Quantum kinetic field theory in curved spacetime: Covariant Wigner function and Liouville-Vlasov equations

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We consider quantum fields in an external potential and show how, by using the Fourier transform on propagators, one can obtain the mass-shell constraint conditions and the Liouville-Vlasov equation for the Wigner distribution function. We then consider the Hadamard function $G_1(x_1,x_2)$ of a real, free, scalar field in curved space. We postulate a form for the Fourier transform $F^{(Q)}(X,k)$ of the propagator with respect to the difference variable $x=x_1-x_2$ on a Riemann normal coordinate centered at Q. We show that $F^{(Q)}$ is the result of applying a certain Q-dependent operator on a covariant Wigner function F. We derive from the wave equations for G_1 a covariant equation for the distribution function and show its consistency. We seek solutions to the set of Liouville-Vlasov equations for the vacuum and nonvacuum cases up to the third adiabatic order. Finally we apply this method to calculate the Hadamard function in the Einstein universe. We show that the covariant Wigner function can incorporate certain relevant global properties of the background spacetime. Covariant Wigner functions and Liouville-Vlasov equations are also derived for free fermions in curved spacetime. The method presented here can serve as a basis for constructing quantum kinetic theories in curved spacetime or for near-uniform systems under quasiequilibrium conditions. It can also be useful to the development of a transport theory of quantum fields for the investigation of grand unification and post-Planckian quantum processes in the early Universe.

I. INTRODUCTION

In the previous paper aimed at providing a framework for the study of nonequilibrium quantum fields in flat and curved spacetimes, Calzetta and Hu[1] developed a functional-integral approach based on the use of closed-time-path effective action and the relativistic Wigner-function formalism. From a two-loop calculation, they derived the Boltzmann equation for the distribution function of interacting quantum fields and the gap equation for the effective mass of the quasiparticles. In this paper we will develop the Wigner-function formalism in curved spacetime and derive the set of Liouville equations for noninteracting quantum fields. This may serve as a basis for quantum kinetic theory in curved spacetime, which is useful for the investigation of quantum transport phenomena as occurring in the early Universe and around black holes.

Consider the propagator of a free scalar field in flat space. Take, for example, the Hadamard function

$$G_1(x_1,x_2)=\langle \{\phi(x_1),\phi(x_2)\}\rangle \ , \tag{1.1}$$

where $\{\ \}$ denotes the symmetrized product and $\langle\ \rangle$ denotes the statistical average with respect to the density matrix ρ. It obeys the Klein-Gordon equation

$$(\Box_{x_{1,2}}+m^2)G_1(x_1,x_2)=0 \ , \tag{1.2}$$

and the symmetry conditions

$$\begin{aligned} G_1(x_1,x_2)&=G_1(x_2,x_1) \ , \\ G_1(x_1,x_2)&=G_1^*(x_1,x_2) \ . \end{aligned} \tag{1.3}$$

If the density matrix is diagonal in some chosen Fock basis, then G_1 will be translation invariant in these states. A general solution to (1.2) with these properties has the representation

$$G_1(x_1,x_2)=\int \frac{d^4k}{(2\pi)^4}e^{ik\cdot(x_1-x_2)}F(k) \ , \tag{1.4}$$

where

$$F(k)=[1+f(k)]\delta(k^2-m^2) \ . \tag{1.5}$$

We see that $f(k)$ is a real, positive function even in k and is readily interpretable as the distribution function in momentum space which gives the occupation number of the kth mode. The δ-function term is, of course, the vacuum contribution.

We note that the existence and validity of the Fourier representation is a consequence of translation invariance. In flat space this is globally well defined. For curved space this is possible only for the very restricted classes of static or stationary homogeneous spaces. If the space is spatially homogeneous, then the Fourier representation of functions of spatial variables can be globally defined (by methods of harmonic analysis[2]), but not in the time variable. This means that dynamical systems can have nonlocal (or history-dependent) behavior. This also means that a statistical system will, in general, evolve away from an otherwise thermal equilibrium condition.[3] For spatially inhomogeneous systems, the Fourier representation is, in general, globally ill defined. One can, however, consider a Fourier (or momentum-space) representation of propagators in local coordinate patches, and attempt to approximate the true propagator by expansion in orders of curvature.[4] But there is no assurance of equivalence or even convergence in this procedure. The approach is nevertheless applicable for the consideration of quasilocal or near-homogeneous systems. The momentum-space representation is useful in the treatment of quantum-field systems because many of the well-developed perturbation techniques (Feynman-Dyson expansion) in quantum-field theory and equilibrium statistical mechanics in flat space can be readily generalized. This can be seen in, e.g., the proof of renormalizability of interacting field theories by Bunch and Parker[4] with momentum-space representation of the Feynman propagators. For the study of statistical properties of quantum fields, where the causal and correlation information is important, the density matrix or the distribution function is of central interest. In the treatment of quasilocal[5] or near-uniform systems, the Wigner-distribution-function technique[6,7] is particularly convenient.

Introduce the difference and center-of-mass spacetime coordinates $x=x_1-x_2$, $X=\frac{1}{2}(x_1+x_2)$. If the system has some approximate translation invariance (in time or space or both) such that the X dependence of the propagators, say G_1, is much weaker than the x dependence, then one can Fourier analyze G_1, with respect to x alone, i.e.,

$$G_1(x_1,x_2)=\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}F(X,k) , \tag{1.6}$$

where now, as different from G_1 of the global system (1.4), the Fourier amplitude for mode k is a slowly varying function of X (Ref. 8). The corresponding nonvacuum part $f(X,k)$ is the Wigner function (see below). It provides information not only on the momentum distribution of the modes, but also on how it changes with spacetime location. It is therefore most suitable in treating transport properties of near-equilibrium or quasiuniform systems. It is easy to see that the wave equation (1.2) for G_1, upon its assuming the form (1.6), now decomposes into two equations—the Liouville equation and a modified mass-shell condition. This provides the basis for a kinetic theory of nonequilibrium quantum fields. However, the weak dependence on X or quasilocal condition dictates that it can only provide the next higher order beyond the (infinite-wavelength, collision-dominated) thermodynamic-hydrodynamic approximation but not the complete statistical theory where long-range excitation and strong correlational behavior can be important. To see how the Wigner-function formalism is suitable for a kinetic theory description, consider the following situation: Suppose one separates the region of spacetime of interest into cells whose characteristic size is much greater than the quantum (Compton) scale of the particles (or quasiparticles,[1] if radiative corrections are included) but much smaller than the interaction scale between particles, then only in the interior of a single cell can quantum corrections be important and only between cells will statistical effects be manifest. Within every cell one can assume a locally flat background and uniform distribution of particles. This would allow the use of Riemann normal coordinates (RNC) for each cell and a momentum-space representation (Fourier transform with respect to x) for functions defined in the local patch. This is the problem discussed by Bunch and Parker and others before. The new aspect in our problem is the change from one cell or local patch to another, where relaxational and correlational phenomena become apparent. The Fourier transform now carries a cell-dependent label, dependent on X. This is where the Wigner transformation is relevant.

In curved spacetime the central task lies in finding the transformation laws for functions defined at different patches in a covariant manner. It is only through covariant transformations that one can seek a description of the dynamics independent of specific coordinate choices or the labeling of cells. Wigner function and the Vlasov equation for free quantum fields in curved spacetime have recently been studied by Winter.[9] He sought a covariant definition of midpoints and distance vectors by constructing geodesics and geodesic deviations. He then introduced Fourier transformation with respect to the difference vector to momentum variables and finally derived the kinetic equations with the curvature correction terms. Calzetta and Hu[10] demonstrated that the same results can be derived with great economy by means of Riemann normal coordinates and momentum representation of propagators. This earlier work then derived the results in specific coordinate constructions. How the distribution function and the dynamical equations may change under variations of the coordinates was not discussed. However, as alluded to above, in curved spacetime the transformation properties of these functions are indeed more important than their specific appearance, because it is through the transformation properties that their intrinsic characters are defined. It is also indispensable for a gauge-independent description of dynamics and for possible development of a covariant formulation of Hamiltonian quantum gravity.[11] On this latter point, the Wigner-function formalism is somewhat of a hybrid between the manifestly covariant formulation based on two-point functions in configuration space[12] which is formally rigorous but less tractable, and that of the momentum-space Feynman propagator approach,[4,13] which is technically versatile but only approximate. We will discuss the Wigner function and symplectic geometry approach to more formal aspects of general relativity and quantum gravity in another context later. In this work we focus on the transformation properties of the Wigner functions and on finding formal solutions to the Liouville equation.

This paper is organized as follows: In Sec. II, as a warm-up for the general curved-spacetime case, we consider quantum fields in an external potential. We show how, by the Wigner transformation and the Fourier representation, one can obtain the mass-shell constraint condition and the Liouville-Vlasov equations from the wave equation governing the propagators. We also discuss the consistency of these iterative equations and their solutions. In Sec. III we consider the Hadamard function $G_1(x_1,x_2)$ of a real free field in curved space. In Sec. III A, we postulate a form for the Fourier transform $F^{(Q)}(X,k)$ of the propagator in a Riemann normal coordinate (RNC) centered at Q and discuss the Q dependence of $F^{(Q)}$. By using the transformation properties of different geometric quantities such as the metric, its determinant, and their derivatives in a RNC, we show that $F^{(Q)}$ is the result of applying a certain Q-dependent operator on a covariantly defined distribution function F. In Sec. III B we derive from the wave equation for G_1 a covariant equation for the distribution function F. In Sec. III C we show the consistency in the set of Liouville equations and seek solutions to these equations for the vacuum and nonvacuum cases up to the third adiabatic order. In Sec. III D we apply our methods to calculate the Hadamard function in the Einstein universe, showing that they can indeed incorporate the relevant global properties of the background spacetime. In Sec. IV we do the same for free fermions. We end with a few closing remarks on the methodology, viewpoint, generalization, and application of the Wigner function approach in curved-space quantum field theory. The Appendix contains some useful formulas for Riemann normal-coordinate expansions of geometric quantities.

II. RELATIVISTIC QUANTUM FIELD IN AN EXTERNAL POTENTIAL

As a useful prelude to the general curved-spacetime case, we will consider in this section how the Hadamard function (1.1) of a real scalar field ϕ in flat space with a varying external potential $V(x)$ may be expressed in terms of a distribution function obeying Vlasov's equation.[8]

The object of interest is

$$G_1(x_1,x_2)=\langle\{\phi(x_1),\phi(x_2)\}\rangle\ , \tag{2.1}$$

where $\langle\ \rangle$ denotes a statistical average with density matrix ρ. It obeys the wave equation

$$[\Box_{x_i}+m^2+V(x_i)]G_1(x_1,x_2)=0\quad (i=1,2) \tag{2.2}$$

and the symmetry condition (signature $+,-,-,-$)

$$\begin{aligned}G_1(x_1,x_2)&=G_1(x_2,x_1)\ ,\\ G_1(x_1,x_2)&=G_1^*(x_1,x_2)\ .\end{aligned} \tag{2.3}$$

For a constant potential $V(x_i)=V_0$, the theory is equivalent to a free field theory with square mass m^2+V_0. The most general solution of (2.2) is then

$$G_1(x_1,x_2)=\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x_1-x_2)}[1+f(k)]\times\delta(k^2-m^2-V_0)\ , \tag{2.4}$$

where again $f(k)$ is a real, positive function even in k [cf. (1.4) and (1.5)]. It is the occupation number of mode $\mathbf{k}$.

