Index theorems and superconducting cosmic strings

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The interaction of vortex lines with chiral fermions in $3+1$ spacetime dimensions is investigated. We construct Dirac operators relevant to the study of superconducting cosmic strings, obtain conditions for the cancellation of triangle anomalies, and examine properties of the corresponding Dirac Hamiltonian. Our analysis is applicable to both cosmic strings arising from spontaneously broken gauged U(1) symmetries and axion vortices in broken global U(1) symmetry groups. We generalize known index theorems to consider angular-momentum-weighted indices and η invariants of the Hamiltonian in the background field of a topological vortex. We further obtain explicit zero modes of the Hamiltonian in a rotationally covariant vortex field. Implications of the results for the quantum numbers of light fermionic excitations and their axial anomalies are discussed. We use the η invariants to derive anomaly equations for charges and angular momenta and find discrepancies with those of effective two-dimensional field theory. Our results indicate a novel anomalous behavior of the angular momentum and suggest a new mechanism for the transfer of energy and momentum between axionic strings.

I. INTRODUCTION

The existence of topological objects such as monopoles, domain walls, and vortices is a spectacular and as yet unconfirmed consequence of the idea that fundamental forces are governed by spontaneously broken non-Abelian gauge theories. Vortexlike objects occur in all grand unified field theories which have spontaneously broken U(1) symmetries or non-Abelian gauge symmetries broken to discrete subgroups.[1,2] Furthermore, many non-Abelian gauge theories which support magnetic monopoles also have unstable vortices which would decay by nucleation of monopole-antimonopole pairs. For various models these could be metastable.

Cosmic strings would be formed as line defects in phase transitions and have been conjectured to play a role in the early Universe[3–5] particularly by providing the density fluctuations necessary for galaxy formation.[5–9] They furthermore have several potentially observable effects such as gravitational lensing[10] or emission of long-wavelength gravitational radiation.[11–14]

Several authors have discussed the interactions of vortices with chiral fermions.[15–20] Jackiw and Rossi[18] showed that when fermions obtain their masses from a particular coupling to the complex scalar field the transverse component of the fermionic Hamiltonian in the background of a rotationally symmetric vortex has zero modes. This result was later generalized to background fields with arbitrary vortex profile by Weinberg[20] who proved the appropriate index theorem.

Zero modes of the transverse Hamiltonian propagate along the string like two-spacetime-dimensional chiral fermions and subsequently Witten[21] used the two-dimensional axial anomaly to argue that in particular grand unified models vortices can act as superconducting wires. Witten further discussed criteria for anomaly cancellation and examined the behavior of cosmic strings in external electromagnetic fields. These provide several observable effects of the presence of cosmic strings with mass densities too small to be observed by their gravitational interactions.

Independently, there have been numerous studies of the interactions of fermions with vortices in the context of three-spacetime-dimensional electrodynamics. In the models studied to date the fermions do not couple to the complex scalar-field component of the vortex. Fermionic loop corrections have been shown to induce a Chern-Simons topological mass term[22–25] in the effective action of the gauge fields. This has been demonstrated by perturbative calculations[25,26] as well as an analysis of the role of zero modes of the fermionic Hamiltonian[27,28] and index-theoretic calculations.[27,29–32] It is further associated with induced fractional charges[27–32] and anomalous spin and statistics[33–37] of vortices in these models.

In this paper I shall examine how some of these anomalous phenomena could appear in the interaction of cosmic strings with fermions. We show that the spectral properties of the Hamiltonian and angular momentum operators which yield induced quantum numbers in $(2+1)$-dimensional electrodynamics also yield anomaly equations in $3+1$ dimensions. We comment on the issue of gauge invariance and angular momentum and find several new index-theoretical results for two-dimensional Dirac operators. We argue that the $(3+1)$-dimensional anomaly equations that we derive indicate a new mechanism for transfer of angular momentum between vortices and antivortices induced by external electromagnetic fields.

We consider a $U(1)\otimes U(\tilde{1})$ gauge theory where the $U(\tilde{1})$ symmetry is spontaneously broken and supports the topological vortex configuration and U(1) is ordinary electrodynamics.[38] The fermions obtain masses from their interaction with the $U(\tilde{1})$ charged scalar fields which stabilize the string solutions of the gauge and scalar-field sectors of the theory. These models have the virtue of coupling the $U(\tilde{1})$ condensate to the fermions in a way analogous to the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity[39,40] and the resulting Dirac Hamiltonian is Fredholm. Moreover, when the $U(\tilde{1})$ symmetry is gauged the requirement of finite energy per unit length of the complex scalar field quantizes the $U(\tilde{1})$ magnetic flux.

We shall also consider vortices arising in spontaneous breaking of ungauged global $U(\tilde{1})$ symmetries relevant to axion models.[41] These have long-range interactions and in isolation they have infinite energy per unit length. Since the $U(\tilde{1})$ symmetries are not gauged these models have weaker anomaly-cancellation constraints and exhibit a richer array of anomalous properties. Gauge and gravitational anomalies of the effective two-dimensional field theory have been discussed by Callan and Harvey.[42]

In Sec. II we analyze the properties of the four-spacetime-dimensional Dirac operator and the transverse component of the Hamiltonian in a vortex background field. We show that the model of Weyl fermions coupled to $U(1)\otimes U(\tilde{1})$ background fields considered by Witten[21] can be written in the form of a Dirac action with coupling to a chiral scalar field. This allows us to identify the Dirac Hamiltonian, the transverse part of which we later use for our index theory analysis. We include several flavors of fermions and review criteria for anomaly cancellation. We identify a gauge-invariant angular momentum operator which commutes with the Hamiltonian when the vortex field is rotationally symmetric. When evaluated in a rotationally symmetric gauge this operator reduces to the conserved canonical angular momentum operator for the fermions. The fact that it is obtained from a gauge-invariant operator resolves a criticism[34] of the results of Ref. 33 by showing that the canonical angular momentum operator has gauge-invariant meaning when the background fields are presented in a rotationally covariant gauge.

In Sec. III we reexamine and generalize the index theorem to compute the index and η invariant of the transverse Hamiltonian. The formulas for the index and η invariant are for arbitrary, not necessarily rotationally covariant vortex fields. We find that the index is determined by the topological winding number of the complex scalar field. We also compute the angular-momentum-weighted index and η invariant for a rotationally symmetric vortex. The angular-momentum-weighted η invariant is a bilinear in the winding number of the complex scalar and the magnetic fluxes of both U(1) and $U(\tilde{1})$.

In Sec. IV we find explicit solutions for zero modes of the transverse Hamiltonian for the special case of rotationally covariant background fields. Results for both the index and the angular-momentum-weighted index are shown to agree with the index theorems derived in Sec. III.

In Sec. V we argue that the zero modes of the transverse Hamiltonian propagate along the string as massless two-spacetime-dimensional fermions which carry U(1) electric charge and angular momentum. We find that the signs of their angular momenta are determined by their direction of motion and the magnitudes are independent of their wave number. The axial anomaly for the two-spacetime-dimensional electric charge results in the string superconductivity. It also results in superconductivity of the angular momentum. For the case of the axionic string the gauge anomalies of the effective two-dimensional model do not cancel. Since currents which couple to gauge fields must be conserved in the full four-dimensional theory, the string anomaly must be canceled by higher-dimensional effects.

In Sec. VI we discuss the relationship between the η invariants and the induced vacuum charge and angular momentum. We argue that the charge and angular momentum densities are ambiguous due to linearly divergent momentum integrals. We show how they can be used to derive anomaly equations and compare these with the two-dimensional anomaly equations found in Sec. V. We find a discrepancy when the $U(\tilde{1})$ magnetic flux does not obey the flux quantization condition necessary to give the vortex finite energy per unit length. This is particularly important for the axion string and implies that whenever the flux quantization condition is not satisfied the two-dimensional anomaly does not accurately represent the charge conservation law for the fermion-vortex system.

We find an anomalous conservation law for the ground-state expectation value of the angular momentum and argue that anomaly cancellation requires a flow of angular momentum onto the string from its exterior. We point out that this anomaly which is an effect of the nontrivial topology of the gauge and scalar fields is different from the energy-momentum anomaly discussed by Callan and Harvey[42] which was due to cancellation of gravitational anomalies and was driven by an external gravitational field. We shall also discuss how the angular momentum anomaly implies an induced current for the full angular momentum operator with gauge and Higgs field contributions included.

II. THE DIRAC OPERATOR

We consider the model studied by Witten[21] of a pair of four-spacetime-dimensional left-handed Weyl fermions interacting with $U(1)\otimes U(\tilde{1})$ gauge fields A_μ and R_μ, respectively. The fermions have charges (q,s) and $(-q,-s-e)$ and also interact with a scalar field ϕ of charge $(0,e)$:

$$S_1=\int d^4x\,\{\,\psi_L^\dagger\sigma^\mu(i\partial_\mu-qA_\mu-sR_\mu)\psi_L+\chi_L^\dagger\sigma^\mu[i\partial_\mu+qA_\mu+(s+e)R_\mu]\chi_L-i(\phi\epsilon^{\alpha\beta}\psi_{L\alpha}\chi_{L\beta}+\phi^*\epsilon^{\alpha\beta}\psi^*_{L\alpha}\chi^*_{L\beta})\}\ , \qquad (2.1)$$

where $\sigma^\mu=(1,\boldsymbol{\sigma})$ and $\boldsymbol{\sigma}$ are the Pauli matrices. We shall consider the general case of several fermion flavors with differing charges and also several flavors of similar pairs of left-handed fermions with charges $(\hat{q},\hat{s})$ and $(-\hat{q},-\hat{s}+e)$ interacting with the complex conjugate of the scalar field:

$$S_2=\int d^4x\{\hat{\psi}_L^\dagger\sigma^\mu(i\partial_\mu-\hat{q}A_\mu-\hat{s}R_\mu)\hat{\psi}_L+\hat{\chi}_L^\dagger\sigma^\mu[i\partial_\mu+\hat{q}A_\mu+(\hat{s}-e)R_\mu]\hat{\chi}_L-i(\phi^*\epsilon^{\alpha\beta}\hat{\psi}_{L\alpha}\hat{\chi}_{L\beta}+\phi\epsilon^{\alpha\beta}\hat{\psi}^*_{L\alpha}\hat{\chi}^*_{L\beta})\}\ . \tag{2.2}$$

