Nonequilibrium quantum fields: Closed-time-path effective action, Wigner function, and Boltzmann equation

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This is the first of a series of papers which describe the functional-integral approach to the study of the statistical and kinetic properties of nonequilibrium quantum fields in flat and curved spacetimes. In this paper we treat a system of self-interacting bosons described by $\lambda\phi^4$ scalar fields in flat space. We adopt the closed-time-path (CTP or "in-in") functional formalism and use a two-particle irreducible (2PI) representation for the effective action. These formalisms allow for a full account of the dynamics of quantum fields, and put the correlation functions on an equal footing with the mean fields. By assuming a thermal distribution we recover the real-time finite-temperature theory as a special case. By requiring the CTP effective action to be stationary with respect to variations of the correlation functions we obtain an infinite set of coupled equations which is the quantum-field-theoretical generalization of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy. Truncation of this series leads to dissipative characteristics in the subsystem. In this context we discuss the nature of dissipation in interacting quantum fields. To one-loop order in a perturbative expansion of the CTP effective action, the 2PI formalism yields results equivalent to the leading $1/N$ expansion for an $O(N)$-symmetric scalar field. To higher-loop order we introduce a two-time approximation to separate the quantum-field effects of radiative correction and renormalization from the statistical-kinetic effects of collisions and relaxation. In the weak-coupling quasiuniform limit, the system of nonequilibrium quantum fields can subscribe to a kinetic theory description wherein the propagators are represented in terms of relativistic Wigner distribution functions. From a two-loop calculation we derive the Boltzmann equation for the distribution function and the gap equation for the effective mass of the quasiparticles. One can define an entropy function for the quantum gas of quasiparticles which satisfies the H theorem. We also calculate the limits to the validity of the binary collision approximation from a three-loop analysis. The theoretical framework established here can be generalized to nonconstant background fields and for curved spacetimes.

I. INTRODUCTION AND SUMMARY

Development in this decade in unified theories of particle interactions with gravity has pointed to the usefulness of the study of high-energy quantum processes in the early universe from the grand-unified-theory (GUT) scale ($\sim 10^{14}$ GeV) to the Planck scale ($\sim 10^{19}$ GeV) and beyond.[1] The early Universe is a crucible where matter and spacetime exist under extreme conditions whereby existing physical laws are scrutinized and new ideas are tested. In turn, the new understanding we gain reshapes our conception of the origin and structure of the Universe. Near the Planck scale matter as described by quantum fields is strongly influenced by gravity.[2] Short of a viable theory of unified interactions including quantum gravity, a reasonable scheme to describe quantum processes in the period just below the Planck scale is provided by quantum field theory in curved spacetime,[3] a discipline fairly well established from the efforts in the past two decades. There are three aspects in the analysis of any such typical process: the geometric and topological aspect of spacetime, the quantum-field aspect of matter, and the statistical aspect of their interactions. Furthermore, in the context of evolutionary cosmology, all three aspects are complicated by their overall dynamical dependence.[4] Therefore quantum processes in the early Universe are intrinsically nonequilibrium in nature and for their description one needs to formulate a quantum statistical and kinetic field theory in curved spacetime. By contrast, we see that the "standard" Friedmann-Robertson-Walker model has classical matter in the hydrodynamic limit as source. To include the effect of weakly interacting particles like collisionless neutrinos, where thermal equilibrium is not available, one needs to use a kinetic theory description and solve the Einstein-Boltzmann equations.[5] To include pair creation and interaction processes (e.g., in e^+e^- or quark-gluon plasmas) one needs a relativistic quantum field treatment.[6] When the interaction times of the dominant processes are faster than the expansion (Hubble) time of the background, one can construct an approximate finite-temperature quantum field theory for such considerations.[7] However, finite-temperature quantum field theory in curved and dynamical spacetime is generally not well defined. Only under restrictive conditions where quasiequilibrium is maintained can one define such a theory up to some adiabatic order.[8] In processes such as particle production and phase transition near the Planck time very few of the above simplifying conditions can hold, in which case one needs to deal directly with nonequilibrium quantum fields.[9]

In this and two companion papers, we aim at laying the foundation for the construction of statistical and kinetic theories for nonequilibrium quantum fields in curved spacetime. We are interested in the quantum statistical and kinetic properties of many particle and field systems as influenced by a dynamical classical gravitational field. The classes of *physical problems* to which this formalism can help to address include (1) particle creation and entropy generation,[10] e.g., the study of the statistical properties of spontaneous and stimulated production, the transition from quantum statistical to kinetic and classical regimes, (2) cosmological back-reaction problems,[11] e.g., self-consistent solutions of the Einstein-Boltzmann equation for the distribution function and the metric function, dynamics of quantum fields, dependence of the temperature parameter and thermal history, (3) quantum dissipative processes,[12] e.g., vacuum viscosity, fluctuation-dissipation relations, cosmic arrow of time, and nonequilibrium thermodynamics of black holes, (4) transport properties of quantum fields in GUT and post-Planckian processes,[13] and (5) critical dynamics,[14] e.g., dynamical aspects of phase transition, quantum tunneling, spinodal decomposition, and cosmological implications. Other topics of interest in particle physics include quark-hadron phase transition.[15] GUT inflationary transition,[16] the thermodynamics of superstrings and the black-hole superstring transition.[17]

The *theoretical framework* of our present work derives from the synthesis of two formal developments. The first as related above is the development of quantum field theory in curved space to encompass statistical and kinetic properties. This is done by adding the consideration of density matrices[18] and distribution functions (Wigner function)[19] for the quantum states. An intermediate step is the formulation of finite-temperature quantum field theory for dynamical spacetimes.[8,9] In a special but important class of spacetimes—those with event horizons such as Schwarzschild, de Sitter, and Rindler spaces—finite-temperature theory in imaginary-time formulation and ideas in equilibrium thermodynamics can be used for the description of Hawking radiation.[20] The nonequilibrium framework developed here is aimed at describing dynamical processes in cosmological spacetimes. It goes beyond the linear response theory and near-equilibrium formalisms which are more suitable for describing thermodynamic processes in black-hole spacetimes.[21] The second formal development is the extension, to curved spacetime, of nonequilibrium statistical field theory. The standard treatment begins with the classic formalism of Kandanoff and Baym and of Abrikosov, Gorkov, and Dzyaloshinski.[22] The development of field-theoretical methods in statistical mechanics by Brezin *et al.*[23] accompanied the advances in critical phenomena through the work of Wilson, Fisher, and others. A recent field-theoretical account of relativistic kinetic theory is given by de Groot, van Leewen, and van Weert.[24] The functional-integral approach to problems in statistical physics is described by Graham, Popov, and others.[25] A more powerful treatment which we shall adopt here is the so-called closed-time-path (CTP, "double-time" or "in-in") functional-integral formalism of Schwinger, Keldysh, Korenman,[26] and others. Its relativistic field-theoretical generalization is summarized in the recent work of Zhou, Su, Hao, and Yu.[27] The CTP formalism has recently been extended to curved spacetime and applied to cosmological back-reaction problems by Jordan[28] and by Calzetta and Hu.[29] Despite the apparent technical complexity, its power in dealing with dynamical and nonequilibrium systems makes it far more desirable than the conventional ("single-time," "in-out") method.

Both developments mentioned above make use of path-integral quantization and perturbative expansion techniques. Another approach towards the construction of a viable quantum-statistical field theory in curved spacetime is the field-theoretical generalization of relativistic statistical mechanics by way of canonical quantization [e.g., the Arnowitt-Deser-Misner (ADM) formalism]. This is rooted in the classical theories of statistical mechanics in general relativity as developed by Anderson, Ehler, Ellis, Israel, Kandrup, Stewart, Thorne, and others.[30] The advantage of taking this route is that it allows for nonperturbative treatment and, in some cases, can encompass global properties of the system. Its disadvantage is perhaps the sacrifice of manifest covariance, which is difficult to achieve in a statistical description of the dynamics anyway. The differences of the path integral and the canonical approaches towards quantum statistical field theory are similar to those already encountered in the construction of quantum field theory in curved space. The object of interest and the central focus of course differ: in statistical field theory it is the two-point functions rather than their coincidence limit (for, say, the construction of the energy-momentum tensor) which are important, and it is the causal and correlational properties of the system rather than the transition amplitudes which are more pertinent. We will amplify these points in later papers.

For the construction of a quantum-statistical field theory here we shall adopt the functional-integral approach. In this paper, we will derive the Boltzmann equation from a two-loop perturbative calculation of a self-interacting quantum field by means of an extension of the two-particle irreducible (2PI) representation[31] of the closed-time-path (CTP) effective action. Since in this scheme the background field and the correlation functions are treated in parallel, one can consider the influence of the fluctuation field beyond the usual (one-loop) mean-field theories. In this paper we shall restrict our attention to fields in flat spacetime in the absence of background fields. Later we will consider statistical field theory with varying background fields and for interacting fields in curved spacetime.[32] They will be useful for discussions of critical dynamics in the early Universe and statistical thermodynamics of black holes.[33] In a companion paper,[34] we will develop a kinetic field theory in curved space by deriving the set of Liouville (collisionless Boltzmann) equations governing the quantum distribution function. This would serve as a basis for developing transport theories of quantum fields useful for the investigation of GUT and post-Planckian quantum processes.

We now summarize the main features and results of this paper. There are three essential elements in our formulation. First is the use of the CTP effective action, the second is the use of the 2PI scheme, and the third is the two-time approximation and the Wigner function description. The advantage of the CTP functional formalism in treating dynamical and nonequilibrium statistical processes is amply discussed in Refs. 26–29. Starting from an initial statistical configuration of quantum fields described by a density matrix ρ, the CTP effective action provides the means to follow the time evolution of either the mean field or the correlation functions of the theory. The basic formalism is laid down in Sec. II A. In Sec. II B we give the example of a simple choice for ρ: For a thermal distribution, the in-in formulation reduces to the usual (Matsubara) finite-temperature quantum field theory.[7] Actually in this limit the methods of both Matsubara and Schwinger are regarded as particular instances of a whole family of formalisms, based on a path-integral representation of the generating functional in which the time integration is distorted in various ways into the complex plane.[35] The in-in formulation is specially adept in that it can handle nonequilibrium as well as equilibrium states.

Many functional formulations of quantum-statistical physics tend to give a privileged role to the mean field. However, mean-field theory as the lowest-order approximation to many physical problems can be very inadequate, especially for treating collective infrared behavior such as critical phenomena.[36] Since phase transition is indeed one of the problems which the present formalism is intended for, we want to adopt the most general framework wherein the evolution of the correlation functions can be considered on an equal footing with the mean field. One useful choice is the so-called two-particle irreducible (2PI) formalism developed by De Dominicis and Martin, Dahmen and Jona-Lasinio, Cornwall, Jackiw, and Tomboulis, and others.[31] It is a generalized effective action in which not only the mean field but also the correlation functions appear as independent variables. This technique is easily adapted to the CTP formulation. The object of interest is a functional of an infinite series of correlation functions. By requiring the generalized effective action to be stationary with respect to variations of the correlation functions we obtain an infinite set of coupled equations which are the Dyson equations in the in-in formalism. This is the relativistic quantum field-theoretical version of BBGKY hierarchy usually found in nonrelativistic statistical mechanics.

As in nonrelativistic theory, to solve any concrete problem the hierarchy must be truncated. The simplest truncation is to consider only the one-loop connected mean-field theory, which leads to semiphenomenological formalisms such as the time-dependent Landau-Ginzburg theory. In Sec. II C we will explore truncation in the next higher order. An interesting consequence of the truncated theory is that the evolution of the background field and the propagators shows dissipative characteristics. This phenomenon is analogous to the appearance of dissipation when only a subsystem of the complete system is considered in detail ("system" interacting with a "bath"). The usual notion of dissipation arises from separating the system from the bath and averaging over the "bath" coordinates.[37] Here it can be shown clearly that a truncation without averaging the "bath" degrees of freedom does not lead to information loss. The full information can be retrieved by computing the higher-order connected Green's functions.[38] This point is discussed again in Sec. V.

From the general properties of the 2PI CTP effective action, which generates the equations of motion for both background fields and propagators, it is easy to see that, in the absence of external fields or inhomogeneous initial conditions, the mean field is zero at all times. In this paper we will consider these cases and discuss the dynamics of the propagators alone. The general case for nonzero background fields entails added technical complexity involving derivative expansion of the functional integral[39] and other approximations. This, together with the formally similar problem of quantum fields propagating in a dynamical classical background, will be discussed in companion papers.