The first nontrivial situation is when $V(x_i)$ is a slowly varying function of x_1 and x_2, whence the dependence of G_1 on X will be much weaker than on x. We may then Fourier analyze G_1 with respect to x as in (1.6):

$$G_1(x_1,x_2)=\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}F(X,k)\ . \tag{2.5}$$

Expanding the wave operator and $V(x_i)$ to first order in $\partial/\partial X$, we may write

$$\Box G_1(x_1,x_2)\sim\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}\left[-k^2+ik\cdot\frac{\partial}{\partial X}\right]F(X,k)\ ,$$

and

$$\begin{aligned}V(x_i)G_1(x_1,x_2)&\sim\left[V(X)\pm\frac{x^\mu}{2}V_{,\mu}(X)\right]G_1(x_1,x_2)\\ &=\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}\left[V(X)\pm\frac{i}{2}V_{,\mu}(X)\frac{\partial}{\partial k_\mu}\right]\times F(X,k)\ ,\end{aligned}$$

where $V_{,\mu}\equiv\partial V/\partial X^\mu$. So Eq. (2.2) becomes

$$[-k^2+m^2+V(X)]F(X,k)+i\left[k^\mu\frac{\partial F}{\partial X^\mu}+\frac{1}{2}V_{,\mu}\frac{\partial F}{\partial k_\mu}\right]=0\ . \tag{2.6}$$

Because F must be real, the real and imaginary parts of (2.6) must vanish independently. We recognize that the real part of (2.6) is just the mass-shell constraint on F; thus F is given by

$$F(X,k)=[1+f(X,k)]\delta(k^2-m^2-V(X))\ . \tag{2.7}$$

The imaginary part will yield the Vlasov equation. Taking into account that the δ function is itself a solution, it reduces to an equation for f:

$$\left[k^\mu\frac{\partial}{\partial X^\mu}+\frac{1}{2}V_{,\mu}(X)\frac{\partial}{\partial k_\mu}\right]f(X,k)=0\ . \tag{2.8}$$

This allows one to interpret f as a distribution function. If we include higher derivatives of the external potential with respect to X, the result will be a relaxation of the mass-shell constraint. The excitations of the field acquire a finite lifetime because of absorption and emission processes. Correspondingly, the spectrum of excitations will have a finite extension around the mass shell, the width of the spectrum being inversely proportional to the mean lifetime. The simplest such "blurred" mass shell has a line width Γ associated with the function

$$\mathrm{Im}\{[k^2-m^2-V(X)]+i\Gamma\}^{-1}\ , \tag{2.9}$$

which gives the state corresponding to a lifetime $\tau\sim\Gamma^{-1}$. If the characteristic lifetime of these excitations is much larger than other characteristic times, we may still search for solutions of (2.2) in the form (2.5) with F concentrated on the mass shell, but with a more general singularity than a δ function. That is, we can look for solutions of the form

$$F(X,k)=\sum_{n=0}^{\infty}F_n(X,k)\delta^{(n)}(k^2-m^2-V(X)) , \quad (2.10)$$

where $\delta^{(n)}(x)=d^n\delta(x)/dx^n$. We may think of (2.10) as an asymptotic development for a function of the general type (2.9) in the limit $\Gamma\to 0$. For example, if we retain second derivatives in X, we will find

$$\left[-k^2+\frac{1}{4}\Box_X+m^2+V(X)-\frac{1}{8}V_{,\mu\nu}(X)\frac{\partial^2}{\partial k_\mu \partial k_\nu}\right]F + i\left[k^\mu\frac{\partial}{\partial X^\mu}+\frac{1}{2}V_{,\mu}\frac{\partial}{\partial k_\mu}\right]F=0 . \quad (2.11)$$

Substituting (2.10) for F and treating all derivatives of the δ function as linearly independent, we obtain

$$\begin{aligned} &F_1+\tfrac{1}{4}\Box F_0-\tfrac{1}{8}V_{,\mu\nu}\frac{\partial^2}{\partial k_\mu\partial k_\nu}F_0=0 ,\\ &2F_2-\tfrac{1}{2}F_0(\Box V)-\tfrac{1}{2}V_{,\mu}\frac{\partial F_0}{\partial X^\mu}-\tfrac{1}{2}k^\mu V_{,\mu\nu}\frac{\partial F_0}{\partial k_\nu}=0 ,\\ &3F_3+\tfrac{1}{4}(\partial_\mu V\partial^\mu V)F_0-\tfrac{1}{2}(V_{,\mu\nu}k^\mu k^\nu)F_0=0 ,\\ &nF_n=0 \quad (n\geq 4) , \end{aligned} \quad (2.12)$$

while from the imaginary parts of (2.11) we find

$$\left[k^\mu\frac{\partial}{\partial X^\mu}+\frac{1}{2}V_{,\mu}\frac{\partial}{\partial k_\mu}\right]F_n=0 \quad (n\geq 0) . \quad (2.13)$$

(2.12) constitutes a simple set of recursion relations which allows one to express all the F_n's in terms of F_0 to all orders. In this way, (2.13) becomes an infinite set of equations which F_0 must satisfy. This may seem inconsistent or at least difficult to achieve. However, the actual situation is not as bad. To begin with, at a given degree of accuracy in derivatives of V, only a finite number of Eqs. (2.13) will be nontrivial. In fact, in the above example, only the $n=0$ equation is of second order. Furthermore, not all the equations in (2.13) are independent. To make a nontrivial test of this assertion, let us improve our formalism by including third derivatives of the potential. The new term in (2.11) is purely imaginary, i.e.,

$$-\frac{i}{48}V_{,\mu\nu\rho}\frac{\partial^3}{\partial k_\mu\partial k_\nu\partial k_\rho}F .$$

So Eqs. (2.12) are unchanged, but instead of (2.13) we have a modified Vlasov equation

$$\left[k^\mu\frac{\partial}{\partial X^\mu}+\frac{1}{2}V_{,\mu}\frac{\partial}{\partial k_\mu}-\frac{1}{48}V_{,\mu\nu\rho}\frac{\partial^3}{\partial k_\mu\partial k_\nu\partial k_\rho}\right]F_0=0 . \quad (2.14)$$

The extra equations read

$$\begin{aligned} &\left[k\frac{\partial}{\partial X}+\frac{1}{2}V_{,\sigma}\frac{\partial}{\partial k_\sigma}\right]\left[\Box F_0-\frac{1}{2}V_{,\mu\nu}\frac{\partial^2 F_0}{\partial k_\mu\partial k_\nu}\right]+\frac{1}{2}V_{,\mu\nu\rho}\left[k^\rho\frac{\partial^2 F_0}{\partial k_\mu\partial k_\nu}+\eta^{\mu\nu}\frac{\partial F_0}{\partial k_\rho}\right]=0 ,\\ &\left[k\frac{\partial}{\partial X}+\frac{1}{2}V_{,\sigma}\frac{\partial}{\partial k_\sigma}\right]\left[(\Box V)F_0+V_{,\mu}F_{0,}{}^{\mu}+k^\mu V_{,\mu\nu}\frac{\partial F_0}{\partial k_\nu}\right]-V_{,\mu\nu\rho}\left[k^\mu k^\nu\frac{\partial F_0}{\partial k_\rho}+\eta^{\mu\nu}k^\rho F_0\right]=0 ,\\ &\left[k\frac{\partial}{\partial X}+\frac{1}{2}V_{,\sigma}\frac{\partial}{\partial k_\sigma}\right](V_{,\mu\nu}k^\mu k^\nu-\tfrac{1}{2}\partial_\mu V\partial^\mu V)F_0-V_{,\mu\nu\rho}k^\mu k^\nu k^\rho F_0=0 , \end{aligned} \quad (2.15)$$

while others are of higher order. It is easy to show that all three Eqs. (2.15) are in fact dependent on (2.14) up to third-order terms. So there are no obstacles in the construction of a solution to (2.2), given by Eqs. (2.5), (2.10), and (2.12), with the only independent function F_0 obeying (2.14).

III. DISTRIBUTION FUNCTION FOR A FREE SCALAR FIELD IN CURVED SPACETIME

We now consider a free scalar field defined in a curved spacetime. Our goal is again to obtain an expression for the Hadamard function in terms of a distribution function. The basic definitions carry over from the flat spacetime case: G_1 is a solution of

$$[\Box_{x_1}+m^2+\xi R(x_1)]G_1(x_1,x_2)=0 , \quad (3.1)$$

where $\Box_x=g^{\mu\nu}\nabla_\mu\partial_\nu$ is the Laplace-Beltrami operator (sign $+,-,-,-$) and R is the scalar curvature. G_1 must be real and even. We assume that the metric is "slowly varying" (at least in the region of interest) as measured by some "adiabatic or derivative order." The adiabatic order of a geometric expression is measured by the number of derivatives of $g_{\mu\nu}$ which enter in its definition. For example, R is of second order, $R_{\mu\nu;\rho}$ of third, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ of fourth, and so on. (Here we do not make the fine distinction between adiabatic and quasilocal or derivative order.[5]) Slow variation of the metric to some order will mean that terms of adiabatic order higher than this may be neglected.

Following Sec. II, the first step is to define center-of-mass and difference variables. To give meaning to expressions such as $X=\frac{1}{2}(x_1+x_2)$, we must impart a vector space structure to the curved space. We achieve this by using systems of Riemann normal coordinates (RNC) with origins at point Q. Then x_1 and x_2 are the RNC of the neighboring points P_1 and P_2. It is customary[4] to choose Q as one of the points, e.g., $Q=P_2$. However, this choice destroys the symmetry of the Hadamard function $G_1(x_1,x_2)$. We prefer to keep Q as an arbitrary third point[13] (Fig. 1). A physically meaningful distribution function must be independent of the choice of Q. An important restriction is that the representation of G_1 will be reliable only if both P_1 and P_2 lie within a normal neighborhood of Q. Therefore our formalism is valid mainly in the ultraviolet domain of G_1. Infrared modes will be influenced by the global characteristics of spacetime in addition to the local curvature. Still, the increased generality of our representation of G_1, as compared with previous formalisms, will allow us to incorporate global properties to some degree (see Sec. III D below).

One can make a tentative ansatz for G_1 in a RNC around the point Q as

$$G_1(x_1,x_2)=\Delta_{\rm VM}^{1/2}(x_1,x_2)\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{\sqrt{-g(X)}}F^{(Q)}(X,k) \ , \tag{3.2}$$

where x_i $(i=1,2)$ are the RNC of some spacetime points P_i, $x=x_1-x_2$ and $X=\frac{1}{2}(x_1+x_2)$ are the numerical difference and average of these RNC, k is a local momentum variable and $\Delta_{\rm VM}$ is the Van Vleck–Morette determinant.[14] The factors $\Delta_{\rm VM}^{1/2}$ and $(-g)^{-1/2}$ are included for later convenience.

The Fourier transform $F^{(Q)}$ depends on the choice of the origin of RNC and thus cannot be simply identified with a distribution function. In the following (Sec. III A), we will "factor out" this unwanted Q dependence in order to obtain an admissible definition of the Wigner function F in curved space. In Sec. III B we will derive a dynamical equation for F; in Sec. III C we discuss the structure of the solutions of this equation. Finally, Sec. III D applies this formalism to the Hadamard function in a static Einstein universe.

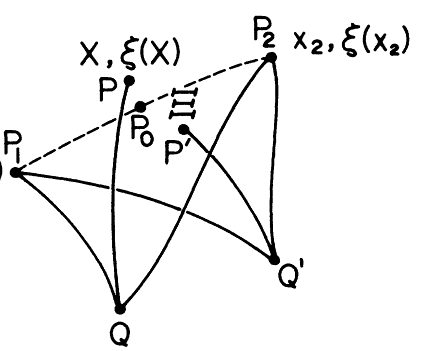

FIG. 1. The x_1 are the Riemann normal coordinates (RNC) of a point P_1 when the origin is at the point Q. When the origin is at Q', the RNC of P_1 are $\xi(x_1)$, analogously for the point P_2 with coordinates x_2 and $\xi(x_2)$. The numerical average of x_1 and x_2 defines the coordinates X of a new point P in the RNC around Q. The numerical average of $\xi(x_1)$ and $\xi(x_2)$ defines the coordinates Ξ of the point P' in the RNC around Q'. P and P' (X and Ξ) are not in general the same spacetime point, rather the RNC of P around Q' is $\xi(X)$. They are also in general different from the geodetic midpoint P_0 (Ref. 9). If Q were chosen to be P_0, then $x_1=-x_2$.

A. Kinematics of covariant distribution functions

Suppose we introduce a second RNC with origin at $Q'\neq Q$. The points $P_{1,2}$ whose coordinates in the old system were $x_{1,2}$ will have in the new system coordinates $\xi_{1,2}$ (Fig. 1). We may regard the transition from the old to the new system as a one-to-one mapping:

$$\xi_i^{\mu'}=\xi^{\mu'}(x_i^{\mu}) \ . \tag{3.3}$$

(We will use primed indices to denote quantities evaluated in RNC around Q', unprimed for those around Q.) In this new system, the same ansatz (3.2) leads to

$$G_1(\xi_1,\xi_2)=\Delta_{\rm VM}^{1/2}(x_1,x_2)\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot\xi}}{\sqrt{-g'(\Xi)}}F^{(Q')}(\Xi,p) \ , \tag{3.4}$$

where $\xi=\xi_1-\xi_2$ and $\Xi=\frac{1}{2}(\xi_1+\xi_2)$. Since $\Delta_{\rm VM}$ is a biscalar, it is independent of the choice of coordinates.

Now, with $x_{1,2}=X\pm(x/2)$, we can expand (3.3) into

$$\xi_{1,2}^{\mu'}=\xi^{\mu'}(X)\pm\frac{1}{2}\frac{\partial\xi^{\mu'}}{\partial X^{\mu}}(X)x^{\mu}+\frac{1}{8}\frac{\partial^2\xi^{\mu'}}{\partial X^{\mu}\partial X^{\nu}}(X)x^{\mu}x^{\nu} \pm\frac{1}{48}\frac{\partial^3\xi^{\mu'}}{\partial X^{\mu}\partial X^{\nu}\partial X^{\rho}}(X)x^{\mu}x^{\nu}x^{\rho}+\cdots \ . \tag{3.5}$$

And so

$$\Xi^{\mu'}=\xi^{\mu'}(X)+\frac{1}{8}\frac{\partial^2\xi^{\mu'}}{\partial X^{\mu}\partial X^{\nu}}(X)x^{\mu}x^{\nu} +\frac{1}{4!16}\frac{\partial^4\xi^{\mu'}}{\partial X^{\mu}\partial X^{\nu}\partial X^{\rho}\partial X^{\sigma}}(X)x^{\mu}x^{\nu}x^{\rho}x^{\sigma}+\cdots \ , \tag{3.6}$$

$$\xi^{\mu'}=\frac{\partial\xi^{\mu'}}{\partial X^{\mu}}(X)x^{\mu}+\frac{1}{24}\frac{\partial^3\xi^{\mu'}}{\partial X^{\mu}\partial X^{\nu}\partial X^{\rho}}(X)x^{\mu}x^{\nu}x^{\rho}+\cdots \ . \tag{3.7}$$

From the usual relationship $\partial_{\mu'}\ln(-g')^{1/2}=\Gamma^{\nu'}_{\mu'\nu'}$ we get

$$[-g'(\Xi)]^{1/2}=[-g'(\xi(X))]^{1/2} \times\left[1+\Gamma^{\nu'}_{\mu'\nu'}(\xi(X))\frac{1}{8}\frac{\partial^2\xi^{\mu'}}{\partial X^{\rho}\partial X^{\sigma}}x^{\rho}x^{\sigma}+\cdots\right] . \tag{3.8}$$

Also

$$[-g'(\xi(X))]^{1/2}=\left|\frac{\partial X}{\partial\xi}\right|(X)[-g(X)]^{1/2} \ ,$$

where $|\partial X/\partial\xi|$ is the Jacobian of the transformation from the new system to the old. We can also expand

$$F^{(Q')}\left[\xi(X)+\frac{1}{8}\frac{\partial^2\xi}{\partial X\partial X}xx+\cdots,p\right]$$
$$=F^{(Q')}(\xi(X),p)+\frac{1}{8}\frac{\partial^2\xi^{\mu'}}{\partial X^\mu\partial X^\rho}x^\nu x^\rho\frac{\partial}{\partial\xi^{\mu'}}F^{(Q')}(\xi(X),p)+\cdots, \quad (3.9)$$

where the ξ derivative acts exclusively on the first argument of $F^{(Q')}$.

