of stable James numbers related to the stunted projective space.

In ending this section, we note that our proposition 2 implies that we have

$$d_{n+1,k+1}=\frac{m}{l}=\frac{n!}{U(n+1,k+1)}\ , \tag{4.17}$$

if $\Pi_{2n+1}(\mathrm{SU}(n+1)/\mathrm{SU}(n-k))=Z\oplus Z_l$ and $\Pi_{2n}(\mathrm{SU}(n-k))=Z_m$ with m being an integral multiple of l^2. This happens for $D=6$: $n=3$, $k=1$, $l=2$, and $m=12$, which lead to $d_{4,2}=6$, in agreement with Eq. (4.15). This fact has been already used in I to establish the global anomaly-free property of SU(2) in $D=6$. Similarly, if we have

$$\Pi_{2n+1}(\mathrm{SU}(n+1)/\mathrm{SU}(n-k)) =Z \quad\text{and}\quad \Pi_{2n}(\mathrm{SU}(n-k))=Z_m\ ,$$

then proposition 2 implies

$$d_{n+1,k+1}=m\ . \tag{4.18}$$

Such a situation happens for the case of $k=1$ and $n=\text{even}\geq 4$ where we have $m=\frac{1}{2}n!$. This again agrees with Eq. (4.15).

V. COMPARISON WITH A DIFFERENT METHOD

In this section, we compare our method with a more general formula derived by Bismut and Freed.[3] They have derived the following formula for the $A(\omega)$:

$$A(\omega)=\exp[i\pi(\mathrm{ind}D_{2n}+2\xi)]\ , \tag{5.1}$$

where $\mathrm{ind}D_{2n}$ denotes the Atiyah-Singer index for the Dirac operator in $2n$-dimensional space M and

$$\xi=\int_{2n+2}\hat{A}\,\mathrm{Ch}F \tag{5.2}$$

is an integral involving the Dirac genus $\hat{A}$ and the Chern character $\mathrm{Ch}F$ in $(2n+2)$-dimensional space $M\times R^2$. Since we are interested only in the pure gauge anomaly, we can effectively set $\hat{A}=1$ for a flat space or a sphere.[27] Then, the integrand inside Eq. (5.2) in $2n+2$ dimensions is proportional to $\mathrm{Tr}^{(\omega)}F^{n+1}|_H$ which vanishes because of the local anomaly-free condition, Eq. (2.5). Hence, we find

$$A(\omega)=(-1)^{\mathrm{ind}D_{2n}}\ , \tag{5.3}$$

which proves that we have at most Z_2 global anomaly for any gauge group H. Moreover, $\mathrm{ind}D_{2n}$ is identically zero for $n=2k+1=\text{odd}$ for a real or pseudoreal rep of any gauge group, since then $\mathrm{ind}D_{2n}$ is in proportion to an integral involving $\mathrm{Tr}F^{\mathrm{odd}}$. In particular, those groups listed in Eq. (2.6) have no global anomalies in $D=4k+2$. This result is difficult to be shown by the method used in the previous sections except for some special cases. However, apart from these general statements, it is in general difficult to explicitly compute $\mathrm{ind}D_{2n}$. In contrast, for the method employed in this paper, the calculation of $A(\omega)$ is relatively simple, although it seems that our method is inapplicable to spinor reps of orthogonal groups as well as higher-dimensional reps of E_6, E_7, E_8 as stated in Sec. III. Nevertheless, both methods are complementary in their predictions. Our results sometimes give information on the index. For instance, our result that any rep of the Sp($2N$) group in $D=8p$ has no global anomaly will imply the fact that the Atiyah-Singer index for the Dirac operator must be even. Similarly, the same index for SU(2) in $D=8p+4$ can be either odd or even, depending upon whether the underlying rep has spin of the form $J=\frac{1}{2}(4l+1)$ or otherwise. In $D=4$, the relation between both methods are expressed as the even-odd rule

$$Q_3(\omega)\equiv Q_2(\omega)\ (\mathrm{mod}\ 2) \tag{5.4}$$

for the case of $H=\mathrm{SU}(N)$ $(N\geq 3)$ as has been explained in Refs. 26 and 28.

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APPENDIX: COMPUTING JAMES NUMBERS

We explain a method of computing the James number $U(n+1,k+1)$ for some values of k. Let $C(n+1,k)$ denote the stable James number of stunted complex projective spaces, where the stunted complex projective spaces[24] are the quotient spaces given by

$$\mathrm{CP}_{n+1+j,j}=\mathrm{CP}_{n+1+j}/\mathrm{CP}_{n+1} \tag{A1}$$

with CP_N denoting the N-dimensional complex projective space. We define a function of an integer N and primes p, $\nu_p(N)$, such that if an integer N contains a factor p^α for a prime p, we set $\nu_p(N)=\alpha$. Now, let $M_j(C)$ denote the Atiyah-Todd number[29] defined by

$$\nu_p(M_j(C))=\begin{cases}\operatorname*{Max}_r\,[r+\nu_p(r)], & 1\leq r\leq\left[\dfrac{j-1}{p-1}\right] \quad \text{if } p\leq j\ ,\\[2ex] 0 \quad \text{if } p>j\ ,\end{cases} \tag{A2}$$

where $[x]$ is the greatest integer not exceeding x. Then, we have the following theorem.

