Quantum kinematics of spacetime. I. Nonrelativistic theory

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The sum-over-histories quantum mechanics of a nonrelativistic particle gives a spacetime formulation of that theory in which the preferred Newtonian time does not enter immediately into the quantum kinematics. We investigate on what spacetime hypersurfaces a Schrödinger-Heisenberg formulation of the theory can be recovered and conclude that, in general, this is possible only on the surfaces of constant preferred Newtonian time. The significance of this for a quantum theory of spacetime is discussed.

I. INTRODUCTION

Time plays a central and peculiar role in Hamiltonian quantum mechanics. The scalar product specifying the Hilbert space of states is defined at one instant of time. States specify directly probabilities of observations carried out at one instant of time. Time is the sole observable not represented by an operator in Hilbert space but rather enters the theory as a parameter describing evolution. In the construction of a quantum theory for a specific system, the identification of the time variable becomes a central issue.[1]

In nonrelativistic classical physics time plays a special role which is unambiguously transferred to nonrelativistic quantum mechanics. In special-relativistic quantum mechanics there is already an issue of the choice of time variable, but there is also a resolution. We can construct special-relativistic quantum mechanics using as the peculiar time variable the time of a particular Lorentz frame. The issue is whether the quantum theory, so constructed, is consistent with the equivalence of Lorentz frames. It is. There is a unitary relation between the quantum theories constructed in different Lorentz frames and physical probabilities are therefore Lorentz invariant.

For the construction of quantum theories of spacetime, the issue of the choice of time variable becomes a fundamental difficulty. This is apparent in the history of the efforts to construct a canonical quantum theory of general relativity.[2] These efforts necessarily begin by choosing a foliation of spacetime surfaces. The variable labeling the surfaces is the time. This or that variable, this or that "intrinsic time" or "extrinsic time," has been proposed as a fitting starting point for the construction of a Hamiltonian quantum theory. As yet, however, we have no evidence that the quantum theories constructed from different choices are physically equivalent.[3] It seems natural to conclude that there is a conflict between the framework of Hamiltonian quantum mechanics and the general covariance of theories of spacetime such as general relativity.

Feynman's sum-over-histories formulation of quantum mechanics is a natural alternative starting point for constructing quantum theories of spacetime. Here the problem of the choice of time is neither as immediate nor as central as it is in the Hamiltonian theory. If, however, the sums over histories are regarded only as calculational tools for specifying scalar products of quantum states then equivalent issues arise. For then, to associate the resulting amplitudes with physical probabilities, one must have in hand the scalar product defining the Hilbert space of states. In the identification of the scalar product "at one instant of time" or "on a spacelike surface" the problem of time reenters the theory.

In this series of papers we shall pursue the point of view that the sum-over-histories formulation of quantum mechanics is a more general framework for quantum theory than the Hamiltonian formulation in either the Schrödinger or the Heisenberg pictures. From this point of view the existence of either a Schrödinger-picture or a Heisenberg-picture formulation of quantum mechanics which is equivalent to the sum-over-histories formulation is an issue for investigation rather than a fundamental assumption. To stress this distinction we shall refer to the sum-over-histories formulation of quantum mechanics on the one hand and the Schrödinger-Heisenberg formulation on the other.

We shall apply the sum-over-histories formulation to describe a quantum kinematical framework for spacetime theories for which there may be no preferred time. Sums over histories will be used to define conditional probabilities for observation directly. An observational interpretation of the quantum theory is thus achieved in which a Hilbert space of states on a surface of preferred time does not enter as a primary element. Neither is there a fundamental notion of "state of the system at a moment of time." In this series we shall apply such a sum-over-histories formulation to three theories: nonrelativistic quantum mechanics (paper I), a model nonrelativistic "quantum cosmology" including realistic clocks (paper II), and the general-relativistic quantum mechanics of closed cosmologies (paper III). We thus progress from a nonrelativistic quantum mechanics which has a preferred time to theories where there is none. In each case we shall argue that the sum-over-histories formulation is sufficient for the prediction of observations. In each case we shall investigate under what circumstances a Schrödinger-Heisenberg formulation of the theory can be recovered. For nonrelativistic quantum mechanics this

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can be achieved on surfaces of the preferred Newtonian time and only on these surfaces. For quantum cosmology, however, we only recover a Schrödinger-Heisenberg formulation *approximately* in situations where the initial conditions are such as to allow the approximately classical behavior of clocks.