Observe that G_1 is a biscalar and thus independent of the choice of coordinates, i.e., Eqs. (3.2) and (3.4) are two equivalent representations of the same function. We may relate these two expressions by substituting Eqs. (3.6)–(3.9) into (3.4). Making the change of variables

$$p_{\mu'}=\frac{\partial X^\mu}{\partial\xi^{\mu'}}(X)k_\mu\ , \quad (3.10)$$

and substituting $x^\mu=i\partial/\partial k_\mu$ into the integrand, we see that the invariance of G_1 then implies the transformation law for $F^{(Q)}$:

$$F^{(Q)}(X^\mu,k_\mu)=\left[1+\frac{1}{8}\Gamma^{\nu'}_{\mu'\nu'}(\xi(X))\frac{\partial^2\xi^{\mu'}}{\partial X^\rho\partial X^\sigma}(X)\frac{\partial^2}{\partial k_\rho\partial k_\sigma}+\frac{1}{24}\frac{\partial X^\mu}{\partial\xi^{\mu'}}(X)\frac{\partial^3\xi^{\mu'}}{\partial X^\nu\partial X^\rho\partial X^\sigma}(X)\frac{\partial^3}{\partial k_\nu\partial k_\rho\partial k_\sigma}k_\mu\right.$$
$$\left.-\frac{1}{8}\frac{\partial^2\xi^{\mu'}}{\partial X^\mu\partial X^\nu}(X)\frac{\partial^2}{\partial k_\nu\partial k_\mu}\frac{\partial}{\partial\xi^{\mu'}}+\cdots\right]F^{(Q')}\left[\xi(X),\frac{\partial X}{\partial\xi}k\right], \quad (3.11)$$

where the dots stand for corrections of higher adiabatic order. An important particular case of Eq. (3.11) is when $Q'=P$ where P is the point whose RNC in the old system (with respect to Q) are exactly X^μ (remember that X is simply the average of x_1 and x_2 in the old system and not necessarily the midpoint of the geodesic from P_1 to P_2). In this case, Eq. (3.11) simplifies to

$$F^{(Q)}(X,k_\mu)=\left[1+\frac{1}{24}(\Gamma^\mu_{\nu\rho,\sigma}+\Gamma^\mu_{\nu\tau}\Gamma^\tau_{\rho\sigma})(X)\frac{\partial^3}{\partial k_\nu\partial k_\rho\partial k_\sigma}k_\mu-\frac{1}{8}\Gamma^\mu_{\nu\rho}(X)\frac{\partial^2}{\partial k_\nu\partial k_\rho}e^a_\mu(X)\frac{\partial}{\partial y^a}+\cdots\right]F^{(P)}(y^a,e^\mu_a(X)k_\mu)\Big|_{y^a=0}, \quad (3.12)$$

where we have introduced y^a to denote RNC with origin at P and a vierbein $e^\mu_a=(\partial X^\mu/\partial y^a)(0)$ (Ref. 15). In Eq. (3.12) the Christoffel symbols are evaluated in the old system (with origin at Q).

Equation (3.12) suggests that we may succeed in factoring the Q dependence out of $F^{(Q)}$ by introducing a "diagonal" kernel (Fig. 2)

$$F(X^\mu,k_\mu)=F^{(X)}(0,e^\mu_a(X)k_\mu)\ . \quad (3.13)$$

For practical purposes, henceforth we will label F with the coordinate X of P instead of P itself. The new function F is independent of Q; in fact, it is defined in any coordinate system and not just in normal ones. Moreover, its transformation law

$$F(X^{\mu'},k_{\mu'})=F\left[X^\mu,\frac{\partial X^{\mu'}}{\partial X^\mu}k_{\mu'}\right] \quad (3.14)$$

is the appropriate one for a distribution function. Of course, F depends on the choice of the vierbein. Even in flat space a distribution function is not Lorentz invariant. The relationship between $F^{(Q)}$ and F is given by (3.12). We must be careful to recognize that in general

$$\frac{\partial}{\partial X^\mu}F(X,k)\neq\frac{\partial}{\partial y}F^{(X)}(y,k)\Big|_{y=0}\ . \quad (3.15)$$

To find an expression for the derivative of $F(X,k)$ we consider $F(X^\mu+dX^\mu,k_\mu)$, where dX is some infinitesimal displacement. From (3.13),

$$F(X^\mu+dX^\mu,k_\mu)=F^{(X+dX)}(0,e^\mu_{a'}(X+dX)k_\mu) \quad (3.16)$$

(primed latin indices are for RNC around $X+dX$). Using (3.11) to relate $F^{(X+dX)}(0)$ to $F^{(X)}(dX)$,

$$F^{(X+dX)}(0,e^\mu_{a'}(X+dX)k_\mu)$$
$$\simeq F^{(X)}(dX,e^{a'}_a(X)e^\nu_{a'}(X+dX)k_\nu)+\cdots, \quad (3.17)$$

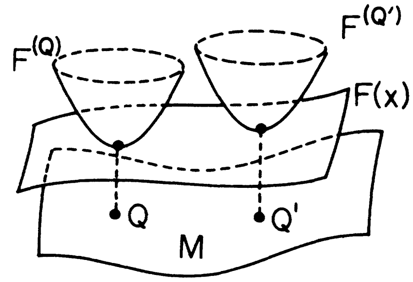

FIG. 2. With each spacetime point Q we associate a function $F^{(Q)}$ in the following way: Set up a RNC with origin at Q and make a partial Fourier transform of $G_1(x_1,x_2)$ with respect to the numerical difference of the RNC of x_1 and x_2 around Q. $F^{(Q)}$ is related to this Fourier transform by Eq. (3.2). If we repeat this construction with origin at a different point Q', we obtain in general a different Fourier transform $F^{(Q')}$. The function $F(X)$ defined in Eq. (3.13) samples all the $F^{(Q)}$ functions according to the prescription that, at the point Q, $F(Q,k)$ is given by $F^{(Q)}(0,k)$ when k is expressed with respect to RNC around Q, and by Eq. (3.14) in a general coordinate system.

we obtain

$$\frac{\partial}{\partial X^\mu}F(X^\mu,k_\mu)=e^a_\mu\frac{\partial}{\partial y^a}F^{(X)}(0,e^\mu_a k_\mu)+e^\nu_{a,\mu}k_\nu\frac{\partial}{\partial k_a}F^{(X)}(0,e^\mu_a k_\mu)+\cdots \quad (3.18)$$

Equation (3.18) can be rewritten as

$$e^a_\mu(X)\frac{\partial}{\partial y^a}F^{(X)}(0,e^\mu_a k_\mu)=\left[\frac{\partial}{\partial X^\mu}+\Gamma^\rho_{\mu\nu}(X)k_\rho\frac{\partial}{\partial k_\nu}\right]F(X,k)-e^\rho_{a;\mu}e^a_\nu k_\rho\frac{\partial}{\partial k_\nu}F(X,k)+\text{terms of higher order}\ , \quad (3.19)$$

where we assumed that the vierbein is constructed such that its covariant derivatives go to zero when the curvature vanishes (thus $e^\rho_{a;\mu}$ is a quantity of second adiabatic order). Finally, (3.12) and (3.19) allow us to formulate an improved ansatz for G_1,

$$G_1(x_1,x_2)=\Delta^{1/2}_{\rm VM}(x_1,x_2)\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{\sqrt{-g(X)}}e^{\mathcal{P}}F(X,k)\ , \quad (3.20)$$

where $\mathcal{P}$ is the differential operator

$$\mathcal{P}=\tfrac{1}{24}(\Gamma^\mu_{\nu\rho,\sigma}+\Gamma^\mu_{\nu\tau}\Gamma^\tau_{\rho\sigma})(X)\frac{\partial^3}{\partial k_\nu\partial k_\rho\partial k_\sigma}k_\mu-\tfrac{1}{8}\Gamma^\mu_{\nu\rho}(X)\frac{\partial^2}{\partial k_\nu\partial k_\rho}\left[\frac{\partial}{\partial X^\mu}+(\Gamma^\lambda_{\mu\sigma}-e^\lambda_{a;\mu}e^a_\sigma)k_\lambda\frac{\partial}{\partial k_\sigma}\right]+\text{terms of higher adiabatic order}\ . \quad (3.21)$$

It is easy to show from (3.14) that $F^{(Q)}=e^{\mathcal{P}}F$ satisfies Eq. (3.11).

B. Covariant dynamical equation for the distribution function

To obtain a dynamical equation for the distribution function F we can substitute the ansatz (3.20) into the Klein-Gordon equation (3.1). Furthermore, it is possible to write down a covariant equation for F. This is because the only property of G_1 which we used to derive the representation (3.20) was that G_1 is a biscalar. The Klein-Gordon equation is also a biscalar identity, and so if we can find a representation of the form

$$[\Box_{x_i}+m^2+\xi R(x_i)]G_1(x_1,x_2)=\Delta^{1/2}_{\rm VM}(x_1,x_2)\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x_1,x_2)}}{\sqrt{-g(X)}}e^{\mathcal{P}}(\mathcal{D}F)\ . \quad (3.22)$$

With $\mathcal{P}$ given by (3.21), then $\mathcal{D}F$ will be independent of the origin of RNC, and the equation $\mathcal{D}F=0$ will be covariant.

In fact, it is straightforward to obtain a representation of the form (3.22). First we observe that

$$[\Box_{x_1}+m^2+\xi R(x_1)]\Delta^{1/2}_{\rm VM}f=\Delta^{1/2}_{\rm VM}[\Box_{x_1}+2g^{\mu\nu}\partial_\mu\ln\Delta^{1/2}_{\rm VM}\partial_\nu+m^2+\xi R(x_1)+\Delta^{-1/2}_{\rm VM}(\Box_{x_1}\Delta^{1/2}_{\rm VM})]f\ , \quad (3.23)$$

where

$$\Delta^{-1/2}_{\rm VM}(\Box_{x_1}\Delta^{1/2}_{\rm VM})=g^{\mu\nu}\left[\frac{\partial^2}{\partial x^\mu\partial x^\nu}\ln\Delta^{1/2}_{\rm VM}+\left[\frac{\partial}{\partial x^\mu}\ln\Delta^{1/2}_{\rm VM}\right]\left[\frac{\partial}{\partial x^\nu}\ln\Delta^{1/2}_{\rm VM}\right]-\Gamma^\rho_{\mu\nu}\frac{\partial}{\partial x^\rho}\ln\Delta^{1/2}_{\rm VM}\right]\ . \quad (3.24)$$

Next we set up a RNC system with origin at some point Q and switch to the average and difference variables. We write

$$x_{1,2}=X\pm\tfrac{1}{2}x,\quad \frac{\partial}{\partial x_{1,2}}=\frac{1}{2}\frac{\partial}{\partial X}\pm\frac{\partial}{\partial x} \quad (3.25)$$

and substitute x by $i\partial/\partial k$, and $\partial/\partial x$ by ik in the integrand. Also we make use of

$$\frac{\partial}{\partial X^\mu}\frac{f}{\sqrt{-g(X)}}=\frac{1}{\sqrt{-g(X)}}[\partial_\mu f-\Gamma^\nu_{\mu\nu}(X)f]\ , \quad (3.26)$$

$$\frac{\partial^2}{\partial X^\mu\partial X^\nu}\frac{f}{\sqrt{-g(X)}}=\frac{1}{\sqrt{-g(X)}}\left[\frac{\partial^2}{\partial X^\mu\partial X^\nu}f-\Gamma^\lambda_{\mu\lambda}(X)\frac{\partial}{\partial X^\nu}f-\Gamma^\lambda_{\nu\lambda}(X)\frac{\partial}{\partial x^\mu}f-(\Gamma^\lambda_{\mu\lambda,\nu}-\Gamma^\lambda_{\mu\lambda}\Gamma^\rho_{\nu\rho})f\right]\ . \quad (3.27)$$

