Spontaneous compactification and coupling constants in R^2 unified gauge theories

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A model problem is solved for a six-dimensional spacetime where ordinary four-space is flat and the two extra dimensions have the geometry of a two-sphere. The geometry is driven by coupled Yang-Mills and Higgs fields. The equations of motion are derived from a geometric theory of the canonical gravitation-Yang-Mills-Higgs fields. The constant radius of the two-sphere is determined. If certain reasonable values are taken for various arbitrary constants in the theory, the radius is of the order of the Planck length and an exact value for the coupling constant of the Yang-Mills field is obtained.

I. INTRODUCTION

Recently there has been an explosion of interest in Kaluza-Klein theories[1] where spacetime is postulated to have more than the usual four dimensions. The main impulse for this work has come from the fact that string and superstring models of field theories have critical dimensions that are greater than four. The embarrassment of these extra dimensions, which goes back to the original ideas of Kaluza and Klein, is that they are not perceived. The solution is just as old and lies in supposing that the extra dimensions have a compact topology. In a space of dimension $d>4$, the topology of the compact extra dimensions is often assumed to be S^{d-4}, where the "radius" associated with these dimensions is small. Modern approaches have the advantage of allowing one to calculate (or at least put limits on) this radius given a series of assumptions about the field equations that the d-dimensional geometry obeys, while in the original work of Kaluza and Klein the radius was given *a priori*.

One of the first modern attempts to calculate the radius of the extra dimensions was due to Cremmer and Scherk.[2] They solved the model problem of gravitation in six dimensions coupled to a Yang-Mills field and a Higgs field, both with SO(3) internal symmetry. They assumed that the physical four-dimensional spacetime was flat and that the two-dimensional compact space was a two-sphere of radius R_0. The simultaneous solution of the Yang-Mills field equations, the equations for the Higgs field, and the six-dimensional Einstein equations $G_{ij}=8\pi T_{ij}$ $(i,j=0,\dots,5)$ shows that $R_0 \le (1/2\kappa e^2)$, where $1/\kappa$ is the Planck length and e is the dimensionless coupling constant of the Yang-Mills field. If e^{-2} is not extremely large, R_0 is quite small.

In the original work of Cremmer and Scherk the action functional was constructed in the usual fashion by summing the Lagrangian for gravity and the Lagrangians for the Yang-Mills and Higgs fields in a curved space. At almost the same time an interesting and more unified method of constructing these Lagrangians based on principal fiber bundles was developed. The idea seems to have originated with Cho.[3] Recently Katanayev and Volovich[4] extended Cho's work to include the possibility of torsion in the fiber-bundle connection. This approach has the advantage of giving an action that automatically includes a Yang-Mills field and the Higgs field as geometric objects that arise naturally from the connection which has torsion on the whole fiber bundle but where the four-dimensional spacetime components of the torsion are zero. The price that one pays for this "natural" appearance of the Higgs fields is that one needs Lagrangians that are quadratic in the fiber-bundle curvature. In sketch form this can be easily seen, because the usual Ricci scalar that leads to the most natural Lagrangian density is quadratic in the torsion, and since the Higgs fields Φ^i are defined in terms of components of the torsion, such Lagrangians densities are at most quadratic in the Φ^i. Terms of the form $R\dots R^{\cdots}$ will have torsion terms of the form $SSSS$ that will give rise to the Φ^4 terms necessary for spontaneous symmetry breaking. This "price" that we pay for having the Higgs fields from the fiber-bundle geometry is actually an advantage in disguise, since Lagrangians quadratic in the curvature arise naturally in the low-energy limit of superstring theory.

The basic idea of this paper is to redo the calculation of Cremmer and Scherk starting from Lagrangian given by the fiber-bundle approach. The most general Lagrangian, as given by Katanayev and Volovich, is a sum of scalar densities formed from geometrical objects on the fiber bundles, each term with an undetermined coefficient. The scalars that are integrated over the invariant volume element are (1) the usual Ricci scalar, (2) the scalars formed from contractions of second-order products of the Riemann and Ricci tensors, (3) the square of the Ricci scalar (this possibility seems to have been overlooked by Katanayev and Volovich), and (4) the scalars formed from contractions of second-order products of the torsion. Linear combinations of the arbitrary coefficients of each of the above terms appear as numerical factors multiplying such quantities as the second-order products of the four-dimensional curvature, quadratic, and quartic terms in the Higgs field, the free Lagrangian density of the Yang-Mills field, etc. These factors must be identified with the usual parameters such as Λ, the cosmological constant; κ, the factor multiplying the Ricci scalar in the Einstein Lagrangian; m, the mass associated with the Higgs field; e, the coupling constant of the Yang-Mills field; and λ, the parameter multiplying the quartic term in the Higgs field. In general there will be enough free parameters in the fiber-bundle Lagrangian to leave all of the physical parameters undetermined (although the factor multiplying one important new term in the Lagrangian is determined as a multiple of λ), but if restrictions are placed on the factors, one ends up with algebraic relations among the physical parameters which lead to physical predictions that we will discuss below.

As we mentioned above, we will redo the calculation of Cremmer and Scherk for our new Lagrangian [the SO(3) symmetry group is achieved by choosing the appropriate structural group for the fiber bundle]. This recalculation turns out to be possible, the only difficulty being slightly more complicated algebraic equations. In the case where no restrictions are placed on the free parameters in the Lagrangian, the only difference between our solution and that of Cremmer and Scherk is a somewhat less restrictive lower limit on R_0. A useful restriction on the parameters of the theory is suggested by the low-energy limit of string theories. It is well known that the Gauss-Bonnet combination of R^2 terms, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}-4R^{\alpha\beta}R_{\alpha\beta}+R^2$, leads to ghost-free nontrivial gravitational self-interactions for dimensions higher than four.[5] If we apply this restriction to the torsion-free part of our Lagrangian, an interesting set of relationships among the physical parameters appears that leads directly to a prediction of the size of the dimensionless coupling constant of the Yang-Mills field (or of the Weinberg-Salam angle if one prefers) that fits the best-known experimental value within 2%.

The organization of the paper is as follows. Section II is devoted to the construction of the most general Lagrangian in terms of the curvature and torsion of the fiber bundle. In Sec. III we do the analogue of the calculation of Cremmer and Scherk. In Sec. IV we discuss the possible restrictions on the general Lagrangian, with special emphasis on the Gauss-Bonnet form of the curvature terms, and their effect on the problem. Section V covers conclusions and suggestions for further research.

II. PRINCIPAL-FIBER-BUNDLE STRUCTURE AND UNIFIED LAGRANGIAN

As pointed out in the Introduction, Cho[3] has developed a comprehensive geometrical treatment for a Kaluza-Klein-type unification of gauge fields and gravitation. The procedure involves in essence a principal fiber bundle $\pi:P\rightarrow M$ with characteristic group G over a spacetime M with metric g. If ω is a connection one-form on P and k is an $\mathfrak{Ad}$-invariant metric on the Lie algebra $\mathcal{G}$ of G, then a metric h can be constructed on P, which depends on g, k, and ω. Furthermore, one can verify that for all $A\in G$, the right action $R_A:P\rightarrow P$ on the fibers is an isometry of (P,h), from which it follows that the scalar curvature $R:P\rightarrow\mathbb{R}$ of h is constant on the fibers and thus yields a well-defined function on the base space M. Integrating $R(g,\omega)$ over an open subset U of M with compact closure, and assuming that this integral is stationary both for independent variations of g and ω, the Einstein and Yang-Mills field equations arise simultaneously from this single variational principle that derives from the scalar curvature of the metric h on P.

One restriction which is contained in Cho's work is that the connection one-form $\theta(h)$, from which the curvature tensor of (P,h) is derived, is a Levi-Civita (torsion-free) connection. This assumption implies that the projection on M of the scalar curvature of P with bundle metric $h=\pi^*g+k\omega$ is the function $\underline{R}(g)+\mathcal{S}(g,\omega)+R_G$, where $\underline{R}(g)$ is the scalar curvature of M, $\mathcal{S}(g,\omega)\equiv-\frac{1}{4}k(\Omega^{hi},\Omega_{hi})$ is the self-action of ω, and R_G is the (constant) scalar curvature of the fibers of P with metric induced by h and which plays the nontrivial role of a cosmological constant in the theory.

For the purpose outlined in the Introduction, we now want to generalize the results of Cho by allowing an arbitrary connection in the formalism. As noted by Katanayev and Volovich,[4] such a generalization makes possible the construction of Lagrangians quadratic in the Riemann and Ricci tensors on (P,h), which when projected on the base manifold provide a dynamical character to the torsion terms that may thus be identified with the Higgs fields. Not only do these scalars appear in the theory in a natural and geometrical fashion, but so does the quartic-type potential needed for the spontaneous symmetry-breaking process which gives mass to the gauge vector bosons.

Although Katanayev and Volovich display in their Letter the forms of the components of the Riemann, Ricci, and torsion terms resulting from a specific connection suited for their construction, we feel that both a more general and more detailed derivation in this section might be useful, since, as far as we know, such material has not been published before or, at least, it is not easily accessible. One additional benefit from this extra work is that we will be able to present to the reader an easy procedure for relating notation and dimensional units commonly used by differential geometers with those appearing in the work of field theorists and particle physicists.

Those readers less interested in the underlying mathematics of this paper may readily skip this section and need only refer to the basic results which are contained in Eqs. (2.35)–(2.49) to be able to follow the remainder of this work.

For notation, we shall rely heavily on the one used by Bleecker[6] (Secs. 6.2 and 9.3). The reader will also find there some of the details which have been omitted here for the sake of brevity.

We thus consider an orthonormal bundle of frames

$\pi: F(P) \rightarrow P$, for which our principal fiber bundle P is a base manifold. A frame at $p \in P$ is given by $\mathbf{E}_1, \ldots, \mathbf{E}_n, \mathbf{E}_{n+1}, \ldots, \mathbf{E}_{n+f}$, where $\mathbf{E}_1, \ldots, \mathbf{E}_n$ [defined on $\pi^{-1}(U) \subset P$] are horizontal lifts, relative to ω, of an orthonormal basis $\overline{\mathbf{E}}_1, \ldots, \overline{\mathbf{E}}_n$ on (M,g) [i.e., $\omega(\mathbf{E}_i)=0$, $\pi_*(\mathbf{E}_i)=\overline{\mathbf{E}}_i$], and $\mathbf{E}_{n+\alpha}=e_\alpha^*$, where e_α^* are fundamental vertical fields on P [i.e., $\omega(e_\alpha^*)=e_\alpha \in \mathcal{G}$] generated by the elements of an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_f$ of $\mathcal{G}$ relative to k. We will use the following ranges for our indices: latin lower-case letters from the middle of the alphabet will have the range $1 \leq i,j,k, \ldots \leq n$, greek lower-case letters will have the range $1 \leq \alpha, \beta, \gamma, \ldots \leq f$, and lower-case latin letters from the beginning of the alphabet will cover the total range of the dimensionality of P, $1 \leq a,b,c,d, \ldots \leq n+f$.