The goal of this series of papers is to present a kinematical framework for quantum cosmology sufficiently general to be free from the usual difficulties associated with the problem of time, but yet sufficiently specific to allow the precise calculation of the *a priori* probabilities used to interpret that theory.[4] This will be discussed in paper III.

Nonrelativistic quantum mechanics is already a sufficient canvas for illustrating the quantum kinematics we shall describe if it is formulated in a spacetime fashion which employs the preferred nonrelativistic time as little as possible. We shall therefore begin in this paper with the sum-over-histories formulation of nonrelativistic quantum mechanics. From a spacetime point of view there is no fundamental distinction between observations made at one instant of the preferred time and observations made at different times. The sum-over-histories formulation of quantum mechanics gives a unified and democratic prescription for the probabilities of both. It does this in a way which does not depend on the introduction of a Hilbert space and a preferred time. The results are fully equivalent to the usual measurement theory.[5] From the sum-over-histories point of view, the preferred Newtonian time enters not in the kinematics which describe observation but in the special property of nonrelativistic histories that they move only forward in that time. Starting from the sum-over-histories formulation we shall investigate on what surfaces it is possible to construct a Hilbert space whose scalar products give the conditional probabilities of a particular set of observations. We shall conclude that probabilities for observations made on a surface of the constant preferred time can be related to an inner product on a Hilbert space while those made at different instants of this time are in general not so related. The possibility of introducing a Schrödinger-Heisenberg formulation of nonrelativistic quantum mechanics on a surface of preferred Newtonian time is thus, in part, a consequence the special property of nonrelativistic particle histories that they move forward in that time.

II. SUM OVER HISTORIES QUANTUM MECHANICS

A. General formulation

There are four general ingredients of the sum-over-histories formulation of the quantum mechanics of a physical system:[6–8] (1) the histories, (2) the action, (3) the measure, and (4) the basic observables for which joint probabilities and the conditional probabilities for exhaustive and exclusive observations can be calculated. In this section we shall describe the general sum-over-histories formulation and indicate each of these ingredients in the case of the nonrelativistic quantum mechanics of a single particle.

(1) The histories: A history $\mathcal{H}$ is the set of observables and auxiliary labels which describe the results of all possible experiments. For a nonrelativistic particle a history is a world line in three dimensions $\mathbf{X}(\tau)$. For a relativistic particle it is a world line in four dimensions, $x^\alpha(\tau)$, and for a relativistic string it is a world sheet $x^\alpha(\tau,\sigma)$. For a gauge field a history is a four-dimensional configuration of a gauge potential $A_\alpha(x)$. For a theory of spacetime a history is a four-geometry. The results of finite numbers of experiments are subsets of history segments—segments of world lines, bounded regions of world sheets, potentials on finite spacetime regions, and four-geometries with boundary.

It is frequently convenient, if not essential, to use unobservable labels to describe a history. Familiar examples are the labels of identical particles, gauge-redundant potentials, and, if spacetime histories as viewed as metrics on manifolds, their coordinate-dependent parts. We shall discuss this distinction between observables and labels more fully in paper II. Labels do not enter in the description of a nonrelativistic particle.

(2) The joint probability amplitude for a history: The joint probability amplitude for a history segment is

$$\Phi(\mathcal{H})=\exp[iS(\mathcal{H})]\ , \tag{2.1}$$

where $S[\mathcal{H}]$ is the action functional for the system.

(3) Joint probability amplitudes for an experiment: The *basic observables* in terms of which any particular experiment[9] can be described are specified as restrictions on the histories. For a particular experiment, the observables and labels of a history, $\mathcal{H}$, can be divided into three classes. (i) Those parts of $\mathcal{H}$ fixed by the experimental design. We call these basic observables the *conditions* $\mathcal{C}$. (ii) The results of the experiment whose probabilities are predicted. We call these basic observables the *observations* $\mathcal{O}$. (iii) The parts of the history which are *undetermined,* neither conditioned nor observed, $\mathcal{U}$. These always include the labels.