Theorem. $C(N,j)$ is equal to $U(lM_j(C)-N,j)$ if $lM_j(C)\geq N+2j-1$ for a positive integer l.

As an example, we calculate

$$C(N,5)=U(lM_5(C)-N,5)\quad\text{for } lM_5(C)\geq N+9\ .$$

The value of $M_5(C)$ can be computed from Eq. (A2) to be

$$M_5(C)=2^6\times 3^2\times 5=2880 \ .$$

For $lM_5(C)-N=n+1$, we have $n\geq 8$ and $N=lM_5(C)-n-1$. Therefore, we obtain

$$U(n+1,5)=C(2880l-n-1,5) \ .$$

On the other hand, we have

$$C(N,5)=C(N,4)\mathrm{denom}[C(N,4)\alpha_c(N,4)]$$

with

$$C(N,4)=C(N,3)=\begin{cases}\dfrac{24}{(N,24)} & \text{if } N\neq 4k+1\ ,\\ \dfrac{12}{(N,3)} & \text{if } N=8k+1\ ,\\ \dfrac{6}{(N,3)} & \text{if } N=8k+5\ ,\end{cases} \quad \text{(A3)}$$

and

$$\alpha_c(N,4)=\frac{N(15N^3+150N^2+485N+502)}{2^7\times 3^2\times 5} \ ,$$

where denom x denotes the denominator of a rational number x when the fraction is expressed in its lowest terms. The explicit values of $C(N,4)$ are given as

$$C(N,4)=\begin{cases}8 & \text{for } n=24l+8,24l+20\ ,\\ 24 & \text{for } n=24l,24l+4,24l+12,24l+16\ ,\\ 12 & \text{for } n=24l+1,24l+6,24l+9,24l+13,24l+21,24l+22\ ,\\ 4 & \text{for } n=24l+5,24l+14,24l+17\ ,\\ 2 & \text{for } n=24l+10,24l+11,24l+18\ ,\\ 6 & \text{for } n=24l+2,24l+3,24l+19\ ,\\ 1 & \text{for } n=24l+23\ ,\\ 3 & \text{for } n=24l+7,24l+15\ ,\end{cases}$$

for $N=2880l-n-1$. If we consider $H=\mathrm{SU}(n-4)$, with $2880>n+1$, we can choose $l=1$. For $l\geq 2$, the results remain the same due to the James periodicity property.

Example 1. For example, for $n=8$ we compute $\mathrm{denom}[8\alpha_c(N,4)]=10$, so that $U(9,5)=80$. Therefore, the global anomaly coefficient for $H=\mathrm{SU}(4)$ in $D=16$ is given by

$$A(\omega)=\exp\left[2\pi i\frac{80}{8!}Q_9(\tilde{\omega})\right]=1 \ ,$$

where we used the local anomaly-free condition Eq. (2.5) for the evaluation of $Q_9(\tilde{\omega})$ as in I.

Example 2. For $n=10$, we calculate $U(11,5)=2^4\times 3\times 5=240$ and we calculate, similarly,

$$A(\omega)=\exp\left[2\pi i\frac{240}{9!}Q_{11}(\tilde{\omega})\right]=1$$

for $H=\mathrm{SU}(6)$ in $D=20$.

Example 3. For $n=12$, we find $U(13,5)=24\times 3\times 5$.

For $U(N,6)$ $[H=\mathrm{SU}(n-5)]$, we use

$$U(n+1,6)=C(lM_6(C)-n-1,6) \ ,$$

if $lM_6(C)\geq lM_6(C)-n-1+2\times 6-1$. Together with the relation

$$C(N,6)=\begin{cases}C(N,5) & \text{if } N=2l,32l+1,32l+11,32l+27\ ,\\ 2C(N,5) & \text{otherwise}\ ,\end{cases}$$

we can calculate $U(n+1,6)$. For example, we have

$$U(11,6)=2^5\times 3\times 5 \quad \text{in } D=20\ ,$$

$$U(13,6)=2U(13,5)=2\times 24\times 3\times 5 \quad \text{in } D=24\ .$$

We can compute formulas for $U(n+1,k)$ with $7\leq k\leq 10$ except for ambiguity in $k=9$ and 10 as in Ref. 24. However, we will not give them here.

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