Note that from the bundle metric

$$h=\pi^* g+k\omega \ , \tag{2.1}$$

and the orthonormality condition on our basis, we have

$$h_{ij} \equiv h(\mathbf{E}_i,\mathbf{E}_j)=(\pi^* g)(\mathbf{E}_i,\mathbf{E}_j)=g(\overline{\mathbf{E}}_i,\overline{\mathbf{E}}_j) \equiv g_{ij}=\pm\delta_{ij}$$

and

$$h_{\alpha\beta} \equiv h(\mathbf{E}_{n+\alpha},\mathbf{E}_{n+\beta})=k\omega(e_\alpha^*,e_\beta^*) \equiv k[\omega(e_\alpha^*),\omega(e_\beta^*)] = k(e_\alpha,e_\beta)=\pm\delta_{\alpha\beta} \ .$$

Now, let $\bar{\phi}^1, \ldots, \bar{\phi}^{n+f}$ be one-forms dual to $\mathbf{E}_1, \ldots, \mathbf{E}_{n+f}$. We can then write the curvature $\Omega \in \overline{\Lambda}^2(P,\mathcal{G})$ of $\omega \in \Lambda^1(P,\mathcal{G})$ in terms of the above bases as $\Omega=\frac{1}{2}\Omega_{ij}^\alpha(\bar{\phi}^i \wedge \bar{\phi}^j) \otimes e_\alpha$, and the torsion two-form $\Theta \in \overline{\Lambda}^2(P,\mathbb{R}^{n+f})$ as $\overline{\Theta}=\frac{1}{2}S^a{}_{bc}(\bar{\phi}^b \wedge \bar{\phi}^c) \otimes e_a$, where $e_a \equiv u^{-1}(\pi_* E_a)$ is the usual basis of $\mathbb{R}^{n+f}$, and the linear isomorphism $u: \mathbb{R}^{n+f} \rightarrow T_p P$ defines a frame at $p \in P$. If we further use the symbol $\theta(h)=[\theta(h)]^a{}_b \in \Lambda^1(F(P),\mathbb{R})$ to denote the real-valued one-form matrix representing a general connection of $F(P)$ relative to the above choice of orthonormal fields and dual forms, we can write

$$\overline{\Theta}^a \equiv \tilde{\sigma}^* D^{\theta(h)}\phi^a = D^{\bar{\theta}(h)}\bar{\phi}^a = d\bar{\phi}^a+\bar{\theta}(h)^a{}_b \wedge \bar{\phi}^b \ . \tag{2.2}$$

Similarly, if $\bar{\phi}_m^1, \ldots, \bar{\phi}_m^n$ are one-forms on M dual to $\overline{\mathbf{E}}_1, \ldots, \overline{\mathbf{E}}_n$, such that $\pi^* \bar{\phi}_M^i=\bar{\phi}^i$, we have

$$d\bar{\phi}_M^i=\Theta_M^i-\bar{\theta}(g)^i{}_j \wedge \bar{\phi}^j{}_M \ ,$$

or

$$d\bar{\phi}^i=\Theta^i(g)-\pi^* \bar{\theta}(g)^i{}_j \wedge \bar{\phi}^j \ , \tag{2.3}$$

where $\Theta^i(g)$ is the pullback with π^* of $\Theta^i{}_M$. Moreover, if the local section $\sigma: U \rightarrow F(M)$ determined by $\overline{\mathbf{E}}_1, \ldots, \overline{\mathbf{E}}_n$ is tangent to the horizontal subspace of $T_{\sigma(x)}F(M)$ relative to $\theta(g)$, we have

$$\theta(g)(\sigma_* \overline{\mathbf{E}}_i)=[\sigma^* \theta(g)](\overline{\mathbf{E}}_i) \equiv \bar{\theta}(g)(\overline{\mathbf{E}}_i)=0 \quad \text{at } x \ ,$$

and, by virtue of (2.3),

$$d\bar{\phi}^i(\mathbf{E}_j,\mathbf{E}_k)=\Theta^i{}_{jk}=S^i{}_{jk} \ . \tag{2.4}$$

Observe also that if we write $\omega=\omega^\alpha e_\alpha$, we have $\omega(e_\beta^*)=e_\beta=\omega^\alpha(e_\beta^*)e_\alpha \Longrightarrow \omega^\alpha(e_\beta^*)=\delta_\beta^\alpha$. But $\bar{\phi}^{n+\alpha}(\mathbf{E}_{n+\beta}) = \bar{\phi}^{n+\alpha}(e_\beta^*)=\delta_\beta^\alpha \Longrightarrow \omega^\alpha=\bar{\phi}^{n+\alpha}$.

Making use of these basic definitions, we are now ready to summarize the relevant steps to be followed for computing the components of the curvature tensor of (P,h), emphasizing those results which involve generalizations of the ones given in Ref. 6.

First, recall that the components of the Riemann tensor are related to the connection one-forms by means of

$$\tfrac{1}{2}R^a{}_{bcd}\bar{\phi}^c \wedge \bar{\phi}^d = d\bar{\theta}(h)^a{}_b+\bar{\theta}(h)^a{}_c \wedge \bar{\theta}(h)^c{}_b \ . \tag{2.5}$$

Thus, in order to evaluate $R^a{}_{bcd}$ we need to calculate first the various matrix terms $\bar{\theta}(h)^a{}_b$ for $1 \leq a,b \leq n+f$. To this end note that (2.2) implies that

$$d\bar{\phi}^{n+\alpha}=\overline{\Theta}^\alpha-\bar{\theta}(h)^{n+\alpha}{}_i \wedge \bar{\phi}^i-\bar{\theta}(h)^{n+\alpha}{}_{n+\gamma} \wedge \bar{\phi}^{n+\gamma} \ . \tag{2.6}$$

Also, from the definition of the curvature two-form we have

$$\begin{aligned}(d\bar{\phi}^{n+\alpha})e_\alpha &= d\omega \\ &= -\tfrac{1}{2}[\omega,\omega]+\Omega \\ &= (-\tfrac{1}{2}c^\alpha{}_{\beta\gamma}\bar{\phi}^{n+\beta} \wedge \bar{\phi}^{n+\gamma}+\tfrac{1}{2}\Omega^\alpha{}_{ij}\bar{\phi}^i \wedge \bar{\phi}^j)e_\alpha \ ,\end{aligned} \tag{2.7}$$

where $c^\alpha{}_{\beta\gamma}$ are the structure constants of G.

Comparing (2.6) and (2.7) we get

$$\overline{\Theta}^\alpha-\bar{\theta}(h)^{n+\alpha}{}_i \wedge \bar{\phi}^i-\bar{\theta}(h)^{n+\alpha}{}_{n+\gamma} \wedge \bar{\phi}^{n+\gamma} = -\tfrac{1}{2}c^\alpha{}_{\beta\gamma}\bar{\phi}^{n+\beta} \wedge \bar{\phi}^{n+\beta}+\tfrac{1}{2}\Omega^\alpha{}_{ij}\bar{\phi}^i \wedge \bar{\phi}^j \ . \tag{2.8}$$

Furthermore, since the connection one-form $\bar{\theta}(h)$ is $\mathcal{O}(r,s)$ valued, the matrix elements $\bar{\theta}(h)^a{}_b$ must satisfy the constraint

$$\bar{\theta}(h)_{ab}+\bar{\theta}(h)_{ba}=0 \ , \tag{2.9}$$

which together with the $\mathfrak{Ad}$-invariance property of the metric k, suggests as a natural choice

$$\bar{\theta}(h)^{n+\alpha}{}_{n+\gamma}=\tfrac{1}{2}c^\alpha{}_{\beta\gamma}\bar{\phi}^{n+\beta} \ . \tag{2.10}$$

We feel that it is important to underline here that this choice is not unique. Clearly the factor in front of the right side of (2.10) is completely arbitrary and is motivated in our approach only by the reasonable demand that in the limit of zero torsion our results should reduce to those given in Ref. 6. This implies, however, that the components of our Riemann and torsion tensors do not agree exactly with those given in Ref. 4.

Note that if we now substitute (2.10) in (2.8) and evaluate on $(\mathbf{E}_{n+\beta},\mathbf{E}_{n+\gamma})$, $(\mathbf{E}_{n+\beta},\mathbf{E}_i)$, and $(\mathbf{E}_i,\mathbf{E}_j)$ we get, respectively,

$$S^\alpha{}_{\beta\gamma}=0 \ , \tag{2.11a}$$

$$S^\alpha{}_{\beta i}=\bar{\theta}(h)^{n+\alpha}{}_i(\mathbf{E}_{n+\beta})=0 \ , \tag{2.11b}$$

and

$$\bar{\theta}(h)^{n+\alpha}{}_i=\tfrac{1}{2}(\Omega^\alpha{}_{ij}-S^\alpha{}_{ij})\bar{\phi}^j-S_{ij}{}^\alpha\bar{\phi}^j \ , \tag{2.12}$$

with $S_{ij}{}^{\alpha}=S_{ji}{}^{\alpha}$.

The expression for $\bar\theta(h)^i{}_{n+\alpha}$ follows immediately from (2.9) and (2.12); it is given by

$$\bar\theta(h)^i{}_{n+\alpha}=-\tfrac{1}{2}(\Omega_\alpha{}^i{}_j-S_\alpha{}^i{}_j)\bar\phi^j+S^i{}_{j\alpha}\bar\phi^j \ . \tag{2.13}$$

To obtain $\bar\theta(h)^i{}_j$, we need to make use of (2.2) and (2.3). When the expressions for $d\bar\phi^i$ resulting from these two equations are compared, and (2.13) is substituted in the process, we get

$$\bar\theta(h)^i{}_j=\pi^*\bar\theta(g)^i{}_j-\tfrac{1}{2}(\Omega_\alpha{}^i{}_j-S_\alpha{}^i{}_j)\bar\phi^{n+\alpha} \tag{2.14}$$

and

$$S^i{}_{\alpha\beta}=0 \ . \tag{2.15}$$

We now have all the ingredients which are needed for evaluating the components relative to our orthonormal basis of the curvature tensor for the metric h on P. The calculation of these components involves substitution into (2.5) of the different connection matrices [Eqs. (2.10)–(2.14)] which we derived above. The actual procedure, although lengthy, is fairly straightforward; therefore, we state only the final results in the Appendix at the end of this paper. We also give there the expressions for the contracted Ricci tensor and Ricci scalar of (P,h) at p in terms of their projected components in the base manifold, the gauge field tensors, and the torsion.