The joint probability amplitude for observing $\mathcal{O}$ and $\mathcal{C}$ is, from the principle of superposition,

$$\Phi(\mathcal{O},\mathcal{C})=\sum_{\mathcal{U}}\Phi(\mathcal{H})\ . \tag{2.2}$$

Note that conditions and observations enter symmetrically into the joint probability amplitude. Together they are the part of the history *determined* by the experiment. A precise *measure* must be specified to carry out each sums. This is just as important as specifying the action.

(4) Probability: The joint probability for an observation $\mathcal{O}$ and conditions $\mathcal{C}$ is given by

$$p(\mathcal{O},\mathcal{C})=|\Phi(\mathcal{O},\mathcal{C})|^2\ . \tag{2.3}$$

Conditional probabilities can be constructed for exhaustive and exclusive sets of observations. These are sets such that, given the conditions $\mathcal{C}$, one member and one member only, is certain to occur. Conditional probabilities are constructed from the joint probabilities (2.3) by normalization according to the usual classical probability laws. For example, if $\{\mathcal{O}_i\}$ are a discrete set of exhaustive and exclusive observations, the conditional probability of $\mathcal{O}_i$ given $\mathcal{C}$ is

$$p(\mathcal{O}_i \mid \mathcal{C}) = \frac{p(\mathcal{O}_i, \mathcal{C})}{\sum_i p(\mathcal{O}_i, \mathcal{C})} . \quad (2.4)$$

Exhaustive and exclusive sets of observations are determined by an analysis of the possible histories consistent with the conditions $\mathcal{C}$. In nonrelativistic quantum mechanics, for example, the positions at a moment of the Newtonian time are an exhaustive and exclusive set of observables because the particle's history intersects a constant time surface at one and only one place. In a similar way the configurations of a field on an arbitrary spacelike surface are an exhaustive and exclusive set because the field history is a single-valued function of spacetime. However, exhaustive and exclusive sets of observations do not have to be confined to spacelike surfaces or moments of time. We shall give examples in nonrelativistic quantum mechanics below. It is to avoid confusion with a complete set of observables *at a moment of time* in the Schrödinger-Heisenberg formulation that we have used the word "exhaustive" rather than "complete."

Not every set of $\mathcal{O}$ and $\mathcal{C}$ correspond to a possible experiment whose probability is predicted by (2.3). There must be restrictions which allow the isolation of a history segment from the totality of a history. The specific form of these restrictions varies from application to application. A common one for the applications we shall consider is that at least one of the observations $\mathcal{O}$ define the end of a history segment. We shall illustrate the others in the subsequent discussion.

The joint probabilities specified by (2.3) are for $\mathcal{O}$ and $\mathcal{C}$ selected from among the basic observables. In the language of Feynman,[6] we assume the conditions and define "interfering alternatives." In realistic situations, however, there is not always complete information as to what $\mathcal{O}$ and $\mathcal{C}$ are. If information is absent or ignored, the probabilities determined by the above rules must be summed over the absent or ignored quantities in the familiar classical way.

An experiment may not contain conditions or observations which isolate a history segment and indeed most experiments fall in this class. Probabilities for these experiments are computed as though there *were* conditions or observations which isolated the history but whose values are unknown. One sums the *probabilities* computed from (2.3) over these unknown values as discussed above. This is a restatement in more general terms of the injunction to "sum probabilities if you could have measured it but didn't." The precise rules for constructing these probabilities depends on the specific form of the restrictions isolating a history. In this series of papers we shall illustrate several different cases.

Such is the general framework of sum-over-histories quantum mechanics. It is not claimed that its four elements may be specified arbitrarily and yield a sensible theory. Rather, this is a context in which the search for a sensible theory can be conducted. The four ingredients must be consistent with each other and may have to satisfy other criteria as well. In the end we hope for a single specific theory of quantum spacetime and general "frameworks" such as this one can be dispensed with. Until that time, however, they are important as guides to research even if unspecified in all precision.