Finally, multiplying by $1=e^{\mathcal{P}}e^{-\mathcal{P}}$ we get

$$\mathcal{D}F=e^{-\mathcal{P}}\Bigg\{g^{\mu\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\left[-k_\mu k_\nu+ik_\mu\frac{\partial}{\partial X^\nu}+\frac{1}{4}\frac{\partial^2}{\partial X^\mu\partial X^\nu}\right]$$
$$+\left[-(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]+2g^{\mu\rho}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\left[\frac{\partial}{\partial x_1^\mu}\ln\Delta_{\rm VM}^{1/2}\right]\left[X\pm\frac{i}{2}\frac{\partial}{\partial k}\right]\right.$$
$$\left.+g^{\mu\rho}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\Gamma^\lambda_{\mu\lambda}(X)\right]\left[ik_\rho+\frac{1}{2}\frac{\partial}{\partial X^\rho}\right]$$
$$+m^2+\xi R\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]+g^{\mu\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\left[\left[\frac{\partial^2}{\partial x_1^\mu\partial x_1^\nu}\ln\Delta_{\rm VM}^{1/2}\right]\left[X\pm\frac{i}{2}\frac{\partial}{\partial k}\right]\right.$$
$$\left.+\left[\frac{\partial}{\partial x_1^\mu}\ln\Delta_{\rm VM}^{1/2}\frac{\partial}{\partial x_1^\nu}\ln\Delta_{\rm VM}^{1/2}\right]\left[X\pm\frac{i}{2}\frac{\partial}{\partial k}\right]\right]$$
$$-(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\left[\frac{\partial}{\partial x_1^\rho}\ln\Delta_{\rm VM}^{1/2}\right]\left[X\pm\frac{i}{2}\frac{\partial}{\partial k}\right]+\tfrac{1}{4}g^{\mu\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right](-\Gamma^\lambda_{\mu\lambda,\nu}+\Gamma^\lambda_{\mu\lambda}\Gamma^\rho_{\nu\rho})(X)$$
$$+\left[(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]-2g^{\mu\rho}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\left[\frac{\partial}{\partial x_1^\mu}\ln\Delta_{\rm VM}^{1/2}\right]\left[X\pm\frac{i}{2}\frac{\partial}{\partial k}\right]\right]\Gamma^\lambda_{\rho\lambda}(X)\Bigg\}e^{\mathcal{P}}F\ . \qquad (3.28)$$

The covariant equation for F is, of course,

$$\mathcal{D}F=0\ . \qquad (3.29)$$

C. The structure of $F(X,k)$ and Liouville-Vlasov equations

Equation (3.28) for the distribution function F is rather complex, and therefore of limited practical interest. Nevertheless, we can achieve considerable simplification if we realize that admissible distribution functions have a particular structure. For example, in Sec. II we have seen that the distribution function F for a scalar field in an external potential obeys (2.11). If we propose that F has the form (2.10), we can express F as a functional of a "reduced" distribution function F_0 which obeys the much simpler Vlasov equation (2.14). Essentially the same strategy may be applied to the curved-space situation.

The task of investigating the solutions of (3.29) can be greatly simplified by making the observation that the transformation law for the quantity $(\mathcal{D}F)$ has the same form as (3.14). Thus, if for some X, $\mathcal{D}F(X,k)=0$ identically in k when evaluated in a RNC around X itself, then $\mathcal{D}F(X,k)=0$ in any coordinate system. If we can construct an F such that $\mathcal{D}F(X,k)\equiv 0$ when evaluated in RNC around X, for any X, then our F will automatically be a solution of (3.29). Now, at the origin of the RNC, $\mathcal{D}F$ is given by

$$\mathcal{D}F(0,k)=\Bigg\{g^{\mu\nu}\left[\frac{i}{2}\frac{\partial}{\partial k}\right]\left[-k_\mu k_\nu+ik_\mu\frac{\partial}{\partial X^\nu}+\frac{1}{4}\frac{\partial^2}{\partial X^\mu\partial X^\nu}\right]$$
$$+\left[-(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\left[\frac{i}{2}\frac{\partial}{\partial k}\right]+2g^{\mu\rho}\left[\frac{i}{2}\frac{\partial}{\partial k}\right]\left[\frac{\partial}{\partial x_1^\mu}\ln\Delta_{\rm VM}^{1/2}\right]\left[\pm\frac{i}{2}\frac{\partial}{\partial k}\right]\right]\left[ik_\rho+\frac{1}{2}\frac{\partial}{\partial X^\rho}\right]$$
$$+m^2+\xi R\left[\frac{i}{2}\frac{\partial}{\partial k}\right]-\tfrac{1}{12}R_{\mu\nu}(0)g^{\mu\nu}\left[\frac{i}{2}\frac{\partial}{\partial k}\right]$$
$$+g^{\mu\nu}\left[\frac{i}{2}\frac{\partial}{\partial k}\right]\left[\frac{\partial^2}{\partial x_1^\mu\partial x_1^\nu}\ln\Delta_{\rm VM}^{1/2}+\frac{\partial}{\partial x_1^\mu}\ln\Delta_{\rm VM}^{1/2}\frac{\partial}{\partial x_1^\nu}\ln\Delta_{\rm VM}^{1/2}\right]\left[\pm\frac{i}{2}\frac{\partial}{\partial k}\right]$$
$$-(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\left[\frac{i}{2}\frac{\partial}{\partial k}\right]\left[\frac{\partial}{\partial x_1^\rho}\ln\Delta_{\rm VM}^{1/2}\right]\left[\pm\frac{i}{2}\frac{\partial}{\partial k}\right]\Bigg\}e^{\mathcal{P}}F\ . \qquad (3.30)$$

We will introduce the notation $\mathcal{D}F=\hat{\mathcal{D}}e^{\mathcal{P}}F$.

Let us consider approximations only up to the fourth adiabatic order. As derivatives of F or the vierbein are themselves at least of the second order, the operator $\mathcal{P}$ in (3.21) is a fourth-order quantity, and one can approximate $e^{\mathcal{P}} \sim 1+\mathcal{P}$. Moreover, as $\mathcal{P}(X=0)=0$, we have

$$\hat{\mathcal{D}}\mathcal{P}F \approx \eta^{\mu\nu} \left[ik_\mu \frac{\partial}{\partial X^\nu} + \frac{1}{4} \frac{\partial^2}{\partial X^\mu \partial X^\nu} \right] \mathcal{P}F \ . \tag{3.31}$$

Rather than looking for the most general F, let us begin by considering the vacuum case. Our trial solution is

$$F_{\text{vac}}(X,k) = \delta(\Omega) + \delta F_{\text{vac}} \ , \tag{3.32}$$

where

$$\Omega = g^{\mu\nu}(X) k_\mu k_\nu - m^2 - (\xi - \tfrac{1}{6}) R(X) \ . \tag{3.33}$$

Clearly F_{vac} has the correct flat-space limit, and $\delta(\Omega)$ by itself obeys the transformation law (3.14). The introduction of a curvature-dependent effective mass leads to an improved asymptotic approximation for G_1 (Ref. 13). Observe that

$$\begin{aligned} \left. \frac{\partial}{\partial X^\mu} \delta(\Omega) \right|_{X=0} &= -(\xi - \tfrac{1}{6}) R_{;\mu}(0) \delta'(\Omega) \ , \\ \left. \frac{\partial^2}{\partial X^\mu \partial X^\nu} \delta(\Omega) \right|_{X=0} &= [-\tfrac{2}{3} R^\rho{}_\mu{}^\sigma{}_\nu(0) k_\rho k_\sigma - (\xi - \tfrac{1}{6}) R_{;\mu\nu}(0)] \delta'(\Omega) + \text{terms of higher than fourth order} \ . \end{aligned} \tag{3.34}$$

Using the property that

$$k_\mu \frac{\partial}{\partial k_\nu} \delta(\Omega) = 2 k_\mu k^\nu \delta'(\Omega) \tag{3.35}$$

is symmetric in (μ,ν), we find

$$e^\lambda_{a;\mu} e^a_\sigma k_\lambda \frac{\partial}{\partial k_\sigma} \delta(\Omega) = 2 e^\lambda_{a;\mu} e^a_\sigma k_\lambda g^{\sigma\rho} k_\rho \delta'(\Omega) = (e^\lambda_a e^{a\rho})_{;\mu} k_\lambda k_\rho \delta'(\Omega) \equiv 0 \ . \tag{3.36}$$

All other terms in Eqs. (3.30) and (3.31) can be computed by using the formulas in the Appendix. After a long but straightforward calculation we arrive at

$$\begin{aligned} \mathcal{D}\delta(X=0) = {} & -\frac{\xi}{2} (\Box R \delta' + R_{;\rho\sigma} k^\rho k^\sigma \delta'') + \tfrac{7}{60} \Box R \delta' + \tfrac{1}{10} (\Box R_{\rho\sigma} + \tfrac{1}{2} R_{;\rho\sigma}) k^\rho k^\sigma \delta'' \\ & - \tfrac{1}{30} (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \delta' + 2 R^{\mu\nu\lambda}{}_\rho R_{\mu\nu\lambda\sigma} k^\rho k^\sigma \delta'') + \tfrac{1}{30} [R^{\mu\nu} R_{\mu\nu} \delta' + 4 (R^\mu{}_\rho R_{\mu\sigma} - \tfrac{1}{2} R_{\mu\nu} R^\mu{}_\rho{}^\nu{}_\sigma) k^\rho k^\sigma \delta''] \ . \end{aligned} \tag{3.37}$$

We observe that $\mathcal{D}\delta$ is real and of fourth order. So we may assume that δF_{vac} in (3.32) is also of fourth order itself. For a fourth-order quantity, $\mathcal{D}\delta F$ reduces to

$$\mathcal{D}\delta F_{\text{vac}} = [-k^2 + m^2 + (\xi - \tfrac{1}{6}) R] \delta F_{\text{vac}} = -\Omega \delta F_{\text{vac}} \ . \tag{3.38}$$

If δF_{vac} is expanded in derivatives of the δ function in analogy with (2.10), i.e.,

$$\delta F_{\text{vac}} = \sum_{n=1}^{\infty} (\delta F_{\text{vac}})_n \delta^{(n)}(\Omega) \tag{3.39}$$

then

$$-\Omega \delta F_{\text{vac}} = \sum_{n=1}^{\infty} n (\delta F_{\text{vac}})_n \delta^{n-1}(\Omega) \ . \tag{3.40}$$

To obtain $\mathcal{D}F=0$ we need

$$\begin{aligned} \delta F_{\text{vac}} = {} & [\tfrac{1}{4}(\xi - \tfrac{7}{30}) \Box R + \tfrac{1}{60} (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - R^{\mu\nu} R_{\mu\nu})] \delta''(\Omega) \\ & + [\tfrac{1}{6}(\xi - \tfrac{1}{10}) R_{;\rho\sigma} - \tfrac{1}{30} \Box R_{\rho\sigma} + \tfrac{1}{45} (R^{\mu\nu\lambda}{}_\rho R_{\mu\nu\lambda\sigma} + R_{\mu\nu} R^\mu{}_\rho{}^\nu{}_\sigma - 2 R_{\mu\rho} R^\mu{}_\sigma)] k^\rho k^\sigma \delta'''(\Omega) \ . \end{aligned} \tag{3.41}$$

In principle, (3.41) would hold only in RNC around X. But as (3.41) satisfies (3.14), we may assert that it and (3.32) give F_{vac} in any coordinate system.

It is easy to show that we obtain a representation for the Feynman propagator if we replace the δ function by $(\Omega + i\epsilon)^{-1}$. If we use $e^{is(\Omega + i\epsilon)}$ instead of $\delta(\Omega)$ we will obtain a representation for the heat kernel.