So far our results are completely general, as they involve no *a priori* choice of the structure group or specialization of the connection coefficients that might restrict the torsion terms. To proceed in this direction, which suits the specific purposes of this paper as discussed in the Introduction, note first that

$$\bar\theta(h)^i{}_{n+\alpha}(\mathbf{E}_j)\equiv\Gamma^i{}_{\alpha j}=[\text{by }(2.13)]$$
$$=-\tfrac{1}{2}(\Omega_\alpha{}^i{}_j-S_\alpha{}^i{}_j)+S^i{}_{j\alpha} \ , \tag{2.16}$$

$$\bar\theta(h)^i{}_{n+\alpha}(\mathbf{E}_{n+\beta})\equiv\Gamma^i{}_{\alpha\beta}=[\text{by }(2.13)]=0 \ , \tag{2.17}$$

$$\bar\theta(h)^i{}_j(\mathbf{E}_{n+\alpha})\equiv\Gamma^i{}_{j\alpha}=[\text{by }(2.14)]$$
$$=-\tfrac{1}{2}(\Omega_\alpha{}^i{}_j-S_\alpha{}^i{}_j) \ , \tag{2.18}$$

$$\bar\theta(h)^{n+\alpha}{}_i(\mathbf{E}_j)=\Gamma^\alpha{}_{ij}=[\text{by }(2.12)]$$
$$=\tfrac{1}{2}(\Omega^\alpha{}_{ij}-S^\alpha{}_{ij})-S_{ij}{}^\alpha \ , \tag{2.19}$$

$$\bar\theta(h)^{n+\alpha}{}_i(\mathbf{E}_{m+\beta})=\Gamma^\alpha{}_{i\beta}=[\text{by }(2.12)]=0 \ . \tag{2.20}$$

If we now assume that $\Gamma^i{}_{j\alpha}=0$, then it follows from (2.18) and (2.16) that

$$S_\alpha{}^i{}_j=\Omega_\alpha{}^i{}_j \ , \tag{2.21a}$$

$$\Gamma^i{}_{\alpha j}=S^i{}_{j\alpha} \ . \tag{2.21b}$$

Further choosing

$$\Gamma^i{}_{\alpha j}=(1/n)\delta^i{}_j\Phi_\alpha\Longrightarrow S^i{}_{j\alpha}=(1/n)\delta^i_j\Phi_\alpha \quad (n=\dim M) \ , \tag{2.22}$$

which, because of (2.19), also implies that

$$\Gamma^\alpha{}_{ij}=-(1/n)g_{ij}\Phi^\alpha \ . \tag{2.23}$$

The characteristic group that we will need for our development in the next section is SO(3). In this case $f=3$, and the group metric is given by

$$k_{\gamma\lambda}=-\tfrac{1}{2}\epsilon_{\beta\gamma\alpha}\epsilon_{\alpha\lambda\beta} \ , \tag{2.24}$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol and greek letter indices range from 1 to 3. The factor in front of the right-hand side of (2.24) has been introduced in order to have an orthonormal basis of $\mathcal{G}$ relative to k, i.e.,

$$k_{\mu\nu}=\delta_{\mu\nu} \ .$$

One remark which has to be made at this point concerns the dimensionality of the structure constants as they enter in our expressions for the components of the Riemann and Ricci tensors. This will allow us to relate the formalism as it usually occurs in the work of differential geometers and that used by field theorists and particle physicists, as well as to guarantee the appropriate units of the unified action integral to be presented at the end of this section and the correctness of the numerical results given in Sec. III. The units of $c^\alpha{}_{\beta\gamma}$ can be readily established by noting that the Riemann tensor has to have units of (length)$^{-2}$, and the connection coefficients are in units of (length)$^{-1}$. Consequently, it becomes evident from (2.10) that $c^\alpha{}_{\beta\gamma}$ has to be also given in units of (length)$^{-1}$. Therefore, to have the proper normalization for the group metric, we need to write

$$c^\alpha{}_{\beta\gamma}=\frac{-i}{\sqrt{2}}\epsilon_{\alpha\beta\gamma}\tau \ , \tag{2.25}$$

where τ is an as-yet undetermined constant factor of dimensions (length)$^{-1}$, which acts as a length gauge in the theory.

To further relate our formalism to the one used in the physics literature, we need to use a local section $\sigma_u : U\to P$ (i.e., make a choice of gauge) to pull back some of our expressions to $U\subset M$. This is perfectly valid since, as mentioned previously, the scalars obtained from the Riemann tensor are constant on the fibers. Thus the pullback of the curvature two-form

$$\Omega^\alpha=\tfrac{1}{2}\Omega^\alpha{}_{ij}\bar\phi^i\wedge\bar\phi^j=d\omega^\alpha+\tfrac{1}{2}c^\alpha{}_{\beta\gamma}\omega^\beta\wedge\omega^\gamma \ , \tag{2.26}$$

is

$$\Omega^\alpha_u=\sigma^*_u\Omega^\alpha=\tfrac{1}{2}\Omega^\alpha{}_{ij}\bar\phi^i_M\wedge\bar\phi^j_M=d\hat A^\alpha+\tfrac{1}{2}c^\alpha{}_{\beta\gamma}\hat A^\beta\wedge\hat A^\gamma \ , \tag{2.27}$$

where $\hat A^\beta\equiv\sigma^*_u\omega^\beta$.

Moreover, for the orthonormal basis at each $x\in U$, we can choose the coordinate basis $\bar{\mathbf{E}}_i=\partial_i$, in which case

$$(\Omega^\alpha_u)_{ij}=\partial_i\hat A^\alpha{}_j-\partial_j\hat A^\alpha{}_i+c^\alpha{}_{\beta\gamma}\hat A^\beta{}_i\hat A^\gamma{}_j \ , \tag{2.28}$$

where $\hat A^\alpha{}_i\equiv\hat A^\alpha(\partial_i)$ is so far a dimensionless gauge vector potential.

If we now let $\tau\hat A^\alpha{}_i\to eA^\alpha{}_i$, then e will be the usual dimensionless coupling constant [$e=\text{charge}/(\hbar c)^{1/2}$], and $A^\alpha{}_i$ the gauge-vector potential in units of (length)$^{-1}$, as they commonly appear in physics papers. Note that we can then write

$$\tau\Omega^{\alpha}{}_{ij}=e\left[\partial_i A^{\alpha}{}_j-\partial_j A^{\alpha}{}_i-\frac{ie}{\sqrt{2}}\epsilon_{\alpha\beta\gamma}A^{\beta}{}_i A^{\gamma}{}_j\right], \tag{2.29}$$

where from here on we will understand that the gauge field tensor $\Omega^{\alpha}{}_{ij}$ has already been pulled back to $U\subset M$, i.e., $\Omega^{\alpha}{}_{ij}\equiv(\Omega^{\alpha}_u)_{ij}$.

We next need to evaluate the directional derivatives which appear in the expressions for the Riemann and Ricci tensors given in the Appendix. For this purpose note first that evaluating (2.26) on $(\mathbf{E}_i,\mathbf{E}_j)$ yields

$$\Omega^{\alpha}{}_{ij}=d\bar{\phi}^{n+\alpha}(\mathbf{E}_i,\mathbf{E}_j)=-\bar{\phi}^{n+\alpha}([\mathbf{E}_i,\mathbf{E}_j]) .$$

This implies that the commutator $[\mathbf{E}_i,\mathbf{E}_j]$ must be vertical and have the value

$$[\mathbf{E}_i,\mathbf{E}_j]=-\Omega^{\alpha}{}_{ij}\mathbf{E}_{n+\alpha} . \tag{2.30}$$

It is easy to verify that this relation is satisfied if we take

$$\mathbf{E}_i=\sigma_*\bar{\mathbf{E}}_i-\hat{A}^{\alpha}{}_i\mathbf{E}_{n+\alpha}=\partial_i-\hat{A}^{\alpha}{}_i\mathbf{E}_{n+\alpha} , \tag{2.31}$$

which is suggested by the fact that $\bar{\phi}^{n+\alpha}(\bar{\mathbf{E}}_i)=0$ and $\omega^{\alpha}(\sigma_*\mathbf{E}_i)=\hat{A}^{\alpha}{}_i$. In verifying (2.30) from (2.31), with $\Omega^{\alpha}{}_{ij}$ given by (2.28), we also need to recall that from the properties of the connection one-forms, the Lie derivative of ω^{α}, relative to $\mathbf{E}_{n+\beta}\equiv e^*_{\beta}$, is given by

$$(\mathcal{L}_{e^*_\beta}\omega^{\alpha})_p=-c^{\alpha}{}_{\beta\gamma}\omega^{\gamma}_{(p)} , \tag{2.32}$$

and that

$$\mathbf{E}_{n+\beta}[\omega^{\alpha}(\sigma_*\bar{\mathbf{E}}_j)]=(\mathcal{L}_{e^*_\beta}\omega^{\sigma})(\sigma_*\bar{\mathbf{E}}_j)=-c^{\alpha}{}_{\beta\gamma}\hat{A}^{\gamma}{}_i . \tag{2.33}$$

Equations (2.31) and (2.33) are all we need to evaluate the directional derivatives which occur in the Riemann and Ricci terms listed in the Appendix. Thus, since the scalar field Φ^{α} derives from a connection, it must transform by means of the adjoint representation of G, i.e., according to (2.33), and we therefore have

$$\mathbf{E}_k[\Phi^{\alpha}]\equiv\Phi^{\alpha}{}_{,k}=\partial_k\Phi^{\alpha}+c^{\alpha}{}_{\beta\gamma}\hat{A}^{\beta}{}_k\Phi^{\gamma}\equiv D_k\Phi^{\alpha} . \tag{2.34}$$

Consequently, with the metric (2.24) and structure constants (2.25), the choice of connection coefficients and torsion components contained in Eqs. (2.15)–(2.23), and the additional assumption that torsion is zero on the base manifold ($S^i{}_{jk}=0$), Eqs. (A1)–(A14) in the Appendix yield (for typographical simplicity, from here on we make the change of index notation $n+\alpha\to\alpha$, etc., in the Riemann tensor components)

$$R=\underline{R}-\tfrac{5}{6}\Phi^{\gamma}\Phi_{\gamma}-\tfrac{3}{4}\tau^2 , \tag{2.35}$$

$$\begin{aligned}R_{jm}&=\underline{R}_{jm}-\tfrac{5}{36}g_{jm}\Phi^{\gamma}\Phi_{\gamma} ,\\ R_{\alpha j}&=-\tfrac{5}{6}D_j\Phi_{\alpha},\quad R_{j\alpha}=0 ,\end{aligned} \tag{2.36}$$

$$R_{\beta\gamma}=\frac{-i\tau}{2\sqrt{2}}\epsilon_{\beta\gamma\alpha}\Phi^{\alpha}-\frac{\tau^2}{4}\delta_{\beta\gamma} , \tag{2.37}$$

$$R^{\alpha}{}_{\beta\delta\gamma}=-\frac{\tau^2}{8}(\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\gamma}\delta_{\beta\delta}) , \tag{2.38}$$

$$R^{\alpha}{}_{\beta ij}=-\frac{ie}{2\sqrt{2}}\epsilon_{\alpha\beta\gamma}\Omega^{\alpha}{}_{ij},\quad R^{\alpha}{}_{\beta\gamma i}=0 , \tag{2.39}$$

$$R^{\alpha}{}_{jki}=-\tfrac{1}{6}g_{ij}D_k\Phi^{\alpha}+\tfrac{1}{6}g_{jk}D_i\Phi^{\alpha} , \tag{2.40}$$

$$R^{\alpha}{}_{j\beta i}=-\frac{i\tau}{12\sqrt{2}}\epsilon_{\alpha\beta\gamma}g_{ij}\Phi^{\gamma},\quad R^{\alpha}{}_{j\beta\gamma}=0 , \tag{2.41}$$

$$R^{i}{}_{jkm}=\underline{R}^{i}{}_{jkm}+\tfrac{1}{36}(\delta^i{}_m g_{jk}-\delta^i{}_k g_{jm})\Phi^{\gamma}\phi_{\gamma} , \tag{2.42}$$

$$R^{i}{}_{jk\alpha}=0,\quad R^{i}{}_{j\beta\alpha}=0 . \tag{2.43}$$

Note that in the above expressions we use the gauge potentials $A^{\alpha}{}_i$ [i.e., properly dimensioned to units of $(\text{length})^{-1}$], and for $\Omega^{\alpha}{}_{ij}$ we use the term inside the large parentheses on the right of (2.29). Also, in (2.35)–(2.43) we have assumed that $n=\dim M=6$, since this is the dimension of the base manifold which we will need for the development to be presented in the next section.