The sum-over-histories framework differs in several aspects from the Schrödinger-Heisenberg formulation. Probabilities for observation are constructed directly from spacetime histories rather than through a notion of state at a moment of time. As a consequence a preferred time is not a prerequisite for the formulation. However, also as a consequence, there is a preferred set of variables for physical description—those basic observables contained in the histories—and it is assumed that all experiments can be described by these variables. There is no immediate transformation theory as there is in the Schrödinger-Heisenberg formulation. From the sum-over-histories perspective, the existence of the familiar features of the Schrödinger-Heisenberg formulation—a preferred time, a Hilbert space of states, transformation theory, unitary evolution—are possibilities for investigation rather than assumed starting points.[10] In this paper we shall give an example of such an investigation with nonrelativistic quantum mechanics.

B. Nonrelativistic quantum mechanics

The nonrelativistic quantum mechanics of a particle in one dimension gives a familiar and straightforward concrete example of the application of the general sum-over-histories framework described above. The histories are the paths $X(\tau)$ which move forward in Newtonian time in the sense of having a unique X for each τ. From the sum-over-histories point of view, this restriction is the characteristic feature of nonrelativistic theory. The action is

$$S[X(\tau)] = \int_{\tau'}^{\tau''} d\tau\left[\tfrac{1}{2}m\dot{X}^2 - V(X)\right] . \quad (2.5)$$

The measure which reproduces the usual theory is discussed in standard texts.[6–8] There remains the specification of the basic observables.

By basic observables in nonrelativistic quantum mechanics we mean observables which can be measured by an external observer with external apparatus. The division of the universe into a single particle and all else external to it is, of course, an approximation whose range of validity can be discussed in the broader context of cosmology (Refs. 4 and paper III). In asserting certain basic observables for nonrelativistic quantum mechanics we are positing the possibility of external detectors which can measure them.

The basic observables of nonrelativistic theory may be taken to be determinations of whether or not the particle crosses a given spacetime region. That is, we imagine a detector associated with a spacetime region which registers if the particle crosses that region at least once but possibly many times. As idealizations, we include spacetime regions of vanishingly small time dimension thus permitting determinations of position at a given moment of time. This choice of basic observables includes all the variables of a history. There are thus no labels. Positing that the basic observables correspond to spacetime regions does not, of course, guarantee that apparatus can be constructed which registers their values.

As restrictions on the observations and conditions which isolate the history it is convenient to require that the end points of a history segment be determined, that is, that measurements of position which determine the ends of a segment of path be among the observations and conditions of those experiments whose probabilities are given by (2.3). These conditions can be considered as possible conditions for the approximation of isolating the system from its cosmological context. There are other possibilities which yield effectively equivalent theories. For example, one could introduce detectors which emit a particle at a given position and time and similarly detectors which destroy it. Experiments whose probabilities are predicted by (2.3) would include one detector of each type. One could also imagine detectors which introduce particles in a definite way analogous to preparation in a particular "state." We shall confine attention, however, to the simple requirement that there be both conditions and observations which determine the ends of a path segment. Probabilities for experiments which do not precisely localize the end points are to be computed as if the detections which determine the end points were present but their locations were ignored to the precision of the experimental arrangement. In fact, it is usually easiest to simply imagine adding position measurements to the experiment in the far future or far past which determine the end points but whose results are completely ignored. The consistency of this procedure is guaranteed by the causality of nonrelativistic quantum mechanics.