In Sec. III we discuss the perturbative loop expansion of the CTP effective action. To the one-loop order, the 2PI formalism yields results equivalent to the leading-order analysis of the O(N)-symmetric scalar field theory in the large-N approximation.[40,41] Beyond one-loop the complexity of the theory greatly increases, whence some approximation based on physical considerations is needed. One such scheme is the two-time approximation. We see that in any quantum-statistical system two different kinds of phenomena are usually involved. On one hand, there are (microscopic) quantum field effects such as the interaction of a bare particle or field quantum with its own cloud of virtual quanta. These interactions cause ultraviolet divergences and require the renormalization of the field parameters. On the other hand, there are (macroscopic) relaxation processes which involve the statistical or kinetic interaction of the renormalized particles and field quanta. The quantum and kinetic processes are in general characterized by two very different time or length scales. In classical kinetic theory they correspond to the interaction range and the mean free path, respectively. We will assume (as in the weak-coupling or dilute-gas approximation) that the statistical-kinetic scale is much larger than the quantum field-theoretical one. This will simplify the problem to the point where the relevant microscopic effects can be accounted for by a suitable renormalization of the field parameters, thus resulting in a quantum theory of statistical interactions among quasiparticles alone. This "macroscopic" theory may be cast in the language of kinetic theory by representing the propagators in terms of the relativistic Wigner function.[19]

We give such a kinetic description in Sec. IV. Under these approximations the nonequilibrium quantum fields become a gas of Bose-Einstein quasiparticles with dressed mass and charges against a background of vacuum fluctuations. The quasiparticles interact with each other through binary collisions. The state of the gas is described by a one-particle Wigner distribution function obeying a relativistic Boltzmann equation. In Secs. IV A and IV B we derive the Boltzmann equation and the gap equation, respectively, from a two-loop perturbation calculation. The mass of the quasiparticles is given implicitly by a gap equation involving the Wigner function. In this approximation the Boltzmann and gap equations form a coupled system which replaces the original wave equations for fields defined at nearby points. They are to be solved consistently for any physical analysis. In Sec. IV C we describe the limitations of the binary collision approximation by performing a three-loop analysis. The two-time approximation is valid away from the critical region, but breaks down as the phase transition is approached, since in this region the microscopic correlation scale diverges. The phase transition occurs when the effective mass of the quasiparticles becomes zero, and proceeds through Bose-Einstein condensation. In Sec. V we return to the discussion of dissipation in interacting quantum fields by demonstrating explicitly how the truncation of the full hierarchy of Green's functions to $O(\lambda^2)$ leads to apparent dissipation.

Finally, as an overall appraisal, a few comments on placing our work in relation to existing work in this and related fields may be appropriate.

(a) Although the closed-time-path functional integral (Schwinger-Keldysh) formalism is quite well known,[26–29] it has seen only limited applications, mainly to nonrelativistic many-body theories in condensed-matter and plasma physics. New in our work is the adaptation of this formalism to deal with the statistical mechanics of a fully relativistic nonlinear quantum field system. (A relativistic free field theory in curved space has been treated with this formalism in Ref. 29.) The $\lambda\phi^4$ model considered here is of interest as it imitates, by its quartic structure, more realistic gauge interactions but without the complications associated with these additional degrees of freedom. It is also suited for investigating critical phenomena.

(b) Wigner function technique has been used before to describe the kinetic properties of quantum fields (e.g., Cooper and Sharp and Carruthers and Zachariasen in Ref. 19 used Wigner's function in $\lambda\phi^4$ theory to derive the relativistic transport equations). New in this work is the inclusion of radiative corrections of quantum fields in the derivation of a gap equation for the effective mass, which, together with the Boltzmann equation, form the basis of a self-consistent statistical quantum-field-theoretical description. It also entails a thorough consideration of the related renormalization problem.

(c) The nonlocal source expansion[31] enables one to incorporate the effect of fluctuation fields consistently and its relativistic field-theoretical formulation (the two-particle irreducible formalism) ensures that the mean field and the correlation functions are treated on the same footing. This field-theoretical technique has been applied to the study of QCD or plasma processes and recently to cosmological phase transitions.[14] It has not been considered fully and formally in a statistical mechanics context.

(d) The nature of dissipation is well understood in classical and quantum-mechanical systems,[38] but a relativistic field-theoretical formulation from first principles, involving the BBGKY hierarchy of propagators and including higher-order radiative effects, has, to our knowledge, not been explored before. (Dissipation from particle creation–back-reaction effects in a cosmological context was discussed in the closed-time-path effective action formalism in Ref. 29.)

In our view the synthesis of the closed-time-path functional formalism (for treating irreversible systems), the nonlocal source theory or two-particle-irreducible representation (for incorporating quantum fluctuations) and the Wigner function techniques (for a kinetic description of nonuniform systems) are the essential elements towards a fundamental, unified and consistent description of the statistical properties of nonequilibrium quantum fields. The strength of each of these elements are demonstrated in the sections that follow, each dealing with a different aspect of the overall problem.

II. FUNCTIONAL FORMALISM

A. The generating functional

We consider a $\lambda\phi^4$ theory with classical action

$$S[\phi]=\int d^4x \left[\tfrac{1}{2}(\partial\phi)^2-\tfrac{1}{2}m^2\phi^2-\frac{1}{4!}\lambda\phi^4\right] . \tag{2.1}$$

(We use metric signature $+---$.) In the absence of external sources, the Heisenberg field $\Phi_H(x)$ obeys

$$\frac{\partial}{\partial t}\Phi_H(x)=i[H,\Phi(x)] , \tag{2.2}$$

while the Heisenberg states are time independent. If an external scalar source $J(x)$ is turned on, then the Heisenberg states acquire a time dependence: the state $|\alpha,0\rangle$ at $t=0$ evolves into

$$|\alpha,t\rangle=\left[T\exp\left[-i\int_0^t dt'd^3x'J(x')\Phi_H(x')\right]\right]|\alpha,0\rangle \tag{2.3}$$

at time t, where T stands for temporal order. The interaction-picture field $\Phi_I(x)$ is defined by

$$\Phi_I(x)=\left[T\exp\left[-i\int_0^t dt'H_I(t')\right]\right]\Phi_H(x) \times\left[\tilde{T}\exp\left[i\int_0^t dt'H_I(t')\right]\right] , \tag{2.4}$$

where

$$H_I(t)=\int d^3x\frac{\lambda}{4!}\Phi_I^4(t,x) \tag{2.5}$$

and $\tilde{T}$ means antitemporal order.

The interaction and Heisenberg pictures coincide at $t=0$. The interaction picture field Φ_I obeys free field equations and can therefore be expanded in terms of creation and destruction operators. These operators define a Fock basis of free-particle states, the in basis. Formally,

$$\Phi_I(x)=\int\frac{d^4p}{(2\pi)^3}\theta(p^0)\delta(p^2-m^2)(e^{-ipx}a_p+e^{ipx}a_p^\dagger) , \tag{2.6}$$

$$a_p|0\rangle=0,\quad \forall p , \tag{2.7}$$

$$|n_{p_1}n_{p_2}\cdots n_{p_n}\rangle = \frac{(a^\dagger_{p_1})^{n_{p_1}}}{(n_{p_1}!)^{1/2}}\frac{(a^\dagger_{p_2})^{n_{p_2}}}{(n_{p_2}!)^{1/2}}\cdots\frac{(a^\dagger_{p_n})^{n_{p_n}}}{(n_{p_n}!)^{1/2}}|0\rangle \ . \tag{2.8}$$

The state of the field is defined by a statistical operator or density matrix ρ. In the Heisenberg representation, without external sources, ρ is time independent. The Heisenberg density matrix ρ agrees with the interaction-picture matrix $\rho_I(t)$ at $t=0$.

The in-in generating functional is[26-29]

$$Z[J_+,J_-,\rho]=\mathrm{Tr}\left\{\left[\tilde{T}\exp\left[-i\int d^4x\, J_-(x)\Phi_H(x)\right]\right]\left[T\exp\left[i\int d^4x\, J_+(x)\Phi_H(x)\right]\right]\rho(0)\right\} , \tag{2.9}$$

where we assume that the external sources vanish for $t<0$. By taking derivatives of Z with respect to the external sources J_+ and J_- we generate expectation values of various products of Heisenberg fields. The connected generating functional is

$$W[J_+,J_-,\rho]=-i\ln Z[J_+,J_-,\rho] \ . \tag{2.10}$$

Z admits a path-integral representation. To derive it, consider a basis of common eigenvectors of the Heisenberg fields at time $t=0$,

$$\Phi_H(\mathbf{x},0)|\varphi,0\rangle=\Phi_I(\mathbf{x},0)|\varphi,0\rangle=\varphi(\mathbf{x})|\varphi,0\rangle \ , \tag{2.11}$$

and at time $t=+\infty$,

$$\Phi_H(\mathbf{x},+\infty)|\psi,+\infty\rangle=\psi(\mathbf{x})|\psi,+\infty\rangle \ . \tag{2.12}$$

We choose a closed time path from $t=0$ to $t=+\infty$ and back, giving

$$Z[J_+,J_-,\rho]=\int D\varphi\, D\varphi' D\psi\left\langle\varphi,0\left|\tilde{T}\exp\left[-i\int d^4x\, J_-\Phi_H\right]\right|\psi,+\infty\right\rangle$$
$$\times\left\langle\psi,+\infty\left|\exp\left[i\int d^4x\, J_+\Phi_H\right]\right|\varphi',0\right\rangle\langle\varphi',0|\rho|\varphi,0\rangle \ . \tag{2.13}$$

The square brackets involving the external sources are just the ordinary transition amplitudes for the Heisenberg states in the presence of J_+ and J_-. So we can write[27-29]

$$Z[J_+,J_-,\rho]=\int D\phi^+D\phi^-\exp\{i[(S[\phi^+]+J_+\phi^+)-(S^*[\phi^-]+J_-\phi^-)]\}\langle\phi^+,0|\rho|\phi^-,0\rangle \ , \tag{2.14}$$

where we are using the shorthand $J_+\phi^+$ for $\int d^4x\, J_+(x)\phi^+(x)$ and $|\phi^\pm,0\rangle$ is the quantum state corresponding to the field configuration $\phi^\pm(\mathbf{x},0)$.

It is convenient to replace the indices + and − by $a=1,2$ and to introduce a "metric tensor"

$$c_{ab}=c^{ab}=\mathrm{diag}(1,-1) \tag{2.15}$$

and a matrix h_{abcd} which is equal to 1 if $a=b=c=d=1$, to -1 if $a=b=c=d=-1$, and zero otherwise. With these conventions we may write

$$S[\phi^+]-S^*[\phi^-]\equiv S[\phi^a]=\int d^4x\left[\tfrac{1}{2}c_{ab}\partial\phi^a\partial\phi^b-\tfrac{1}{2}m^2c_{ab}\phi^a\phi^b-\frac{\lambda}{4!}h_{abcd}\phi^a\phi^b\phi^c\phi^d\right] \tag{2.16}$$

and

$$Z[J_+,J_-,\rho]=\int D\phi^a e^{i(S[\phi^a]+J_a\phi^a)}\langle\phi^1,0|\rho|\phi^2,0\rangle \ . \tag{2.17}$$

The kernel $\langle\phi^1,0|\rho|\phi^2,0\rangle$ is some functional of the field configurations $\phi^1(\mathbf{x},0)$ and $\phi^2(\mathbf{x},0)$. We can write

$$\langle\phi^1,0|\rho|\phi^2,0\rangle=\exp(iK[\phi^a]) \tag{2.18}$$

and expand $K[\phi^a]$ functionally as

$$K[\phi^a]=K+\int d^4x\, K_a(x)\phi^a(x)+\tfrac{1}{2}\int d^4x\, d^4x' K_{ab}(x,x')\phi^a(x)\phi^b(x')+\cdots \ , \tag{2.19}$$

where the indices are contracted with the metric tensor c_{ab}. We have written the integrals as in four dimensions, although in practice the K kernels will be concentrated at $t=0$ in all their entries.

Clearly the sequence of kernels K, K_a, K_{ab}, etc., contains as much information as the original density matrix, since the $|\phi,0\rangle$ states form a basis. So we can rewrite Z as a functional of an infinite number of nonlocal sources[31]

$$Z[J_+,J_-,\rho]=Z[J_a,K_{ab},K_{abc},\ldots]$$
$$=\int D\phi^a\exp[i(S[\phi^a]+J_a\phi^a+\tfrac{1}{2}K_{ab}\phi^a\phi^b+\tfrac{1}{6}K_{abc}\phi^a\phi^b\phi^c+\cdots)] \ , \tag{2.20}$$

where we have absorbed K in the normalization and K_a into J_a. This form of the generating functional will be the starting point of our study.

B. Finite-temperature generating functional

Before proceeding with the formal development of the theory, we want to compute the generating functional Eq. (2.20) for a few familiar choices of the initial state ρ.

In the first example, we choose ρ to be a thermal distribution of in states. That is,

$$\rho = C\exp(-\beta H_0) , \tag{2.21}$$

where

$$H_0 = \int d^3x\,[\tfrac{1}{2}(\partial_t\Phi_I)^2 + \tfrac{1}{2}(\nabla\Phi_I)^2 + \tfrac{1}{2}m^2\Phi_I^2] . \tag{2.22}$$

The kernel $\langle\phi^1,0|\rho|\phi^2,0\rangle$ has a well-known path-integral representation

$$\langle\phi^1,0|\rho|\phi^2,0\rangle = \int D\phi^3\exp(-S_{E_0}[\phi^3]) , \tag{2.23}$$

where S_{E_0} is the Euclidean free action, and the integral is over all Euclidean field configurations satisfying the boundary conditions

$$\phi^3(\mathbf{x},0) = \phi^2(\mathbf{x},0) , \tag{2.24a}$$

$$\phi^3(\mathbf{x},-i\beta) = \phi^1(\mathbf{x},0) . \tag{2.24b}$$

The superscript a on ϕ^a labels the fields, not the power. As the free Euclidean action is quadratic in ϕ^3, the path integral in (2.23) can be solved exactly. We obtain

$$\langle\phi^1,0|\rho|\phi^2,0\rangle = [\mathrm{Det}(-\Box_E + m^2)^{-1/2}] \times\exp(-S_{E_0}[\tilde{\phi}^3]) , \tag{2.25}$$

where $\Box_E$ is the Euclidean D'Alambertian, and $\tilde{\phi}^3$ is the solution of the Euclidean Klein-Gordon equation with boundary conditions (2.24). Introducing the partial Fourier transform

$$\phi^a(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}\phi^a_{\mathbf{k}}(t) , \tag{2.26}$$

we have

$$\tilde{\phi}^3_{\mathbf{k}}(-it) = (\sinh\beta\omega_k)^{-1}[\phi^1_{\mathbf{k}}(0)\sinh\omega_k t + \phi^2_{\mathbf{k}}(0)\sinh\omega_k(\beta - t)] , \tag{2.27}$$

where $\omega_k = (\mathbf{k}^2 + m^2)^{1/2}$. Therefore,

$$S_{E_0}[\tilde{\phi}^3] = \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{\sinh\beta\omega_k}[(|\phi^1_{\mathbf{k}}|^2 + |\phi^2_{\mathbf{k}}|^2)\cosh\beta\omega_k - (\phi^1_{\mathbf{k}}\phi^{2*}_{\mathbf{k}} + \phi^{1*}_{\mathbf{k}}\phi^2_{\mathbf{k}})] . \tag{2.28}$$

Using the inverse transform of Eq. (2.26) we could easily find the kernels K, K_a, K_{ab}, etc. For example, we have

$$K = -\tfrac{1}{2}\mathrm{Tr}\ln(-\Box_E + m^2), \quad K_a = 0 . \tag{2.29}$$

Observe that the Taylor series terminates with the quadratic term.