To complete the summary of the basic expressions that we will be needing next for the construction of a Lagrangian density, we give the formula for the various components of the torsion tensor, which already incorporate the assumptions discussed above:

$$S^{\alpha}{}_{\beta\gamma}=0,\quad S^{i}{}_{\alpha\beta}=0,\quad S^{i}{}_{jk}=0 , \tag{2.44}$$

$$S_{ji\alpha}=S_{ij\alpha}=\tfrac{1}{6}g_{ij}\Phi_{\alpha},\quad S^{j}{}_{\alpha i}=-\tfrac{1}{6}\delta^j{}_i\Phi_{\alpha} , \tag{2.45}$$

$$S_{\alpha}{}^{i}{}_{j}=\Omega_{\alpha}{}^{i}{}_{j} . \tag{2.46}$$

Choice of a general Lagrangian density

The most general Lagrangian on P that can be constructed, up to quadratic terms in the Riemann, Ricci, and torsion terms as well as the Ricci scalar, is of the form

$$\begin{aligned}\mathcal{L}=\frac{\sqrt{-g}}{V_I}[&\alpha_0 R-\tfrac{1}{4}S_{abc}(\beta_1 S^{abc}+\beta_2 S^{cab}+\beta_3 g^{ac}S^{b})\\&-\tfrac{1}{4}R_{abcd}(\alpha_1 R^{abcd}+\alpha_2 R^{cdab}+\alpha_3 R^{acbd})\\&-\tfrac{1}{4}R_{ac}(\alpha_4 R^{ac}+\alpha_5 R^{ca})+\alpha_6 R^2] ,\end{aligned} \tag{2.47}$$

where $S^b\equiv S_a{}^{ba}$, and V_I is the volume of the internal coordinates of the base manifold, as described in Sec. III.

Note that our Lagrangian density has been divided by $\hbar$ so that everything on the right-hand side has units of powers of length. Thus the torsion tensor components are in units of $(\text{length})^{-1}$, the arbitrary parameters $\alpha_0,\beta_1,\beta_2,\beta_3$ have units of $(\text{length})^{-2}$, while the remaining (also arbitrary) parameters are dimensionless.

If we now substitute Eqs. (2.35)–(2.46) into the above expression, we get

$$\begin{aligned}\mathcal{L}=\frac{\sqrt{-g}}{V_I}\Bigg[&(\alpha_0-\tfrac{3}{2}\alpha_6\tau^2)\underline{R}-\left[\tfrac{5}{6}\alpha_0+\frac{\beta_1}{12}-\frac{\beta_2}{24}+\frac{\beta_3}{4}-\frac{\alpha_1\tau^2}{96}+\frac{\alpha_2\tau^2}{96}+\tfrac{1}{16}\tau^2\alpha_4-\tfrac{1}{16}\tau^2\alpha_5-\tfrac{5}{4}\tau^2\alpha_6\right]\Phi^\gamma\Phi_\gamma+\alpha_6\underline{R}^2\\
&-\frac{3}{4}\left[\alpha_0+\frac{\alpha_1\tau^2}{16}+\frac{\alpha_2\tau^2}{16}+\frac{\alpha_3\tau^2}{32}+\frac{\alpha_4\tau^2}{16}+\frac{\alpha_5\tau^2}{16}-\tfrac{3}{4}\tau^2\alpha_6\right]\tau^2\\
&-\frac{1}{4}\left[\frac{\beta_1}{\tau^2}-\frac{\alpha_1}{4}\right]e^2\Omega_{aij}\Omega^{aij}-\frac{5}{72}\left[\alpha_1+\frac{\alpha_3}{2}+\tfrac{5}{2}\alpha_4\right](D^k\Phi^\alpha)(D_k\Phi_\alpha)\\
&-\frac{1}{4}\left[\alpha_1+\alpha_2+\frac{\alpha_3}{2}\right]\underline{R}_{ijkm}\underline{R}^{ijkm}-\tfrac{1}{4}(\alpha_4+\alpha_5)\underline{R}_{jm}\underline{R}^{jm}+\frac{1}{36}\left[\alpha_1+\alpha_2+\frac{\alpha_3}{2}+\tfrac{5}{2}\alpha_4+\tfrac{5}{2}\alpha_5-60\alpha_6\right]\underline{R}\,\Phi^\gamma\Phi_\gamma\\
&-\frac{5}{432}\left[\alpha_1+\alpha_2+\frac{\alpha_3}{2}+\tfrac{5}{2},\alpha_4+\tfrac{5}{2}\alpha_5-60\alpha_6\right](\Phi^\gamma\Phi_\gamma)^2\Bigg]\ .\end{aligned}\tag{2.48}$$

The action integral from which we will set up a unified variational principle for the Einstein-Yang-Mills-Higgs system in the following section, is given in terms of (2.48) by

$$I=\int_U \mathcal{L}\mu\ ,\tag{2.49}$$

where μ is the element of volume of our six-dimensional base space M, and the subset $U\subset M$ is assumed to have compact closure.

III. SPONTANEOUS COMPACTIFICATION OF THE BASE MANIFOLD

The Lagrangian density given by (2.48) describes a generalized Einstein-Yang-Mills-Higgs system, with a potential containing quartic terms of the form leading to spontaneous symmetry breaking. The Higgs fields in this formalism have a geometrical origin since they are derived from the connections on the principal fiber bundle P, when one assumes that torsion on the fibers is nonvanishing, and acquire a dynamical character when quadratic terms in the curvature tensor are admitted. In addition, the theory is of the Kaluza-Klein type, since our base manifold M is a spacetime of six dimensions. For this scenario to be considered appropriate to describe the real physical world, it is necessary that the extra dimensions of M compactify into a size of the order of the Planck length via spontaneous symmetry breaking of the Poincaré invariance of M, so that the symmetries of these extra dimensions can then correspond to internal symmetries.

Although spontaneous compactification of space for the Einstein-Yang-Mills-Higgs system with the usual Einstein gravitational Lagrangian has been considered by Cremmer and Scherk, no such study has been undertaken for quadratic Einstein-Cartan gravitational Lagrangians, and it is not obvious, *a priori,* that compact solutions do exist for such a case. We have already commented in the Introduction that theories with quadratic curvature terms in the Lagrangian on a six-dimensional space have attracted interest, since they seem to occur as the bosonic low-energy limit of superstring theories (four-dimensional Minkowski spacetime can be obtained by compactification).[7] This consideration and the fact that in our formalism there is no need for nongeometrical fields to trigger compactification, and that some of the so far arbitrary and most physically interesting parameters in the theory may be fixed by geometrical arguments, provides in our view a good motivation for undertaking such a program.

We therefore proceed by first making the obvious identifications which are needed to bring (2.48) into the usual form of Einstein-Cartan gravity coupled to Yang-Mills and Higgs fields. We take

$$(\alpha_0-\tfrac{3}{2}\alpha_6\tau^2)=\kappa\tag{3.1}$$

(the proportionality factor in the Einstein Hilbert Lagrangian),

$$\left[\tfrac{5}{6}\alpha_0+\frac{\beta_1}{12}-\frac{\beta_2}{24}+\frac{\beta_3}{4}-\frac{\alpha_1\tau^2}{96}+\frac{\alpha_2\tau^2}{96}+\tfrac{1}{16}\alpha_4\tau^2-\tfrac{1}{16}\alpha_5-\tfrac{5}{4}\alpha_6\tau^2\right]=\frac{m^2}{2}\tag{3.2}$$

(half of the Compton wavelength associated with the mass of the Higgs boson),

$$-\frac{3}{4}\left[\alpha_0+\left[\alpha_1+\alpha_2+\frac{\alpha_3}{2}\right]\tau^2+\frac{\alpha_4\tau^2}{16}+\frac{\alpha_5\tau^2}{16}-\tfrac{3}{4}\alpha_6\tau^2\right]\tau^2=\Lambda\tag{3.3}$$

(the cosmological constant),

$$-\frac{5}{432}\left[\alpha_1+\alpha_2+\frac{\alpha_3}{2}+\tfrac{5}{2}\alpha_4+\tfrac{5}{2}\alpha_5-60\alpha_6\right]=\lambda\tag{3.4}$$

(the parameter in the quartic term in the potential), to fix the physical parameters, and

$$e^2\left[\frac{\beta_1}{\tau^2}-\frac{\alpha_1}{4}\right]=\frac{1}{2}\tag{3.5}$$

(e^2 the dimensionless coupling constant of the Yang-Mills field),

$$\left[\alpha_1+\frac{\alpha_3}{2}+\tfrac{5}{2}\alpha_4\right]=-\tfrac{36}{5}\ . \tag{3.6}$$

in order to normalize the free Lagrangian of the Yang-Mills and Higgs fields to their customary values.