In the general (nonvacuum) case, we will look again for a solution of the form

$$F(X,k)=\sum_{n=0}^{\infty} F_n \delta^{(n)}(\Omega) \ . \tag{3.42}$$

In computing $\mathcal{D}F$ we will encounter X derivatives of the F_n's. Under the assumption that any position dependence of the F_n's is induced solely by that of the metric, we will count these derivatives of the F_n's as contributing to the adiabatic order. Keeping only terms of third adiabatic order we find

$$\begin{aligned}(\mathcal{D}F_0\delta)(X=0)=&\left[\frac{1}{4}\eta^{\mu\nu}\frac{\partial^2 F_0}{\partial X^\mu \partial X^\nu}-\frac{1}{12}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\frac{\partial^2 F_0}{\partial k_\rho \partial k_\sigma}k_\mu k_\nu+\frac{1}{6}R^{\mu}{}_{\sigma}\left[\frac{\partial F_0}{\partial k_\sigma}\right]k_\mu\right]\delta\\
&+i\left[k^\nu\frac{\partial F_0}{\partial X^\nu}-\frac{1}{24}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma;\lambda}k_\mu k_\nu\frac{\partial^3 F_0}{\partial k_\rho \partial k_\sigma \partial k_\lambda}+\frac{1}{24}R_{\rho\sigma}{}^{;\nu}k_\nu\frac{\partial^2 F_0}{\partial k_\rho \partial k_\sigma}\right.\\
&\left.+\frac{1}{6}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}k_\mu\frac{\partial^2}{\partial k_\rho \partial k_\sigma}\frac{\partial F_0}{\partial X^\nu}-\frac{1}{12}R^{\mu}{}_{\nu}\frac{\partial}{\partial k_\nu}\frac{\partial F_0}{\partial X^\mu}+\frac{1}{2}\left[\xi-\frac{1}{4}\right]R_{;\mu}\frac{\partial F_0}{\partial k_\mu}\right]\delta\\
&+i\left[-\frac{1}{12}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma;\lambda}k_\mu k_\nu k^\lambda\frac{\partial^2 F_0}{\partial k_\rho k \partial k_\sigma}+\frac{1}{3}R^{\mu}{}_{\sigma;\lambda}k_\mu k^\lambda\frac{\partial F_0}{\partial k_\sigma}\right.\\
&\left.-\frac{1}{4}R^{\mu}{}_{\sigma;\lambda}\frac{\partial F_0}{\partial k_\lambda}k_\mu k^\sigma+\frac{1}{3}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}k_\mu k^\sigma\frac{\partial F_0}{\partial k_\rho \partial X^\nu}+\frac{1}{6}R^{\mu}{}_{\lambda}k^\lambda\frac{\partial F_0}{\partial X^\mu}\right]\delta' \ .\end{aligned} \tag{3.43}$$

We conclude that to this order only F_1 is nonzero, and its value at the origin is given by

$$F_1(0,k)=(-1)\left[\frac{1}{4}\eta^{\mu\nu}\frac{\partial^2 F_0}{\partial X^\mu \partial X^\nu}-\frac{1}{12}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}k_\mu k_\nu\frac{\partial^2 F_0}{\partial k_\rho \partial k_\sigma}+\frac{1}{6}R^{\mu}{}_{\sigma}k_\mu\frac{\partial F}{\partial k_\sigma}\right] . \tag{3.44}$$

Our task is to find a function F_1 obeying (3.14) and (3.44). From (3.14) we conclude that

$$\frac{\partial}{\partial k_{\nu'}}F_0'(X^{\mu'},k_{\mu'})=\frac{\partial X^{\nu'}}{\partial X^\nu}\frac{\partial}{\partial k_\nu}F_0\left[X^\mu,\frac{\partial X^{\mu'}}{\partial X^\mu}k_{\mu'}\right] , \tag{3.45}$$

so expressions such as $R^{\mu}{}_{\sigma}k_\mu \partial F_0/\partial k_\sigma$ already are in covariant form. On the other hand, also from (3.14),

$$\frac{\partial}{\partial X^{\mu'}}F_0'(X^{\mu'},k_{\mu'})=\left\{\frac{\partial X^\mu}{\partial X^{\mu'}}\frac{\partial}{\partial X^\mu}F_0+\left[\frac{\partial}{\partial X^{\mu'}}\left[\frac{\partial X^{\lambda'}}{\partial X^\lambda}\right]\right]k_{\lambda'}\frac{\partial F_0}{\partial k_\lambda}\right\}\left[X^\mu,\frac{\partial X^{\mu'}}{\partial X^\mu}k_{\mu'}\right] . \tag{3.46}$$

From this we conclude that

$$F_{0;\mu}=\frac{\partial}{\partial X^\mu}F_0+\Gamma^{\nu}_{\mu\rho}k_\nu\frac{\partial}{\partial k_\rho}F_0 \tag{3.47}$$

transforms as a vector (modulo the shift in the momentum variable)

$$F'_{0;\mu'}(X^{\mu'},k_{\mu'})=\frac{\partial X^\mu}{\partial X^{\mu'}}F_{0;\mu}\left[X^\lambda,\frac{\partial X^{\lambda'}}{\partial X^\lambda}k_{\lambda'}\right] . \tag{3.48}$$

Analogously, we find that

$$F_{0;\mu\nu}=\frac{\partial}{\partial X^\nu}F_{0;\mu}-\Gamma^{\lambda}_{\mu\nu}F_{0;\lambda}+\Gamma^{\rho}_{\nu\sigma}k_\rho\frac{\partial}{\partial k_\sigma}F_{0;\mu} \tag{3.49}$$

transforms "almost" like a tensor

$$F'_{0;\mu'\nu'}=\frac{\partial X^\mu}{\partial X^{\mu'}}\frac{\partial X^\nu}{\partial X^{\nu'}}F_{0;\mu\nu}\left[X,\frac{\partial X^{\lambda'}}{\partial X^\lambda}k_{\lambda'}\right] . \tag{3.50}$$

At the origin of RNC,

$$F_{0;\mu\nu}(0,k)=\frac{\partial^2}{\partial X^\mu \partial X^\nu}F_0+\tfrac{1}{3}(R^{\lambda}{}_{\mu\rho\nu}+R^{\lambda}{}_{\rho\mu\nu})k_\lambda\frac{\partial}{\partial k_\rho}F_0 \ . \tag{3.51}$$

From these we obtain the desired F_1:

$$F_1(X,k)=(-1)\left[\tfrac{1}{4}g^{\mu\nu}(X)F_{0;\mu\nu}(X,k)+\tfrac{1}{12}R^{\mu}{}_{\rho}k_{\mu}\frac{\partial F_0}{\partial k_\rho}-\tfrac{1}{12}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}k_\mu k_\nu\frac{\partial^2 F_0}{\partial k_\rho \partial k_\sigma}\right] . \tag{3.52}$$

As in the flat spacetime case, it would seem that the condition of the imaginary part of $\mathcal{D}F$ being zero leads to two equations for F_0. One equation is the covariant generalization of (3.43) with the coefficient of δ set to zero:

$$g^{\mu\nu}k_\mu F_{0;\nu}+\tfrac{1}{6}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}k_\mu\frac{\partial^2}{\partial k_\rho\partial k_\sigma}F_{0;\nu}-\tfrac{1}{12}R^{\mu}{}_{\nu}\frac{\partial}{\partial k_\nu}F_{0;\mu}-\tfrac{1}{24}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma;\lambda}k_\mu k_\nu\frac{\partial^3 F_0}{\partial k_\rho\partial k_\sigma\partial k_\lambda}$$
$$+\tfrac{1}{24}R_{\rho\sigma}{}^{;\nu}k_\nu\frac{\partial^2 F_0}{\partial k_\rho\partial k_\sigma}+\tfrac{1}{2}(\xi-\tfrac{1}{4})R_{;\mu}\frac{\partial F_0}{\partial k_\mu}=0 . \tag{3.53}$$

This equation was first obtained by Winter[9] through a different method. Recall that there (see Fig. 1) the midpoint of two spacetime points P_1 and P_2 is defined to be the point equidistant along the unique geodesic connecting them, and the difference of P_1 and P_2 as proportional to the tangent vector to such a geodesic at P_0. Here we introduce Riemann normal coordinates at any point Q in the vicinity of P_1 and P_2 with coordinates x_1 and x_2. Our Wigner function $F^{(Q)}(X,k)=F^{(X)}(0,k)$ is the Fourier transform of the Hadamard function with respect to $x=x_1-x_2$. The point P with coordinate $X=\frac{1}{2}(x_1+x_2)$ is in general not the same as P_0. Our approach, therefore, is not *a priori* based on the choice of the geodesic midpoint. This saves much work in solving the covariant geodesic and geodesic deviation equations as is needed in Winter's method. The generality of our method is retained by devising transformations of functions defined in RNC's at Q and a different nearby point Q'. Winter's result is regained when Q' is chosen at the geodesic midpoint P_0 of P_1 and P_2, in which case they have equal and opposite RNC's.

The second equation is the condition that $k\cdot\partial F_1/\partial X$ cancels the coefficient of δ' in Eq. (3.43). However, it can be shown that this second condition is a consequence of Eqs. (3.52) and (3.53). Thus, we conclude that to the third adiabatic order inclusive, the general form of F is

$$F(X,k)=F_0(X,k)\delta(\Omega)-\frac{1}{4}\left[g^{\mu\nu}F_{0;\mu\nu}+\frac{1}{3}\left[R^{\mu}{}_{\rho}k_\mu\frac{\partial F_0}{\partial k_\rho}-R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}k_\mu k_\nu\frac{\partial^2}{\partial k_\rho\partial k_\sigma}F_0\right]\right]\delta'(\Omega) , \tag{3.54}$$

where F_0 satisfies the Liouville-Vlasov equation (3.53). Eventually we may write $F_0=1+f(X,k)$; the constant will drop out of (3.53), and we may identify f as the curved-space distribution function. Unfortunately, the accuracy of Eq. (3.54) is not satisfactory. For example, (3.54) will not predict the trace anomaly of the energy-momentum tensor, which is of fourth adiabatic order. But we know that because $\mathcal{D}F$ is a linear equation, if F_0 were constant it would drop out of the equation and reduce to the vacuum case, which we have already solved. So we can write

$$F(X,k)=F_0(X,k)\{\delta(\Omega)+[\tfrac{1}{4}(\xi-\tfrac{7}{30})\Box R+\tfrac{1}{60}(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-R^{\mu\nu}R_{\mu\nu})]\delta''(\Omega)$$
$$+[\tfrac{1}{6}(\xi-\tfrac{1}{10})R_{;\rho\sigma}-\tfrac{1}{30}\Box R_{\rho\sigma}+\tfrac{1}{45}(R^{\mu\nu\lambda}{}_{\rho}R_{\mu\nu\lambda\sigma}+R_{\mu\nu}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}-2R_{\mu\rho}R^{\mu}{}_{\sigma})]k^\rho k^\sigma\delta'''(\Omega)\}$$
$$-\frac{1}{4}\left[g^{\mu\nu}F_{0;\mu\nu}+\frac{1}{3}\left[R^{\mu}{}_{\rho}k_\mu\frac{\partial F_0}{\partial k_\rho}-R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}k_\mu k_\nu\frac{\partial^2 F_0}{\partial k_\rho\partial k_\sigma}\right]\right]\delta'(\Omega)$$
$$+(\text{fourth-order terms involving derivatives of } F_0)+(\text{terms of fifth order or higher}) . \tag{3.55}$$

The missing fourth-order terms in Eq. (3.55) do not contribute to the ultraviolet divergences (at least for any physical F_0) and are numerically much smaller than the terms already included, at least in the ultraviolet region in which (3.55) is reliable. So we may safely ignore them and take (3.55) together with (3.53) as our fundamental equations. Clearly, Eq. (3.53) is a vast improvement over the original Eq. (3.29).

D. Covariant distribution function in the Einstein universe

In this section we will consider the application of the above formalism to the particular case in which spacetime is given by the static Einstein universe $R\times S^3$. This is an interesting example in that while the conventional RNC expansion of the vacuum solution works exceedingly well (it actually terminates, providing an exact solution of the Klein-Gordon equation), it fails to give the true vacuum Hadamard function because it cannot incorporate the global boundary conditions[13,16] (the true vacuum propagator may be obtained from the local RNC expansion by a reasoning involving a sum over different images).

The line element of Einstein's universe is given by

$$ds^2=dt^2-a^2[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\, d\varphi^2)] , \tag{3.56}$$

where $-\infty<t<\infty$, $0\le\chi$, $\theta\le\pi$, $0\le\varphi\le 2\pi$, and $a=$const is the radius of S^3. The Christoffel symbols and the Riemann tensor are zero whenever one of its indices is zero. When all indices are spatial we have

$$R^{\alpha}_{\ \beta\gamma\delta}=a^{-2}(\delta^{\alpha}_{\gamma}g_{\beta\delta}-\delta^{\alpha}_{\delta}g_{\beta\gamma}) \tag{3.57}$$

from which $R_{\beta\delta}=2a^{-2}g_{\beta\gamma}$ and $R=6a^{-2}=$const ($\alpha,\beta,\ldots$ run from 1 to 3). Evaluating the geometric elements, (3.55) reduces to

$$F(X,k)=F_0\delta(\Omega)-\frac{1}{4}\left[F_{0;00}-g^{\alpha\beta}F_{0;\alpha\beta}+\frac{1}{3a^2}\left[2k^{\alpha}\frac{\partial F_0}{\partial k^{\alpha}}-(g^{\alpha\beta}g_{\gamma\delta}-\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma})k_{\alpha}k_{\beta}\frac{\partial^2F_0}{\partial k_{\gamma}\partial k_{\delta}}\right]\right]\delta'(\Omega)+\cdots . \tag{3.58}$$

If $F_0=$const, the expansion (3.58) has only the first term. The transport equation (3.53) now reads

$$k^0F_{0,0}-g^{\alpha\beta}k_{\alpha}F_{0;\beta}+\frac{1}{6a^2}k^{\alpha}\frac{\partial^2}{\partial k_{\beta}\partial k^{\beta}}F_{0;\alpha}-\frac{1}{6a^2}k^{\alpha}\frac{\partial^2}{\partial k^{\alpha}\partial k_{\beta}}F_{0;\beta}-\frac{1}{6a^2}\frac{\partial}{\partial k^{\alpha}}F_{0;\alpha}=0 . \tag{3.59}$$