It is necessary to adjust the charges of the fermions to cancel chiral gauge anomalies. There are four kinds of triangle anomaly: AAA, AAR, ARR, and RRR. The charges have already been chosen to cancel the AAA anomaly for each pair of fermions separately. When we consider nontrivial coupling to both gauge fields A_μ and R_μ we must choose the charges so that the AAR, ARR, and RRR anomalies cancel:

$$\sum_\alpha q_\alpha^2=\sum_\beta \hat{q}_\beta^2\ , \tag{2.3a}$$

$$\sum_\alpha q_\alpha(2s_\alpha+e)=\sum_\beta \hat{q}_\beta(2\hat{s}_\beta-e)\ , \tag{2.3b}$$

$$\sum_\alpha(3s_\alpha^2+3es_\alpha+e^2)=\sum_\beta(3\hat{s}_\beta^2-3e\hat{s}_\beta+e^2)\ . \tag{2.3c}$$

The minimal model with anomaly cancellation and not all q_α and $\hat{q}_\beta$ equal to zero has two pairs of left-handed fermions and $\hat{q}=q$, $\hat{s}=s+e$ or $\hat{q}=-q$, $\hat{s}=-s$. We shall remain with the most general case of several flavor copies of the actions (2.1) and (2.2). If we set $R_\mu=0$ and consider the string as arising in an axionlike model with ungauged global $U(\tilde{1})$ symmetry, (2.1) and (2.2) has a U(1) gauge-invariant renormalization with no restrictions on the charges q_α and $\hat{q}_\beta$.

With charge conjugation $\chi_L^c=\sigma^2\chi_L^*$ and $\bar{\sigma}^\mu=(1,-\boldsymbol{\sigma})$ we can rewrite (2.1) and (2.2) as

$$S_1=\int d^4x\,(\psi_L^\dagger,\chi_L^{c\dagger})\begin{bmatrix}\sigma^\mu(i\partial_\mu-qA_\mu-sR_\mu) & -\phi^*\\ -\phi & \bar{\sigma}^\mu[i\partial_\mu-qA_\mu-(s+e)R_\mu]\end{bmatrix}\begin{bmatrix}\psi_L\\ \chi_L^c\end{bmatrix} \tag{2.4}$$

and

$$S_2=\int d^4x\,(\hat{\psi}_L^\dagger,\hat{\chi}_L^{c\dagger})\begin{bmatrix}\sigma^\mu(i\partial_\mu-\hat{q}A_\mu-\hat{s}R_\mu) & -\phi\\ -\phi^* & \bar{\sigma}^\mu[i\partial_\mu-\hat{q}A_\mu-(\hat{s}-e)R_\mu]\end{bmatrix}\begin{bmatrix}\hat{\psi}_L\\ \hat{\chi}_L^c\end{bmatrix}\ . \tag{2.5}$$

Using the chiral representation of the Dirac matrices

$$\gamma^0=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \boldsymbol{\gamma}=\begin{bmatrix}0&-\boldsymbol{\sigma}\\ \boldsymbol{\sigma}&0\end{bmatrix},\quad \gamma^5=\begin{bmatrix}1&0\\0&-1\end{bmatrix}, \tag{2.6}$$

and the definitions

$$\Psi=\begin{bmatrix}\psi_L\\ \chi_L^c\end{bmatrix},\quad \hat{\Psi}=\begin{bmatrix}\hat{\psi}_L\\ \hat{\chi}_L^c\end{bmatrix}, \tag{2.7}$$

$$\phi(x)=\rho(x)e^{i\sigma(x)}\ , \tag{2.8}$$

we can present (2.4) and (2.5) in the form of a Dirac action with coupling to a chiral scalar field:

$$S_1=\int d^4x\,\bar{\Psi}(x)[i\gamma^\mu D_\mu-\rho(x)e^{i\gamma^5\sigma(x)}]\Psi(x)\ , \tag{2.9}$$

$$S_2=\int d^4x\,\bar{\hat{\Psi}}(x)[i\gamma^\mu D_\mu-\rho(x)e^{-i\gamma^5\sigma(x)}]\hat{\Psi}(x)\ , \tag{2.10}$$

where

$$iD_\mu=i\partial_\mu-qA_\mu-\left[s+\frac{e}{2}-\frac{e}{2}\gamma^5\right]R_\mu\ ,$$

$$i\hat{D}_\mu=i\partial_\mu-\hat{q}A_\mu-\left[\hat{s}-\frac{e}{2}+\frac{e}{2}\gamma^5\right]R_\mu\ .$$

Anomaly cancellation implies that the gauge symmetries are realized at the quantum level. When the Higgs-boson field $\phi(x)$ has a vacuum expectation value the $U(\tilde{1})$ gauge symmetry and also the corresponding phase symmetry are spontaneously broken. The theory whose fermionic sector is governed by (2.9) and (2.10) exhibits a superconducting phase where $\phi(x)$ is either a fundamental or a composite field representing the pair condensate. The fermion spectrum is obtained by setting $\phi(x)$ equal to its vacuum expectation value. When $|\phi(x)|=|\phi|_\infty=$const (2.9) and (2.10) describe Dirac fermions with mass $|\phi|_\infty$ coupled to vector and axial gauge fields.

As $|\mathbf{x}|\to\infty$ a topological vortex configuration approaches the ground state $|\phi(x)|=|\phi|_\infty$, $\mathbf{A}(x)=\mathbf{R}(x)=0$. A single vortex with winding number n is characterized by

$$\frac{1}{2\pi}\oint dl\cdot\frac{\phi^* i\partial\phi}{\phi^*\phi}=n\ , \tag{2.11}$$

where the integration is over a loop which encircles the string. [$\phi(x)$ must have a zero inside the loop.] If the energy density

$$\mathcal{E}_\phi(x)=|(i\boldsymbol{\partial}+e\mathbf{R})\phi(x)|^2 \tag{2.12}$$

damps rapidly enough in directions transverse to the vortex ($\sim r^{-2-\epsilon}$ as $r\to\infty$) so that the energy per unit length is finite we would have the following additional condition:

$$(i\boldsymbol{\partial}+e\mathbf{R})\phi(x)=O(r^{-1-\epsilon})\ . \tag{2.13}$$

This would quantize the $U(\tilde{1})$ magnetic flux of the vortex

$$\Phi_R=\frac{e}{2\pi}\oint dl\cdot\mathbf{R}=-n\ , \tag{2.14}$$

where the loop encircles the vortex and is outside the region where $|\phi(x)|$ differs from its constant vacuum value $|\phi|_\infty$. Although this constraint could be imposed by a finite-energy requirement in the bosonic sector of the theory it is unnecessary when we restrict our attention to the fermions. In the following we shall remain with the more general case of arbitrary $U(\tilde{1})$ magnetic flux and comment on the effect of the constraints (2.13) and (2.14). Furthermore, we shall apply our analysis to the case where the $U(\tilde{1})$ symmetry is ungauged by setting $R_\mu=0$ and consequently $\Phi_R=0$.

Particular rotationally covariant field configurations for a vortex lying along the x^3 axis have the zeroth and third components of all vectors vanishing, all quantities independent of x^0 and x^3, and

$$\phi(\mathbf{r})=e^{-in\theta}f(r) , \tag{2.15a}$$

$$eR^i(\mathbf{r})=\epsilon^{ij}\hat{\mathbf{r}}^jR(r) , \tag{2.15b}$$

$$A^i(\mathbf{r})=\epsilon^{ij}\hat{\mathbf{r}}^jA(r) . \tag{2.15c}$$

The vectors are two dimensional in the x-y plane, $\mathbf{r}=(r\cos\theta, r\sin\theta)$ and the asymptotic conditions are

$$f(r)\rightarrow f_0r^{|n|},\quad R(r)\rightarrow 0,\quad A(r)\rightarrow 0 , \tag{2.16a}$$

as $r\rightarrow 0$, and

$$f(r)\rightarrow |\phi|_\infty,\quad R(r)\rightarrow -\frac{\Phi_R}{r},\quad A(r)\rightarrow -\frac{\Phi_A}{r} , \tag{2.16b}$$

as $r\rightarrow\infty$. Here

$$\Phi_A=\frac{1}{2\pi}\int d^2x\,\epsilon^{ij}\partial_iA_j,\quad \Phi_R=\frac{e}{2\pi}\int d^2x\,\epsilon^{ij}\partial_iR_j ,$$

are the U(1) and $U(\tilde{1})$ magnetic flux, respectively. The finite-energy condition (2.12) and (2.13) would set $\Phi_R=-n$.

The single-particle Hamiltonians corresponding to (2.9) and (2.10) are

$$h=\boldsymbol{\alpha}\cdot\left[i\boldsymbol{\partial}-q\,\mathbf{A}-\left[s+\frac{e}{2}-\frac{e}{2}\gamma^5\right]\mathbf{R}\right]+\beta\rho e^{i\gamma^5\sigma}+qA_0+\left[s+\frac{e}{2}-\frac{e}{2}\gamma^5\right]R_0 \tag{2.17a}$$

and

$$\hat{h}=\boldsymbol{\alpha}\cdot\left[i\boldsymbol{\partial}-\hat{q}\,\mathbf{A}-\left[\hat{s}-\frac{e}{2}+\frac{e}{2}\gamma^5\right]\mathbf{R}\right]+\beta\rho e^{-i\gamma^5\sigma}+\hat{q}A_0+\left[\hat{s}-\frac{e}{2}+\frac{e}{2}\gamma^5\right]R_0 , \tag{2.17b}$$

where $\boldsymbol{\alpha}=\gamma^0\boldsymbol{\gamma}$ and $\beta=\gamma^0$. Note that (2.17b) is obtained from (2.17a) by setting $q,s,e,\sigma(x)\rightarrow\hat{q},\hat{s},-e,-\sigma(x)$. The transverse Hamiltonian is

$$h_\kappa=\boldsymbol{\alpha}\cdot\left[i\boldsymbol{\partial}-q\,\mathbf{A}-\left[s+\frac{e}{2}-\frac{e}{2}\gamma^5\right]\mathbf{R}\right]+\alpha^3\kappa+\beta\rho^{i\gamma^5\sigma} \tag{2.18a}$$

and

$$\hat{h}_\kappa=\boldsymbol{\alpha}\cdot\left[i\boldsymbol{\partial}-\hat{q}\,\mathbf{A}-\left[\hat{s}-\frac{e}{2}+\frac{e}{2}\gamma^5\right]\mathbf{R}\right]+\alpha^3\kappa+\beta\rho e^{-i\gamma^5\sigma} , \tag{2.18b}$$

where we take the string to lie along the x^3 axis, for later reference we have included a wave number κ for propagation in the longitudinal direction, and vectors are from now on two dimensional and in the x-y plane. Both h_κ and $\hat{h}_\kappa$ share the property

$$\alpha^3h_\kappa\alpha^3=-h_{-\kappa} \tag{2.19}$$

and when $\kappa=0$ their spectrum is symmetric, i.e., if $h_0\Psi_E=E\Psi_E$, $h_0\alpha^3\Psi_E=-E\alpha^3\Psi_E$.