The above basic observables include the familiar "ideal" measurements of position at a moment of time and localization of the particle to a position interval at a moment of time. The latter case corresponds to a spacetime region of zero extent in time which the particle can cross once and only once. As an example of an experiment described in terms of these familiar observables we might consider conditions such that the particle be located at a definite position X_1 at time τ_1 and in an interval Δ_2 at time τ_2. For observations we might ask whether the particle is inside or outside an interval Δ_3 at time τ_3 and an observation of its position X_4 at time τ_4. These four conditions and observations are examples of the types we shall need to discuss (Fig. 1). The joint amplitude for the observations given the conditions is

$$\Phi(X_4\tau_4,\Delta_3\tau_3,\Delta_2\tau_2,X_1\tau_1)=\sum_{\text{paths}} \exp\{iS[X(\tau)]\} \ . \tag{2.6}$$

Here, the action is (2.5) and the sum is over paths which start at X_1 at time τ_1, pass through the intervals Δ_2 and Δ_3 at time τ_2 and τ_3, and end at X_4 at time τ_4. This sum as others for the nonrelativistic particle may be given a definite meaning as the limit of approximate sums defined on a spacetime lattice as we shall illustrate in Sec. IV.

In this experiment an exhaustive and exclusive set of observations might be the position at time τ_4 and whether the particle was in the interval Δ_3 at time τ_3 or outside it in $R-\Delta_3$. A nonrelativistic particle must be at one position at a given time and cannot be at two. The probability for any one observation described in this way is zero because the position at τ_4 has a continuous range. In the familiar way, however, one can ask not for the probability that the precise measurement of position at τ_4 will yield a definite value but rather a value in one or another of a set of small disjoint intervals Δ_4 which comprise the X_4 axis. These and the observations at τ_3 also constitute a complete and exclusive set of observations. The joint probability for each is the sum of the joint probability for definite position over the small intervals. If the intervals Δ_4 have size $|\Delta_4|$ small compared to the characteristic scales of the problem we have

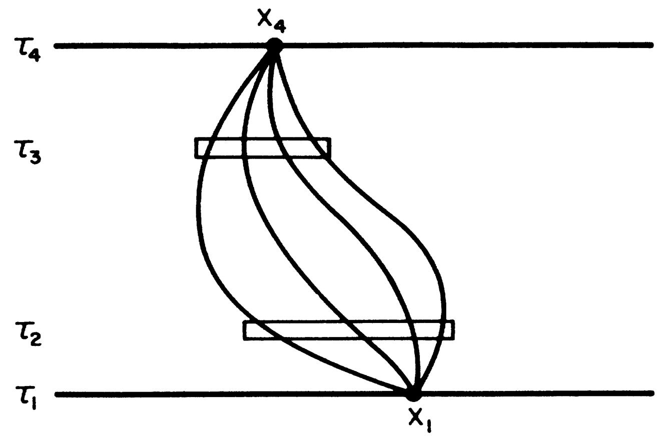

FIG. 1. A simple experiment illustrating the construction of conditional probabilities in nonrelativistic sum-over-histories quantum mechanics. A particle starts localized at τ_1, registers or does not register in detectors at τ_2 and τ_3 and has its position determined at τ_4. The detectors at τ_2 and τ_3 localize the particle to intervals Δ_2 and Δ_3. The joint amplitude to start at X_1 pass through both detectors and arrive at X_4 is the sum of $\exp(iS)$ over all paths satisfying these restrictions. Possible conditions defining an experiment are that the particle start at X_1 at τ_1 and definitely register in the detector at τ_2. An exhaustive and exclusive set of observations are the possible values of X_4 *and* whether the detector at τ_3 registers or does not. Conditional probabilities for these outcomes are the normalized squares of the joint amplitudes computed from the sum over paths corresponding to these possibilities.

$$p(\Delta_4\tau_4,\Delta_3\tau_3\,|\,\Delta_2\tau_2,X_1\tau_1) =N^{-2}|\Delta_4|\ |\Phi(X_4\tau_4,\Delta_3\tau_3,\Delta_2\tau_2,X_1\tau_1)|^2 \ , \tag{2.7}$$

where X_4 is the center of Δ_4 and, in an obvious notation,

$$N^2=\int_R dX_4[\,|\Phi(X_4\tau_4,\Delta_3\tau_3,\Delta_2\tau_2,X_1\tau_1)|^2 +|\Phi(X_4\tau_4,(R-\Delta_3)\tau_3,\Delta_2\tau_2,X_1\tau_1)|^2] \ . \tag{2.8}$$

As we shall show in the next section this is equivalent to the results of the standard quantum-mechanical measurement theory.