For the sake of convenience we also set

$$\rho=\alpha_1+\alpha_2+\frac{\alpha_3}{2}\ . \tag{3.7}$$

The final action becomes

$$I=\frac{1}{V_I}\int\sqrt{-g}\left[\kappa\underline{R}-\tfrac{1}{4}\rho\underline{R}_{ijkm}\underline{R}^{ijkm}-\tfrac{216}{25}(-\lambda-\tfrac{5}{432}\rho+\tfrac{25}{36}\alpha_6)\underline{R}_{jm}\underline{R}^{jm}+\Lambda+\alpha_6\underline{R}^2-\tfrac{1}{8}\Omega_{\alpha ij}\Omega^{\alpha ij}\right.$$
$$\left.+\tfrac{1}{2}(D^k\Phi^\alpha)(D_k\Phi_\alpha)-\frac{m^2}{2}\Phi^\alpha\Phi_\alpha-\tfrac{12}{5}\lambda\underline{R}\,\Phi^\alpha\Phi_\alpha+\lambda(\Phi^\alpha\Phi_\alpha)^2\right]d^6x\ . \tag{3.8}$$

Note that since we have chosen $\dim M=6$, and $G=SO(3)$, lower-case latin indices will have the range $0\le i,j,k,\ldots\le 5$ while lower-case greek indices will have the range $1\le\alpha,\beta,\gamma,\ldots\le 3$. Recall that the Yang-Mills gauge field tensor $\Omega^\alpha{}_{ij}$ was defined as the term inside the large parentheses on the right of (2.29), while $D_k\Phi^\alpha$ is given by (2.34) in Sec. II. An interesting point to notice is the existence of the term $\frac{12}{5}\lambda\underline{R}\,\Phi^\alpha\Phi_\alpha$, a curvature "mass" term where the coefficient $\frac{12}{5}\lambda$ is given by the dimensionality of the fibers and base space in our fiber-bundle approach. This term will be important in our later development of the theory. Given the action (3.8) we can carry out the calculation analogous to that of Cremmer and Scherk which is based in turn, on the magnetic monopole solutions of 't Hooft,[8] van Nieuwenhuizen, Wilkinson, and Perry.[9] Thus our metric will be given by

$$\begin{aligned}ds^2&=g_{ij}dx^idx^j\\&=\eta_{\mu\nu}dx^\mu dx^\nu+R_0^2(d\theta^2+\sin^2\theta\,d\phi^2)\ ,\end{aligned} \tag{3.9}$$

i.e., four-dimensional spacetime is taken to be flat Minkowski space, while the two extra internal dimensions have the geometry and topology of a two-sphere of constant radius R_0. Note that in separating ordinary spacetime from internal coordinates in (3.9), we use the indices $\mu,\nu=0,1,2,3$ to denote the former. We will maintain this notation for the remainder of the paper and reserve the letters μ,ν for four-dimensional spacetime, while internal coordinates will be designated by capital latin indices and use, as we have so far, indices from the beginning of the greek alphabet to denote group-related components of our fields. Also note that with the metric (3.9), $V_I=4\pi R_0^2$.

If we now vary the action (3.8) with respect to g_{ij}, we get the following set of equations of motion:

$$\delta g_{ij}\to G_{ij}=8\pi T_{ij}$$

which breaks up naturally into three equations:

$$G_{\mu\nu}=\eta_{\mu\nu}\left[\frac{\kappa}{R_0^2}-\frac{216}{25R_0^4}(-\lambda+\tfrac{25}{54}\alpha_6+\tfrac{5}{108}\rho)\right.$$
$$\left.-\frac{12}{5}\frac{\lambda}{R_0^2}\Phi^\alpha\Phi_\alpha+\tfrac{1}{2}\Lambda\right]=8\pi T_{\mu\nu}\ , \tag{3.10}$$

where

$$T_{\mu\nu}=\frac{1}{8\pi}\left[\frac{m^2}{4}\Phi^\alpha\Phi_\alpha+\tfrac{1}{16}\Omega_{\alpha\ AB}\Omega^{\alpha\ AB}-\tfrac{1}{4}[D^A\Phi^\alpha(D_A\Phi_\alpha)]\right.$$
$$\left.-\frac{\lambda}{2}(\Phi^\alpha\Phi_\alpha)^2\right]\eta_{\mu\nu}\ , \tag{3.11}$$

where $A,B=4,5$, and $dx^4=R_0d\theta$, $dx^5=R_0\sin\theta\,d\phi$;

$$G_{\mu A}=8\pi T_{\mu A}\ , \tag{3.12}$$

which is satisfied identically as both sides are zero, and

$$G_{AB}=\left[\frac{216}{25R_0^4}(-\lambda+\tfrac{25}{54}\alpha_6+\tfrac{5}{108}\rho)+\tfrac{1}{2}\Lambda\right]g_{AB}=8\pi T_{AB}\ , \tag{3.13}$$

where

$$T_{AB}=-\frac{1}{16\pi}\left[-\tfrac{1}{8}\Omega_{\alpha CD}\Omega^{\alpha CD}+\tfrac{1}{2}(D^C\Phi^\alpha)(D_C\Phi_\alpha)\right.$$
$$\left.-\frac{m^2}{2}\Phi^\alpha\Phi_\alpha+\lambda(\Phi^\alpha\Phi_\alpha)^2\right]g_{AB}$$
$$+\frac{1}{8\pi}[\tfrac{1}{4}\Omega^\alpha{}_{CA}\Omega_\alpha{}^C{}_B-\tfrac{1}{2}(D_A\Phi_\alpha)(D_B\Phi^\alpha)]\ . \tag{3.14}$$

The field equations resulting from varying $A^\alpha{}_i$ are essentially the same as those given in Refs. 2, 8, and 9 except for a relative factor and sign which stem from our somewhat different definition of the gauge potentials $[-(i/\sqrt{2})A^\alpha{}_i=W^\alpha{}_i$, where $W^\alpha{}_i$ is the quantity used in the references cited above]. Thus we get

$$-\frac{1}{2\sqrt{-g}}\partial_k(\Omega_\alpha{}^{ik}\sqrt{-g}\,)+\frac{ie}{2\sqrt{2}}\epsilon_{\gamma\alpha\beta}A^\beta{}_k\Omega^{\gamma ik}$$
$$-\frac{e}{\sqrt{2}}\epsilon_{\beta\gamma\alpha}\Phi_\beta(D^i\Phi^\gamma)=0\ . \tag{3.15}$$

Finally, variation of (3.8) with respect to Φ^α yields

$$-\partial^k\partial_k\Phi_\alpha+\frac{ie}{\sqrt{2}}\partial^k(A^\beta{}_k\epsilon_{\gamma\alpha\beta}\Phi_\gamma)-\left[\frac{\partial^k\sqrt{-g}}{\sqrt{-g}}\right]\left[\partial_k\Phi_\alpha-\frac{ie}{\sqrt{2}}A^\beta{}_k\epsilon_{\gamma\alpha\beta}\Phi_\gamma\right]$$
$$-\frac{i}{\sqrt{2}}eA^{\beta k}\epsilon_{\alpha\gamma\beta}\left[\partial_k\Phi_\gamma-\frac{ie}{\sqrt{2}}A^\lambda{}_k\epsilon_{\delta\gamma\lambda}\Phi_\delta\right]-\tfrac{24}{5}\lambda\underline{R}\,\Phi_\alpha-m^2\Phi_\alpha+4\lambda\Phi_\alpha(\Phi_\beta\Phi^\beta)=0\ . \tag{3.16}$$

Continuing along the lines of Cremmer and Scherk, we assume that both the gauge and Higgs fields are independent of the first four spacetime coordinates while their dependence on θ and ϕ is of the form

$$\begin{aligned}&A^\alpha{}_\mu=0\ ,\\&A^\alpha{}_\theta=-i\sqrt{2}a_\theta(-\sin\phi,\cos\phi,0)\ ,\\&A^\alpha{}_\phi=-i\sqrt{2}a_\phi(-\cos\phi\cos\phi,-\sin\phi\cos\theta,\sin\theta)\sin\theta\ ,\\&\phi^\alpha=p(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)\ ,\end{aligned}\tag{3.17}$$

where a_θ, a_ϕ, and p are constants since in the ansatz for the internal coordinate metric no radial variables have been included. Substituting (3.17) into (3.15) it is clear that this field equation is identically satisfied for $i,k=\mu,\nu$ ($A^\alpha{}_\mu=0$) and, given the explicit dependence of the $A^\alpha{}_\mu$ and Φ^α on θ and ϕ of Eqs. (3.17), the $i,k=A,B$ equations reduce to algebraic conditions on the constants a_θ, a_ϕ, and p. A solution only exists if $a_\theta=a_\phi$, and p is given by

$$p^2=\frac{a}{eR_0}(2-eR_0a)\ , \tag{3.18}$$

where we define $a=a_\theta=a_\phi$.

Similarly, substituting (3.17) into (3.16) yields, for each component of Φ^α, the unique solution

$$\frac{1}{R_0^2}[(1-eR_0a)^2-\tfrac{24}{5}\lambda]=-2\lambda p^2+\frac{m^2}{2}\ . \tag{3.19}$$

The remaining algebraic constraints on the constants that appear in the model come from replacing the expressions for $A^\alpha{}_i$ and Φ^α given in (3.17) into (3.10)–(3.14) which thus reduce to the following set of equations:

$$\frac{\kappa}{R_0^2}-\frac{216}{25R_0^4}(-\lambda+\tfrac{5}{108}\rho+\tfrac{25}{54}\alpha_6)-\frac{12}{5}\frac{\lambda}{R_0^2}p^2+\tfrac{1}{2}\Lambda=-\frac{a^2}{4R_0^2}(2-eR_0a)^2-\frac{p^2}{2R_0^2}(1-eR_0a)^2+\tfrac{1}{4}m^2p^2-\tfrac{1}{2}\lambda p^4\ , \tag{3.20}$$

$$\frac{R_0^2}{2}\Lambda+\frac{216}{25R_0^2}(-\lambda+\tfrac{5}{108}\rho+\tfrac{25}{54}\alpha_6)=-\tfrac{3}{4}a^2(2-eR_0a)^2+\tfrac{1}{4}m^2R_0^2p^2-\tfrac{1}{2}R_0^2\lambda p^4-p^2(1-eR_0a)^2\ . \tag{3.21}$$

The set of Eqs. (3.18)–(3.21) must be solved for p, a, R_0, and Λ in terms of the rest of the constants of the problem. Since a is relatively unimportant, we will not give the final solution for it, although it may be easily found from the solutions for p and R_0 which are

$$p^2=\frac{2\kappa-\dfrac{216}{25R_0^2}(-\lambda+\tfrac{5}{108}\rho+\tfrac{25}{54}\alpha_6)}{1+\tfrac{24}{5}\lambda} \tag{3.22}$$

and

$$\frac{1}{R_0^2}=\frac{\dfrac{2(e^2-2\lambda)\kappa}{1+\tfrac{24}{5}\lambda}+\dfrac{m^2}{2}}{(1-\tfrac{24}{5}\lambda)+\tfrac{216}{25}(1+\tfrac{24}{5}\lambda)^{-1}(-\lambda+\tfrac{5}{108}\rho+\tfrac{25}{54}\alpha_6)(e^2-2\lambda)}\ . \tag{3.23}$$

We will consider the cosmological constant Λ and the additional contribution to it which arises from the spontaneous symmetry-breaking process in the next section.