It admits a variety of solutions. We are interested in those which are commensurate with the symmetries of the Einstein universe, in particular, the static solutions satisfying

$$F_0(X,k)=F_0(g^{\alpha\beta}(X)k_{\alpha}k_{\beta}) . \tag{3.60}$$

Obviously $F_{0,0}=0$; also $F_{0;\alpha}=0$, and thus (3.59) is automatically satisfied. In fact, (3.58) simplifies to

$$F(X,k)=F_0(g^{\alpha\beta}(X)k_{\alpha}k_{\beta})\delta(\Omega) . \tag{3.61}$$

The question is whether the class of solutions (3.60) indeed contains the true complete Hadamard function. Consider two points $x_{1,2}$. Without loss of generality we may assume that $t_1=t_2$, and we can always choose a coordinate system in which x_1 and x_2 are symmetric around the north pole of S^3. Choosing this north pole as the origin (P) of RNC we find

$$G_1(x_1,x_2)=\Delta_{\rm VM}^{1/2}(x_1,x_2)\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{2\omega_k}F_0(k^2) , \tag{3.62}$$

where we have used the fact that in this RNC, $X=0$, $\omega_k^2=k^2+m^2+(6\xi-1)a^{-2}$, and we have absorbed a factor of (2π) into F_0. G_1 depends only on the angle χ between x_1 and x_2:

$$G_1(\chi)=\frac{i}{(2\pi)^2\sin\chi}\int_{-\infty}^{\infty}dk\frac{k}{2\omega_k}F_0(k^2)e^{-ik\chi} . \tag{3.63}$$

Equation (3.63) is valid in principle for $0\le\chi\le\pi$ only, but it can be used to extend G_1 to a function on the covering space $-\infty<\chi<\infty$. A physically acceptable propagator must then be even in χ and be invariant under a shift of χ by 2π. Equation (3.63) satisfies the first condition, and the second one leads to

$$\frac{k}{2\omega_k}F_0(k^2)(e^{-2\pi ik}-1)=0 . \tag{3.64}$$

In other words,

$$F_0(k^2)=\sum_{n=0}^{\infty}a_n\delta(k^2-n^2) . \tag{3.65}$$

Substituting back in (3.63) this gives

$$G_1(\chi)=\frac{1}{(2\pi)^2}\sum_{n=1}^{\infty}\frac{a_n}{2[n^2+m^2+(6\xi-1)a^{-2}]}\left[\frac{\sin n\chi}{\sin\chi}\right] . \tag{3.66}$$

The class of solutions Eqs. (3.65) and (3.66) includes the usual vacuum function, which corresponds to $a_n=2n$. In this way by using our local but covariant formalism (in contrast to all previous local but noncovariant methods), we are able to deduce the exact form of G_1 with the global properties of the underlying space fully accounted for. Generalization of this method to group manifolds and homogeneous spaces will be discussed in later communications.

IV. THE FORMALISM FOR SPINOR FIELDS

The methods developed in the previous sections will now be applied to a spin-$\frac{1}{2}$ field. Since the formalism does not differ much from that for scalar fields, we shall be content in restricting our analysis to second adiabatic order. The procedure may be extended to higher orders in a straightforward, though somewhat tedious manner.

We start with the two-point function $S_1(x_1,x_2)$ that satisfies

$$(\gamma^{\mu}\nabla_{\mu}-m)_{x_{1,2}}S_1(x_1,x_2)=0 . \tag{4.1}$$

The covariant derivative is that appropriate for spin-$\frac{1}{2}$ fields:

$$\nabla_{\mu}=\partial_{\mu}-\Gamma_{\mu} , \tag{4.2}$$

where

$$\Gamma_{\mu}=\tfrac{1}{2}\Sigma^{ab}e_{a\nu}e_b{}^{\nu}{}_{;\mu} \tag{4.3}$$

and

$$\Sigma^{ab}=\tfrac{1}{4}[\gamma^a,\gamma^b] . \tag{4.4}$$

The γ^{μ} are the curved spacetime generalizations of the Dirac matrices γ^a. They are linked by the defining equation

$$\gamma^\mu = e^\mu{}_a \gamma^a \ . \tag{4.5}$$

Rather than work with $S_1(x_1,x_2)$ directly we define

$$S_1(x_1,x_2) \equiv (\gamma^\nu \nabla_\nu + m)_{x_1} G_1(x_1,x_2) \ . \tag{4.6}$$

Substitution of (4.6) into the original equation (4.1) produces,

$$\left[\Box_{x_1} + m^2 - \frac{R}{4}(x_1)\right] G_1(x_1,x_2) = 0 \ , \tag{4.7}$$

where we exploit the fact that the covariant derivative of the Dirac matrices vanishes and that $-\gamma^\mu\gamma^\nu\gamma^\eta\gamma^\lambda R_{\mu\nu\eta\lambda} = 2R\,1$. The $\Box$ operator includes spin connection terms. Written explicitly in terms of coordinates, (4.7) takes the form

$$\left[g^{\mu\nu}[\partial_\mu\partial_\nu - (\Gamma^\lambda_{\mu\nu} + 2\Gamma_\nu\delta^\lambda_\mu)\partial_\lambda - \Gamma_{\nu,\mu} + \Gamma_\lambda(\Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu\Gamma_\nu)] + m^2 - \frac{R}{4} \right]_{x_1} G_1(x_1,x_2) = 0 \ . \tag{4.8}$$

It is important to remember that all the equations being considered are really matrix equations—the spinor indices have been suppressed. $G_1(x_1,x_2)$ is a 4×4 matrix that transforms under coordinate transformations as a biscalar whereas under Lorentz transformations of the vierbeins at x_1 and x_2 it transforms as $\psi(x_1)\bar{\psi}(x_2)$ (Ref. 4). We note that

$$\bar{\psi} = \psi^\dagger \gamma \tag{4.9}$$

and

$$\gamma\gamma^a\gamma^{-1} = -(\gamma^a)^T \ . \tag{4.10}$$

Finally, we take the γ and γ^μ matrices to be real. This can always be accomplished.

In order to ensure the retention of the transformation properties of $G_1(x_1,x_2)$ we assume a representation of the form

$$G_1(x_1,x_2) = \Delta_{\rm VM}^{1/2}(x_1,x_2) \times \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{\sqrt{-g(X)}} A^{(Q)}(x_1,X) \times F^{(Q)}(X,k) A^{(Q)}(x_2,X) \ , \tag{4.11}$$

where the $A^{(Q)}(y_1,y_2)$ are the parallel transport matrices. They transport a spinor at y_1 to one at the point y_2 along a specified path. In order that there be an unambiguous notion of what $A^{(Q)}(y_1,y_2)\psi(y_1)$ is to mean we have to uniquely fix the curve connecting the points y_1 and y_2. Traditionally, this choice has been a geodesic.[4,12] In order to simplify computation we pick instead, in RNC based at Q, the "straight line," $y_2^\mu + t(y_1 - y_2)^\mu$ with $0 \le t \le 1$. Of course if y_2 and Q are taken to coincide then this curve will be a geodesic. The parallel transport equation is

$$(x_1 - X)^\mu A^{(Q)}_{;\mu}(x_1,X) = 0 \ . \tag{4.12}$$

Successively differentiating the above with respect to x_1^μ, treating X^μ as fixed, and then setting $x_1^\mu = X^\mu$ we arrive at

$$A^{(Q)}_{;\alpha}(X,X) = 0 \tag{4.13}$$

and

$$A^{(Q)}_{;(\alpha,\beta,\gamma,\ldots,\delta)}(X,X) = 0 \ . \tag{4.14}$$

Next we introduce a Taylor expansion for $A^{(Q)}(x_1,X)$ around the point X^μ:

$$A^{(Q)}(x_1,X) = A^{(Q)}(X,X) + (x_1 - X)^\alpha A^{(Q)}_{,\alpha}(X,X) + \frac{1}{2!}(x_1 - X)^\alpha (x_1 - X)^\beta A^{(Q)}_{,\alpha\beta}(X,X) + \cdots \ . \tag{4.15}$$

Noting that $A^{(Q)}(X,X) = 1$ and

$$\nabla^{x_1}_\mu A^{(Q)}(x_1,X) = (\partial_\mu - \Gamma_\mu)_{x_1} A^{(Q)}(x_1,X) \ ,$$

Eqs. (4.13)–(4.15) enable us to write

$$A^{(Q)}(x_{1,2},X) = 1 \pm \frac{x^\alpha}{2}\Gamma_\alpha(X) + \frac{x^\alpha x^\beta}{8}(\Gamma_{\alpha;\beta} + \Gamma^\lambda_{\alpha\beta}\Gamma_\lambda + \Gamma_\lambda\Gamma_\beta)(X) + \cdots \tag{4.16}$$

which is a concrete expression for the parallel transport matrices correct to second adiabatic order. The quantities Γ_α and $\Gamma^\lambda_{\alpha\beta}$ are evaluated in RNC with origin at Q.

In order to study the kinematics of the object $F^{(Q)}(X,k)$ we introduce a second RNC with origin at Q' just as in the scalar case. The analog of Eq. (3.4) will be

$$G_1(x_1,x_2) = \Delta_{\rm VM}^{1/2}(x_1,x_2) \times \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot\xi}}{\sqrt{-g'(\Xi)}} A^{(Q')}(\xi_1,\Xi) \times F^{(Q')}(\Xi,p) A^{(Q')}(\xi_2,\Xi) \ , \tag{4.17}$$

where

$$A^{(Q')}(\xi_1,\Xi) = 1 + \frac{\xi^{\alpha'}}{2}\Gamma_{\alpha'}(\Xi) + \frac{\xi^{\alpha'}\xi^{\beta'}}{8}(\Gamma_{\alpha';\beta'} + \Gamma^{\lambda'}_{\alpha'\beta'}\Gamma_{\lambda'} + \Gamma_{\alpha'}\Gamma_{\beta'})(\Xi) + \cdots \ . \tag{4.18}$$

Our task is to express the right-hand side of (4.17) in terms of quantities evaluated in the original coordinate system with origin at Q. We write

$$\Gamma_{\alpha'}(\Xi) = \Gamma_{\alpha'}\left[\xi^{\mu'}(X) + \Delta X^\nu \frac{\partial \xi^{\mu'}}{\partial X^\nu}(X) + \cdots\right] = \Gamma_{\alpha'}(\xi^{\mu'}(X)) + \Delta X^\nu \frac{\partial \xi^{\mu'}}{\partial X^\nu}\frac{\partial \Gamma_{\alpha'}}{\partial \xi^{\mu'}}[\xi(X)] + \cdots \ . \tag{4.19}$$

From (3.6) we recall that

$$\Delta X^{\nu}=\frac{1}{8}\frac{\partial X^{\nu}}{\partial \xi^{\mu'}}\frac{\partial^2 \xi^{\mu'}}{\partial X^{\eta}\partial X^{\lambda}}(X)x^{\eta}x^{\lambda} . \tag{4.20}$$

Under coordinate transformations Γ_α transforms as a vector:

$$\Gamma_{\alpha'}(\xi(X))=\frac{\partial x^{\lambda}}{\partial \xi^{\alpha'}}\Gamma_{\lambda}(X) . \tag{4.21}$$

Now we make our special choice of origin by taking Q' to P. Equations (4.19)–(4.21) enable us to write (4.18) as

$$\Gamma_{\alpha'}(\Xi)\,|_{Q'=P}=\frac{\partial X^{\lambda}}{\partial \xi^{\alpha'}}\Gamma_{\lambda}(X) + \tfrac{1}{8}\Gamma^{\mu'}_{\lambda\eta}(X)x^{\eta}x^{\lambda}\left[\frac{\partial^2 X^{\sigma}}{\partial\xi^{\mu'}\partial\xi^{\alpha'}}\Gamma_{\sigma}(X) + \frac{\partial X^{\alpha}}{\partial \xi^{\alpha'}}\frac{\partial X^{\beta}}{\partial \xi^{\mu'}}\frac{\partial \Gamma_{\alpha}(X)}{\partial X^{\beta}}\right] + \cdots . \tag{4.22}$$

The second term in (4.22) is higher order and may be dropped. Recall from (3.7) that, to second order,

$$\xi^{\mu'}=\frac{\partial \xi^{\mu'}}{\partial X^{\mu}}(X)x^{\mu} .$$

Thus to this order,

$$\xi^{\alpha'}\Gamma_{\alpha'}(\Xi)=x^{\lambda}\Gamma_{\lambda}(X) . \tag{4.23}$$

In a similar fashion we can check that

$$\Gamma_{\alpha';\beta'}(\Xi)=\frac{\partial X^{\lambda}}{\partial \xi^{\alpha'}}\frac{\partial X^{\sigma}}{\partial \xi^{\beta'}}\Gamma_{\lambda;\sigma}(X)+ \cdots . \tag{4.24}$$

Substituting (4.23) and (4.24) in (4.22) and keeping only second-order terms we arrive at

$$A^{(P)}(\xi_1,\Xi)=A^{(Q)}(x_1,X) . \tag{4.25}$$

This fortuitous circumstance does not hold at higher orders. In fact, at the very next order correction terms will have to be introduced. All other origin dependences are the same as for the scalar case. These have already been calculated and found to be of fourth order.

In analogy with the approach of Sec. III we postulate an improved ansatz for $G_1(x_1,x_2)$:

$$G_1(x_1,x_2)=\Delta^{1/2}_{\rm VM}(x_1,x_2)\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{\sqrt{-g(X)}}e^{P}F(X,k) . \tag{4.26}$$

To our order of approximation P is produced purely from the transport term. Because of (4.25) P can be found directly to be

$$PF=\frac{i}{2}\frac{\partial}{\partial k_{\alpha}}[\Gamma_{\alpha}(X),F]-\frac{1}{8}\frac{\partial^2}{\partial k_{\alpha}\partial k_{\beta}}\{F,\Gamma_{\alpha;\beta}(X)\} + \cdots , \tag{4.27}$$

by explicitly evaluating

$$A^{(Q)}(x_1,X)F(X,k)A^{(Q)}(x_2,X) .$$

We observe that at higher orders P will include contributions such as those from (3.21) as well as terms compensating for the change in the parallel transport matrix.