A field configuration is rotationally symmetric if a rotation generates a gauge copy of the field

$$(\mathbf{r}\times\boldsymbol{\partial}\delta^{ij}+\epsilon^{ij})A_j(\mathbf{r})=\partial^i\nu_A(\mathbf{r}) , \tag{2.20}$$

$$(\mathbf{r}\times\boldsymbol{\partial}\delta^{ij}+\epsilon^{ij})R_j(\mathbf{r})=\partial^i\nu_R(\mathbf{r}) , \tag{2.21}$$

$$\mathbf{r}\times\boldsymbol{\partial}\phi(\mathbf{r})=-ie\nu_R(\mathbf{r})\phi(\mathbf{r}) \tag{2.22}$$

[when $\mathbf{r}$ and $\boldsymbol{\partial}$ are restricted to the x-y plane $\mathbf{r}\times\boldsymbol{\partial}$ is the third component of the three-dimensional vector $(\mathbf{r}\times\boldsymbol{\partial})^i$] with the asymptotic conditions

$$\nu_A(\mathbf{r})\rightarrow 0 \text{ as } |\mathbf{r}|\rightarrow\infty , \tag{2.23}$$

$$\nu_R(\mathbf{r})\rightarrow\frac{n}{e} \text{ as } |\mathbf{r}|\rightarrow\infty . \tag{2.24}$$

This symmetry of the background field implies a conserved Noether charge and the single-particle Hamiltonian h_κ commutes with the rotation generator

$$J=i\mathbf{r}\times\boldsymbol{\partial}-\tfrac{1}{2}\gamma^5\alpha^3-q\nu_A(\mathbf{r})-\left[s+\frac{e}{2}-\frac{e}{2}\gamma^5\right]\nu_R(\mathbf{r}) \tag{2.25}$$

and $\hat{h}_\kappa$ with

$$\hat{J}=i\mathbf{r}\times\boldsymbol{\partial}-\tfrac{1}{2}\gamma^5\alpha^3-\hat{q}\nu_A(\mathbf{r})-\left[\hat{s}-\frac{e}{2}+\frac{e}{2}\gamma^5\right]\nu_R(\mathbf{r}) . \tag{2.26}$$

The first terms in (2.25) and (2.26) effect a rotation of the coordinates about the x^3 axis, the second is the spin operator and the third and fourth terms implement compensating gauge transformations. Solving (2.20)–(2.24) for ν_A and ν_R leads to

$$\mathcal{J}=i\mathbf{r}\times\mathbf{D}-\tfrac{1}{2}\gamma^5\alpha^3-\left[\frac{s}{e}+\frac{1}{2}-\frac{1}{2}\gamma^5\right]n+\frac{q}{2\pi}\int_{\mathbf{x}^2\le\mathbf{r}^2}d^2x\,\epsilon^{ij}\partial_iA_j+\left[s+\frac{e}{2}-\frac{e}{2}\gamma^5\right]\frac{1}{2\pi}\int_{\mathbf{x}^2\le\mathbf{r}^2}d^2x\,\epsilon^{ij}\partial_iR_j \tag{2.27}$$

and

$$\hat{\mathcal{J}}=i\mathbf{r}\times\hat{\mathbf{D}}-\tfrac{1}{2}\gamma^5\alpha^3-\left[\frac{\hat{s}}{e}-\frac{1}{2}+\frac{1}{2}\gamma^5\right]n+\frac{\hat{q}}{2\pi}\int_{\mathbf{x}^2\le\mathbf{r}^2}d^2x\,\epsilon^{ij}\partial_iA_j+\left[\hat{s}-\frac{e}{2}+\frac{e}{2}\gamma^5\right]\frac{1}{2\pi}\int_{\mathbf{x}^2\le\mathbf{r}^2}d^2x\,\epsilon^{ij}\partial_iR_j . \tag{2.28}$$

These operators are manifestly gauge invariant.

However, they are not uniquely determined by the Noether prescription. Indeed we can add to them any constant times the unit matrix which also commutes with the single-particle Hamltonian and represents the conserved electric charge. The particular mixture of angular momentum and charge that we consider here is determined by the boundary conditions (2.23) and (2.24). The implicit assumption is that the electromagnetic field $\mathbf{A}(\mathbf{r})$ can be brought to a rotationally covariant form by a gauge transformation with compact support. If this is not possible we would have to modify the boundary condition (2.23) and a further constant would appear in J and $\hat{J}$.

For rotationally symmetric fields in a rotationally covariant gauge,

$$\mathbf{r}\times\mathbf{A}(\mathbf{r})=\frac{1}{2\pi}\int_{\mathbf{x}^2\leq\mathbf{r}^2}d^2x\,\epsilon^{ij}\partial_i A_j(x) \tag{2.29}$$

and

$$\mathbf{r}\times\mathbf{R}(\mathbf{r})=\frac{1}{2\pi}\int_{\mathbf{x}^2\leq\mathbf{r}^2}d^2x\,\epsilon^{ij}\partial_i R_j(x) \tag{2.30}$$

are the magnetic fluxes inside radius r and (2.27) and (2.28) reduce to

$$J=i\mathbf{r}\times\partial-\tfrac{1}{2}\gamma^5\alpha^3-\left[\frac{s}{e}+\frac{1}{2}-\frac{1}{2}\gamma^5\right]n\ , \tag{2.31}$$

$$J=i\mathbf{r}\times\partial-\tfrac{1}{2}\gamma^5\alpha^3-\left[\frac{\hat{s}}{e}-\frac{1}{2}+\frac{1}{2}\gamma^5\right]n\ . \tag{2.32}$$

Finally we note that spectral properties of operators similar to (2.31) and (2.32) were shown in Ref. 37 to represent the vacuum matrix elements of the full angular-momentum operator of the gauge and fermion fields which is constructed from the symmetric gauge-invariant energy-momentum tensor in (2 + 1)-dimensional electrodynamics. We expect that this result generalizes to the four-dimensional cosmic-string models we consider here and the J and $\hat{J}$ weighted spectral asymmetries which we shall introduce in Sec. III represents the eigenvalues of the full angular-momentum operator on the Fock vacuum of the second-quantized fermions. The operators (2.31) and (2.32) have gauge-invariant meaning only when the gauge is fixed to be rotationally covariant.

III. INDEX THEOREMS

In this section we present a proof of the index theorem and the angular-momentum-weighted index theorem for the transverse Hamiltonian h_0 and a computation of the spectral asymmetry and the angular-momentum-weighted spectral asymmetry of h_κ. We shall find that they are determined by asymptotic properties of the background fields. Zero modes of the two-dimensional Dirac operator in background magnetic gauge fields have been known for some time[15–17] and those of Hamiltonians of the type discussed here were first discovered by Jackiw and Rossi.[18] Index theorems for the model considered by them were proven by Weinberg[20] and by Niemi and Semenoff.[29] The results we derive are in full agreement with those found previously when specialized to the appropriate models and applicable to the more general cases relevant to cosmic strings.

Since h_0 anticommutes with α^3 it has a symmetric spectrum and its zero modes are also eigenstates of α^3. Furthermore, the interaction of fermions with the complex scalar field renders them massive and the continuum spectrum of the Dirac operator is bounded away from zero. Consequently the zero modes of h_0 must be square integrable in the x-y plane.

We define

$$\text{index}(h_0)=\lim_{\delta\to\infty}\text{tr}\alpha^3e^{-\delta h_0^2}\ , \tag{3.1a}$$

$$J-\text{index}(h_0)=\lim_{\delta\to\infty}\text{tr}\alpha^3Je^{-\delta h_0^2}\ . \tag{3.1b}$$

The limit $\delta\to\infty$ projects on the zero modes and α^3 or α^3J in the trace weights them by their α^3 eigenvalue or their α^3 eigenvalue and J expectation value. (This would be the J eigenvalue when the vortex is rotationally symmetric.) The latter quantity (3.1b) is an example of a character-weighted index.

In this section we consider only h_κ and J. In Sec. V we shall deduce the pertinent quantities for $\hat{h}_\kappa$ and $\hat{J}$ by the substitution $e,q,s,\sigma(x)\to -e,\hat{q},\hat{s},-\sigma(x)$. For computation of index (h_0) the profile of the vortex fields need not be rotationally covariant. However, in calculating J-index (h_0) we shall need to assume that $[J,h_0]=0$, i.e., that the background fields are rotationally symmetric. In that case, for convenience, we also choose a rotationally symmetric gauge.

The Atiyah-Singer index theorem[43] is applicable to Dirac operators on compact manifolds. If the present index problem were defined on a compact space, or if the background fields were suitably short ranged so that they could be projected on a compact space by conformal mapping, the traces in (3.1a) and (3.1b) could be shown to be independent of δ and evaluated in the limit $\delta\to 0$ where they are accessible to asymptotic expansion of the heat kernel.[44] The index would be proportional to a characteristic class, the U($\tilde{1}$) magnetic flux.