Our second example is a nonequilibrium generalization of Eq. (2.21):

$$\rho = C\exp\left[-\int d^3k\,\beta_k a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\right] , \tag{2.30}$$

where the $a_{\mathbf{k}}$'s are the in-particle destruction operators, and β_k is any non-negative function of $\mathbf{k}$, [if $\beta_k = \beta\omega_k$, we regain Eq. (2.21)]. To compute the kernel $\langle\phi^1,0|\rho|\phi^2,0\rangle$, we simply observe that mode by mode, Eq. (2.30) is a thermal state (the system as a whole is out of equilibrium because different modes may have different temperatures). So we can split the in field into its spatial modes and carry on the same analysis which led us to Eq. (2.28). In this case we get

$$\langle\phi^1,0|\rho|\phi^2,0\rangle = \exp\left\{-\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\sum_{n=1}^{\infty} V\ln\left[\left[\frac{2\pi n\omega_n}{\beta_k}\right]^2 + \omega_k^2\right] - \tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\omega_k}{\sinh\beta_k}[(|\phi^1_{\mathbf{k}}|^2 + |\phi^2_{\mathbf{k}}|^2)\cosh\beta_k - (\phi^1_{\mathbf{k}}\phi^{2*}_{\mathbf{k}} + \phi^{1*}_{\mathbf{k}}\phi^2_{\mathbf{k}})]\right\} , \tag{2.31}$$

where we have written explicitly the constant term. As in the previous case, the expansion of $\ln\langle\phi^1,0|\rho|\phi^2,0\rangle$ terminates with the second order.

As a matter of fact, Eq. (2.30) is the most general initial state which is diagonal in the in Fock basis and for which the kernels K_{abc}, K_{abcd}, etc., are all zero. To show this it is sufficient to prove that if these higher-order kernels are zero, then the expectation value of products of in fields may be computed using Wick's theorem, since it is known that this property and its being diagonal imply that the statistical operator has the form of Eq. (2.30).[43] To show in turn the validity of Wick's theorem, observe that, for any n-interaction fields $\Phi_I(x_1)\cdots\Phi(x_n)$,

$$\begin{aligned}\langle T(\Phi_I(x_1)\cdots\Phi_I(x_n))\rangle &= \int D\phi^2 D\phi^1\langle\phi^2,0|T(\Phi_I(x_1)\cdots\Phi_I(x_n))|\phi^1,0\rangle\langle\phi^1,0|\rho|\phi^2,0\rangle \\ &= e^{iK}\int D\phi^2 D\phi^1(\phi^1(x_1)\cdots\phi^1(x_n))\exp[i(S_f[\phi^a] + J_a\phi^a + \tfrac{1}{2}K_{ab}\phi^a\phi^b)] .\end{aligned} \tag{2.32}$$

As the integral is the nth moment of a Gaussian measure, it can be computed using Wick's theorem.

Finally, let us consider as an initial state a thermal state with respect to the complete Hamiltonian:

$$\rho = C\exp(-\beta H) , \tag{2.33}$$

$$H = \int d^3x \left[\tfrac{1}{2}(\partial_t \Phi_H)^2 + \tfrac{1}{2}(\nabla \Phi_H)^2 + \tfrac{1}{2}m^2\Phi_H^2 + \frac{\lambda}{4!}\Phi_H^4 \right] . \tag{2.34}$$

The kernel $\langle \phi^1, 0 | \rho | \phi^2, 0 \rangle$ has a path-integral representation as in Eq. (2.23),[7] only now the interaction term should be included in the Euclidean action. If this representation is used in the expressions for the generating functional Eqs. (2.14) or (2.17), we find a path-integral representation for Z in which the time integral is along a three-branched path in the complex plane (branch I goes from 0 to $+\infty$, branch II from $+\infty$ to 0, and branch III—the new one—from 0 to $-i\beta$). The integral is over all continuous field configurations defined on the path, such that the field is periodic between $t=0$ on branch I and $t=-i\beta$ on branch III. It can be proven that the choice of time path is actually inmaterial as long as the path starts at $t=0$, ends at $t=-i\beta$, and the imaginary part of t along the path is never increasing.[35] The particular choice of path depends therefore on the problem at hand. For example, if the time development of the system is not of interest, the simplest choice is to go straight from 0 to $(-i\beta)$ along the imaginary axis; this gives the conventional imaginary-time (Matsubara) finite-temperature quantum field theory.[7] We prefer a "closed time path" over other choices of time paths (e.g., thermofield dynamics, etc.)[9] because it is the only formalism which does not put the finite-temperature condition in a special role and can encompass nonthermal states on general grounds.

For the initial state (2.33), $\langle \phi^1, 0 | \rho | \phi^2, 0 \rangle$ cannot be computed in closed form. However, it is easy to develop a perturbative expansion, either in powers of λ or $\hbar$, along the usual lines in quantum field theory.

C. The generalized effective action and dissipation

We return now to the generating functional Z and its logarithm W in Eq. (2.10). The nonlocal sources K_{ab}, K_{abc}, etc., do not in general have any direct physical meaning. One can work instead with the effective action functional by performing a multiple Legendre transform on W.

Let us first define the mean field $\hat{\phi}^a$,

$$\hat{\phi}^a(x) = \partial W / \partial J_a(x) , \tag{2.35}$$

and the nonlocal kernels $G^{ab}(x,x')$, $G^{abc}(x,x',x'')$ by

$$\frac{\partial W}{\partial K_{ab}(x,x')} = \tfrac{1}{2}[G^{ab}(x,x') + \hat{\phi}^a(x)\hat{\phi}^b(x')] ,$$
$$\frac{\partial W}{\partial K_{abc}} = \tfrac{1}{6}(G^{abc} + 3G^{(ab}\hat{\phi}^{c)} + \hat{\phi}^a\hat{\phi}^b\hat{\phi}^c), \quad \text{etc} . \tag{2.36}$$

From Eq. (2.20) it is clear that $\partial W / \partial K$ are the expectation values of products of Heisenberg fields with respect to the state described by ρ in the presence of the sources J_a. Under these conditions, $\hat{\phi}^a$'s are the mean fields, and the G's are the connected or irreducible expectation values of the same products of Heisenberg fields. The physical situation corresponds to having a single external source $J = J_1 = J_2$. In this case, $\phi^1 = \phi^2 = \phi$ is the physical mean field.[26–29] The generalized effective action is the multiple Legendre transform of W (Ref. 31):

$$\Gamma[\hat{\phi}^a, G^{ab}, G^{abc}, G^{abcd}, \ldots] = W[J_a, K_{ab}, K_{abc}, \ldots] - J_a\hat{\phi}^a - \tfrac{1}{2}K_{ab}(G^{ab} + \hat{\phi}^a\hat{\phi}^b) - \tfrac{1}{6}K_{abc}(G^{abc} + 3G^{ab}\hat{\phi}^c + \hat{\phi}^a\hat{\phi}^b\hat{\phi}^c) - \cdots . \tag{2.37}$$

The (ϕ, G) and (J, K) series are related by Eqs. (2.35) and (2.36) and their inverses:

$$\frac{\partial \Gamma}{\partial \hat{\phi}^a} = -J_a - K_{ab}\hat{\phi}^b - \tfrac{1}{2}K_{abc}(G^{bc} + \hat{\phi}^b\hat{\phi}^c) - \cdots ,$$
$$\frac{\partial \Gamma}{\partial G^{ab}} = -\tfrac{1}{2}K_{ab} - \tfrac{1}{2}K_{abc}\hat{\phi}^c - \cdots , \tag{2.38}$$
$$\frac{\partial \Gamma}{\partial G^{abc}} = -\tfrac{1}{6}K_{abc} - \cdots , \quad \text{etc} .$$

Observe that because the K kernels are concentrated at $t=0$ in all their variables, in general it will be possible to absorb their contributions into the boundary conditions for Eq. (2.38), so that in practice only the first (mean-field) equation is truly inhomogeneous.

Equations (2.38) are the quantum-field-theoretical analog of the BBGKY hierarchy. To proceed towards a workable model we must truncate the hierarchy. This truncation leads to the appearance of irreversible behavior in the part of the system one is interested in. Let us dwell on this aspect somewhat.

To reduce the infinite system (2.38) to a finite one, we will assume that all the K kernels from K_{abc} on vanish. In fact, we will make the stronger assumption that the initial state has the form (2.30). In this particular case we have

$$\Gamma[\hat{\phi}^a, G^{ab}, G^{abc}, G^{abcd}, \ldots] = W[J_a, K_{ab}, 0, 0, \ldots] - J_a\hat{\phi}^a - \tfrac{1}{2}K_{ab}(G^{ab} + \hat{\phi}^a\hat{\phi}^b) , \tag{2.39}$$

so that we are only performing the Legendre transform of W with respect to J and K_{ab}. Equations (2.38) now read

$$\frac{\partial\Gamma}{\partial\hat{\phi}^a}=-J_a-K_{ab}\hat{\phi}^b,\quad \frac{\partial\Gamma}{\partial G^{ab}}=-\tfrac{1}{2}K_{ab}\ ,$$
$$\frac{\partial\Gamma}{\partial G^{abc}}=0,\quad \text{etc}\ . \tag{2.40}$$

Observe that we still have an infinite system, although only the first two equations contain information about the initial state, and that the evolution remains unitary. Therefore the initial entropy in the states (2.30) will not increase in the evolution.[44]

In order to obtain a finite system, we must eliminate altogether the kernels G^{abc}, G^{abcd}, etc., as dynamical variables. That is, we must introduce a new functional

$$\Gamma[\hat{\phi}^a,G^{ab}]=\Gamma[\hat{\phi}^a,G^{ab},\tilde{G}^{abc},\tilde{G}^{abcd},\dots]\ , \tag{2.41}$$

where the $\tilde{G}$'s are functionals of ϕ^a and G^{ab} given by

$$\frac{\partial\Gamma}{\partial G^{abc}}[\hat{\phi}^a,G^{ab},\tilde{G}^{abc},\dots]=0\ ,$$
$$\frac{\partial\Gamma}{\partial G^{abcd}}[\hat{\phi}^a,G^{ab},\tilde{G}^{abc},\dots]=0,\quad \text{etc}\ . \tag{2.42}$$

Here $\Gamma[\hat{\phi}^a,G^{ab}]$ is just the double Legendre transform of $W[J_1,K_{ab},0,0,\dots]$. For any state of the form (2.30), Eqs. (2.42) are easy to solve. We have already seen that the G's are the connected expectation values of products of Heisenberg fields. These can be rewritten as perturbative series of products of interaction-picture fields. But as we saw earlier, for a state of the form (2.30), expectation values of interaction fields can be computed using Wick's theorem. So the solution of Eqs. (2.42) boils down to the construction of the connected expectation values from $\hat{\phi}^a$ and G^{ab} using ordinary perturbation theory.

In order to understand how the truncation and averaging [from Eqs. (2.39) and (2.40) to Eqs. (2.41) and (2.42)] introduce dissipation, it is perhaps easier to first describe a toy model of a typical dissipative system.[37,38] Consider first the simplest case of two coupled harmonic oscillators a and b with natural frequencies Ω_a and Ω_b, and coupling parameter λ. Oscillator a is also driven by an external source $J(t)$. Their equations of motion are

$$\ddot{x}_a+\Omega_a^2x_a-\lambda x_b=J(t)\ ,$$
$$\ddot{x}_b+\Omega_b^2x_b-\lambda x_a=0\ . \tag{2.43}$$

Expressed in frequency domain, these become

$$(\omega^2-\Omega_a^2)x_a+\lambda x_a=-J(\omega)\ , \tag{2.44a}$$

$$(\omega^2-\Omega_b^2)x_b+\lambda x_a=0\ . \tag{2.44b}$$

The system of Eqs. (2.44) is, of course, time-reversal invariant, and shows no dissipation. However, suppose one tries to eliminate one variable, say b, by explicitly solving Eq. (2.44b). To get a solution one needs to impose certain boundary conditions. The natural choice corresponding to physical situations is the causal one

$$x_b(\omega)=-[(\omega-i\epsilon)^2-\Omega_b^2]^{-1}\lambda x_a(\omega)\ . \tag{2.45}$$

This breaks the time-reversal symmetry. When Eq. (2.45) is used in Eq. (2.44) we get the relation

$$\{\omega^2-\Omega_a^2-\lambda^2P[\omega^2-\Omega_b^2]^{-1}-\lambda^2\pi i\,\mathrm{sgn}(\omega)\delta(\omega^2-\Omega_b^2)\}x_a=-J(\omega)\ . \tag{2.46}$$

The appearance of an imaginary part here usually signifies dissipative effects. If b is considered to be an array of oscillators, it then constitutes a "bath" to the "system" a of interest.[37,38] Separating the system from the bath and making specific choices of boundary conditions for the bath is what leads to the appearance of dissipation as observed in the system.