We are now in a position to obtain a dynamical equation for F. In the same way as (3.22) we seek a representation of the form

$$\left[\Box_{x_1}+m^2-\frac{R}{4}(x_1)\right]G_1(x_1,x_2) =\Delta^{1/2}_{\rm VM}(x_1,x_2)\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{\sqrt{-g(X)}}e^{P}(\mathcal{D}F) . \tag{4.28}$$

Writing

$$G_1(x_1,x_2)\equiv\frac{\Delta^{1/2}_{\rm VM}(x_1,x_2)}{\sqrt{-g(X)}}f(x_1,x_2) \tag{4.29}$$

and using the results (3.23)–(3.27) we obtain

$$\begin{aligned}\mathcal{D}F=e^{-P}\Bigg[g^{\mu\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\Bigg\{&-k_{\mu}k_{\nu}+ik_{\mu}\frac{\partial}{\partial X^{\nu}}+\frac{1}{4}\frac{\partial^2}{\partial X^{\mu}\partial X^{\nu}}+\frac{1}{4}[\Gamma^{\beta}_{\beta\mu}(X)\Gamma^{\alpha}_{\alpha\nu}(X)-\Gamma^{\alpha}_{\alpha\nu,\mu}(X)]-\Gamma^{\alpha}_{\alpha\nu}(X)\left[ik_{\mu}+\frac{1}{2}\frac{\partial}{\partial X^{\mu}}\right]\\ &+2\left[\left[\frac{\partial}{\partial x_1^{\mu}}\ln\Delta^{1/2}_{\rm VM}\right]\delta^{\lambda}_{\nu}-\Gamma_{\mu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\delta^{\lambda}_{\nu}-\frac{1}{2}\Gamma^{\lambda}_{\mu\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\right]\left[\frac{\partial}{\partial x_1^{\lambda}}-\frac{1}{2}\Gamma^{\alpha}_{\alpha\lambda}(X)\right]\\ &+\left[\frac{\partial^2}{\partial x_1^{\mu}\partial x_1^{\nu}}\ln\Delta^{1/2}_{\rm VM}\right]+\left[\frac{\partial}{\partial x_1^{\mu}}\ln\Delta^{1/2}_{\rm VM}\right]\left[\frac{\partial}{\partial x_1^{\nu}}\ln\Delta^{1/2}_{\rm VM}\right]\\ &-\left[\Gamma^{\lambda}_{\mu\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]+2\Gamma_{\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\delta^{\lambda}_{\mu}\right]\left[\frac{\partial}{\partial x_1^{\lambda}}\ln\Delta^{1/2}_{\rm VM}\right]-\frac{\partial}{\partial x_1^{\mu}}\Gamma_{\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\\ &+\Gamma_{\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\left[\Gamma^{\lambda}_{\mu\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]+\delta^{\lambda}_{\mu}\Gamma_{\nu}\left[X+\frac{i}{2}\frac{\partial}{\partial k}\right]\right]\Bigg\}+m^2-\frac{R}{4}\Bigg]e^{P}F .\end{aligned} \tag{4.30}$$

Of course directly solving $\mathcal{D}F=0$ in its present form would be a rather difficult task. However, we can write (4.30) evaluated in RNC at X to get the much simpler equation

$$\mathcal{D}F(0,k)=e^{-P}\left[g^{\mu\nu}\left[\frac{i}{2}\frac{\partial}{\partial k}\right]\left\{-k_\mu k_\nu+ik_\mu\frac{\partial}{\partial X^\nu}+\frac{1}{4}\frac{\partial^2}{\partial X^\mu\partial X^\nu}-\frac{1}{4}\Gamma^\alpha_{\alpha\nu,\mu}(0)\right.\right.$$
$$+2\left[\left[\frac{\partial}{\partial x_1^\mu}\ln\Delta_{\rm VM}^{1/2}\right]\delta^\lambda_\nu-\delta^\lambda_\nu\Gamma_\mu\left[\frac{i}{2}\frac{\partial}{\partial k}\right]-\frac{1}{2}\Gamma^\lambda_{\mu\nu}\left[\frac{i}{2}\frac{\partial}{\partial k}\right]\right]\left[ik_\lambda+\frac{1}{2}\frac{\partial}{\partial X^\lambda}\right]$$
$$\left.\left.+\left[\frac{\partial^2}{\partial x_1^\mu\partial x_1^\nu}\ln\Delta_{\rm VM}^{1/2}\right]-\frac{\partial}{\partial x_1^\mu}\Gamma_\nu\left[\frac{i}{2}\frac{\partial}{\partial k}\right]\right\}+m^2-\frac{R}{4}(0)\right]e^P\,. \tag{4.31}$$

Introduce the operator D by

$$\mathcal{D}F=e^{-P}De^PF \tag{4.32}$$

and approximate e^P by $1+P$,

$$\mathcal{D}F=DF+D(PF)-P(DF)\,. \tag{4.33}$$

Substituting (4.27) in (4.33) and then using that to evaluate (4.32) we obtain

$$\left\{\eta^{\mu\nu}\left[-k_\mu k_\nu+ik_\mu\left[\frac{\partial}{\partial X^\nu}-2\Gamma_\nu\right]+\frac{1}{4}\frac{\partial^2}{\partial X^\mu\partial X^\nu}-\Gamma_{\mu;\nu}+k_\mu\Gamma_{\nu;\alpha}\frac{\partial}{\partial k_\alpha}\right]+\frac{1}{6}k^\mu R_{\mu\alpha}\frac{\partial}{\partial k_\alpha}-\frac{1}{12}k_\mu k_\nu R^{\mu\ \nu}_{\ \rho\ \sigma}\frac{\partial^2}{\partial k_\rho\partial k_\sigma}\right.$$
$$\left.+m^2-\frac{R}{12}\right\}F+k^\alpha\left[i[\Gamma_\alpha,F]+\frac{1}{2}\frac{\partial}{\partial k_\beta}[F,\Gamma_{\beta;\alpha}]\right]-\frac{1}{4}\left[\eta^{\alpha\beta}+k^\alpha\frac{\partial}{\partial k_\beta}+k^\beta\frac{\partial}{\partial k_\alpha}\right]\{\Gamma_{\alpha;\beta},F\}=0\,, \tag{4.34}$$

where we have used the RNC expansion formulas of the Appendix, as well as

$$\Gamma_\mu(x_1)\,|\,_{X=0}=\Gamma_\mu(0)+\frac{i}{2}\Gamma_{\mu;\nu}(0)\frac{\partial}{\partial k_\nu}+\cdots\,. \tag{4.35}$$

Note that everything in (4.34) is evaluated at the origin. Our rationale for writing (4.34) is the same as that for the scalar case—if $\mathcal{D}F=0$ in RNC at X then this is automatically true in any coordinate system because of the transformation properties of $\mathcal{D}F$. Thus it is sufficient to look for solutions of $\mathcal{D}F=0$ in RNC around X.

We start with the vacuum case assuming a solution of the form

$$F_{\rm vac}=1\delta(\Omega)+\delta F_{\rm vac}\,, \tag{4.36}$$

where

$$\Omega=g^{\mu\nu}(X)k_\mu k_\nu-m^2+\tfrac{1}{12}R\,. \tag{4.37}$$

With the aid of (3.34) and (3.35) we have

$$\mathcal{D}\delta(\Omega)=\eta^{\mu\nu}(-2ik_\mu\Gamma_\nu-\tfrac{3}{2}\Gamma_{\mu;\nu})\delta(\Omega)\,. \tag{4.38}$$

Since $\delta F_{\rm vac}$ is already a second-order term,

$$\mathcal{D}\delta F_{\rm vac}=(-\eta^{\mu\nu}k_\mu k_\nu+m^2-\tfrac{1}{12}R)\delta F_{\rm vac}=-\Omega\delta F_{\rm vac}\,. \tag{4.39}$$

If $\delta F_{\rm vac}$ is expanded in derivatives of δ functions as in Sec. II,

$$\delta F_{\rm vac}=\sum_{n=1}^{\infty}(\delta F_{\rm vac})_n\delta^{(n)}(\Omega) \tag{4.40}$$

and

$$-\Omega\delta F_{\rm vac}=\sum_{n=1}^{\infty}n(\delta F_{\rm vac})_n\delta^{(n-1)}(\Omega)\,. \tag{4.41}$$

Therefore in order to satisfy $\mathcal{D}F_{\rm vac}=0$ we must have

$$(\delta F_{\rm vac})_1=\eta^{\mu\nu}(2ik_\mu\Gamma_\nu+\tfrac{3}{2}\Gamma_{\mu;\nu})\,. \tag{4.42}$$

All other $(\delta F_{\rm vac})_n$ with $n>1$ vanish at this order.

In the general case we will assume a solution of the form (3.42). Then,

$$\mathcal{D}F_0\delta(\Omega)(X=0)=ik^\nu\left[\frac{\partial F_0}{\partial X^\nu}-\{\Gamma_\nu,F_0\}\right]\delta(\Omega)$$
$$+\left[\frac{1}{4}\eta^{\mu\nu}\frac{\partial^2F_0}{\partial X^\mu\partial X^\nu}-\eta^{\mu\nu}\Gamma_{\mu;\nu}F_0+\eta^{\mu\nu}k_\mu\Gamma_{\nu;\alpha}\frac{\partial F_0}{\partial k_\alpha}+\frac{1}{6}k^\mu R_{\mu\alpha}\frac{\partial F_0}{\partial k_\alpha}\right.$$
$$-\frac{1}{12}k_\mu k_\nu R^\mu{}_\rho{}^\nu{}_\rho\frac{\partial^2F_0}{\partial k_\rho\partial k_\sigma}+\frac{1}{2}k^\alpha\frac{\partial}{\partial k_\beta}[F_0,\Gamma_{\beta;\alpha}]-\frac{1}{4}\eta^{\alpha\beta}\{\Gamma_{\alpha;\beta},F_0\}$$
$$\left.-\frac{1}{4}\left[k^\alpha\frac{\partial}{\partial k_\beta}+k^\beta\frac{\partial}{\partial k_\alpha}\right]\{\Gamma_{\alpha;\beta},F_0\}\right]\delta(\Omega)\ . \quad (4.43)$$

Again, to this order, only $F_1(0,k)$ is nonzero:

$$F_1(0,k)=\eta^{\mu\nu}\left[\Gamma_{\mu;\nu}F_0+\frac{1}{4}\{\Gamma_{\mu;\nu},F_0\}-\frac{1}{6}k_\mu R_{\nu\alpha}\frac{\partial F_0}{\partial k_\alpha}-\frac{1}{4}\frac{\partial^2F_0}{\partial X^\mu\partial X^\nu}\right]$$
$$+\frac{1}{12}k_\mu k_\nu R^\mu{}_\rho{}^\nu{}_\sigma\frac{\partial^2F_0}{\partial k_\rho\partial k_\sigma}-k^\alpha\Gamma_{[\alpha;\beta]}\frac{\partial F_0}{\partial k_\beta}+\frac{k^\alpha}{2}\left[\frac{\partial F_0}{\partial k_\beta},\Gamma_{[\alpha;\beta]}\right]\ . \quad (4.44)$$

The final step is to write all equations in covariant form. The covariant derivative constructed in the scalar case may be extended to the spinorial case by adding on the spin connection,

$$F_{0;\mu}=\frac{\partial F_0}{\partial X^\mu}+\Gamma^\nu_{\mu\rho}k_\nu\frac{\partial F_0}{\partial k_\rho}-\{\Gamma_\mu,F_0\}\ , \quad (4.45)$$

and

$$F_{0;\mu\nu}=\frac{\partial}{\partial X^\nu}(F_{0;\mu})-\Gamma^\lambda_{\mu\nu}F_{0;\lambda}+\Gamma^\eta_{\nu\sigma}k_\eta\frac{\partial}{\partial k_\sigma}(F_{0;\mu})$$
$$-\{\Gamma_\nu,F_{0;\mu}\}\ . \quad (4.46)$$

It follows that, at the origin,

$$F_{0;\mu}(0,k)=\frac{\partial F_0}{\partial X^\mu}-\{\Gamma_\mu,F_0\}\ , \quad (4.47a)$$

$$F_{0;\mu\nu}(0,k)=\frac{\partial^2F_0}{\partial X^\mu\partial X^\nu}+\tfrac{1}{3}(R^\beta{}_{\mu\rho\nu}+R^\beta{}_{\rho\mu\nu})k_\beta\frac{\partial F_0}{\partial k_\rho}$$
$$-\{\Gamma_{\mu;\nu},F_0\}\ . \quad (4.47b)$$

Setting the imaginary part of (4.43) to zero,

$$k^\nu\left[\frac{\partial F_0}{\partial X^\nu}-\{\Gamma_\nu,F_0\}\right]=0\ ,$$

or

$$g^{\mu\nu}k_\mu F_{0;\nu}=0\ , \quad (4.48)$$

which is the covariant Liouville-Vlasov equation to second order. From (4.44) and (4.48) we obtain the general covariant form of F:

$$F(X,k)=F_0(X,k)\delta(\Omega)-\frac{1}{4}\left\{\left[g^{\mu\nu}F_{0;\mu\nu}+\frac{1}{3}\left[R^\beta{}_\alpha k_\beta\frac{\partial F_0}{\partial k_\alpha}-k_\mu k_\nu R^\mu{}_\rho{}^\nu{}_\sigma\frac{\partial^2F_0}{\partial k_\rho\partial k_\sigma}\right]\right]\right.$$
$$\left.+g^{\mu\nu}\Gamma_{\mu;\nu}F_0-k^\alpha\left[\Gamma_{[\alpha;\beta]}\frac{\partial F_0}{\partial k_\beta}-\frac{1}{2}\left[\frac{\partial F_0}{\partial k_\beta},\Gamma_{[\alpha;\beta]}\right]\right]\right\}\delta'(\Omega)\ , \quad (4.49)$$

with F_0 being the solution of the Liouville-Vlasov equation (4.48).