However, vortex fields are long ranged and the index problem must be considered on the open infinite space. The relevant theorem is a generalization of the Atiyah-Patodi-Singer index theorem[45] considered by Callias[46] and Niemi and Semenoff[29–32] and contains besides the characteristic classes of the gauge fields a contribution from the asymptotic surface. This can be isolated by rewriting (3.1a) as

$$\text{index}(h_0)=\lim_{\epsilon\to 0}\lim_{\delta\to\infty}\left[\text{Tr}\alpha^3 e^{-\epsilon h_0^2}+\int_\epsilon^\delta d\nu \frac{d}{d\nu}\text{Tr}\alpha^3 e^{-\nu h_0^2}\right] \tag{3.2}$$

$$=\lim_{\epsilon\to 0}\lim_{\delta\to\infty}\left[\text{Tr}\alpha^3 e^{-\epsilon h_0^2}+\tfrac{1}{2}\int d^2x\, i\partial \text{tr}\left\{x\left|\alpha\alpha^3\frac{e^{-\epsilon h_0^2}-e^{-\delta h_0^2}}{h_0}\right|x\right\}\right] \tag{3.3}$$

$$=\lim_{\epsilon\to 0}\left[\text{Tr}\alpha^3 e^{-\epsilon h_0^2}+\tfrac{1}{2}\int d^2x\, i\partial \text{tr}\left\{x\left|\alpha\alpha^3\frac{e^{-\epsilon h_0^2}}{h_0}\right|x\right\}\right], \tag{3.4}$$

where Tr represents a function-space trace and tr represents a pointwise trace over spinor indices. The second term is a surface integral. This justifies taking δ to infinity in (3.3) since the zero modes of h_0 are L^2 functions and do not contribute to the trace when integrated on the asymptotic surface.

Similarly, with explicit use of $[J,h_0]=0$ for rotationally symmetric fields,

$$J\text{-index}(h_0)=\lim_{\epsilon\to 0}\left[\text{Tr}\alpha^3 J e^{-\epsilon h_0^2}+\tfrac{1}{2}\int d^2x\, i\partial\cdot \text{tr}\left\{x\left|\alpha\alpha^3 J\frac{e^{-\epsilon h_0^2}}{h_0}\right|x\right\}\right]. \tag{3.5}$$

Both (3.4) and (3.5) can be evaluated by straightforward methods. We shall return with details after we examine the η invariant of h_κ,

$$\eta(h_\kappa)=\lim_{\epsilon\to 0}\text{Tr}\,\text{sgn}(h_\kappa)e^{-\epsilon h_0^2}, \tag{3.6}$$

and the J-weighted η invariant

$$\eta_J(h_\kappa)=\lim_{\epsilon\to 0}\text{Tr}J\text{sgn}(h_\kappa)e^{-\epsilon h_0^2}, \tag{3.7}$$

where

$$\text{sgn}(E)=\begin{cases}1, & E\geq 0,\\ -1, & E<0.\end{cases} \tag{3.8}$$

These measure antisymmetric moments of the spectrum of h_κ and are regulated by a manifestly gauge-invariant exponential cutoff.

Using the identity

$$\text{sgn}(h_\kappa)=\int\frac{d\omega}{\pi}\frac{h_\kappa}{h_\kappa^2+\omega^2} \tag{3.9}$$

we rewrite (3.6) as

$$\eta(h_\kappa)=\lim_{\epsilon\to 0}\int\frac{d\omega}{\pi}\text{Tr}\left[\frac{h_\kappa}{h_\kappa^2+\omega^2}e^{-\epsilon h_0^2}\right] \tag{3.10}$$

$$=\lim_{\epsilon\to 0}\int\frac{d\omega}{\pi}\frac{\kappa}{\kappa^2+\omega^2}\text{Tr}\left[\alpha^3\frac{\kappa^2+\omega^2}{h_0^2+\kappa^2+\omega^2}e^{-\epsilon h_0^2}\right] \tag{3.11}$$

$$=\lim_{\epsilon\to 0}\int\frac{d\omega}{\pi}\frac{\kappa}{\kappa^2+\omega^2}\left[\text{Tr}\alpha^3 e^{-\epsilon h_0^2}+\tfrac{1}{2}\int d^2x\, i\partial\cdot\text{tr}\left\{x\left|\alpha^3\alpha\frac{h_0}{h_0^2+\kappa^2+\omega^2}e^{-\epsilon h_0^2}\right|x\right\}\right], \tag{3.12}$$

where we have used the fact that α^3 anticommutes with h_0. Similarly, assuming $[J,h_0]=0$,

$$\eta_j(h_\kappa)=\lim_{\epsilon\to 0}\int\frac{d\omega}{\pi}\frac{\kappa}{\kappa^2+\omega^2}\left[\text{Tr}\alpha^3 J e^{-\epsilon h_0^2}+\tfrac{1}{2}\int d^2x\, i\partial\cdot\text{tr}\left\{x\left|\alpha\alpha^3 J\frac{h_0}{h_0^2+\kappa^2+\omega^2}e^{-\epsilon h_0^2}\right|x\right\}\right]. \tag{3.13}$$

The similarity of (3.12), (3.13) and (3.4), (3.5) is explicit in the limit $\kappa\to 0$ with the identity

$$\lim_{\kappa\to 0}\frac{\kappa}{\kappa^2+\omega^2}=\text{sgn}(\kappa)\pi\delta(\omega), \tag{3.14}$$

which implies

$$\lim_{\kappa\to 0}\eta(h_\kappa)=\text{sgn}(\kappa)\lim_{\epsilon\to 0}\left[\text{Tr}\alpha^3 e^{-\epsilon h_0^2}+\tfrac{1}{2}\int d^2x\, i\partial\cdot\text{tr}\left\{x\left|\alpha\alpha^3\frac{e^{-\epsilon h_0^2}}{h_0}\right|x\right\}\right] \tag{3.15a}$$

and

$$\lim_{\kappa\to 0}\eta_J(h_\kappa)=\text{sgn}(\kappa)\lim_{\epsilon\to 0}\left[\text{Tr}\alpha^3 J e^{-\epsilon h_0^2}+\tfrac{1}{2}\int d^2x\, i\partial\cdot\text{tr}\left\langle x\left|\alpha\alpha^3 J\frac{e^{-\epsilon h_0^2}}{h_0}\right|x\right\rangle\right] . \tag{3.15b}$$

By comparing with (3.4) and (3.5),

$$\lim_{\kappa\to 0^\pm}\eta(h_\kappa)=\pm\text{index}(h_0) , \tag{3.16a}$$

$$\lim_{\kappa\to 0^\pm}\eta_J(h_\kappa)=\pm J\text{-index}(h_0) . \tag{3.16b}$$

The $\epsilon\to 0$ limit of the trace of the heat kernel can be evaluated using an asymptotic expansion. We first represent the trace pointwise and in a plane-wave base:

$$\text{Tr}\alpha^3 e^{-\epsilon h_0^2}=\int d^2x \lim_{y\to x}\text{tr}\alpha^3 e^{-\epsilon h_0^2(x,i\partial)}\mathcal{I}\delta(x-y) \tag{3.17}$$

$$=\int d^2x \lim_{y\to x}\int\frac{d^2k}{(2\pi)^2}\text{tr}\alpha^3 e^{-\epsilon h_0^2(\mathbf{x},i\partial)}\mathcal{I}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{3.18}$$

$$=\int d^2x\int\frac{d^2k}{(2\pi)^2}\text{tr}\alpha^3 e^{-\epsilon h_0^2(\mathbf{x},i\partial+\mathbf{k})}\mathcal{I} , \tag{3.19}$$

where $\mathcal{I}$ is the unit matrix,

$$h_0^2(\mathbf{x},i\partial+\mathbf{k})=\mathbf{k}^2+2i\mathbf{k}\cdot\mathbf{D}+h_0^2(\mathbf{x},i\partial) \tag{3.20}$$

and

$$h_0^2(\mathbf{x},i\partial)=(i\mathbf{D})^2+\gamma^5\alpha^3\epsilon^{ij}\left[q\partial_i A_j+\left[s+\frac{e}{2}-\frac{e}{2}\gamma^5\right]\partial_i R_j\right]+\rho^2-\beta\boldsymbol{\alpha}\cdot(i\partial+e\gamma^5\mathbf{R})\rho e^{i\gamma^5\sigma} . \tag{3.21}$$

Consequently,

$$\text{Tr}\alpha^3 e^{-\epsilon h_0^2}=\int d^2x\int\frac{d^2k}{(2\pi)^2}e^{-k^2}\sum_n\frac{(-1)^n\epsilon^{n-1}}{n!}\text{tr}\alpha^3\left[h_0^2+2i\frac{\mathbf{k}}{\sqrt{\epsilon}}\cdot\mathbf{D}\right]^n\mathcal{I} \tag{3.22}$$

and

$$\lim_{\epsilon\to 0}\text{Tr}\alpha^3 e^{-\epsilon h_0^2}=\frac{e}{2\pi}\int d^2x\,\epsilon^{ij}\partial_i R_j=\Phi_R , \tag{3.23}$$

which is the U($\tilde{1}$) magnetic flux of the string. When the finite-energy asymptotic conditions (2.13) and (2.14) are satisfied it also is proportional to the vortex number n. However, in the absence of that constraint it is an arbitrary real number. On a compact two-dimensional space it would be an integer.

Similarly, it is straightforward to evaluate

$$\text{tr}\alpha^3 J e^{-\epsilon h_0^2}=\int d^2x\int\frac{d^2k}{(2\pi)^2}e^{-k^2}\sum_n\frac{(-1)^n\epsilon^{n-1}}{n!}\text{tr}\alpha^3\left[J+\mathbf{r}\times\frac{\mathbf{k}}{\sqrt{\epsilon}}\right]\left[h_0^2+2i\frac{\mathbf{k}}{\sqrt{\epsilon}}\cdot\mathbf{D}\right]^n\mathcal{I} \tag{3.24}$$

and

$$\lim_{\epsilon\to 0}\text{Tr}\alpha^3 J e^{-\epsilon h_0^2}=\frac{1}{4\pi}\int d^2x\left\{n\epsilon^{ij}\left[q\partial_i A_j+\left[s+\frac{e}{2}\right]\partial_i R_j\right]+e\left[s+\frac{e}{2}\right]\mathbf{r}\times\mathbf{R}\epsilon^{ij}\partial_i R_j+eq\epsilon^{ij}\partial_i A_j\mathbf{r}\times\mathbf{R}\right.$$
$$\left.+eq\epsilon^{ij}\partial_i R_j\mathbf{r}\times\mathbf{A}\right\}-\left[\frac{s}{e}+\frac{1}{2}\right]n\Phi_R , \tag{3.25}$$

where we have used a rotationally covariant gauge. Note that (3.25) is invariant under rotationally symmetric gauge transformations. Here and in the following we indicate the contribution of the term in J which is proportional to the unit matrix [and $(s/e+\frac{1}{2})n$] by writing it separately and at the end of the formula. It is this term which is sensitive to the boundary condition (2.23).