Suppose we tune the external source $J(\omega)$ to the resonance frequency Ω_b. Then the "system" remains unaffected: all the energy is transferred to the bath. This physical process is of course the same whether one is using the model (2.44) or (2.46). If one views the situation through (2.44), where a and b are considered on equal footing, one knows that there is a second spring which is storing the energy of the source. However, if one considers the same situation from the point of view of the "system" via (2.46) which is only a part of the complete picture, one would say that the energy is being dissipated away. The answer to the question of whether or not one knows about the existence of the second oscillator is not subjective in nature. It is discernible in principle by an analysis of the experimental apparatus being used to monitor the whole arrangement.

The analysis of the toy model can be carried over unaltered to the quantum field system discussed above. Formally, the time-reversal invariance is broken because Eqs. (2.42) carry with them specified boundary conditions. For example, when all coordinates fall on the surface $t=0$, the G kernels must reduce to the connected expectation values of in fields, which are all zero except for the two-point functions, according to Wick's theorem. Physically, one sees dissipative effects appearing in the truncated system because one does not know that information about the initial state and external sources is being shared by higher correlation functions. We will continue this discussion in Sec. V.

III. PERTURBATION THEORY

A. Perturbative expansion of the effective action

In Sec. II we set up a functional description of nonequilibrium quantum fields. The state at time $t=0$ is given by a density matrix of the form (2.30), which admits a representation (2.18) in terms of a single nonlocal source K_{ab}. As a consequence, the generating functional Z in Eq. (2.20) depends only on local sources J_a and nonlocal ones K_{ab}. The dynamics of the background field ϕ^a and the propagators G^{ab} is generated by an effective action which is the double Legendre transform of $W=-i\ln Z$. In this section we will try to substantiate this framework by studying the results of a perturbative expansion of the effective action (2.37) in loop orders.

One obvious task in perturbation theory is the treatment of infinites which appear at each order. To absorb the pole terms arising in the perturbative calculations we must regularize the (bare) parameters $\tilde{m}_B^2, \tilde{\lambda}_B$ and the field $\hat{\phi}_B$, which are infinite. We shall use dimensional regularization here. With the wave-functional renormalization

$$\hat{\phi}_B = Z_B^{1/2}\hat{\phi} \tag{3.1}$$

we can define the new bare parameters m_B^2, λ_B as

$$\tilde{m}_B^2 = Z_B^{-1} m_B^2, \quad \tilde{\lambda}_B = Z_B^{-2}\lambda_B \ . \tag{3.2}$$

In terms of these parameters, the classical action in Eq. (2.1) with $\tilde{m}_B$, $\tilde{\lambda}_B$, and $\hat{\phi}_B$ becomes

$$S[\hat{\phi}^a] = \int d^4x \left\{ \tfrac{1}{2} Z_B c_{ab} \partial\hat{\phi}^a \partial\hat{\phi}^b - \tfrac{1}{2} m_B^2 c_{ab} \hat{\phi}^a \hat{\phi}^b - \frac{\lambda_B}{4!} h_{abcd} \hat{\phi}^a \hat{\phi}^b \hat{\phi}^c \hat{\phi}^d \right\} . \tag{3.3}$$

The calculation of the double Legendre transform of W has been extensively discussed in the literature. The result is[31]

$$\Gamma[\hat{\phi}^a, G^{ab}] = S[\hat{\phi}^a] + \frac{i}{2}\ln \mathrm{Det}(G^{ab})^{-1} + \frac{1}{2}\frac{\partial^2 S}{\partial\hat{\phi}^a \partial\hat{\phi}^b} G^{ab} + \Gamma_2[\hat{\phi}^a, G^{ab}] + \mathrm{const} \ , \tag{3.4}$$

where Γ_2 is the sum of the two-particle-irreducible vacuum graphs of a theory with propagators G^{ab}, a cubic interaction with strength $\lambda_B h_{abcd}\hat{\phi}^d$, and a quartic interaction with strength $\lambda_B h_{abcd}$. Each graph carries a weight computed as in ordinary perturbation theory.

The relationship

$$2V_4 + V_3 = 2(l-1) \tag{3.5}$$

between the number of quartic (V_4) and cubic (V_3) vertices and loops (l) in a given graph shows that $\hat{\phi}$ appears only in even powers in Γ_2, and therefore Γ is even in $\hat{\phi}$. It follows that, if the initial conditions are homogeneous, and there are no external sources, $\hat{\phi}$ will remain zero at all times. Setting $\hat{\phi}=0$ and keeping terms up to $O(\lambda_B^2)$ only, we find

$$\Gamma[G^{ab}] = \frac{i}{2}\ln \mathrm{Det}(G^{-1}) - \frac{1}{2}\int d^4x \, c_{ab}(Z_B\Box + m_B^2) G^{ab}(x,x) - \tfrac{1}{8}\lambda_B h_{abcd} \int d^4x \, G^{ab}(x,x) G^{cd}(x,x)$$
$$+ \frac{i}{48}\lambda_B^2 h_{abcd} h_{efgh} \int d^4x \, d^4x' G^{ae}(x,x') G^{bf}(x,x') G^{cg}(x,x') G^{dh}(x,x') \ . \tag{3.6}$$

The equation of motion for G^{ab} is

$$i(G^{-1})_{ab} + c_{ab}(Z_B\Box + m_B^2)\delta(x-x') + \tfrac{1}{2}\lambda_B h_{abcd} G^{cd}(x,x)\delta(x-x')$$
$$- \frac{i}{6}\lambda_B^2 h_{aecd} h_{bfgh} G^{ef}(x,x') G^{cg}(x,x') G^{dh}(x,x') = -\tfrac{1}{2}K_{ab} \ . \tag{3.7}$$

It takes a more familiar form upon multiplication on the right and the left by G. Absorbing the source term into the boundary conditions we find

$$[Z_B\Box_x + m_B^2 + \tfrac{1}{2}\lambda_B G^{aa}(x,x')] G^{ab}(x,x') - \frac{i}{6}\lambda_B^2 \int d^4x'' c_{de} \Sigma^{ad}(x,x'') G^{eb}(x'',x') = -ic^{ab}\delta(x-x') \tag{3.8}$$

and

$$[Z_B\Box_{x'} + m_B^2 + \tfrac{1}{2}\lambda_B G^{bb}(x',x')] G^{ab}(x,x') - \frac{i}{6}\lambda_B^2 \int d^4x'' c_{de} G^{ad}(x,x'') \Sigma^{eb}(x'',x') = -ic^{ab}\delta(x-x') \tag{3.9}$$

(no sum on a or b), where we have introduced the kernel $\Sigma^{ab}(x,x') = G^{ab}(x,x')^3$.

B. Free and one-loop propagators

In order to gain more insight into the content of Eqs. (3.8) and (3.9) we will consider their solution in the free field case, and when only $O(\lambda)$ terms are retained. For a free field, the propagators are solutions of the Klein-Gordon equation

$$(\Box_x + m^2) G^{11} = -(\Box_x + m^2) G^{22}$$
$$= -i\delta(x-x') \ , \tag{3.10}$$

$$(\Box_x + m^2) G^{12} = (\Box_x + m^2) G^{21} = 0 \ . \tag{3.11}$$

To solve for G^{ab} in Eqs. (3.10) and (3.11) we must take into account the conditions imposed by the initial state [which we take to have the form Eq. (2.30)]. In particular, because ρ is diagonal, the propagators are translation invariant and we will have no need for the adjoint equations to Eqs. (3.10) and (3.11). A second set of restrictions comes from the structure of G^{ab} as the expectation value of the ordered products of Heisenberg fields. Specifically, we have the Feynman, negative-frequency Wightman, positive-frequency Wightman, and Dyson propagators given, respectively, by

$$\begin{aligned} G^{11}(x,x') &= \langle \phi^1(x)\phi^1(x')\rangle \\ &= \langle T(\Phi_H(x)\Phi_H(x'))\rangle\ , \\ G^{12}(x,x') &= \langle \phi^1(x)\phi^2(x')\rangle \\ &= \langle \Phi_H(x')\Phi_H(x)\rangle\ , \\ G^{21}(x,x') &= \langle \phi^2(x)\phi^1(x')\rangle \\ &= \langle \Phi_H(x)\Phi_H(x')\rangle\ , \\ G^{22}(x,x') &= \langle \phi^2(x)\phi^2(x')\rangle \\ &= \langle \tilde{T}(\Phi_H(x)\Phi_H(x'))\rangle\ . \end{aligned} \tag{3.12}$$

When translation invariance and Eqs. (3.12) are taken into account, we find the most general solution of Eqs. (3.11) to be

$$G^{ab}(x,x') = i\int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} G^{ab}(k) \tag{3.13}$$

with

$$G^{11}(k) = (k^2-m^2+i\varepsilon)^{-1} - 2\pi i f(k)\delta(k^2-m^2)\ , \tag{3.14a}$$

$$\begin{aligned} G^{12}(k) &= G^{21}(-k) \\ &= (-2\pi i)[\theta(k^0)+f(k)]\delta(k^2-m^2)\ , \end{aligned} \tag{3.14b}$$

$$\begin{aligned} G^{22}(k) &= (-1)(k^2-m^2-i\varepsilon)^{-1} \\ &\quad -2\pi i f(k)\delta(k^2-m^2)\ . \end{aligned} \tag{3.14c}$$

The function $f(k)$ must be the same for all four propagators. This follows from the fact that, for free fields, the commutator $[\Phi_H(x),\Phi_H(x')]$ is a c number. Therefore,

$$\begin{aligned} G(x,x') &= i\langle [\Phi_H(x),\Phi_H(x')]\rangle \\ &= i(G^{21}-G^{12})(x,x') \end{aligned} \tag{3.15}$$

and the advanced and retarded propagators

$$\begin{aligned} G_{\text{adv}}(x,x') &= -\theta(t'-t)G(x,x') \\ &= -i(G^{22}-G^{12})\ , \end{aligned} \tag{3.16}$$

$$\begin{aligned} G_{\text{ret}}(x,x') &= \theta(t-t')G(x,x') \\ &= i(G^{11}-G^{12}) \end{aligned} \tag{3.17}$$

must be independent of the state. By considering

$$\begin{aligned} G_1(x,x') &= \langle \{\Phi_H(x),\Phi_H(x')\}\rangle \\ &= G^{12}+G^{21}\ , \end{aligned} \tag{3.18}$$

we see that $f(k)$ must be real and even. Finally, from the inequality valid for any function h,

$$\int d^4x\, d^4x' h^*(x') G^{12}(x,x') h(x) \geq 0\ , \tag{3.19}$$

we conclude that $f(k)$ must be positive. Actually, for a state of the form (2.30) it is simply the occupation number of mode k:

$$f(k) = (e^{\beta_k}-1)^{-1}\ . \tag{3.20}$$

In this sense we may identify f with a distribution function.

The propagators (3.14) are solutions of the $O(\lambda)$ equations as well. Now for the problem of renormalization. Using translation invariance and that $G^{ab}(x,x') = G^{11}(x,x)$ for any combination of (a,b) indices, the equations read

$$[\Box_x + m_B^2 + \tfrac{1}{2}\lambda G^{11}(0)] \begin{bmatrix} G^{11} \\ G^{22} \\ G^{12} \\ G^{21} \end{bmatrix} = \begin{bmatrix} -i \\ +i \\ 0 \\ 0 \end{bmatrix} \delta(x-x')\ , \tag{3.21}$$

where we have used the fact that, to this order, $\lambda_B = \lambda$.is finite and $Z_B = 1$. From Eq. (3.14), after computing the integral in $d = 4+\epsilon$ dimensions, we find

$$G^{11}(0) = (4\pi)^{-2}\left\{\frac{2m^2}{\epsilon} + m^2\left[\ln\left[\frac{m^2}{4\pi\mu^2}\right] - \psi(2)\right]\right\} + M_f^2\ , \tag{3.22}$$

where μ is the renormalization scale parameter (with units of mass) $\psi(z) = \partial \ln\Gamma(z)/\partial z$, and

$$M_f^2 = \int \frac{d^4p}{(2\pi)^4}\delta(p^2-m^2)[2\pi f(p)]\ . \tag{3.23}$$

To remove the infinities in m_B^2 in (3.21) we use a modified minimal subtraction,[45]

$$m_B^2 = m_r^2 - \frac{\lambda}{(4\pi)^2} m_r^2[\epsilon^{-1} - \tfrac{1}{2}\psi(2)]\ , \tag{3.24}$$

where m_r^2 is a finite, μ-dependent, renormalized mass satisfying $m^2 = m_r^2 + O(\lambda)$. The exact relationship between m^2 and m_r^2 comes from (3.14) being a solution of (3.21), and reads

$$m^2\left[1 - \frac{\lambda}{2(4\pi)^2}\ln\frac{m^2}{4\pi\mu^2}\right] = m_r^2 + \frac{\lambda}{2}M_f^2\ . \tag{3.25}$$

Note the differences among the five parameters with units of mass in this problem: the unobservable, infinite bare mass m_B, the finite but μ-dependent renormalized mass m_r, the renormalization scale fixing parameter μ, the state-dependent mass M_f, and finally the physical, observable mass of the system, which includes the effects of fluctuations and interaction. Also called the effective mass or the inverse susceptibility function, m being the pole of the regularized propagators, fixes the correlation length for fluctuations, and gives the spectrum of quasiparticles.