For comparison purposes we reproduce the analogous solution of Cremmer and Scherk for $1/R_0^2$, which in our notation reads

$$\frac{1}{R_0^2}=2e^2\kappa+\left[\frac{m^2}{2}-4\lambda\kappa\right]\ .$$

It is evident from this that the analysis of compactified solutions is somewhat more complicated for our case. In fact, if we rewrite (3.18) as

$$p^2=\frac{1}{e^2R_0^2}[1-(1-eaR_0)^2]\ , \tag{3.24}$$

it follows immediately that $p^2 \geq 0 \Longrightarrow (1-eaR_0)^2 \leq 1$. On the other hand, substituting (3.24) into (3.19), gives an alternative expression for p^2:

$$p^2 = \frac{\frac{1}{R_0^2}(1-\frac{24}{5}\lambda) - \frac{m^2}{2}}{e^2 - 2\lambda} , \tag{3.25}$$

which when combined with (3.22) yields

$$\frac{1}{R_0^2} = \frac{(e^2-2\lambda)\left[2\kappa - \frac{216}{25R_0^2}(-\lambda+\frac{5}{108}\rho+\frac{25}{54}\alpha_6)\right]}{(1+\frac{24}{5}\lambda)(1-\frac{24}{5}\lambda)} + \frac{m^2}{2(1-\frac{24}{5}\lambda)} . \tag{3.26}$$

If we now substitute p^2 from (3.22) in (3.24) and make use of (3.26) in the result, we find

$$[1-(1-eaR_0)^2]^{-1} = \frac{1-\frac{2\lambda}{e^2}}{1-\frac{24}{5}\lambda} + \frac{(1+\frac{24}{5}\lambda)m^2}{2e^2(1-\frac{24}{5}\lambda)\left[2\kappa - \frac{216}{25R_0^2}(-\lambda+\frac{5}{108}\rho+\frac{25}{54}\alpha_6)\right]} . \tag{3.27}$$

We thus see that since $(1-eaR_0) \leq 1$ the right-hand side of (3.27) must be larger than or equal to 1, and we have the inequality

$$1-\frac{24}{54}\lambda \leq 1 - \frac{2\lambda}{e^2} + \frac{(1+\frac{24}{5}\lambda)\frac{m^2}{2e^2}}{2\kappa - \frac{216}{25R_0^2}(-\lambda+\frac{5}{108}\rho+\frac{25}{54}\alpha_6)} . \tag{3.28}$$

Finally, multiplying (3.28) through by

$$e^2\left[2\kappa - \frac{216}{25R_0^2}(-\lambda+\frac{5}{108}\rho+\frac{25}{54}\alpha_6)\right] ,$$

dividing by $(1-\frac{24}{5}\lambda)(1+\frac{24}{5}\lambda)$ and comparing the result with (3.26), we conclude that

$$R_0^2 \leq \frac{1+\frac{24}{54}\lambda}{2\kappa e^2}\left[1 + \frac{216e^2}{25(1+\frac{24}{54}\lambda)}(-\lambda+\frac{5}{108}\rho+\frac{25}{54}\alpha_6)\right] . \tag{3.29}$$

Note that the expressions corresponding to (3.28) and (3.29) in the mode of Cremmer and Scherk are

$$1 \leq 1 - \frac{2\lambda}{e^2} + \frac{m^2}{4\kappa e^2}, \quad R_0^2 \leq \frac{1}{2e^2\kappa} .$$

Consequently, provided e^2 is not extremely small, compactification of the radius of the two-sphere follows in their case and its dimension is determined by the inverse of the Planck length.

In our case the combination of constants $(-\lambda+\frac{5}{108}\rho+\frac{25}{54}\alpha_6)$ plays an important role, and if this combination is allowed in principle to have any value, there is little if any restriction on the size of R_0. As we mentioned in the Introduction, an interesting combination of the quadratic curvature terms is the Gauss-Bonnet form, $\underline{R}^{ijkm}\underline{R}_{ijkm} - 4R^{ij}\underline{R}_{ij} + R^2$, which in our model implies additional restrictions on λ, ρ, and α_6. If we impose the Gauss-Bonnet form on the quadratic curvature terms in our Lagrangian, then we have

$$\rho = -4\alpha_6 \text{ and } \tfrac{216}{25}(-\lambda - \tfrac{5}{432}\rho + \tfrac{25}{26}\alpha_6) = 4\alpha_6 , \tag{3.30}$$

from which it follows that

$$\alpha_6 = \tfrac{18}{5}\lambda . \tag{3.31}$$

Substituting (3.30) and (3.31) in the combination $(-\lambda+\frac{5}{108}\rho+\frac{25}{54}\alpha_6)$ we find that it becomes zero. We will devote the next section to exploring the consequence of this restriction for our solution.

IV. THE GAUSS-BONNET FORM FOR THE GRAVITATIONAL LAGRANGIAN

It is well known for four-dimensional spacetime the term

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\kappa\lambda\tau\zeta}\underline{R}_{\mu\nu}{}^{\kappa\lambda}\underline{R}_{\rho\sigma}{}^{\tau\zeta} = \underline{R}^{\mu\nu\rho\sigma}\underline{R}_{\mu\nu\rho\sigma} - 4\underline{R}^{\mu\nu}\underline{R}_{\mu\nu} + \underline{R}^2$$

is a topological invariant, and that its variation relative to the metric leads only to a total divergence.

For dimensions higher than four this is no longer true however, as has been noted by Zwiebach;[5] the above Gauss-Bonnet combination of quadratic terms in the curvature is still special, since it is the only action of this type found so far which is free of ghost particles, and it also forms an important part of a series pattern for the bosonic low-energy limit of superstring theory.

These arguments provide a strong justification for the choice of parameters as given in Eqs. (3.30) and (3.31) at the end of the last section. We now consider some further physical implications in our model which derive from these choices.

First, Eqs. (3.22), (3.23), (3.28), and (3.29) reduce to

$$p^2 = \frac{2\kappa}{1+\frac{24}{5}\lambda} , \tag{4.1}$$

$$R_0^2=\frac{1-\frac{24}{5}\lambda}{\frac{2(e^2-2\lambda)\kappa}{1+\frac{24}{5}\lambda}+\frac{m^2}{2}}\ , \tag{4.2}$$

$$1-\tfrac{24}{5}\lambda\le 1-\frac{2\lambda}{e^2}+\frac{m^2(1+\frac{24}{5}\lambda)}{4\kappa e^2}\ , \tag{4.3}$$

and

$$R_0^2\le\frac{1+\frac{24}{5}\lambda}{2\kappa e^2}\ . \tag{4.4}$$

Thus the internal space becomes properly compactified provided e^2 is not extremely small (the same requirement as in Cremmer and Scherk), and λ is not extremely large.

This point is where the curvature "mass" term in our Lagrangian becomes decisive since it leads to rather definite predictions on the values of λ and the coupling constant e. Indeed, from the inequality (4.3) we get

$$\lambda\le\frac{m^2}{4\kappa\left[2-\frac{24}{5}e^2-\frac{6}{5}\frac{m^2}{\kappa}\right]}\ . \tag{4.5}$$

Hence, unless we set

$$\left[2-\tfrac{24}{5}e^2-\frac{6}{5}\frac{m^2}{\kappa}\right]=O\left[\frac{m^2}{\kappa}\right],$$

the parameter λ will turn out to be extremely small (we assume that $m^2<<\kappa$). However, λ need not be so small. For example, in the electroweak model and for the lower search limit for the masses of the Higgs bosons $\lambda\sim 0.005$. Consequently, we take

$$\left[2-\tfrac{24}{5}e^2-\frac{6}{5}\frac{m^2}{\kappa}\right]=O\left[\frac{m^2}{\kappa}\right],$$

which in turn implies that

$$e^2=\frac{5}{12}-\sigma\left[\frac{m^2}{\kappa}\right]\ . \tag{4.6}$$

The lower limit of σ is determined when one substitutes (4.6) back into (4.5) to get

$$\lambda\le\frac{5}{24(4\sigma-1)}\ . \tag{4.7}$$

Since $\lambda\ge 0$, we must have $\sigma>\frac{1}{4}$. We will have more to say about the upper limit of σ later on. What we find interesting here is that to zeroth order in m^2/κ, Eq. (4.6) predicts the value

$$e\approx(\tfrac{5}{12})^{1/2}\approx 0.645 \tag{4.8}$$

for the coupling constant e.

This is remarkably close to the value of the coupling constant g for the SU(2)$\times$U(1) electroweak model, for which one gets $g=0.637\pm0.005$. However suggestive this agreement might seem to be, it is important to remark at this point that some of our assumptions leading to it require additional justification. More specifically, since our Lagrangian is not a four-dimensional effective Lagrangian, we cannot as yet relate some of our parameters such as the gauge coupling constant e and the constant κ to observable quantities. In order to be able to do so, we adopt the same line of reasoning as that followed by Cremmer and Scherk, that is, we consider fluctuations of our fields around the solutions given in Sec. III. Thus for the gravitational field we examine the perturbations in our solutions induced by the metric

$$ds^2=g_{\mu\nu}(x)dx^\mu dx^\nu+R_0^2(d\theta^2+\sin^2\theta\, d\phi^2)\ .$$

In this case we will have $\underline{R}=R_s+2/R_0^2$, where R_s is the ordinary spacetime Ricci scalar. Moreover, noting that changes in the action of the gravitational field can originate only from terms in (3.8) containing $\underline{R}$ or $\underline{R}^2$ (the Gauss-Bonnet terms in four-dimensional spacetime contribute only with a total divergence to the action principle), we get

$$\begin{aligned}\delta I_g&=\frac{1}{V_I}\left[\kappa+\frac{4\alpha_6}{R_0^2}-\tfrac{12}{5}\lambda p^2\right]\int\sqrt{-g}\,R_s d^6x\\&=\left[\kappa+\frac{4\alpha_6}{R_0^2}-\tfrac{12}{5}\lambda p^2\right]\int\sqrt{-g}\,R_s d^4x\ .\end{aligned} \tag{4.9}$$

Hence, the quantity to be identified with the inverse square of the Planck length $\kappa_{[4]}$ ($\equiv c_3/16\pi\hbar G_N=0.762\times10^{64}\ \mathrm{cm}^{-2}$) is

$$\kappa+\frac{4\alpha_6}{R_0^2}-\tfrac{12}{5}\lambda p^2\equiv\kappa_{[4]}\ . \tag{4.10}$$

By (4.1), (4.16), and (4.18), $\frac{12}{5}\lambda p^2=\kappa/4\sigma'\approx\kappa/2$, so

$$\kappa_{[4]}=\frac{\kappa}{2}+\frac{72}{5}\frac{\lambda}{R_0^2}\ .$$

This means that our assumption $m^2<<\kappa$ would be physically reasonably provided that $\kappa\sim4\alpha_6/R_0^2\sim\kappa_{[4]}$. It turns out that such a situation can be made possible and compatible with the constraints on the cosmological terms discussed below by fine-tuning the parameter σ which we still have at our disposal.

As for the gauge coupling constant, note that if instead of taking $A_\mu^\alpha=0$ as in (3.17), we introduce a nonzero and spacetime-dependent value for these components of the Yang-Mills fields, then an argument similar to the one used for the gravitational field leads to the conclusion that $e\equiv e_{[4]}$, i.e., the value given in (4.6) for our gauge coupling constant is indeed the observable value for this quantity.

In light of these arguments and the fact that even though our model is based on the structure group SO(3) and not SU(2)$\times$U(1), the factor SU(2) is a covering of SO(3), so one might reasonably expect that some of the salient features of the electroweak model should already be contained in ours.