For rotationally covariant background fields the identities (2.32) and (2.33) yield

$$\frac{1}{2\pi}\int d^2x\,\mathbf{r}\times\mathbf{R}\epsilon^{ij}\partial_i R_j=\frac{1}{2}\left[\frac{1}{2\pi}\int d^2x\,\epsilon^{ij}\partial_i R_j\right]^2 . \tag{3.26}$$

Using a similar identify for the R-A cross terms we obtain the gauge-invariant result

$$\lim_{\epsilon\to 0}\alpha^3 J e^{-\epsilon h_0^2}=(n+\Phi_R)\left[q\Phi_A+\left[\frac{s}{e}+\frac{1}{2}\right]\Phi_R\right]-\left[\frac{s}{e}+\frac{1}{2}\right]n\Phi_R\ . \tag{3.27}$$

By similar methods, the surface terms in (3.12) and (3.13) are

$$\lim_{\epsilon\to 0}\int d^2x\, i\partial\cdot\mathrm{Tr}\left[x\left|\alpha\alpha^3\frac{h_0 e^{-\epsilon h_0^2}}{h_0^2+\kappa^2+\omega^2}\right|x\right]=-\frac{1}{2\pi}\oint dl\cdot\frac{\phi^*(i\partial+\mathbf{R})\phi}{\phi^*\phi+\kappa^3+\omega^2}=-\frac{|\phi|_\infty^2}{|\phi|_\infty^2+\kappa^2+\omega^2}(n+\Phi_R) \tag{3.28}$$

and

$$\lim_{\epsilon\to 0}\int d^2x\, i\partial\cdot\mathrm{Tr}\left[x\left|\alpha J\alpha^3\frac{h_0 e^{-\epsilon h_0^2}}{h_0^2+\kappa^2+\omega^2}\right|x\right]=\frac{2|\phi|_\infty^2}{|\phi|_\infty^2+\kappa^2+\omega^2}\left\{(n+\Phi_R)\left[q\Phi_A+\left[\frac{s}{e}+\frac{1}{2}\right]\Phi_R\right]-\frac{1}{2}\left[\frac{s}{e}-\frac{1}{2}\right]n\Phi_R\right\}. \tag{3.29}$$

Using (3.4a), (3.4b), (3.12a), (3.12b), (3.28), and (3.29) we deduce

$$\eta(h_\kappa)=\mathrm{sgn}(\kappa)\left\{\Phi_R\left[\frac{\kappa^2}{\kappa^2+|\phi|_\infty^2}\right]^{1/2}-n\left[1-\left[\frac{\kappa^2}{\kappa^2+|\phi|_\infty^2}\right]^{1/2}\right]\right\}, \tag{3.30}$$

$$\eta_J(h_\kappa)=\mathrm{sgn}(\kappa)(n+\Phi_R)\left[q\Phi_A+\left[\frac{s}{e}+\frac{1}{2}\right]\Phi_R\right]\left[\frac{\kappa^2}{|\phi|_\infty^2+\kappa^2}\right]^{1/2}-\left[\frac{s}{e}+\frac{1}{2}\right]n\eta(h_\kappa)\ , \tag{3.31}$$

and using (3.16a) and (3.16b) we obtain

$$\mathrm{index}(h_0)=-n \tag{3.32}$$

and

$$J\text{-index}(h_0)=\left[\frac{s}{e}+\frac{1}{2}\right]n^2$$
$$=-\left[\frac{s}{e}+\frac{1}{2}\right]n\ \mathrm{index}(h_0)\ . \tag{3.33}$$

If the asymptotic conditions (2.13) and (2.14) were satisfied we would have

$$\eta(h_\kappa)_{\Phi_R=-n}=-n\ \mathrm{sgn}(\kappa) \tag{3.34}$$

and

$$\eta_J(h_\kappa)_{\Phi_R=-n}=\left[\frac{s}{e}+\frac{1}{2}\right]n^2\mathrm{sgn}(\kappa)\ . \tag{3.35}$$

For axion strings $\Phi_R=0$ and we would have

$$\eta(h_\kappa)_{R=0}=-\mathrm{sgn}(\kappa)n\left[1-\left[\frac{\kappa^2}{|\phi|_\infty^2+\kappa^2}\right]^{1/2}\right] \tag{3.36}$$

and

$$\eta_J(h_\kappa)_{R=0}=\mathrm{sgn}(\kappa)\left[\frac{\kappa^2}{|\phi|_\infty^2+\kappa^2}\right]^{1/2}nq\Phi_A$$
$$-\left[\frac{s}{e}+\frac{1}{2}\right]n\eta(h_\kappa)_{R=0}\ . \tag{3.37}$$

We note that the indices and η invariants are determined by the asymptotic properties of the background fields. The index always depends only on the winding number of the string n. The η invariant also can depend on the magnetic flux. When the finite-energy condition (2.13) and (2.14) is satisfied η and η_J are proportional to the index and the J-index, respectively. On the other hand, when the $U(\tilde{1})$ magnetic flux vanishes they remain nontrivial functions of κ and are linear and bilinear in the winding number n.

Note that η is proportional to the index when $\kappa\to 0$ and is proportional to the characteristic class Φ_R when $\kappa\to\infty$. Furthermore, these limits are identical when $\Phi_R=-n$ and the vortex is localized.

Finally, notice that the J-index obtains nontrivial contributions only from the unit matrix component of J proportional to $(s/e+\frac{1}{2})n$. In the next section we shall confirm the present results for the index and J-index by explicit solution of the zero-mode problem. No such explicit confirmation is available for the η invariants we have computed.

IV. EXPLICIT SOLUTIONS

In this section we shall present explicit solutions for the zero modes of the Dirac Hamiltonian with a rotationally symmetric vortex background field. We shall find that, regardless of the value of the gauge fields, as long as they decrease at $r\to\infty$ at least as r^{-1}, the number of zero modes and their angular momentum eigenvalues depend only on the winding number of the scalar field.

This is in contrast with the zero modes of the Hamiltonian in the absence of coupling to the scalar fields. In that case, consider the zero-eigenvalue problem for the 2×2 Hamiltonian:

$$\sigma\cdot[i\partial-\mathbf{A}(\mathbf{r})]\psi_0(\mathbf{r})=0\ , \tag{4.1}$$

where $\sigma=(\sigma^1,\sigma^2)$ are the first two Pauli matrices. The gauge field is transverse and can therefore be written

$$A^i(\mathbf{r})=\epsilon^{ij}\partial_j A(\mathbf{r}) , \tag{4.2}$$

where

$$A(\mathbf{r})=\frac{1}{-\partial^2}B(\mathbf{r})=\frac{1}{2\pi}\int d\mathbf{r}_1 \ln|\mathbf{r}-\mathbf{r}_1|B(\mathbf{r}_1) \tag{4.3}$$

and where $B=\epsilon^{ij}\partial_i A_j$ is the magnetic field. Asymptotically, as $r\to\infty$,

$$A(\mathbf{r})\to\Phi\ln r \tag{4.4}$$

with magnetic flux Φ. Using the identity $\sigma^i\epsilon^{ij}=-i\sigma^j\sigma^3$ the eigenvalue problem (4.1) can be rewritten

$$\sigma\cdot[\partial+\partial A(\mathbf{r})]\psi_0(\mathbf{r})=0 . \tag{4.5}$$

This equation is solved by

$$\psi_0(\mathbf{r})=\begin{bmatrix} e^{-A(\mathbf{r})}u(\mathbf{r}) \\ e^{A(\mathbf{r})}v(\mathbf{r}) \end{bmatrix} \tag{4.6}$$

and

$$\left[\partial_r+\frac{i}{r}\partial_\theta\right]u(r,\theta)=0 , \tag{4.7a}$$

$$\left[\partial_r-\frac{i}{r}\partial_\theta\right]v(r,\theta)=0 . \tag{4.7b}$$

Equations (4.7a) and (4.7b) have the solutions

$$u(r,\theta)=e^{i\nu\theta}r^\nu , \tag{4.8a}$$

$$v(r,\theta)=e^{-i\nu\theta}r^\nu , \tag{4.8b}$$

where ν is an integer. Regularity at the origin requires that $\nu\geq 0$. (Regularity, rather than integrability, is required at the origin so that the Hamiltonian is self-adjoint.) Since by (4.4) $e^{\pm A(r)}\to r^{\pm\Phi}$ as $r\to\infty$, normalizability requires that $\nu<\Phi-1$ for $u(r,\phi)$ and $\nu<-\Phi-1$ for $v(r,\theta)$. Therefore the number of zero modes are given by the largest integer which is less than $|\Phi|$ and their helicity by $\text{sgn}(\Phi)$. This agrees with the index theorem[32] for the Hamiltonian operator on an open space and explicitly verifies a vanishing theorem—the zero modes are all of a given helicity. In the special case of rotationally symmetric gauge field this result was obtained in Refs. 15, 16, and 28.