Although we have derived Eq. (3.25) using one-loop perturbation theory, its validity goes beyond $O(\lambda)$. By summing an infinite set of graphs for N component fields with leading $1/N$ contributions, the so-called daisy diagrams,[40] one obtains the same Eq. (3.25) for the effective mass. The renormalized mass m_r scales according to the renormalization-group equation in the usual manner:

$$\mu\frac{d}{d\mu}m_r^2=\frac{\lambda m_r^2}{(4\pi)^2}+O(\lambda^2)\ . \tag{3.26}$$

If we change μ by μ' in Eq. (3.25) we find

$$m^2\left[1-\frac{\lambda}{2(4\pi)^2}\ln\frac{m^2}{4\pi\mu'^2}\right]=\left[m_r^2(\mu)+\frac{\lambda m_r^2}{2(4\pi)^2}\ln\frac{\mu'^2}{\mu^2}\right]+\frac{\lambda}{2}M_f^2\ , \tag{3.27}$$

which, to $O(\lambda)$, is equivalent to Eq. (3.25).

For $m_r^2+(\lambda/2)M_f^2=0$, Eq. (3.25) has two solutions

$$m^2=0$$

and (3.28)

$$m^2\equiv m_0^2=(4\pi\mu^2)\exp[(4\pi)^2(2/\lambda)]\ .$$

They give rise to two branches of solutions for positive values of $m_r^2+(\lambda/2)M_f^2$ (Ref. 46): one (the lower one) increasing and the other decreasing. The two branches meet at

$$m_r^2+\frac{\lambda}{2}M_f^2=\left[\frac{\lambda m_0^2}{2(4\pi^2)e}\right]\ ,$$

at which value

$$m^2=m_0^2/e\ . \tag{3.29}$$

There are no positive solutions beyond this point. We will argue that only the lower branch is physical. This is in part because the lower branch is the only one accessible to perturbation theory. Since all our results are perturbative, to adopt the upper branch as a physical solution may infringe on consistency. More seriously, for a thermal state $f(k)=(e^{\beta\omega_k}-1)^{-1}$, M_f^2 can be computed to give

$$M_f^2\simeq T^2/12\quad(T^2\gg m^2)\ . \tag{3.30}$$

If we took the upper branch as the physically relevant one, we find that the correlation length m^{-1} increases with increasing temperature, a rather unphysical behavior. The breakdown of the theory at large scales, on the other hand, is an expected pathological trait of the $\lambda\phi^4$ theory, which is not asymptotically free.

We conclude this section with the following observation. By its very definition, the physical mass m^2 must be nonnegative. Otherwise, small perturbations of the field would propagate faster than light. But no such restriction applies to m_r^2. If m_r^2 is negative it only indicates that the system has a critical point, at which the correlation length can become infinite (or equivalently the quasiparticles become massless). The critical point T_c is given by[7,40,41]

$$m_r^2+\frac{\lambda}{24}T_c^2=0\ . \tag{3.31}$$

Because the correlation length is infinite, this is a second-order phase transition. Since in this discussion we have assumed the constant background field averages out to $\hat{\phi}=0$ (symmetric state) at all times, the transition is in the nature of a Bose-Einstein condensation of quasiparticles. Below T_c, we find that the equilibrium distribution becomes

$$f(p)=(2\pi)^3M^2(T)\,|p^0|\,\delta(\mathbf{p})+f'(p)\ , \tag{3.32}$$

where f' is a thermal distribution at temperature T for the massless quasiparticles above the condensate, and

$$M^2(T)=(T_c^2-T^2)/12\ . \tag{3.33}$$

The physical picture will be further clarified in the following sections.

IV. KINETIC THEORY

A. Wigner function and the Boltzmann equation

It does not take a very careful analysis to see that the one-loop approximation of Sec. III B is really not very satisfactory: to this order, the field does not even thermalize. In order to obtain a more realistic picture we need to proceed to higher orders in perturbation theory, beginning with the two-loop equations we derived in Sec. III A. Already at this first nontrivial order the theory becomes quite involved: Equations (3.8) and (3.9) are nonlinear, nonlocal integrodifferential equations, not readily solvable in closed form.

To progress further we need to introduce reasonable approximations based on physical considerations. One such consideration is to recognize the two-time nature of the system. The main phenomena are characterized by two different time (or length) scales: quantum-field-theoretical (microscopic) and statistical-kinetic theoretical (macroscopic). The first measures the range of radiative corrections to the Compton wavelength of particles and the second measures the range of interaction among particles. In the more familiar classical kinetic or reaction theory, similar distinction is made between the scattering length (or reaction time) and the mean free path (or time between collisions).

This separation of two scales is already present in the one-loop calculation. Here the microscopic scale is given by the correlation length m^{-1} while the statistical scale is actually infinite, because the only radiative correction is in the mass of the particles which obey equations as in a free field theory. If launched off-equilibrium, the system can never reach a thermal state. When higher orders in perturbation theory are included, we expect to obtain a finite relaxation time. But still it is a reasonable assumption that for weak-enough coupling the kinetic scale will be much larger than the quantum one (cf. the weak coupling or dilute-gas approximation in classical kinetic theory or the Born or adiabatic approximation in nuclear reaction theory).

This observation will allow us to recast the quantum-field-theoretical problem into the much simpler forms of kinetic theory, using well-known techniques from nonrelativistic many-body theory.[42] Suppose we could separate spacetime into "cells" whose characteristic size is intermediate between the kinetic and quantum scales. As the correlation between different cells will be negligible by design, the only interesting case is when the two arguments of a given propagator lie in the same cell. In the interior of a single cell, relaxation phenomena are negligible, which means that it is sufficient to use the one-loop approximation of Sec. III B. More concretely, the propagators may be Fourier transformed over a cell, and for weak enough coupling, the Fourier transforms will take the one-loop form (3.14). The difference is that now the Fourier transforms also carry a new, cell-dependent label. Relaxation phenomena become apparent as we move from cell to cell.

In mathematical language, if x and x' are the arguments of a propagator G^{ab}, we Fourier transform G^{ab} with respect to $x-x'$ (or rather, we transform a function which coincides with G^{ab} inside a cell, and is zero outside). As a cell-dependent label we choose the midpoint $X=\frac{1}{2}(x+x')$ (Ref. 42). The Fourier transform reads

$$G^{ab}(x,x')=i\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-x')}G^{ab}(X,k)\ , \tag{4.1}$$

where the $G^{ab}(X,k)$ are given by Eq. (3.14) plus $O(\lambda^2)$ terms. m^2 and f therein are now functions of X, but the X dependence of these quantities arises only at the $O(\lambda^2)$ level. In this formulation, $f(X,k)$ becomes a distribution function, giving the spectrum of quasiparticles at the position X. In this sense, f may be considered a relativistic Wigner function.

The basic equations of the theory are still (3.8) and (3.9). Let us introduce the transform of the kernel $\Sigma^{ab}(x,x')$:

$$\Sigma^{ab}(x,x')=(-i)\int\frac{d^4k}{(2\pi)^4}e^{ik(x-x')}\Sigma^{ab}(X,k)\ , \tag{4.2}$$

where

$$\Sigma^{ab}(X,k)=\int\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}[(2\pi)^4\delta(p_1+p_2+p_3-k)]G^{ab}(X,p_1)G^{ab}(X,p_2)G^{ab}(X,p_3)\ . \tag{4.3}$$

The nonlocal term in Eq. (3.8), for example, becomes

$$\frac{-i}{6}\lambda_B^2c_{de}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\int d^4x''e^{ip(x-x'')}\Sigma^{ad}\left[\frac{x+x''}{2},p\right]e^{iq(x''-x')}G^{eb}\left[\frac{x''+x'}{2},q\right]\ . \tag{4.4}$$

The integrand will be appreciably different from zero only when x'' belongs to the same cell as x and x', and, in that case, $(x+x'')/2\sim(x'+x'')/2\sim X$ [the difference will appear to $O(\lambda^2)$, but Eq. (4.4) is itself $O(\lambda^2)$ already]. In this approximation, Eq. (4.4) reduces to

$$\frac{-i}{6}\lambda_B^2c_{de}\int\frac{d^4p}{(2\pi)^4}e^{ip(x-x')}\Sigma^{ad}(X,p)G^{eb}(X,p)\ . \tag{4.5}$$

After a similar approximation in the tadpole term, we find the equations for $G^{ab}(X,k)$ to be

$$\left[Z_B\left[-k^2+ik\frac{\partial}{\partial X}+\tfrac{1}{4}\Box_X\right]+m_B^2+\frac{i\lambda_B}{2}\int\frac{d^4p}{(2\pi)^4}G^{aa}(X,p)\right]G^{ab}(X,k)-\frac{\lambda_B^2}{6}c_{de}\Sigma^{ad}(X,k)G^{eb}(X,k)=-c^{ab} \tag{4.6}$$

and

$$\left[Z_B\left[-k^2+ik\frac{\partial}{\partial X}+\tfrac{1}{4}\Box_X\right]+m_B^2+\frac{i\lambda_B}{2}\int\frac{d^4p}{(2\pi)^4}G^{bb}(X,p)\right]G^{ab}(X,k)-\frac{\lambda_B^2}{6}c_{de}G^{ad}(X,k)\Sigma^{eb}(X,k)=-c^{ab}\ . \tag{4.7}$$

Equivalently, taking the average and the difference of these last two equations[42] we have

$$\left[Z_B(-k^2+\tfrac{1}{4}\Box_X)+m_B^2+\frac{i\lambda_B}{2}\int\frac{d^4p}{(2\pi)^4}G^{aa}(X,p)\right]G^{ab}(X,k)-\frac{\lambda_B^2}{12}c_{de}[\Sigma^{ad}(X,k)G^{eb}(X,k)+\Sigma^{eb}(X,k)G^{ad}(X,k)]=-c^{ab} \tag{4.8}$$

and

$$Z_Bik\frac{\partial}{\partial X}G^{ab}(X,k)-\frac{\lambda_B^2}{12}c_{de}(\Sigma^{ad}G^{eb}-\Sigma^{eb}G^{ad})(X,k)=0\ . \tag{4.9}$$

Equation (4.9) is our passage to a Boltzmann equation for the Wigner function f. The second term describes binary interactions (see later). Observe that both terms in it are $O(\lambda^2)$ since $(\partial G^{ab}/\partial X)(X,k)=0$ to $O(\lambda)$. This allows us to set $Z_B=1$, $\lambda_B=\lambda$, and to use directly the forms Eq. (3.14) for the propagators. For example, if we choose $a=1$, $b=2$, we obtain

$$k\frac{\partial}{\partial X}f=\frac{\lambda^2}{12}(2\pi)^3\int\left[\prod_{i=1}^{3}\frac{d^4p_i}{(2\pi)^4}\delta(p_i^2-m^2)\right][(2\pi)^4\delta(p_1+p_2+p_3-k)]$$
$$\times\{[\theta(-p_1^0)+f(p_1)][\theta(-p_2^0)+f(p_2)][\theta(-p_3^0)+f(p_3)][1+f(k)]$$
$$-[\theta(p_1^0)+f(p_1)][\theta(p_2^0)+f(p_2)][\theta(p_3^0)+f(p_3)]f(k)\}\ , \tag{4.10}$$

where $k^2=m^2$ and we have assumed $k^0>0$ (remember that f is even in k). To satisfy the mass-shell and momentum-conservation constraints, two of the p_1^0,p_2^0,p_3^0 components must be positive, and the third, negative. We may relabel them so that p_3^0 is always the negative component. Changing further p_3 to $-p_3$ we get

$$k\frac{\partial}{\partial X}f=\frac{\lambda^2}{4}(2\pi)^3\int\left[\prod_{i=1}^{3}\frac{d^4p_i}{(2\pi)^4}\theta(p_i^0)\delta(p_i^2-m^2)\right][(2\pi)^4\delta(p_1+p_2-p_3-k)]$$
$$\times\{[1+f(p_3)][1+f(k)]f(p_1)f(p_2)-f(p_3)f(k)[1+f(p_1)][1+f(p_2)]\}\ . \tag{4.11}$$

This is the relativistic Boltzmann equation for a Bose-Einstein gas of quasiparticles.[44] Observe that the collision integral is proportional to λ^2 instead of λ. The microscopic or quantum radiative interactions will be taken into account by a correct renormalization of the wave function, mass, and coupling constant of the quantum field, as we will see in the next section.

Since Eq. (4.11) is the standard Boltzmann equation obeyed by the relativistic Wigner function f, it is obvious that it admits thermal states as solutions.[47] This can be seen from Eq. (4.9) by applying the KMS condition

$$G^{12}(X,k)=e^{-\beta k^0}G^{21}(X,k)\ , \tag{4.12}$$

which holds for thermal propagators. This in turn suggests that the KMS condition may be understood as a detailed balance condition for equilibrium quantum fields. Another well-known property of the kinetic equation (4.11) is that if one defines the entropy flux as

$$S^\mu(X)=\int\frac{d^4k}{(2\pi)^4}\theta(k^0)\delta(k^2-m^2)k^\mu\{[1+f(X,k)]\ln[1+f(X,k)]-f(X,k)\ln f(X,k)\}\ , \tag{4.13}$$

then the relativistic H theorem holds, i.e., $\partial_\mu S^\mu\geq 0$.