In particular it would be worthwhile to test whether the agreement in the value of the coupling constants noted above would be maintained by a generalization of our formalism to SU(2)$\times$U(1) as a characteristic group of the principal fiber bundle, and possibly allowing for a more general spacetime dependence of the gravitational, gauge, and Higgs fields. If this were so, then we would have a unified theory where some of the most important physical parameters would be determined solely by the geometry of our structures. [Recall that the factor $\frac{5}{12}$ which appears in (4.6), originates from the curvature "mass" term $\frac{12}{5}\lambda \underline{R}\,\Phi^\alpha\Phi_\alpha$ in our Lagrangian (3.8), and that its value is fixed by the dimensionality of the base space M and the structure constants of the group.]

In contrast, note that in the linear curvature case of Cremmer and Scherk, one gets $\lambda \leq m^2/8\kappa_{[4]}$ instead of (4.5). Thus λ will be extremely small and, since the e^2 factor cancels out, no further constraints on its value can be derived from that model.

We now give an argument for fine-tuning the value of σ in (4.6) and (4.7) which will allow us to determine the values of all but one of the remaining arbitrary parameters originally introduced in (2.48). In the end we will have both the mass m of the Higgs boson and the cosmological constant Λ as functions of the undetermined parameter α_5 or, inverting the argument, given α_5 we will have a relation between the cosmological constant and the mass of the Higgs boson.

Observe first that substituting (3.18) and (3.21) and using (4.1) yields (for the Gauss-Bonnet Lagrangian)

$$R_0^2 = \frac{2\kappa}{\dfrac{\kappa^2(e^2-2\lambda)}{1+\frac{24}{5}\lambda} + \dfrac{\kappa m^2}{2} - \dfrac{\Lambda}{2}(1+\frac{24}{5}\lambda)} . \tag{4.11}$$

Comparing this expression with (4.2), we find

$$\Lambda' = \frac{\dfrac{2\kappa(e^2-2\lambda)(3+\frac{24}{5}\lambda)}{1+\frac{24}{5}\lambda} + m^2(1+\frac{24}{5}\lambda)}{1-(\frac{24}{5})^2\lambda^2} , \tag{4.12}$$

where we have set

$$\Lambda' \equiv \frac{\Lambda}{\kappa} . \tag{4.13}$$

Note that Λ' has units of (length)$^{-2}$, so it is this quantity which we should appropriately call a cosmological term. On the other hand, Λ' is not the only contribution to the "observable" cosmological constant, there is also a contribution given by

$$\Lambda'' = \frac{\lambda}{\kappa}\left[\frac{m^2}{4\lambda} + \frac{12}{5R_0^2}\right]^2 , \tag{4.14}$$

where the quantity $m^2/4\lambda + 12/5R_0^2$ is just the vacuum value of $\Phi^\alpha\Phi_\alpha$, resulting from the spontaneous symmetry-breaking process in the quartic term $\lambda(\Phi^\alpha\Phi_\alpha)^2$ in the effective Lagrangian and using our unperturbed solution for the Ricci scalar. In addition, there are other terms that contribute to the cosmological constant which arise in the effective Lagrangian from all the terms in (3.8) which are four-spacetime independent.

We thus have that the "observable" cosmological constant is given by

$$\Lambda_{\text{obs}} = \frac{2}{R_0^2} + \frac{4\kappa(e^2-2\lambda)}{(1+\frac{24}{5}\lambda)^2}\,\Lambda' + \Lambda'' . \tag{4.15}$$

It is known,[10] however, from estimates on small clusters of galaxies that $|\Lambda_{\text{obs}}| \lesssim 10^{-57}\ \text{cm}^{-2}$. We will show next that if we require that $\Lambda_{\text{obs}} = O(m^2/\kappa)^r$, for sufficiently large r, we can satisfy this constraint and simultaneously have that $\kappa \sim 8\alpha_6/R_0^2 \sim \kappa_{[4]}$, by additional fine-tuning of the value of σ.

In fact, substituting (4.2) into (4.15) making use of (4.6) and taking

$$\lambda = \frac{5}{24(4\sigma'-1)}, \quad \sigma' \geq \sigma > \tfrac{1}{4} , \tag{4.16}$$

we find

$$\frac{5}{3}\frac{4\sigma'-1}{(4\sigma')^2}(6\sigma'-\tfrac{1}{2}) - \frac{m^2}{\kappa}\frac{(4\sigma'-1)^2}{(4\sigma')(4\sigma'-2)}\left[6\sigma-1-\frac{4\sigma'}{4\sigma'-1}\right] + \frac{24}{5}\left[\frac{m^2}{\kappa}\right]^2(\sigma'-\sigma)^2\frac{(4\sigma'-1)^3}{(4\sigma')^2(4\sigma'-2)^2} \leq 10^{-122} . \tag{4.17}$$

One obvious solution to this expression is to take $\sigma' = \frac{1}{4}$. This however implies that $\lambda = \infty$, so it must be discarded. Another solution comes from substituting into (4.17) the series expansion

$$\sigma' = \tfrac{1}{2} + \gamma_1\left[\frac{m^2}{\kappa}\right] + \gamma_2\left[\frac{m^2}{\kappa}\right]^2 + \cdots , \tag{4.18}$$

and equating to zero the coefficients of the various powers of m^2/κ, up to a sufficiently high order in this parameter. In this way, the zeroth-order contribution from (4.17) gives a quadratic expression for γ_1, with the following two possible solutions:

$$\gamma_1 = \tfrac{1}{6}(2\sigma-1)(1 \pm \tfrac{4}{5}) . \tag{4.19}$$

Proceeding in a similar fashion, one can get solutions for the remainder of the parameters in (4.18) expressed in terms of σ.

Note however that if we substitute (4.18) into (4.16) and use the result in the expression for the vacuum value of the Higgs fields

$$\Phi_0 \equiv |(\Phi^\alpha\Phi_\alpha)_0|^{1/2} = \left[\frac{m^2}{4\lambda} + \frac{12}{5R_0^2}\right]^{1/2} ,$$

we get

$$\Phi_0^2 = \frac{3\kappa}{10\gamma_1}\left[\tfrac{10}{3}\gamma_1 - 2\sigma + 1 + \frac{m^2}{\kappa}(10\gamma_1^2 - 12\sigma\gamma_1 + 8\gamma_1)\right] .$$

Consequently, no sensible solutions for the vacuum are obtained unless we discard the smallest root for γ_1, and get

$$\Phi_0^2=\left[-\frac{9\sigma}{5}+\frac{3}{2}\right]m^2 . \tag{4.20}$$

Furthermore is we substitute (4.6), (4.16), and (4.18) into (4.2), we find that

$$\frac{4\alpha_6}{R_0^2}=3\left[1+\frac{5}{12(2\sigma-1)}\right]\kappa . \tag{4.21}$$

Thus this term is indeed of the same order of magnitude as κ, and by virtue of (4.10) of the same order of magnitude as the inverse square of the Planck length provided $(2\sigma-1)$ is not very small.

Recalling that (4.16) and (4.18) allows us to set $\frac{11}{32}>\sigma>\frac{1}{4}$ as the acceptable range for the values of σ, and

$$\kappa=\frac{\kappa_{[4]}}{\frac{7}{2}+\frac{5}{4(2\sigma-1)}} , \quad \frac{11}{32}>\sigma>\frac{1}{4} . \tag{4.22}$$

To conclude this section, we now outline the procedure for obtaining the values of the as-yet undetermined parameters in our Lagrangian (3.8).

To derive τ^2, note that (3.14), (3.17), and (3.30) together imply

$$\alpha_4+\alpha_5=\tfrac{288}{5}\lambda , \tag{4.23}$$

which, in turn, enables us to simplify (3.3) to the form

$$\Lambda'=-\tfrac{3}{4}(\kappa-\tfrac{81}{10}\lambda\tau^2)\frac{\tau^2}{\kappa} . \tag{4.24}$$

Comparing (4.24) with (4.12) and making use of (4.16), (4.18), and (4.19), yields

$$\frac{\tau^2}{\kappa}=\frac{1\pm\left[1+\frac{4}{3}(8\sigma-1)\frac{m^2}{\kappa}\left[\frac{m^2}{\kappa}(2\sigma-1)\frac{81}{160}-\frac{27}{32}\right]\right]^{1/2}}{2\left[\frac{27}{32}-\frac{m^2}{\kappa}(2\sigma-1)\frac{81}{160}\right]} . \tag{4.25}$$

Thus, to first order in m^2/κ, the length factor introduced in (2.25) has the following two possible values:

$$\frac{\tau^2}{\kappa_{1,2}}\simeq\begin{cases}\frac{1}{3}(8\sigma-1)\frac{m^2}{\kappa} ,\\ \frac{32}{27}\left[1-\frac{m^2}{2\kappa}(\frac{51}{80}+\frac{21}{10}\sigma)\right] .\end{cases} \tag{4.26}$$

Substituting (4.26) back into (4.24) gives a relation between the mass of the Higgs scalar and the cosmological term Λ'.

As for the mass term itself, recall that it was given in (3.2) in term of the parameters β_1, β_2, β_3, α_1, α_2, α_4, and α_6. We can use (3.5), (3.6), (3.7), (3.30), and (3.31) to solve for some of these parameters in terms of the others and λ. We do not have, however, any relations for β_2 and β_3. For the purpose of constraining our system a bit more, we can set $\beta_1=\beta_2=\beta_3=0$. In essence this involves assuming that torsion does not occur explicitly in the Lagrangian and that it only appears through its contribution to the curvature terms. Our theory would then be akin to the more commonly used Einstein-Cartan models, but with a Gauss-Bonnet form for the curvature terms.

Adopting these restrictions, we find that

$$\frac{m^2}{2}=\frac{5}{6}\kappa+\left[\frac{1}{48e^2}+\tfrac{3}{40}+\tfrac{99}{20}\lambda-\tfrac{29}{192}\alpha_5\right]\tau^2 \tag{4.27}$$

and replacing in this expression the values previously found for λ and τ^2 yields (to first order in m^2/κ)

$$(\alpha_5)_{1,2}\simeq\begin{cases}\frac{480}{29(8\sigma-1)}\frac{\kappa}{m^2} ,\\ 15.62+\frac{m^2}{\kappa}(6.884-10.704\sigma) .\end{cases} \tag{4.28}$$

Clearly the two alternative solutions in (4.28) for α_5 correspond to the two possible values for the length gauge found in (4.26).

V. CONCLUSIONS

We have studied the compactification of the extra dimensions in a six-dimensional Kaluza-Klein theory by means of a calculation similar to that of Cremmer and Scherk[2] where the six-dimensional geometrodynamics is driven by Yang-Mills and Higgs fields. The internal group of the gauge fields is SO(3). We differ from Crammer and Scherk[2] in that we use a geometric theory of the coupled gravitation, Yang-Mills, and Higgs fields based on principal fiber bundles with torsion that was developed by Cho[3] and Katanayev and Volovich.[4] The geometrical character of the theory leads naturally to Lagrangians quadratic in the curvature. The modifications of the Einstein-Yang-Mills equations have interesting consequences, even though the calculation is only slightly more difficult than the original one of Cremmer and Scherk. Our result, if no conditions are imposed, implies that there is essentially no compactification of the extra dimensions. If, however, the Gauss-Bonnet combination of R^2 terms, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-4R^{\alpha\beta}R_{\alpha\beta}+R^2$ is taken in the Lagrangian, a number of arbitrary constants in the theory can be determined, including the dimensionless coupling constant of the Yang-Mills field. The value we obtain for this last constant is extremely close to the present experimental value for the electroweak model.