Finally we would like to point out certain relationships with previous work. For example, if one chooses P_1 or P_2 as the origin itself and picks the tetrad as being parallel transported along a geodesic joining P_1 and P_2, then it can be shown that $A^{x_2}(x_1,x_2)=1$ (Ref. 4). Our formula (4.18) is consistent with this result. By using the appropriate RNC expansions for e^λ_α and $\Gamma^\eta_{\lambda\beta}$ it is possible to show that the higher-order terms in (4.18) cancel. Another result of not choosing P_2 as the origin is the appearance of extra terms involving Γ_λ and its derivatives. A large number of such terms vanish when one makes the special coordinate origin and tetrad field choice of Ref. 4, and our results reduce to theirs.

V. REMARKS

As concluding remarks, we will just mention a few parallel developments related to the present work which are perhaps worthy of further pursuit. In terms of (1) methodology, the relationship of the Wigner function approach with the quasilocal functional approach,[5] the eikonal approach,[17] and the point-splitting approach;[12] (2) viewpoint, its relationship with the sympletic geometry view of general relativity[18] and with the geometric quantization[19] approach to quantum gravity; (3) generalization, construction of Wigner functions in group manifolds and homogeneous spaces, and relationship with the mathematics of pseudodifferential operators;[7] and finally, (4) application, the description of transport phenomena of nonequilibrium quantum fields in extreme conditions such as occurring in the early Universe and late stages of black-hole collapse.

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APPENDIX: CONVENTIONS AND USEFUL IDENTITIES

We list below the main conventions and some useful identities which have been used throughout [the signature for the metric is $(+\,-\,-\,-)$]:

$$\Gamma^{\mu}{}_{\nu\rho}=\tfrac{1}{2}g^{\mu\sigma}(g_{\nu\sigma,\rho}+g_{\rho\sigma,\nu}-g_{\nu\rho,\sigma})\ , \tag{A1}$$

$$R^{\mu}{}_{\nu\rho\sigma}=\Gamma^{\mu}{}_{\nu\rho,\sigma}-\Gamma^{\mu}{}_{\nu\sigma,\rho}+\Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\nu\rho}-\Gamma^{\mu}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\nu\sigma}\ , \tag{A2}$$

$$R_{\mu\nu}=R^{\lambda}{}_{\mu\lambda\nu}\ , \tag{A3}$$

$$R=g^{\mu\nu}R_{\mu\nu}\ . \tag{A4}$$

The rule for commuting covariant derivatives is

$$A^{\rho}{}_{;\mu\nu}=A^{\rho}{}_{;\nu\mu}+R_{\mu\nu}{}^{\rho}{}_{\sigma}A^{\sigma}\ . \tag{A5}$$

The following identities hold:

$$R^{\rho\sigma\lambda}{}_{\mu}R_{\rho\nu\lambda\sigma}=\tfrac{1}{2}R^{\rho\sigma\lambda}{}_{\mu}R_{\rho\sigma\lambda\nu}\ , \tag{A6}$$

$$R^{\mu}{}_{\rho\sigma\lambda;\mu}=R_{\rho\lambda;\sigma}-R_{\rho\sigma;\lambda}\ , \tag{A7}$$

$$R^{\rho}{}_{\sigma;\rho}=\tfrac{1}{2}R_{;\sigma}\ . \tag{A8}$$

A Riemann normal coordinate (RNC) system is constructed as follows. Consider a point z to become the origin of coordinates. Now given a second point x, consider the tangent vector t^{μ} at z to the (unique) geodesic joining z and x. Then the RNC of x are $y^{\mu}=\alpha t^{\mu}$, where α is chosen so that $\eta_{\alpha\beta}y^{\alpha}y^{\beta}$ equals the geodesic distance from z to x.

An equivalent definition is that a RNC system is a coordinate frame in which the following identities hold:

$$\begin{gathered} g_{\mu\nu}(0)=\eta_{\mu\nu},\quad \Gamma^{\mu}_{\nu\rho}(0)=0\ , \\ \Gamma^{\mu}_{(\nu\rho,\sigma)}=0,\quad \Gamma^{\mu}_{(\nu\rho,\sigma\lambda)}=0,\quad \text{etc.}\ , \end{gathered} \tag{A9}$$

where ( ) stands for symmetrization. In a RNC system the Taylor expansion of the metric is given by the curvature at the origin. Concretely,

$$\begin{aligned} g_{\mu\nu}(x)=\eta_{\mu\nu}&+\tfrac{1}{3}R_{\mu\rho\nu\sigma}x^{\rho}x^{\sigma}+\tfrac{1}{6}R_{\mu\rho\nu\sigma;\lambda}x^{\rho}x^{\sigma}x^{\lambda} \\ &+(\tfrac{1}{20}R_{\mu\rho\nu\sigma;\lambda\tau}+\tfrac{2}{45}R_{\xi\lambda\mu\sigma}R^{\xi}{}_{\rho\nu\tau})x^{\rho}x^{\sigma}x^{\lambda}x^{\tau} \\ &+\cdots\ . \end{aligned} \tag{A10}$$

The following expansions also hold:

$$g^{\mu\nu}(x)=\eta^{\mu\nu}-\tfrac{1}{3}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}x^{\rho}x^{\sigma}-\tfrac{1}{6}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma;\lambda}x^{\rho}x^{\sigma}x^{\lambda}-(\tfrac{1}{20}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma;\lambda\tau}-\tfrac{1}{15}R^{\mu}{}_{\rho\xi\sigma}R^{\xi}{}_{\lambda}{}^{\nu}{}_{\tau})x^{\rho}x^{\sigma}x^{\lambda}x^{\tau}+\cdots\ , \tag{A11}$$

$$\begin{aligned} z^{\nu}z^{\rho}\Gamma^{\mu}_{\nu\rho}(x)=z^{\nu}z^{\rho}\{&\tfrac{2}{3}R^{\mu}{}_{\nu\rho\sigma}x^{\sigma}+\tfrac{1}{12}(5R^{\mu}{}_{\nu\rho\sigma;\lambda}+R^{\mu}{}_{\sigma\nu\lambda;\rho})x^{\sigma}x^{\lambda} \\ &+\tfrac{1}{6}[\tfrac{9}{10}R^{\mu}{}_{\rho\nu\sigma;\lambda\tau}+\tfrac{3}{20}(R^{\mu}{}_{\sigma\nu\lambda;\rho\tau}+R^{\mu}{}_{\sigma\nu\lambda;\tau\rho}) \\ &\quad+\tfrac{1}{60}(21R^{\mu}{}_{\lambda\xi\rho}R^{\xi}{}_{\sigma\nu\tau}+48R^{\mu}{}_{\xi\rho\lambda}R^{\xi}{}_{\sigma\nu\tau}-37R^{\mu}{}_{\sigma\xi\lambda}R^{\xi}{}_{\nu\rho\tau})]x^{\sigma}x^{\lambda}x^{\tau}+\cdots\}\ , \end{aligned} \tag{A12}$$

$$(g^{\rho\nu}\Gamma^{\mu}_{\rho\nu})(x)=-\tfrac{2}{3}R^{\mu}{}_{\sigma}x^{\sigma}+\tfrac{1}{12}(R_{\lambda\sigma;}{}^{\mu}-6R^{\mu}{}_{\sigma;\lambda})x^{\sigma}x^{\lambda}+\tfrac{1}{30}(\tfrac{3}{2}R_{\lambda\sigma;}{}^{\mu}{}_{\tau}-6R^{\mu}{}_{\lambda;\sigma\tau}-\tfrac{8}{3}R^{\mu}{}_{\xi\rho\lambda}R^{\xi}{}_{\sigma}{}^{\rho}{}_{\tau}+\tfrac{7}{3}R_{\xi\tau}R^{\xi}{}_{\lambda}{}^{\mu}{}_{\sigma})x^{\sigma}x^{\lambda}x^{\tau}\ , \tag{A13}$$

$$\ln[-g(x)]=\tfrac{1}{3}R_{\mu\nu}x^{\mu}x^{\nu}+\tfrac{1}{6}R_{\mu\nu;\rho}x^{\mu}x^{\nu}x^{\rho}+(\tfrac{1}{20}R_{\mu\nu;\rho\sigma}-\tfrac{1}{90}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R_{\alpha\rho\beta\sigma})x^{\mu}x^{\nu}x^{\rho}x^{\sigma}\ , \tag{A14}$$

$$\eta^{\lambda\tau}z^{\nu}z^{\rho}z^{\sigma}\Gamma^{\mu}_{\nu\rho,\sigma\lambda\tau}(0)=\tfrac{1}{15}(12R^{\mu}{}_{\nu;\rho\sigma}-\tfrac{21}{2}R_{\nu\rho;}{}^{\mu}{}_{\sigma}+7R^{\xi}{}_{\nu}R^{\mu}{}_{\rho\xi\sigma}-\tfrac{16}{3}R^{\mu}{}_{\xi\rho\eta}R^{\xi}{}_{\sigma}{}^{\eta}{}_{\nu})z^{\nu}z^{\rho}z^{\sigma}\ . \tag{A15}$$

The Van Vleck–Morette determinant is defined as

$$\Delta_{\rm VM}(x_1,x_2)=[-g(x_1)]^{-1/2}[-g(x_2)]^{-1/2}\det\frac{\partial^2\sigma(x_1,x_2)}{\partial x_1^{\mu}\partial x_2^{\nu}}\ , \tag{A16}$$

where σ is one-half of the geodesic distance from x_2 to x_1. Equivalently, in RNC with the origin at x_2,

$$\Delta_{\rm VM}(x_1,x_2)=[-g(x_1)]^{-1/2}\ . \tag{A17}$$

We have made use of the expansions (in RNC)

$$\frac{\partial}{\partial x_1^\mu}\ln\Delta_{\rm VM}\left[\frac{x}{2},\frac{-x}{2}\right]=(-\tfrac{1}{2})[\tfrac{2}{3}R_{\mu\nu}x^\nu+\tfrac{1}{6}R_{\nu\rho;\mu}x^\nu x^\rho$$
$$+(\tfrac{1}{60}R_{\mu\nu;\rho\sigma}+\tfrac{1}{20}R_{\nu\rho;\sigma\mu}-\tfrac{1}{30}R_{\nu\rho;\mu\sigma}-\tfrac{2}{45}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R_{\alpha\rho\beta\sigma}-\tfrac{1}{12}R^{\lambda}_{\rho\mu\sigma}R_{\lambda\nu})x^\nu x^\rho x^\sigma+\cdots]\,, \tag{A18}$$

$$\eta^{\mu\nu}\frac{\partial^2}{\partial x_1^\mu\partial x_1^\nu}\ln\Delta_{\rm VM}\left[+\frac{x}{2},-\frac{x}{2}\right]=(-\tfrac{1}{2})[\tfrac{2}{3}R+\tfrac{1}{3}R_{;\rho}x^\rho+(\tfrac{1}{20}R_{;\rho\sigma}+\tfrac{1}{10}\Box R_{\rho\sigma}+\tfrac{1}{45}R_{\xi\sigma}R^{\xi}{}_{\rho}-\tfrac{1}{15}R^{\alpha\beta\gamma}{}_{\rho}R_{\alpha\beta\gamma\sigma}$$
$$-\tfrac{1}{90}R_{\xi\eta}R^{\xi}{}_{\rho}{}^{\eta}{}_{\sigma})x^\rho x^\sigma+\cdots]\,. \tag{A19}$$

Finally, we give the RNC x^a around z of the point P in terms of the difference x^μ of the coordinates of P and Z in the original frame. They are

$$x^a=e^a_\mu(z)[x^\mu+\tfrac{1}{2}\Gamma^\mu_{\nu\rho}(z)x^\nu x^\rho+\tfrac{1}{6}(\Gamma^\mu_{\nu\rho,\sigma}+\Gamma^\mu_{\xi\nu}\Gamma^\xi_{\rho\sigma})(z)x^\nu x^\rho x^\sigma+\cdots]\,. \tag{A20}$$

Because of the conditions (A9) the e^a_μ are the components of a vierbein, that is, they satisfy

$$g^{\mu\nu}(z)e^a_\mu e^b{}_\nu=\eta^{ab} \tag{A21}$$

and

$$\eta^{ab}e^\mu_a e^\nu_b=g^{\mu\nu}\,. \tag{A22}$$

A vierbein is any collection of four vector fields satisfying Eqs. (A21) and (A22).

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