From this we learn that by including two-loop terms in the effective-action functional and by introducing a separation (by cells or coarse graining) of the macroscopic and microscopic scales we begin to see relaxation phenomena due to binary interactions as in (4.11). We may use a simple argument to estimate the statistical relaxation time. Suppose the population of mode k were to depart from equilibrium by an amount $\delta f(X,k)$. The subsequent evolution would be

$$k\frac{\partial}{\partial X}\delta f(X,k)\approx\frac{-\lambda^2}{4}(2\pi)^3\int\frac{d^4p_3}{(2\pi)^4}\theta(p_3^0)\delta(p_3^2-m^2)f(X,p_3)$$
$$\times\left[\int\frac{d^4p_1}{(2\pi)^4}\theta(p_1^0)\delta(p_1^2-m^2)\frac{d^4p_2}{(2\pi)^4}\theta(p_2^0)\delta(p_2^2-m^2)(2\pi)^4\delta(p_1+p_2-p_3-k)\right]\delta f(X,k)\ .$$

The p_1 and p_2 integrals are dimensionless functions of (p_3+k) and m. Taking this to be of order 1 we get

$$k\frac{\partial}{\partial X}\delta f(X,k)\approx-\frac{\lambda^2}{8}(2\pi)^2M_f^2\delta f(X,k)\ .$$

From this the relaxation time is given by $\tau\sim\omega_k/\lambda^2M_f^2$. The two-time approximation assumes

$$\omega_k(\lambda^2M_f^2)^{-1}\gg m^{-1}\simeq\left[m_r^2+\frac{\lambda}{2}M_f^2\right]^{-1/2}\ , \tag{4.14}$$

which is satisfied by all modes (with wave number k) if m_r^2 is positive and λ is small enough, or for any temperature if k is large enough. However, it will not hold for long wavelengths close to the critical temperature. This regime needs separate treatment.

In this section we have assumed that f becomes position independent when λ goes to zero. However, the formalism can be easily generalized to the case in which small inhomogeneities persists even to zeroth order (for example, if the quantum field is placed in a weak temperature gradient).[42] Consider the contribution of the tadpole term in Eq. (3.8):

$$\frac{i\lambda}{2}\int\frac{d^4p}{(2\pi)^4}G^{11}(x,p)\sim\frac{i\lambda}{2}\int\frac{d^4p}{(2\pi)^4}G^{11}(X,p)+\frac{(x-x')^\mu}{2}\frac{i\lambda}{2}\frac{\partial}{\partial X^\mu}\int\frac{d^4p}{(2\pi)^4}G^{11}(X,p)\ .$$

After Fourier transforming, the new term reads

$$\frac{i\lambda}{4}\left[\frac{\partial}{\partial X^{\mu}} i\int \frac{d^4p}{(2\pi)^4} G^{11}(X,p)\right]\frac{\partial}{\partial k_{\mu}} G^{ab}(X,k) .$$

This term also appears but with reversed sign in Eq. (3.9), and therefore it enters into the kinetic equation. We observe that this term represents a Vlasov-type correction to the Boltzmann equation.

B. Two-loop renormalization and the gap equation

In the previous section we saw that the nonequilibrium quantum field can be pictured as a relativistic gas of quasiparticles whose distribution function obeys a Boltzmann equation. This contains a quantum-field generalization of existing methods and results from nonrelativistic many-body theory. In this section we will study the quantum-field-theoretical aspects, specifically, problems of the renormalization of the theory and the relationship between the parameters of the quasiparticle theory (m^2 and λ) and the renormalized parameters of the underlying quantum-field theory. Our analysis will culminate with the formulation of a state-dependent gap equation for the mass m^2. In the kinetic approximation, the original wave equations for two particles are replaced by the gap equation and the Boltzmann equation, which are to be solved self-consistently.

Our goal is to obtain, out of the free propagators, Eqs. (3.14), the best possible approximation to the complete Green's functions. The problem is, of course, that real quasiparticles are not free. However, we may distinguish two levels at which the self-interaction operates. On the one hand, there is a Hartree-type self-consistent potential generated by the whole distribution of quasiparticles and vacuum fluctuations, and on the other there are the collisions between individual quasiparticles, as accounted for by the right-hand side of the Boltzmann equation (4.11). By a "free" quasiparticle we mean that binary interactions (e.g., collisions) have been neglected, but not so for quantum interactions giving rise to the self-consistent potential in which the quasiparticles propagate. For example, in the one-loop theory there is no collision term in the Boltzmann equation, but the state-dependent contribution to the effective mass m^2 exists.

We will make the assumption that the propagation of free quasiparticles is determined by the advanced and retarded propagators, Eqs. (3.16) and (3.17). The rationale for this choice is that these propagators depend on the distribution function much more weakly than, for example, the Feynman or Dyson propagators. (For a free theory, they are altogether state independent.) By looking at $G_{\rm ret}$ or $G_{\rm adv}$ rather than G_F or G_D, the effects of collisions are attenuated. We will indeed work with $G_{\rm ret}$.

For a vacuum theory (zero temperature, $T=0$), or for a nonvacuum theory in the $O(\lambda)$ approximation, the propagators are Lorentz invariant. As a function of a complex variable $z=k^2$, they have a single pole on the real axis. This pole determines the asymptotic characteristics of the propagation of small disturbances of the quantum field. Thus the position $z=m^2$ of the pole can be unambiguously identified as the physical mass of the field, and of the quasiparticles with distribution f. The wave-function renormalization Z_B follows from the unit residue of the propagators at the pole. Both m^2 and Z_B defined in this way are renormalization scale (μ) independent quantities.

This scheme cannot be applied to the nonvacuum theory to $O(\lambda^2)$ because to this order the propagators are no longer Lorentz invariant in the sense that they are no longer functions of k^2 alone (the manifold in which they are zero is no longer a mass hyperboloid). However, it may be argued that the exact $O(\lambda^2)$ equations for the propagators can be written as a Lorentz-invariant part plus an inhomogeneous interaction term. Then an approximate effective mass can be defined as the zero of the Lorentz-invariant part.

Although this splitting of $G_{\rm ret}^{-1}$ into a Lorentz-invariant part and the rest can be done in several ways, a few minimal requirements must be met. In particular, the Lorentz-invariant part must be finite, μ independent, and contain all $O(\lambda)$ and vacuum terms. Any two prescriptions meeting these conditions would have the same leading terms and describe the same effects. Let us introduce the Fourier transform

$$G_{\rm ret}(x,x')=i\int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} G_{\rm ret}(X,k) . \tag{4.15}$$

We get, within the same approximations as in Sec. III,

$$\left[Z_B(-k^2+\tfrac{1}{4}\Box_X)+m_B^2+\frac{i\lambda_B}{2}\int\frac{d^4p}{(2\pi)^4}G^{11}(p)-\frac{\lambda_B^2}{6}(\Sigma^{11}-\Sigma^{12})(X,k)\right]G_{\rm ret}(X,k)=-i , \tag{4.16}$$

$$ik\frac{\partial}{\partial X}G_{\rm ret}(X,k)=0 . \tag{4.17}$$

Equation (4.17) is no surprise. We already know that for free fields $G_{\rm ret}$ does not depend on f and for interacting fields it depends on f at $O(\lambda)$. From (4.11) we see that the dependence of f on X is $O(\lambda^2)$. Therefore the dependence of $G_{\rm ret}$ on X begins at $O(\lambda^3)$, which lies beyond our present approximation. The same argument shows that the $\Box_X$ term in Eq. (4.16) can be safely neglected. An analysis based on adiabatic approximation suggests that $\Box_X G_{\rm ret}$ is, to lowest nontrivial order, a dynamically induced mass term. To this order we can safely neglect it. However, this approximation would not be possible at the critical point, where $m^2\to 0$.

Using the free form of the propagators, Eqs. (3.14), to compute the kernels in Eq. (4.16) we find

$$\Sigma^{11}-\Sigma^{12}=F(k^2)+3(-2\pi i)\int\frac{d^4p}{(2\pi)^4}\delta(p^2-m^2)f(X,p)F_1(k-p)$$
$$+3(-2\pi i)^2\int\frac{d^4p}{(2\pi)^4}\delta(p^2-m^2)f(X,p)\int\frac{d^4q}{(2\pi)^4}\delta(q^2-m^2)f(X,q)F_2(k-p-q)\;, \tag{4.18}$$

where

$$F(k^2)=\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\{(p^2-m^2+i\varepsilon)^{-1}(q^2-m^2+i\varepsilon)^{-1}[(k-p-q)^2-m^2+i\varepsilon]^{-1}$$
$$-(-2\pi i)^3\theta(p^0)\delta(p^2-m^2)\theta(q^0)\delta(q^2-m^2)\theta((k-p-q)^0)\delta((k-p-q)^2-m^2)\}\;, \tag{4.19}$$

$$F_1(k)=\int\frac{d^dq}{(2\pi)^d}\{(q^2-m^2+i\varepsilon)^{-1}[(k-q)^2-m^2+i\varepsilon]^{-1}-(-2\pi i)^2\theta(q^0)\delta(q^2-m^2)\theta((k-q)^0)\delta((k-q)^2-m^2)\}\;, \tag{4.20}$$

$$F_2(k)=(k^2-m^2+i\varepsilon)^{-1}-(-2\pi i)\theta(k^0)\delta(k^2-m^2)\;. \tag{4.21}$$

All three F functions have the structure of a difference between a Feynman graph and a "cut" graph, in which all internal moments are on shell and have positive energy.[48] The closed time-path formalism makes it easy to relate these graphs: Because of the close connection between the ordinary and the "cut" graphs, only the former really needs to be computed. For example, we can write F_2 as

$$F_2(k)=[(k^0-i\varepsilon)^2-\mathbf{k}^2-m^2]^{-1}\;. \tag{4.22}$$

Henceforth we shall use the shorthand $(k-i\varepsilon)^2=(k^0-i\varepsilon)^2-\mathbf{k}^2$. The only effect of the "cut" graph was to flip the position of the pole from Feynman to a causal prescription. The same is true of F and F_1. We will only follow the argument for F, the corresponding argument for F_1 is much simpler.

Observe that (F) is nothing but the partial Fourier transform of

$$\{\langle 0|T(\Phi_H(x)\Phi_H(x'))|0\rangle^3-\langle 0|\Phi_H(x')\Phi_H(x)|0\rangle^3\}\;. \tag{4.23}$$

This clearly vanishes if $t'>t$. Correspondingly the real and imaginary part of F are related by a Kramers-Kronig relation, so that either one defines the other. On the other hand, the "cut" graph in F is clearly purely imaginary for real k^2, so that the real part of F is just the real part of the Feynman graph (in this case, the two-loop "setting-sun" graph).[49] It follows that F is the only causal kernal whose real part coincides with that of the setting-sun graph for real k^2.

From a consideration of their analytical structure, it is easy to show that any Feynman graph with two external legs allows a representation of the form[50]

$$S[k^2]=\sum_{j=0}^{n}a_n(k^2)^n-\frac{(k^2)^{n+1}}{2\pi i}\int_{M^2}^{\infty}\frac{d\sigma^2}{(\sigma^2)^{n+1}}\frac{H(\sigma^2)}{(k^2-\sigma^2+i\varepsilon)}\;, \tag{4.24}$$

where the a_n, M^2 are real constants and $H(\sigma^2)$ a given function. F admits a similar representation, with the same a_n, M^2, and H (because of the condition on the real part), but with causal boundary conditions

$$F[k^2]=\sum_{j=0}^{n}a_n(k^2)^n-\frac{(k^2)^{n+1}}{2\pi i}\int_{M^2}^{\infty}\frac{d\sigma^2}{(\sigma^2)^{n+1}}\frac{H(\sigma^2)}{(k-i\varepsilon)^2-\sigma^2}\;. \tag{4.25}$$

So a computation of the ordinary Feynman graph automatically yields F. In our particular case we have, writing $z=(k-i\varepsilon)^2$ (Ref. 49),

$$F_1(z)=\frac{(-i)\mu^{\epsilon}}{(4\pi)^2}\left[\frac{2}{\epsilon}+\ln\frac{m^2}{4\pi\mu^2}-\psi(1)+z\int_{4m^2}^{\infty}\frac{d\sigma^2h_1(\sigma^2)}{\sigma^2(\sigma^2-z)}\right]\;, \tag{4.26}$$

$$F(z)=\frac{(-1)\mu^{2\epsilon}}{(4\pi)^4}\left\{3m^2\left[\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\left[\ln\frac{m^2}{4\pi\mu^2}-\frac{3}{2}-\psi(1)\right]+\ln^2\frac{m^2}{4\pi\mu^2}-[3+2\psi(1)]\ln\frac{m^2}{4\pi\mu^2}+A\right]\right.$$
$$\left.-z\left[\frac{1}{2\epsilon}+\frac{1}{2}\ln\frac{m^2}{4\pi\mu^2}+B\right]+z^2\int_{9m^2}^{\infty}\frac{d\sigma^2h(\sigma^2)}{(\sigma^2)^2(\sigma^2-z)}\right\}\;, \tag{4.27}$$

where A and B are pure numbers, $d=4+\epsilon$ is the number of dimensions, and

$$h_1(\sigma^2)=[1-(4m^2/\sigma^2)]^{1/2} , \tag{4.28}$$

$$h(\sigma^2)=16(\sigma^2)^{-1}\int_{4m^2}^{(\sigma-m)^2} dt\, h_1(t)[(\sigma^2+m^2-t)^2-4\sigma^2m^2]^{1/2} . \tag{4.29}$$