The prediction of the Yang-Mills coupling constant seems to imply that the fiber-bundle Lagrangian with torsion that we have derived is a candidate for a realistic model of the gravitation-Yang-Mills-Higgs systems, especially if the Gauss-Bonnet form for the curvature squared terms is taken. The major difficulty in proposing our exact Lagrangian as the true one is that the fundamental group which we used is not the usual SU(2)×U(1) characteristic group of the electroweak models. It should be possible, however, to redo the calculation for this second group and compare the new value for e to that obtained above.

Another possible way of extending the present work would be to investigate the more realistic possibility of "dynamic" compactification, where one takes the four-dimensional "external" manifold to be a cosmological model and the "internal" space to be a compact manifold whose "radius" is large near the singularity of the four-dimensional cosmology and then decays to small values as the external universe expands.

APPENDIX: THE RIEMANN CURVATURE TENSOR COMPONENTS ON THE PRINCIPAL FIBER BUNDLE

Relative to the noncoordinate bases $\mathbf{E}_1, \ldots, \mathbf{E}_{n+f}$ introduced in Sec. II, the components of the curvature term for the metric (2.1) on the principal fiber bundle P described in that section are obtained by substituting Eqs. (2.10)–(2.14) for the one-form-valued connection matrices into (2.5) which relates these connections to the Riemann tensor components on P. The resulting expressions, in a notation that closely follows that of Ref. 6, are

$$R^{n+\alpha}{}_{n+\beta\, n+\delta\, n+\lambda}=\tfrac{1}{4}c^{\alpha}{}_{\beta\gamma}c^{\gamma}{}_{\delta\lambda}\ , \tag{A1}$$

$$\begin{aligned} R^{n+\alpha}{}_{n+\beta\, ij}=&\tfrac{1}{2}c^{\alpha}{}_{\beta\gamma}\Omega^{\gamma}{}_{ij}-\tfrac{1}{4}(\Omega^{\alpha}{}_{k[i}\Omega_{|\beta|}{}^{k}{}_{j]}-\Omega^{\alpha}{}_{k[i}S_{|\beta|}{}^{k}{}_{j]}-S^{\alpha}{}_{k[i}\Omega_{|\beta|}{}^{k}{}_{j]}+S^{\alpha}{}_{k[i}S_{|\beta|}{}^{k}{}_{j]})\\ &+\tfrac{1}{2}[S^{k}{}_{i\beta}(\Omega^{\alpha}{}_{kj}-S^{\alpha}{}_{kj})-S_{ki}{}^{\alpha}(\Omega_{\beta}{}^{k}{}_{j}-S_{\beta}{}^{k}{}_{j})]\\ &+\tfrac{1}{2}[S^{k}{}_{j}{}^{\alpha}(\Omega_{\beta}{}^{k}{}_{i}-S_{\beta}{}^{k}{}_{i})-S^{k}{}_{j\beta}(\Omega^{\alpha}{}_{ki}-S^{\alpha}{}_{ki})]-S_{k[i}{}^{\alpha}S^{k}{}_{j]\beta} \end{aligned} \tag{A2}$$

(where we have made use of the conventional notation $\Omega^{\alpha}{}_{k[i}\Omega_{|\beta|}{}^{k}{}_{j]}\equiv\Omega^{\alpha}{}_{ki}\Omega_{\beta}{}^{k}{}_{j}-\Omega^{\alpha}{}_{kj}\Omega_{\beta}{}^{k}{}_{i}$),

$$R^{n+\alpha}{}_{n+\beta n+\gamma i}=0\ , \tag{A3}$$

$$R^{n+\alpha}{}_{jki}=\tfrac{1}{2}(\Omega^{\alpha}{}_{ji,k}-\Omega^{\alpha}{}_{jk,i})-\tfrac{1}{2}(S^{\alpha}{}_{ji,k}-S^{\alpha}{}_{jk,i})-(S_{ji}{}^{\alpha}{}_{,k}-S_{jk}{}^{\alpha}{}_{,i})+\tfrac{1}{2}(\Omega^{\alpha}{}_{jl}-S^{\alpha}{}_{jl})S^{l}{}_{ki}-S_{jl}{}^{\alpha}S^{l}{}_{ki} \tag{A4}$$

(where $\Omega^{\alpha}{}_{ji,a}\equiv\mathbf{E}_a[\Omega^{\alpha}{}_{ji}]$ (i.e., directional derivative)),

$$R^{n+\alpha}{}_{jn+\beta\, i}=\tfrac{1}{2}(\Omega^{\alpha}{}_{ji,\beta}-S^{\alpha}{}_{ji,\beta}-2S_{ji}{}^{\alpha}{}_{,\beta})+\tfrac{1}{4}(\Omega^{\alpha}{}_{ki}-S^{\alpha}{}_{ki}-2S_{ki}{}^{\alpha})(\Omega_{\beta}{}^{k}{}_{j}-S_{\beta}{}^{k}{}_{j})+\tfrac{1}{4}c^{\alpha}{}_{\beta\gamma}(\Omega^{\gamma}{}_{ji}-S^{\gamma}{}_{ji}-2S_{ji}{}^{\gamma})\ , \tag{A5}$$

$$R^{n+\alpha}{}_{jn+\beta\, n+\gamma}=0\ , \tag{A6}$$

$$R^{i}{}_{jkm}=\underline{R}^{i}{}_{jkm}+\tfrac{1}{2}(\Omega_{\alpha}{}^{i}{}_{j}-S_{\alpha}{}^{i}{}_{j})\Omega^{\alpha}{}_{mk}-\tfrac{1}{4}(\Omega_{\gamma}{}^{i}{}_{[k}-S_{\gamma}{}^{i}{}_{[k}-2S^{i}{}_{[k|\gamma|})(\Omega^{\gamma}{}_{|j|m]}-S^{\gamma}{}_{|j|m]}-2S_{|j|m]}{}^{\gamma}) \tag{A7}$$

[where $\underline{R}^{i}{}_{jkm}$ are the components of the curvature tensor of (M,g)],

$$R^{i}{}_{jkn+\alpha}=-\tfrac{1}{2}(\Omega_{\alpha}{}^{i}{}_{j,k}-S_{\alpha}{}^{i}{}_{j,k})\ , \tag{A8}$$

$$R^{i}{}_{j\, n+\beta\, n+\alpha}=-\tfrac{1}{2}(\Omega_{[\alpha}{}^{i}{}_{|j|,\beta]}-S_{[\alpha}{}^{i}{}_{|j|,\beta]})+\tfrac{1}{2}c^{\gamma}{}_{\beta\alpha}(\Omega_{\gamma}{}^{i}{}_{j}-S_{\gamma}{}^{i}{}_{j})+\tfrac{1}{4}(\Omega_{[\beta}{}^{i}{}_{|k|}-S_{[\beta}{}^{i}{}_{|k|})(\Omega_{\alpha]}{}^{k}{}_{j}-S_{\alpha]}{}^{k}{}_{j})\ . \tag{A9}$$

To evaluate the Ricci tensor of (P,h) at p relative to $\mathbf{E}_1,\ldots,\mathbf{E}_{n+f}$, note that $R_{jm}=R^{i}{}_{jim}+R^{n+\alpha}{}_{jn+\alpha m}$. Thus, making use of (A5) and (A7), we find

$$\begin{aligned} R_{jm}=&\underline{R}_{jm}+\tfrac{1}{2}(\Omega^{\alpha}{}_{jm,\alpha}-S^{\alpha}{}_{jm,\alpha}-2S_{jm}{}^{\alpha}{}_{,\alpha})+\tfrac{1}{4}(c^{\alpha}{}_{\alpha\gamma}+2S^{i}{}_{i\gamma})(\Omega^{\gamma}{}_{jm}-S^{\gamma}{}_{jm}-2S_{jm}{}^{\gamma})+S^{i}{}_{m\gamma}S_{ji}{}^{\gamma}\\ &+\tfrac{1}{2}(\Omega_{\gamma}{}^{i}{}_{j}-S_{\gamma}{}^{i}{}_{j})\Omega^{\gamma}{}_{mi}-\tfrac{1}{2}S_{ji}{}^{\gamma}(\Omega_{\gamma}{}^{i}{}_{m}-S_{\gamma}{}^{i}{}_{m})\ . \end{aligned} \tag{A10}$$

Proceeding similarly with the other components of the Ricci tensor, we find

$$R_{n+\alpha j}=-\tfrac{1}{2}(\Omega_{\alpha}{}^{i}{}_{j,i}-S_{\alpha}{}^{i}{}_{j,i})+(S^{i}{}_{j\alpha,i}-S^{i}{}_{i\alpha,j})-\tfrac{1}{2}(\Omega_{\alpha kl}-S_{\alpha kl})S^{lk}{}_{j}+S_{kl\alpha}S^{lk}{}_{j}\ , \tag{A11}$$

$$R_{jn+\alpha}=-\tfrac{1}{2}(\Omega_{\alpha}{}^{i}{}_{j,i}-S_{\alpha}{}^{i}{}_{j,i})\ , \tag{A12}$$

$$R_{n+\gamma\, n+\lambda}=\tfrac{1}{4}(\Omega_{\gamma ki}-S_{\gamma ki}-2S_{ki\gamma})(\Omega_{\lambda}{}^{ki}-S_{\lambda}{}^{ki})-S^{i}{}_{i\gamma,\lambda}-\tfrac{1}{2}c_{\gamma\lambda\beta}S^{i}{}_{i}{}^{\beta}+\tfrac{1}{4}c^{\alpha}{}_{\gamma\beta}c^{\beta}{}_{\alpha\lambda}\ . \tag{A13}$$

Finally, for the Ricci scalar we have $R=g^{jm}R_{jm}+k^{\beta\gamma}R_{n+\gamma\, n+\beta}$, and inserting (A10) and (A13) leads to

$$R=\underline{R}-2S_{j}{}^{j\alpha}{}_{,\alpha}-S_{j}{}^{j\gamma}(c^{\alpha}{}_{\alpha\gamma}+S^{i}{}_{i\gamma})+S^{ij}{}_{\gamma}S_{ij}{}^{\gamma}+\tfrac{1}{4}c^{\alpha}{}_{\gamma\beta}c^{\beta}{}_{\alpha}{}^{\gamma}-\tfrac{1}{4}\Omega_{\alpha}{}^{i}{}_{j}\Omega^{\alpha}{}_{i}{}^{j}+\tfrac{1}{4}S_{\gamma}{}^{i}{}_{j}S^{\gamma}{}_{i}{}^{j}\ . \tag{A14}$$

Note that this last expression generalizes to the case of nonzero torsion the theorem by Cho relating the scalar curvature of a principal fiber bundle to the resulting projected function on the base manifold.

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