We now consider the transverse Hamiltonian (2.18a) with the rotationally covariant background fields (2.15) and (2.16). With the identities

$$A^i=\epsilon^{ij}\partial_j\mathcal{A}(r),\quad \mathcal{A}(r)=-\frac{1}{\partial^2}B_A(r) , \tag{4.9a}$$

$$R^i=\epsilon^{ij}\partial_j\mathcal{R}(r),\quad \mathcal{R}(r)=-\frac{1}{\partial^2}B_R(r) , \tag{4.9b}$$

and, since $i,j=1,2$,

$$\alpha^i\epsilon^{ij}=-i\alpha^j\Sigma_3,\quad \Sigma_3=\gamma^5\alpha_3 , \tag{4.10}$$

the Hamiltonian (2.18a) can be presented as

$$h_0=K[i\alpha\cdot\partial+\beta f(r)e^{i\gamma^5 n\phi-2q\mathcal{A}\Sigma_3+(2s+e)\mathcal{R}\Sigma_3}]K , \tag{4.11}$$

where

$$K=\exp\left[q\mathcal{A}\Sigma_3+\left[s+\frac{e}{2}-\frac{e}{2}\gamma^5\right]\mathcal{R}\Sigma_3\right]$$

and the zero-eigenvalue problem for h_0 reduces to

$$[i\alpha\cdot\partial+\beta f(r)e^{i\gamma^5 n\phi-2q\mathcal{A}\Sigma_3-(2s+e)\mathcal{R}\Sigma_3}]\zeta(\mathbf{r})=0 . \tag{4.12}$$

With

$$\zeta(\mathbf{r})=\sum_{l=-\infty}^{\infty}e^{il\phi}\begin{bmatrix} u_l^+(r) \\ u_l^-(r) \\ v_l^+(r) \\ v_l^-(r)\end{bmatrix} , \tag{4.13}$$

(4.12) reduces to the coupled equations

$$i\left[\partial_r+\frac{l}{r}\right]u_l^-(r)+f(r)e^{-2q\mathcal{A}(r)-(2s+e)\mathcal{R}(r)}v_{l+n-1}^+(r)=0 , \tag{4.14a}$$

$$-i\left[\partial_r-\frac{l+n-1}{r}\right]v_{l+n-1}^+(r)+f(r)e^{2q\mathcal{A}(r)+(2s+e)\mathcal{R}(r)}u_l^-(r)=0 , \tag{4.14b}$$

$$i\left[\partial_r-\frac{l}{r}\right]u_l^+(r)+f(r)e^{2q\mathcal{A}(r)+(2s+e)\mathcal{R}(r)}v_{l+n+1}^-(r)=0 , \tag{4.14c}$$

$$-i\left[\partial_r+\frac{l+n+1}{r}\right]v_{l+n+1}^-(r)+f(r)e^{-2q\mathcal{A}(r)-(2s+e)\mathcal{R}(r)}u_l^+(r)=0 . \tag{4.14d}$$

Asymptotically,

$$e^{\pm[2q\mathcal{A}(r)+(2s+e)\mathcal{R}(r)]}\sim r^{\pm[2q\Phi_A+(2s/e+1)\Phi_R]}\quad\text{at } r\to\infty , \tag{4.15a}$$

$$e^{\pm[2q\mathcal{A}(r)+(2s+e)\mathcal{R}(r)]}\sim\text{const}\quad\text{at } r=0 . \tag{4.15b}$$

Each pair of equations [(4.14a), (4.14b), and (4.14c), (4.14d)] has two solutions and the normalizable solution for $u^\pm$, $v^\pm$ near $r\to\infty$ has the asymptotic form $e^{-|\phi|_\infty r}$

modified by powers of r. Near $r=0$,

$$u_l^-(r)\sim r^{-l},\quad v_{l+n-1}^+\sim r^{l+n-1}\,, \tag{4.16a}$$

or

$$u_l^+(r)\sim r^{l},\quad v_{l+n+1}^-(r)\sim r^{-l-n-1}\,. \tag{4.16b}$$

Since we can choose only one solution at ∞, we must have both solutions regular near 0. Then either (4.16a) or (4.16b) (but not both) yield solutions with

$$1-n\le l\le 0 \quad \text{if } n\ge 1 \tag{4.17a}$$

or

$$0\le l\le -n-1 \quad \text{if } n\le -1\,, \tag{4.17b}$$

respectively. Therefore, if $n\ge 1$,

$$\zeta_l(r,\phi)=K^{-1}(r)\begin{bmatrix}0\\ e^{il\phi}u_l^-(r)\\ e^{i(l+n-1)\phi}v_{l+n-1}^+(r)\\ 0\end{bmatrix} \tag{4.18a}$$

with $l=1-n,2-n,\ldots,0$. If $n\le -1$,

$$\zeta_l(r,\phi)=K^{-1}(r)\begin{bmatrix}e^{il\phi}u_l^+(r)\\ 0\\ 0\\ e^{i(l+n+1)\phi}v_{l+n+1}^-(r)\end{bmatrix} \tag{4.18b}$$

with $l=0,1,\ldots,-n-1$. These are eigenstates of α_3 with eigenvalues -1 and $+1$, respectively.

Furthermore (4.18a) and (4.18b) are eigenstates of J with eigenvalues

$$\frac{1-n-2l}{2}-\left[\frac{s}{e}+\frac{1}{2}\right]n,\quad l=1-n,\ldots,0$$

or

$$\frac{-n+1+2l}{2}-\left[\frac{s}{e}+\frac{1}{2}\right]n,\quad l=0,1,\ldots,-n-1\,,$$

respectively. We see that the eigenvalues of $J+(s/e+\frac{1}{2})n$ are symmetric about zero in both cases.

These results agree with the index theorems which state that $\text{index}(h_0)=-n$ and $J\text{-index}(h_0)=-(s/e+\frac{1}{2})n\,\text{index}(h_0)$.

V. SUPERCONDUCTING COSMIC STRINGS

The index determines the difference between the number of zero modes with positive and negative eigenvalues of α^3. In the absence of a vanishing theorem which would imply that all zero modes have the same helicity there can be further nongeneric pairs of zero modes whose existence are not guaranteed by the index theorem and whose energy is therefore susceptible to perturbations of the vortex background. For a rotationally symmetric vortex the explicit solution found in Sec. IV indicates that the number of zero modes is given by the index. Proof of a vanishing theorem for the more general case of arbitrary background fields remains an open problem. (Note that the results of Sec. IV for the Dirac operator with no coupling to the scalar field imply both the index theorem and vanishing theorem for that operator.)

Here we shall consider a rotationally symmetric vortex lying along the x^3 axis and with winding number $n>0$. We assume that the transverse components of the gauge fields and the scalar are independent of x_0 and x_3 and that $R_0=R_3=0$ and we take account the possibility of a longitudinal U(1) electric field by the time-dependent but $\mathbf{r}$-independent gauge potentials $A_0(x^3,t)$ and $A_3(x^3,t)$ and $R_0(x^3,t)$ and $R_3(x^3,t)$.

The transverse Hamiltonian h_0 has n zero modes with negative helicity and $\hat{h}_0$ has n zero modes with positive helicity. We shall consider the low-energy effective dynamics of the fermions. If we assume that only the zero modes of the transverse Hamiltonian are relevant and with

$$\Psi_\alpha(\mathbf{r},x^3,t)=\sum_{l=1-n}^{0}\zeta_{\alpha l}(\mathbf{r})u_{\alpha l}(x^3,t)\,, \tag{5.1a}$$

$$\Psi_\beta(\mathbf{r},x^3,t)=\sum_{l=0}^{n-1}\hat{\zeta}_{\beta l}(\mathbf{r})\hat{u}_{\beta l}(x^3,t)\,, \tag{5.1b}$$

where $\zeta_{\alpha l}$ and $\hat{\zeta}_{\beta l}$ are the zero-mode wave functions, the effective action resulting from (2.9) and (2.10) represents a collection of right- and left-handed two-spacetime-dimensional fermions

$$S_{\text{eff}}=\int dx^3dt\left[\sum_{\alpha,l=1-n}^{0}u_{\alpha l}^\dagger(x^3,t)iD_-^\alpha u_{\alpha l}(x^3,t)+\sum_{\beta,l=0}^{n-1}\hat{u}_{\beta l}^\dagger(x^3,t)i\hat{D}_+^\beta\hat{u}_{\beta l}(x^3,t)\right], \tag{5.2}$$

where

$$iD_\pm^\alpha=i(\partial_0\pm\partial_3)-q_\alpha[A_0(x^3,t)\pm A_3(x^3,t)]\,, \tag{5.3a}$$

$$i\hat{D}_\pm^\beta=i(\partial_0\pm\partial_3)-\hat{q}_\beta[A_0(x^3,t)\pm A_3(x^3,t)]\,. \tag{5.3b}$$

Chiral gauge anomalies for this effective theory are guaranteed to cancel by the conditions (2.3) for the four-dimensional theory. The gauge fields couple to the conserved two-dimensional right- and left-handed currents j_a and $\hat{j}_a$, respectively, where

$$j_a(x^3,t)=\sum_{\alpha,l=1-n}^{0}q_\alpha u_{\alpha l}^\dagger(x^3,t)u_{\alpha l}(x^3,t)(1,1)\,, \tag{5.4a}$$

$$\hat{j}_b(x^3,t)=\sum_{\beta,l=0}^{n-1}\hat{q}_\beta\hat{u}_{\beta l}^\dagger(x^3,t)\hat{u}_{\beta l}(x^3,t)(1,-1)\,. \tag{5.4b}$$

The anomaly equation for j_a is

$$\partial_a j^a=-\frac{n}{2\pi}\sum_\alpha q_\alpha^2(\partial_0A_3-\partial_3A_0) \tag{5.5a}$$

and for $\hat{j}_a$ is

$$\partial_a \hat{j}^a = \frac{n}{2\pi} \sum_\beta \hat{q}_\beta^2 (\partial_0 A_3 - \partial_3 A_0) \ . \tag{5.5b}$$

The anomaly for the gauge current $j_a + \hat{j}_a$ would vanish if condition (2.3a) is satisfied. Equations (5.4a), (5.4b), (5.5a), and (5.5b) also imply

$$\partial_t (j_3 + \hat{j}_3) - \partial_3 (j^0 + \hat{j}^0) = -\frac{n}{2\pi} \left[\sum_\alpha q_\alpha^2 + \sum_\beta \hat{q}_\beta^2 \right] \times (\partial_0 A_3 - \partial_3 A_0) \ . \tag{5.6}$$

In particular, for $A_0 = 0$ and constant electric field $A_3 = Et$,

$$\partial_t [j_3(t) + \hat{j}_3(t)] = -\frac{n}{2\pi} \left[\sum_\alpha q_\alpha^2 + \sum_\beta \hat{q}_\beta^2 \right] E \ , \tag{5.7}$$

which implies superconductivity of the vortex. In an electric field the electric current changes linearly with time:

$$j_3(t) + \hat{j}_3(t) = -\frac{n}{2\pi} \left[\sum_\alpha q_\alpha^2 + \sum_\beta \hat{q}_\beta^2 \right] Et + \text{const} \ . \tag{5.8}$$