It will be convenient later on to write

$$z^2\int_{9m^2}^{\infty}\frac{d\sigma^2 h(\sigma^2)}{(\sigma^2)^2(\sigma^2-z)}=3Cm^2+3D(z-m^2)+(z-m^2)^2 I(z) , \tag{4.30}$$

where C and D are pure numbers, and

$$I=\int_{9m^2}^{\infty}\frac{d\sigma^2 h(\sigma^2)}{(\sigma^2-z)(\sigma^2-m^2)^2} . \tag{4.31}$$

Because of overlapping divergences, in computing the $O(\lambda)$ term it will be necessary to retain $O(\epsilon)$ terms in the finite part of the tadpole. The extended result is

$$G^{11}(X,X)=\frac{m^2\mu^\epsilon}{(4\pi)^2}\left[\frac{2}{\epsilon}-\psi(2)-\frac{\epsilon}{2}\left[\frac{\pi^2}{3}+\psi^2(2)-\psi'(2)\right]+\left[\ln\frac{m^2}{4\pi\mu^2}\right]\left[1-\frac{\epsilon}{2}\psi(2)\right]+\frac{\epsilon}{4}\left[\ln^2\frac{m^2}{4\pi\mu^2}\right]\right]+\mu^\epsilon M_f^2(d) , \tag{4.32}$$

where formally we have also extended the distribution-dependent term to d dimensions. This extension can be done in several ways, all of which lead to the same physical results. The condition of μ independence of the whole expression leads to the renormalization-group equation

$$\mu\frac{d}{d\mu}M_f^2(d)=-\epsilon M_f^2(d) . \tag{4.33}$$

We can therefore write

$$M_f^2(d)=M_f^2+\frac{\epsilon}{2}\tilde{M}_f^2+O(\epsilon^2) , \tag{4.34}$$

where $\tilde{M}_f^2=M_f^2\ln(\mu_0^2/4\pi\mu^2)+(\mu\text{-independent terms})$. In particular, if we call p the four-dimensional momentum, q the momentum along the ϵ extra dimensions, and define a d-dimensional distribution function

$$f(X,(p,q))=f(X,p)\mu_0^\epsilon\delta^\epsilon(q) , \tag{4.35}$$

then $\tilde{M}_f^2$ is given just by the logarithmic term. Other choices of the d-dimensional form of f are also possible. Equation (4.16) for G_{ret} now reads

$$\Bigg[Z_Bk^2-m_B^2-\frac{\lambda_B}{2}G^{11}(X,X)+\frac{\lambda_B^2}{6}F((k-i\varepsilon)^2)-\frac{\lambda_B^2\mu^{2\epsilon}}{2(4\pi)^2}\left[\frac{2}{\epsilon}+\ln\frac{m^2}{4\pi\mu^2}-\psi(1)\right]\left[M_f^2+\frac{\epsilon}{2}\tilde{M}_f^2\right]$$
$$+\frac{\lambda_B^2}{2(4\pi)^2}\int\frac{d^4p}{(2\pi)^4}\delta(p^2-m^2)2\pi f(X,p)(k-p)^2\int_{4m^2}^{\infty}\frac{d\sigma^2 h_1(\sigma^2)}{\sigma^2((k-p-i\varepsilon)^2-\sigma^2)}$$
$$-\frac{\lambda_B^2}{2}\int\frac{d^4p}{(2\pi)^4}\delta(p^2-m^2)\frac{d^4q}{(2\pi)^4}\delta(q^2-m^2)\frac{(2\pi f(X,p))(2\pi f(X,q))}{(k-p-q-i\varepsilon)^2-m^2}\Bigg]G_{\text{ret}}=i . \tag{4.36}$$

As expected, the last two terms break Lorentz invariance. Observe, however, that for positive m^2 the invariance breaking terms are finite and μ independent by themselves (as $m^2\to 0$ they develop infrared divergences). So as long as $m^2>0$ (far away from the critical region), the simplest approximation is to neglect the last two terms in Eq. (4.36). This results in a finite, μ-independent equation containing all $O(\lambda)$ terms. Consequently, we define the physical mass m^2 by the condition

$$(G_{\text{ret}}^L)^{-1}(k^2=m^2)=\Bigg[Z_Bk^2-m_B^2-\frac{\lambda_B}{2}G^{11}(X,X)+\frac{\lambda_B^2}{6}F((k-i\varepsilon)^2)$$
$$-\frac{\lambda_B^2\mu^{2\epsilon}}{2(4\pi)^2}\left[\frac{2}{\epsilon}+\ln\frac{m^2}{4\pi\mu^2}-\psi(1)\right]\left[M_f^2+\frac{\epsilon}{2}\tilde{M}_f^2\right]\Bigg](k^2=m^2)=0 , \tag{4.37}$$

where G_{ret}^L is the Lorentz-invariant approximation to G_{ret}. The wave-function renormalization is obtained from

$$\frac{\partial}{\partial(k^2)}[(G_{\text{ret}}^L)^{-1}](k^2=m^2)=1 . \tag{4.38}$$

In this way Eqs. (3.14) are the best possible approximation for a free propagator. Equations (4.37) and (4.38) are still written in terms of bare parameters. We can renormalize m_B^2 and λ_B using the same methods as in the conventional (in-out effective-action) theory.[27] Using modified minimal subtraction,[45] we obtain

$$\lambda_B=\mu^{-\epsilon}\left[\lambda-\frac{\lambda^2}{(4\pi)^2}\left[\frac{3}{\epsilon}-\psi(1)-\tfrac{1}{2}\psi(2)\right]\right], \tag{4.39}$$

which leads to the renormalization-group equation

$$\mu\frac{d\lambda}{d\mu}=\epsilon\left[\lambda-\frac{\lambda^2}{(4\pi)^2}[\psi(1)+\tfrac{1}{2}\psi(2)]\right]+\frac{3\lambda^2}{(4\pi)^2}, \tag{4.40}$$

and

$$m_B^2=m_r^2\left\{1-\frac{\lambda}{2(4\pi)^2}\left[\frac{2}{\epsilon}-\psi(2)\right]\right. \\ \left.+\frac{\lambda^2}{2(4\pi)^4}\left[\frac{4}{\epsilon^2}+\frac{1}{\epsilon}[3-4\psi(2)]-\left[\frac{\pi^2}{3}+\psi^2(2)-\psi^{1\prime}(2)\right]+\psi(2)[\psi(1)+\tfrac{1}{2}\psi(2)]-A-C+D\right]\right\}. \tag{4.41}$$

Equations (4.39) and (4.41) yield

$$Z_B=1-\frac{\lambda^2}{6(4\pi)^4}\left[\frac{1}{2\epsilon}+\frac{1}{2}\left[\ln\frac{m^2}{4\pi\mu^2}\right]+B-3D\right], \tag{4.42}$$

which is infinite but μ independent. Using the one-loop result

$$m_r^2=m^2-\frac{\lambda_B m^2\mu^\epsilon}{2(4\pi)^2}\left[\ln\frac{m^2}{4\pi\mu^2}-\frac{\epsilon}{2}\left[\frac{\pi^2}{3}+\psi^2(2)-\psi^{1\prime}(2)\right]+\psi(2)\ln\frac{m^2}{4\pi\mu^2}-\frac{1}{2}\ln^2\frac{m^2}{4\pi\mu^2}\right] \\ -\frac{\lambda_B\mu^\epsilon}{2}\left[M_f^2+\frac{\epsilon}{2}\tilde{M}_f^2\right]+O(\lambda_B^2), \tag{4.43}$$

we finally derive the gap equation as

$$m^2\left[1-\frac{\lambda}{2(4\pi)^2}\ln\frac{m^2}{4\pi\mu^2}-\frac{\lambda^2}{4(4\pi)^4}\left[\ln^2\frac{m^2}{4\pi\mu^2}-4\ln\frac{m^2}{4\pi\mu^2}\right]\right]=m_r^2+\frac{\lambda}{2}M_f^2\left[1+\frac{\lambda}{(4\pi)^2}\ln\frac{m^2}{4\pi\mu^2}\right]. \tag{4.44}$$

We could have used a combination of dimensional arguments and properties of the renormalization-group theory to guess the form of the leading-logarithmic terms in Eq. (4.44). The key observation is that μ^2 only appears in the combination

$$L=\frac{\lambda}{(4\pi)^2}\ln\frac{m^2}{4\pi\mu^2}.$$

If we write, from dimensional considerations,

$$(G_{\rm ret}^L)^{-1}(k^2=m^2)=m^2-m_r^2-m^2F(L,\lambda) \\ -\frac{\lambda M_f^2}{2}G(L,\lambda), \tag{4.45}$$

then the leading logarithms are given by $F(L,0)$ and $G(L,0)$. (In principle, F and G may also depend on M_f^2/m^2.) The condition of μ independence of Eq. (4.45) leads to

$$(3L-2)\frac{\partial F}{\partial L}(L,0)=\frac{L}{2}-1, \tag{4.46}$$

$$(3L-2)\frac{\partial G}{\partial L}(L,0)+3G(L,0)=1, \tag{4.47}$$

where we have used the one-loop renormalization-group equation

$$\mu\frac{d}{d\mu}m_r^2=\frac{\lambda m_r^2}{(4\pi)^2}=\frac{\lambda m^2}{(4\pi)^2}-\frac{\lambda}{2(4\pi)^2}(m^2L+\lambda M_f^2). \tag{4.48}$$

From the classical equations of motion we get the boundary conditions $F(0,0)=0$, $G(0,0)=1$, so the solutions for F and G are

$$F(L,0)=\tfrac{1}{6}[L-\tfrac{4}{3}\ln(1-\tfrac{3}{2}L)], \tag{4.49}$$

$$G(L,0)=\tfrac{1}{3}+\tfrac{2}{3}(1-\tfrac{3}{2}L)^{-1}, \tag{4.50}$$

whose expansion leads us back to Eq. (4.44). Equations (4.49) and (4.50) also suggest that as we approach the critical region ($m^2\to 0$, $L\to-\infty$), the gap equation can be approximated by

$$m^2\left[1-\frac{\lambda}{6(4\pi)^2}\ln\frac{m^2}{4\pi\mu^2}\right]=m_r^2+\frac{\lambda}{6}M_f^2\ , \qquad (4.51)$$

which would yield qualitatively similar results as the one-loop gap equation (3.25). At the same time, because order by order in λ the perturbation theory is dominated by logarithmic terms, we expect that the neglect of the non-Lorentz-invariant terms will be justified in this limit.

At the critical point, the approximation that the distribution function does not change appreciably over distances of order m^{-1} breaks down, as we saw in Sec. IV A. Therefore the present analysis is not expected to hold in the critical region.

C. Limitations of the binary collision approximation from three-loop analysis

In the above we showed that to $O(\lambda^2)$, the fluctuation field is describable by a distribution function obeying Boltzmann's equation. This corresponds to a binary collision approximation in classical kinetic theory. One way to ascertain the limitations of this approximation is to estimate the $O(\lambda^3)$ corrections to it from the closed-time-path effective action, which we will now carry out.

In the absence of background fields, there is only one $O(\lambda^3)$ contribution to the effective action. In the notation of Sec. II it reads

$$\frac{\lambda_B^3}{48}h_{abcd}h_{efgh}h_{ijkl}\int d^4x\, d^4x'd^4x''G^{ae}(x,x')G^{bf}(x,x')G^{ci}(x,x'')G^{dj}(x,x'')G^{gk}(x',x'')G^{hl}(x',x'')\ . \qquad (4.52)$$

The corresponding correction to the equations of motion (3.8) is

$$\frac{\lambda_B^3}{8}c_{de}\int d^4x''\Theta^{ad}(x,x'')G^{eb}(x'',x')\ , \qquad (4.53)$$

and, for (3.9),

$$\frac{\lambda_B^3}{8}c_{de}\int d^4x''G^{ad}(x,x'')\Theta^{eb}(x'',x')\ . \qquad (4.54)$$

Here

$$\Theta^{ab}(x,x')=h_{ijkl}\int d^4x''G^{ab}(x,x')G^{ai}(x,x'')G^{aj}(x,x'')G^{kb}(x'',x')G^{lb}(x'',x')\ . \qquad (4.55)$$

The correction to the kinetic equation is the difference between Eqs. (4.53) and (4.54), which, in the approximation of translation-invariant propagators, is in fact independent of (a,b):

$$\frac{\lambda_B^3}{8}\int d^4x''[G^{12}(x,x'')\Theta^{21}(x'',x')-\Theta^{12}(x,x'')G^{21}(x'',x')]\ . \qquad (4.56)$$

Introducing Fourier transforms with respect to the fast variables and neglecting the position dependence of the Fourier coefficients, we get

$$\frac{\lambda_B^3}{8}\int\frac{d^4p}{(2\pi)^4}e^{ip(x-x')}\int\frac{d^4q}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}\left[\int\frac{d^4r}{(2\pi)^4}[G^{11}(r)G^{11}(s-r)-G^{22}(r)G^{22}(s-r)]\right]$$
$$\times[G^{12}(p-s)G^{12}(q)G^{12}(s-q)G^{21}(p)-G^{21}(p-s)G^{21}(q)G^{21}(s-q)G^{12}(p)]\ , \qquad (4.57)$$

where we recognize, in the terms containing four propagators, the same index structure which led us to Boltzmann's collision term. Using the ansatz

$$\begin{aligned}G^{11}(r)G^{11}(s-r)-G^{22}(r)G^{22}(s-r)=&\{(r^2-m^2+i\varepsilon)^{-1}[(s-r)^2-m^2+i\varepsilon]^{-1}\\&-(r^2-m^2-i\varepsilon)^{-1}[(s-r)^2-m^2-i\varepsilon]^{-1}\}\\&+2(2\pi i)f(X,r)\delta(r^2-m^2)\{[(s-r)^2-m^2+i\varepsilon]^{-1}+[(s-r)^2-m^2-i\varepsilon]^{-1}\}\ ,\end{aligned} \qquad (4.58)$$

we see that Eq. (4.57) involves two different physical processes: the first one, involving the first term in Eq. (4.58), reduces essentially to an (infinite) correction to the effective coupling constant, depending on the momentum transfer s [the pole term is canceled when λ_B^2 is renormalized according to Eq. (4.39)]. The second term can also be thought of as a correction to the effective interaction, but as this correction involves a third particle (of momentum r) it lies beyond the binary collision approximation. If $s^2 \ll m^2$ we may estimate the order of magnitude of this correction as $\Delta\lambda\sim\lambda^3(M_f^2/m^2)$. Therefore the binary collision model will be accurate for

$$\lambda^2 \gg \lambda^3(M_f^2/m^2)\ , \qquad (4.59)$$

which is usually satisfied in regions remote from the critical region [in critical regions $m^2\sim(\lambda/2)(M_f^2-T_C^2/12)$.]