The fermions also carry angular momentum with the left- and right-handed charge currents:

$$\mathcal{J}_a(x^3,t) = \sum_{\alpha, l = 1-n}^{0} \left[\frac{1-n-2l}{2} + \left[\frac{s}{e} + \frac{1}{2} \right] n \right] u_{\alpha l}^\dagger (x^3,t) u_{\alpha l}(x^3,t)(1,1) \ , \tag{5.9a}$$

$$\hat{\mathcal{J}}_a(x^3,t) = \sum_{\beta, l = 0}^{n-1} \left[\frac{-n+1+2l}{2} + \left[\frac{\hat{s}_\beta}{e} - \frac{1}{2} \right] \right] \hat{u}_{\beta l}^\dagger (x^3,t) \hat{u}_{\beta l}(x^3,t)(1,-1) \ , \tag{5.9b}$$

which have the axial anomalies

$$\partial_a \mathcal{J}^a = -\frac{n^2}{2\pi} \sum_\alpha q_\alpha \left[\frac{s_\alpha}{e} + \frac{1}{2} \right] (\partial_0 A_3 - \partial_3 A_0) \ , \tag{5.10a}$$

$$\partial_a \hat{\mathcal{J}}^a = \frac{n^2}{2\pi} \sum_\beta \hat{q}_\beta \left[\frac{\hat{s}_\beta}{e} - \frac{1}{2} \right] (\partial_0 A_3 - \partial_3 A_0) \ . \tag{5.10b}$$

With the anomaly cancellation constraint (2.3b) the angular momentum current would be conserved:

$$\partial_a (\mathcal{J}^a + \hat{\mathcal{J}}^a) = 0 \ . \tag{5.11}$$

The dual to $\mathcal{J}^a + \hat{\mathcal{J}}^a$ has the anomaly

$$\partial_a \epsilon^{ab} (\mathcal{J}_b + \hat{\mathcal{J}}_b) = -\frac{n^2}{2\pi} \left[\sum_\alpha q_\alpha \left[\frac{s_\alpha}{e} + \frac{1}{2} \right] + \sum_\beta \hat{q}_\beta \left[\frac{\hat{s}_\beta}{e} - \frac{1}{2} \right] \right] \times (\partial_0 A_3 - \partial_3 A_0) \tag{5.12}$$

which results in superconductivity of the angular momentum.

Note that both anomalies for the angular momentum in (5.10a) and (5.10b) arise from the component of J proportional to the unit matrix; i.e., it is a result of the particular mixture of kinetic angular momentum and charges required by the boundary conditions (2.23) and (2.24) and is therefore a direct result of the axial anomaly for the charges. Also note that this differs from the results of Callan and Harvey[42] who discussed a gravitational anomaly for the covariant energy-momentum tensor and therefore also the covariant angular momentum in a background gravitational field. Theirs is an anomaly for the covariant energy-momentum tensor of the fermions and is induced by an external gravitational field. The present case is an anomaly for the conserved canonical angular momentum of the fermions and is induced by external electromagnetic fields.

Finally, in the case of the axionic string where the anomaly cancellation constraints (2.3) need not apply the angular momentum would have a two-dimensional anomaly. We shall discuss this effect in more detail in the following section.

VI. DISCUSSION

We have considered the dynamics of fermions in the background gauge and complex scalar fields of a cosmic string. We have shown the existence of fermion zero modes by proving an index theorem and finding explicit solutions. Our results agree with previous analyses when specialized to the appropriate models.

We have also obtained the η invariant and J-weighted η invariant of the transverse Hamiltonian. These contain information about the quantum numbers of the fermionic ground state.[30] The close relationship between η invariants and indices indicates a similar relationship between ground-state expectation values of the electric charge and angular momentum and the anomalies of the effective two-dimensional currents. In this section we shall use the results of Sec. III for the η invariant and J-weighted η invariant to obtain anomaly equations for the electric charge and angular momentum densities, respectively.

In two spacetime dimensions the chiral anomaly has a well-known physical interpretation.[47–49] Massless fermions propagate at the speed of light with direction determined by their helicity. An external electric field flips helicities resulting in nonconservation of the electric charge for each helicity separately. For example, consider the one-dimensional Weyl Hamiltonian

$$h_W = i\partial_x \tag{6.1}$$

whose spectrum is given by the continuous parameter κ. The wave function for a fermion with energy and momentum κ is a plane wave, $\psi_\kappa = \exp(-i\kappa x)$, and the probability density is a constant $\psi_\kappa^\dagger(x)\psi_\kappa(x)d\kappa=d\kappa$. The vacuum charge of a one-dimensional system of fermions is obtained by summing the charge of the filled Dirac sea. The charge density per unit length per unit wave number is given by

$$\rho(x,\kappa)=q\theta(-\kappa)\ . \tag{6.2}$$

A natural alternative definition of vacuum charge density would be the antisymmetric average over the positive- and negative-energy states

$$\rho(x,\kappa)=-\frac{q}{2}\text{sgn}(\kappa)\ . \tag{6.3}$$

This is odd under charge conjugation, $\kappa\to-\kappa$ and corresponds to a commutator normal ordering of the second-quantized charge-density operator:

$$\rho(x)=\langle\tfrac{1}{2}[\psi^\dagger(x),\psi(x)]\rangle\ . \tag{6.4}$$

In either case the total charge density per unit length is undefined, in (6.2) it is infinite, in (6.3) it is the ambiguous difference of two linearly diverging integrals over κ.

However if we consider the same Hamiltonian in a constant electric field in $A_0=0$ gauge

$$h_W=i\partial_x-qEt \tag{6.5}$$

has the time-dependent spectrum $\kappa-Et$ and the time derivative of the charge density is

$$\begin{aligned}\frac{\partial}{\partial t}\rho(x)&=\int\frac{d\kappa}{2\pi}\frac{\partial}{\partial t}\rho(x,\kappa-qEt)\\&=\frac{-qE}{2\pi}[\rho(x,\infty)-\rho(x,-\infty)]\\&=\frac{1}{2\pi}q^2E\end{aligned} \tag{6.6}$$

using either definition (6.2) or (6.3) of $\rho(x,\kappa)$. This is precisely the chiral Schwinger model anomaly equation for constant electric field.

In general the expectation value of the electric charge operator is proportional to the η invariant of the Dirac Hamiltonian. (For a review of how this is derived see Ref. 30.) In a static rotationally symmetric vortex field the charge per unit length of the vortex and per unit of wave number κ is

$$\begin{aligned}\langle Q_\alpha(\kappa)\rangle&=-\tfrac{1}{2}q_\alpha\eta(h_\kappa(q_\alpha))\\&=-\tfrac{1}{2}q_\alpha\text{sgn}(\kappa)\left\{\Phi_R\left[\frac{\kappa^2}{\kappa^2+|\phi|_\infty^2}\right]^{1/2}-n\left[1-\left[\frac{\kappa^2}{\kappa^2+|\phi|_\infty^2}\right]^{1/2}\right]\right\},\end{aligned} \tag{6.7}$$

$$\begin{aligned}\langle\hat{Q}_\beta(\kappa)\rangle&=-\tfrac{1}{2}\hat{q}_\beta\eta(\hat{h}_\kappa(\hat{q}_\beta))\\&=\tfrac{1}{2}\hat{q}_\beta\text{sgn}(\kappa)\left\{\Phi_R\left[\frac{\kappa^2}{\kappa^2+|\phi|_\infty^2}\right]^{1/2}-n\left[1-\left[\frac{\kappa^2}{\kappa^2+|\phi|_\infty^2}\right]^{1/2}\right]\right\}.\end{aligned} \tag{6.8}$$

The charge per unit length is obtained by integrating (6.7) and (6.8) over the wave number κ. This integration is ambiguous—it is essentially the difference of two linearly divergent quantities.

In a constant longitudinal electric field we would replace the wave number κ by the time-dependent eigenvalue of the covariant derivative $i\partial_3-q_\alpha A_3$, $\kappa-q_\alpha Et$. Then the time derivative of (6.7) and (6.8) has a well-defined integral over κ and yields

$$\begin{aligned}\partial_t\langle Q_\alpha(t)\rangle&=-\tfrac{1}{2}\int\frac{d\kappa}{2\pi}\partial_t q_\alpha\eta(h_{\kappa-q_\alpha Et}(q_\alpha))\\&=\frac{q_\alpha^2E}{4\pi}[\eta(h_\infty)-\eta(h_{-\infty})]\\&=q_\alpha^2\Phi_R\frac{E}{2\pi}\ .\end{aligned} \tag{6.9}$$

Similarly

$$\partial_t\langle\hat{Q}_\beta(t)\rangle=-\hat{q}_\beta^{\,2}\Phi_R\frac{E}{2\pi}\ . \tag{6.10}$$

Note that (6.9) and (6.10) are just what we would expect from the mixed anomaly equation

$$\partial_\mu j_\alpha^\mu(\mathbf{x},t)=\frac{q_\alpha^2}{4\pi^2}\mathbf{E}(\mathbf{x},t)\cdot\mathbf{B}_R(\mathbf{x},t)$$

which we would obtain for the AAR triangle anomaly.

The total electric charge is conserved because of the anomaly cancellation condition (2.3a) $\partial_t(\sum_\alpha\langle Q_\alpha(t)\rangle+\sum_\beta\langle\hat{Q}_\beta(t)\rangle)=0$. For the electric charges of individual species of fermions (6.9) and (6.10) agrees with what we would expect from the two-dimensional anomaly computation (5.5) only when the asymptotic conditions (2.13) and (2.14) are satisfied and $\Phi_R=-n$. When the asymptotic condition is not satisfied the discrepancy between the anomalies (6.9), (6.10), and (5.5) must be compensated by four-dimensional effects. This is particularly important in the axion model where $\Phi_R=0$ and $(\partial/\partial t)\langle Q_\alpha(t)\rangle_{R=0}=0$. Our results imply that when $\Phi_R\neq-n$ the anomaly of the effective two-dimensional model does not accurately represent the chiral anomalies of the full theory. In particular the effective two-dimensional field-theory analysis is inadequate to conclude that the axionic string superconducts electric charge.