V. DISSIPATION IN INTERACTING QUANTUM FIELDS

In Sec. II C we pointed out that the behavior of background fields and propagators usually displays dissipative effects. We argued that this fact does not contradict unitarity but is the result of an incomplete description of the state of the quantum field. In the language of generalized effective actions, if all the connected Green's functions were retained as dynamical variables then no dissipation phenomena would be apparent. There we gave an example of a classical system which displays such a behavior. In this section we will show explicitly by way of perturbation theory how this behavior arises in vacuum interacting quantum fields. We shall assume that all odd connected Green's functions remain zero for all times, and restrict ourselves to $O(\lambda^2)$. To this order, the equation for the retarded propagator in vacuum reads

$$\left[k^2-m^2+\frac{\lambda^2}{6}(k^2-m^2)^2\int_{9m^2}^{\infty}\frac{d\sigma^2 h(\sigma^2)}{[\sigma^2-(k-i\epsilon)^2](\sigma^2-m^2)^2}\right]G_{\rm ret}(k)=i\ . \tag{5.1}$$

The nonlocal term acquires an imaginary part whenever k^2 crosses the three-particle threshold $9m^2$, showing dissipation through radiation damping, as discussed in Sec. II C.

In order to see how this apparent dissipation is related to the reduction of the full hierarchy of Green's functions to only the propagators, we must begin our discussion from a more general viewpoint than that adopted in the Sec. IV. Here we will reinstate a four-point nonlocal source[31] $K_{abcd}(x,x',x'',x''')$ in the closed-time-path generating functional. It will prove convenient to introduce a new kind of index $A=(a,x)$, where $a=1,2$ and x denotes a spacetime point. Repeated capital indices indicate sum over a and integration over x. Thus, for example, the classical action will read

$$S[\phi^A]=\tfrac{1}{2}c_{AB}\partial_\mu\phi^A\partial^\mu\phi^B-\tfrac{1}{2}m_B^2c_{AB}\phi^A\phi^B-\frac{\lambda_B}{4!}h_{ABCD}\phi^A\phi^B\phi^C\phi^D \tag{5.2}$$

[cf. Eq. (2.16)], where

$$c_{AB}=c_{ab}\delta(x-x')\ , \tag{5.3}$$

$$h_{ABCD}=h_{abcd}\delta(x'-x)\delta(x''-x)\delta(x'''-x) \tag{5.4}$$

with c_{ab} and h_{abcd} as in Sec. II. The closed-time-path generating functional W becomes, in this notation,

$$W=-i\ln\int D\phi^A\exp\left[i\left[S[\phi^A]+\tfrac{1}{2}K_{AB}\phi^A\phi^B+\frac{1}{4!}K_{ABCD}\phi^A\phi^B\phi^C\phi^D\right]\right]\ . \tag{5.5}$$

Observe that W has the form of a generating functional with only a two-point source and an effective interaction constant $(\lambda_B h_{ABCD}-K_{ABCD})$. Therefore the Legendre transform $\Gamma[G^{AB},K_{ABCD}]$ of W with respect to K_{AB} is immediate [cf. Eq. (3.6)] (in this section we will omit the "bare" subscripts for simplicity):

$$\begin{aligned}\Gamma[G^{AB},K_{ABCD}]=&\frac{i}{2}\ln \mathrm{Det}(G^{AB})^{-1}-\tfrac{1}{2}c_{AB}(Z\Box+m^2)G^{AB}-\tfrac{1}{8}(\lambda h_{ABCD}-K_{ABCD})G^{AB}G^{CD}\\&+\frac{i}{48}(\lambda h_{ABCD}-K_{ABCD})(\lambda h_{EFGH}-K_{EFGH})G^{AE}G^{BF}G^{CG}G^{DH}+\cdots\ .\end{aligned} \tag{5.6}$$

To Legendre transform with respect to K_{ABCD}, we observe that

$$\begin{aligned}\left.\frac{\partial\Gamma[G^{AB},K_{ABCD}]}{\partial K_{ABCD}}\right|_{G^{AB}=\mathrm{const}}&=\left.\frac{\partial W[K_{AB},K_{ABCD}]}{\partial K_{ABCD}}\right|_{K_{AB}=\mathrm{const}}\\&=\frac{1}{4!}(G^{ABCD}+G^{AB}G^{CD}+G^{AC}G^{BD}+G^{AD}G^{BC})\ .\end{aligned} \tag{5.7}$$

Differentiating Eq. (5.6) we find

$$G^{ABCD}=i(K_{EFGH}-\lambda h_{EFGH})G^{AE}G^{BF}G^{CG}G^{DH}\ . \tag{5.8}$$

Finally, the closed-time-path effective action becomes[31]

$$\begin{aligned}\Gamma[G^{AB},G^{ABCD}]&=\Gamma[G^{AB},K_{ABCD}]-\frac{1}{4!}K_{ABCD}(G^{ABCD}+3G^{AB}G^{CD})\\&=\frac{i}{2}\ln \mathrm{Det}(G^{AB})^{-1}-\tfrac{1}{2}c_{AB}(Z\Box+m^2)G^{AB}-\tfrac{1}{8}\lambda h_{ABCD}G^{AB}G^{CD}-\frac{1}{4!}\lambda h_{ABCD}G^{ABCD}\\&\quad+\frac{i}{48}G^{ABCD}(G^{-1})_{AE}(G^{-1})_{BF}(G^{-1})_{CG}(G^{-1})_{DH}G^{EFGH}\ .\end{aligned} \tag{5.9}$$

The equations of motion follow from the variation of Γ:

$$i(G^{-1})_{AB}+C_{AB}(Z\Box+m^2)+\frac{\lambda}{2}h_{ABCD}G^{CD}+\frac{i}{6}(G^{-1})_{AI}G^{IJKL}(G^{-1})_{JM}(G^{-1})_{KN}(G^{-1})_{LO}G^{MNOP}(G^{-1})_{PB}=0 \tag{5.10}$$

and

$$i(G^{-1})_{AE}(G^{-1})_{BF}(G^{-1})_{CG}(G^{-1})_{DH}G^{EFGH}=\lambda h_{ABCD} , \tag{5.11}$$

where we have absorbed the nonlocal sources in the initial conditions. With an algebraic replacement we may rewrite Eq. (5.10) as

$$i(G^{-1})_{AB}+c_{AB}(Z\Box+m^2)+\frac{\lambda}{2}h_{ABCD}G^{CD}+\frac{\lambda}{6}(G^{-1})_{AI}G^{IJKL}h_{JKLB}=0 . \tag{5.12}$$

Finally we observe that to lowest order in λ, $(G^{-1})_{AB}=ic_{AB}(Z\Box+m^2)$, so that, using c_{AB} to lower indices, we may write (5.11) and (5.12) as

$$i(G^{-1})_{AB}+c_{AB}(Z\Box+m^2)+\frac{\lambda}{2}h_{ABCD}G^{CD}+\frac{i\lambda}{6}[(\Box+m^2)G_A^{JKL}]h_{BJKL}=0 , \tag{5.13}$$

$$(\Box+m^2)(\Box'+m^2)(\Box''+m^2)(\Box'''+m^2)G_{ABCD}=-i\lambda h_{ABCD} . \tag{5.14}$$

Equation (5.13) and (5.14) are the evolution equations for G^{AB} and G^{ABCD}. No dissipation effect is apparent. However if we solve Eq. (5.14),

$$G^{ABCD}=-i\lambda h_{EFGH}G^{AE}G^{BF}G^{CG}G^{DH} , \tag{5.15}$$

and use the result in Eq. (5.13), we obtain

$$i(G^{-1})_{AB}+c_{AB}(Z\Box+m^2)+\frac{\lambda}{2}h_{ABCD}G^{CD}-\frac{i\lambda}{6}h_{AFGH}h_{BJKL}G^{JF}G^{KG}G^{LH}=0 . \tag{5.16}$$

We see that this has the same form as Eq. (3.7), where the propagator in Eq. (5.1) becomes dissipative upon acquiring an imaginary term.

It is interesting to observe that the solution, Eq. (5.15), for G^{ABCD} corresponds to homogeneous boundary conditions. In conventional notation, we have

$$G^{abcd}(x,x',x'',x''')=-i\lambda\int d^4y[G^{a1}(x,y)G^{b1}(x',y)G^{c1}(x'',y)G^{d1}(x''',y)-G^{a2}(x,y)G^{b2}(x',y)G^{c2}(x'',y)G^{d2}(x''',y)] , \tag{5.17}$$

where the integral extends over the half-space $y^0>0$. By inspection one can verify that (5.17) satisfies (5.14) for any $x^0,x'^0;x''^0,x'''^0>0$. This means that, as x^0 (or any other 0 component) goes to zero, x will be in the past of y. But then

$$G^{11}(x,y)=\langle 0 \mid T(\Phi_H(x)\Phi_H(y)) \mid 0\rangle=G^{12}(x,y)$$

and also $G^{21}(x,y)=G^{22}(x,y)$, so that Eq. (5.17) vanishes for any set of indices (a,b,c,d).

VI. REMARKS

In this paper we have laid down the foundation of a formalism for the description of nonequilibrium phenomena in quantum-field theory. The basic element of this formalism is the use of the path-integral quantization with in-in boundary conditions; the state of the quantum field is described by the set of all connected Green's functions and the dynamics is derived from a generalized effective action (in the sense of the De Dominicis, Jona-Lasinio, and others). This description is equivalent to the usual one in terms of statistical operators obeying Heisenberg equations of motion. It is exact and can handle systems arbitrarily far from equilibrium. In practice, meaningful model solving requires the use of approximations. We explore here the consequences of truncating the hierarchy of Green's functions to just the two-point propagators. One immediate consequence of the truncation is the onset of dissipative processes.

In this paper we consider a perturbative (loop) expansion of the generalized effective action. To establish a connection with known results we consider first the one-loop approximation. To this order the nonequilibrium aspects are totally lost, as the relaxation time of the system is infinite. The simplest nontrivial physical result comes from the two-loop approximation to the effective action. Here one begins to observe relaxation phenomena with finite characteristic times. By a straightforward application of methods of nonrelativistic statistical mechanics the quantum field can be reduced to a many-body system described by a kinetic equation. In this new language two-loop accuracy corresponds to a binary collision approximation and the kinetic equation is simply the Boltzmann equation. The specific quantum-field-theoretical nature of the system shows up at the kinetic theory level in the nontrivial relationship between the masses and charges of the quasiparticles, and the bare parameters of the underlying quantum field. Finally, we discussed the conditions and mechanism which allows an unitary field theory to display dissipative behavior.

Our formulation provides a simple and direct connection between first principles (quantum-field theories) and semiphenomenological (kinetic) description of nonequilibrium quantum fields. Many aspects can be explored further from this unified scheme. Among some generalizations we may mention the following. (1) Consider more general initial states by including higher nonlocal external sources in the generating functional. (2) The formulation can be generalized to encompass inhomogeneous external conditions, in particular, for nonequilibrium fields in curved space. (3) By including a nontrivial background field it may be used to study the time-dependent Landau-Ginzburg equation or Fokker-Planck equation for nonequilibrium problems. (4) It is possible to go beyond the binary collision approximation, or to improve the theory presented here by a systematic use of renormalization-group arguments. (5) One may want to consider more realistic quantum-field theories by including spinor and gauge fields. This last point alone is more than sufficient argument for the advantage of a path-integral formulation of nonequilibrium quantum fields.

There are many interesting applications of these methods. In particular, a formalism of nonequilibrium quantum fields including gauge fields in curved space is needed for a complete study of critical dynamics in the early universe. It is also an essential tool for analyzing quantum-statistical processes in relativistic cosmology and astrophysics. We will pursue these matters in forthcoming papers.

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