The vacuum expectation value of the angular momentum per unit length and per unit of wave number is proportional to the angular-momentum-weighted η invariant

$$\langle\mathcal{J}_\alpha(\kappa)\rangle=-\tfrac{1}{2}\eta_J(h_\kappa(q_\alpha)) \tag{6.11}$$

and, in a constant longitudinal electric field E,

$$\partial_t\langle \mathcal{J}_\alpha(t)\rangle=\frac{E}{2\pi}(n+\Phi_R)\left[q_\alpha^2\Phi_A+q_\alpha\left[\frac{s_\alpha}{e}+\frac{1}{2}\right]\Phi_R\right]-\frac{E}{2\pi}q_\alpha\left[\frac{s_\alpha}{e}+\frac{1}{2}\right]n\Phi_R\ . \tag{6.12}$$

Similarly,

$$\partial_t\langle \hat{\mathcal{J}}_\beta(t)\rangle=-\frac{E}{2\pi}(n+\Phi_R)\left[\hat{q}_\beta^{\,2}\Phi_A+\hat{q}_\beta\left[\frac{\hat{s}_\beta}{e}-\frac{1}{2}\right]\Phi_R\right]+\frac{e}{2\pi}\hat{q}_\beta\left[\frac{\hat{s}_\beta}{e}-\frac{1}{2}\right]n\Phi_R\ . \tag{6.13}$$

When $\Phi_R=-n$ these agree with the anomaly formulas (5.10)–(5.13) according to which angular momentum for each species of fermions is created along the string. For the gauged cosmic string anomaly cancellation (2.3a) and (2.3b) implies that the total angular momentum is conserved

$$\partial_t\left[\sum_\alpha\langle \mathcal{J}_\alpha(t)\rangle+\sum_\beta\langle \hat{\mathcal{J}}_\beta(t)\rangle\right]=0\ . \tag{6.14}$$

However for the global axionic string we set $\Phi_R=0$ and the anomalies do not necessarily cancel:

$$\partial_t\left[\sum_\alpha\langle \mathcal{J}_\alpha(t)\rangle+\sum_\beta\langle \hat{\mathcal{J}}_\beta(t)\rangle\right]=\left[\sum_\alpha q_\alpha^2-\sum_\beta\hat{q}_\beta^{\,2}\right]\frac{E}{2\pi}n\Phi_A\ . \tag{6.15}$$

Note that this result is insensitive to the value of the term in J in (2.31) and (2.32) proportional to the unit matrix and is therefore independent of the boundary condition (2.23) and (2.24). Furthermore since the anomaly persists we do expect that the axion string superconducts an angular momentum current.

Since the full four-dimensional angular momentum current must be conserved, $\partial_\mu\mathcal{J}^\mu=0$, and by translation invariance in the x^3 direction $\partial_3\mathcal{J}^3=0$, the anomaly (6.12) implies

$$\partial_t\langle \mathcal{J}(t)\rangle=-\int d^2r\,\partial\langle \mathcal{J}(\mathbf{r})\rangle=-\oint d\mathbf{l}\cdot\langle \mathcal{J}(\mathbf{r})\rangle \tag{6.16}$$

and there must be an angular momentum flux through the asymptotic surface. This is a result of the long range of the vortex field.

It is intriguing that U(1) electric and magnetic fields aligned with the string would induce a transverse angular momentum flow resembling an angular momentum Hall effect. We conjecture that this would yield a mechanism for the transfer of energy and momentum of the vortex and the external fields into radiative modes of the electromagnetic and fermion fields. It would further provide a mechanism where angular momentum is transferred between partners in a vortex-antivortex pair with an accompanied absorption of the energy of external electromagnetic fields.

It is straightforward and instructive to demonstrate consistency of our result by estimating the asymptotic spatial angular momentum current which must enter Eq. (6.16). To do so, we must first consider the difference between the canonical angular momentum which we have studied here and the more familiar gauge-invariant symmetric angular momentum operator. Both are Noether currents corresponding to a rotation of the spatial coordinates. If we consider the spacetime transformation

$$\delta_f x^\mu=f^\mu(x)\ , \tag{6.17}$$

where the condition that $f^\mu(x)$ is a Poincaré transformation is that it obeys the Minkowski-space Killing equation $\partial_\mu f_\nu(x)+\partial_\nu f_\mu(x)=0$, the gauge field transforms by its Lie derivative

$$\delta_f A_\mu(x)=f^\nu(x)\partial_\nu A_\mu(x)+\partial_\mu f^\nu(x)A_\nu(x) \tag{6.18}$$

and the scalar and the fermion by

$$\delta_f\psi(x)=f^\mu\partial_\mu\phi(x)\ ,\qquad \delta_f\psi(x)=(f^\nu\partial_\nu+\partial_\mu f_\nu\Sigma^{\mu\nu})\psi(x)\ , \tag{6.19}$$

where $\Sigma^{\mu\nu}$ is the spin matrix. The fermionic component of the resulting Noether current is the canonical energy-momentum tensor (here we consider a single flavor of fermions)

$$\theta_c^{\mu\nu}f_\nu=\bar\psi(x)\gamma^\mu\delta_f\psi(x)\ . \tag{6.20}$$

The transformations of the fields in (6.18) and (6.19) are not gauge invariant. However, they can be augmented by a gauge transformation so that the resulting total transformation refers only to gauge-invariant quantities:

$$\tilde\delta_f A_\mu(x)=f^\nu(x)\partial_\nu A_\mu(x)+\partial_\mu f^\nu(x)A_\nu(x)-\partial_\mu(f^\nu(x)A_\nu(x))=f^\nu(x)F_{\nu\mu}(x) \tag{6.21}$$

and

$$\tilde\delta_f\phi(x)=f^\mu D_\mu\phi(x)\ ,\qquad \tilde\delta_f\psi_\alpha(x)=[f^\nu(x)D_\nu(x)+\partial_\mu f_\nu(x)\Sigma^{\mu\nu}]\psi(x)\ . \tag{6.22}$$

The corresponding Noether current is the symmetric gauge-invariant energy-momentum tensor whose fermionic component

$$\theta_s^{\mu\nu}(x)f_\nu(x)=\theta_c^{\mu\nu}(x)f_\nu(x)+gf^\nu(x)A_\nu(x)j^\mu(x) \tag{6.23}$$

differs from the canonical energy-momentum tensor by the generator of the gauge transformation. The symmetric energy-momentum tensor is not conserved but has the continuity equation[34]

$$\partial_\mu\theta_s^{\mu\nu}(x)=qj_\mu F^{\mu\nu}(x)\ , \tag{6.24}$$

where $j^\mu(x)$ is the total electric current. For the canonical energy-momentum tensor this implies

$$\partial_\mu\theta_c^{\mu\nu}(x)f_\nu(x)=-j^\mu(x)\delta_f A_\mu(x)\ . \tag{6.25}$$

The canonical energy-momentum tensor of the fermions is conserved if the gauge field is symmetric.

It has been argued that it is the matrix elements of this canonical generator of rotations which are relevant to the induced quantum numbers of the fermionic vacuum.[37] (The matrix elements of the full gauge-invariant symmetric angular momentum operator of the gauge and matter fields in a semiclassical approximation are given by the expectation value of the canonical angular momentum operator of the matter fields.[37]) It has also been argued that the induced symmetric angular momentum vanishes in the pertinent limit of infinite mass.[34] With this result, an asymptotic form of the canonical angular momentum current can then be deduced from (6.23) and a result quoted by Callan and Harvey for the asymptotic induced current:

$$\langle j^{\mu}\rangle = -i\frac{q^2}{16\pi^2}\epsilon^{\mu\nu\lambda\rho}\frac{\phi^*\partial_\nu\phi-\phi\partial_\nu\phi^*}{\phi^*\phi}F_{\lambda\rho}$$

which can be derived from the standard axion coupling

$$L_{\text{eff}}=\int d^4x\frac{q^2}{32\pi^2}\sigma(x)\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}\ ,$$

$$\begin{aligned}\langle \theta_c^{\mu\nu}f_\nu\rangle &= q[\mathbf{r}\times\mathbf{A}(\mathbf{r})]_3\langle j^{\mu}\rangle \\ &= -i\frac{q^3}{16\pi^2}\epsilon^{\mu\nu\lambda\rho}\frac{\phi^*\partial_\nu\phi-\phi\partial_\nu\phi^*}{\phi^*\phi}F_{\lambda\rho}[\mathbf{r}\times\mathbf{A}(\mathbf{r})]_3\ .\end{aligned} \tag{6.26}$$

Using $\lim_{r\to\infty} q\mathbf{r}\times\mathbf{A}=\Phi_A$, $F_{03}=E$, $\phi\to|\phi|_\infty e^{-in\theta}$, and considering several flavors of fermions we get

$$\oint dl\cdot\langle \mathcal{J}\rangle = \left[\sum_\alpha q_\alpha^2-\sum_\beta \hat{q}_\beta^{\,2}\right]\frac{E}{2\pi}n\Phi_A \tag{6.27}$$

in agreement with (6.15) and (6.16).

We have assumed the specialized case of a straight-line vortex aligned with the x^3 axis. However our results depend only on global characteristics of the field configurations, the winding number, and the magnetic flux. We therefore expect that an analysis similar to that in Ref. 37 could be used to show that our results are more general—it is likely that for arbitrary background fields the quantum numbers of the fermionic sector of the theory do not depend on the details of the gauge field configurations but only on their topological characteristics. Furthermore, we expect that the semiclassical arguments of Ref. 37 can also be applied here. If we take as a semiclassical ansatz for the ground state a direct product of the fermionic Fock vacuum and a Gaussian wave functional of the gauge and Higgs-boson fields centered about a rotationally symmetric vortex configuration, we would expect that $\langle \mathcal{J}(t)\rangle$ is the expectation value of the full angular momentum operator constructed from the gauge-invariant symmetric energy-momentum tensor of the field theory including gauge and Higgs-field contributions.

Finally, we have not considered gravitational interactions of the fermions with the string. Recent studies of the gravitational fields in topologically massive three-spacetime-dimensional gravity[50–52] have uncovered several interesting features common to the gravitational field of a cosmic string[53,54] and which may also be induced by interactions with chiral fermions. These are currently under investigation.

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