This example illustrates the distinction between experiments in which information is complete and those in which it is not. The measurement at τ_3 was assumed to distinguish only whether the particle was in or out of the integral Δ_3; we therefore summed amplitudes over these ranges and squared to find the probability. Should the distinction have been made by actually measuring the position at this time we should have squared and summed probabilities. Similarly, the localization at τ_2 is assumed to be by passing the particle through a suitable "slit" and not by a measurement of position. Thus we summed amplitudes not probabilities. The final measurement of position at τ_4 was assumed to be precise but the result was unknown to an accuracy Δ_4. We therefore summed the joint probability of a measurement of position over an interval Δ_4 about X_4. However, should the measurement at τ_4 been carried out with an imprecise detector as those used at τ_2 and τ_3 the probability would still have been computed in the *same* way since this is the measurement which isolates the history from the future.

What if our experiment had consisted only of the detectors at the three times τ_1, τ_2, and τ_3? The detector at τ_3 does not define the end of a path so the probabilties are not given by (2.3). Rather we compute probabilities with an assumed detector at any later τ_4 which does localize the particle and integrate over the results. That is, we integrate (2.7) over X_4. The result is independent of τ_4.

Positing that the basic observables for physical description correspond to a particle's presence or absence in spacetime regions is an assertion which has an analogous status to the assertion in the familiar theory that observables correspond to the Hermitian operators in Hilbert space. However, it is important to understand that the two assertions do not necessarily coincide. In one sense, the sum-over-histories observables are more restricted. It is assumed in particular that every measurement has a spacetime description. This is generally the case. For example, the measurement of the momentum of a nonrelativistic particle can be carried out[7] by passing the particle through a slit of width Δ and then detecting its position at time T later. If the detected displacement from the slit's position is D, the classical value of the momentum is mD/T with a classical uncertainty due to the width of the slit $m(\Delta/T)$. There is also the quantum-mechanical uncertainty $\hbar/\Delta$. By making Δ large and Δ/T small an accurate measurement of momentum can be achieved and the probabilities of the outcome of such a measurement calculated in the above framework. The results are, of course, the same as the usual transformation theory of ordinary quantum mechanics.

In this framework, however, it becomes difficult to ask a question such as "Given a preparation what is the amplitude for the particle to have a certain value of the observable X^3P+PX^3?" because the spacetime description of the apparatus which measures this quantity is unclear. However, in the usual formulation of quantum mechanics while it is easy to ask such a question, it is less clear what the answer means for the same reason. In the sum-over-histories formulation spacetime occupies a preferred place in the description of observation.

In another sense, the sum-over-histories observables are broader than the usual "observables" of the Schrödinger-Heisenberg formulation. There are observables associated with spacetime regions which find no simple expressions as operators. These are the measurements which involve spacetime regions with an extent in time. They appear to be no more idealized than measurements of position "at an exact instant of time." Including them makes the set of basic observables symmetric in space and time. This is necessary in order that a preferred time not be distinguished at the outset by the choice of basic observables.

In neither the sum-over-histories formulation nor the Schrödinger-Heisenberg version does positing a set of basic observables guarantee that there is an apparatus which actually registers their values. To ensure this one needs to invent and discuss the apparatus.

III. EQUIVALENCE WITH SCHRÖDINGER-HEISENBERG QUANTUM MECHANICS

In this section we shall review the equivalence of the sum-over-histories formulation of the quantum mechanics of a nonrelativistic particle with the more standard Schrödinger-Heisenberg formulation of that theory. We shall be brief because this connection has been explored very clearly by Feynman[6] and more recently for multitime observations by Caves and others.[5,11] To investigate this equivalence we shall restrict attention to basic observables defined at a moment of time for these are the quantities considered in the Schrödigner-Heisenberg formulation.

In the Schrödinger formulation of quantum mechanics probabilities for the results of experiments carried out at a particular instant of the preferred Newtonian time are calculated from the normalized vector in Hilbert space, $|\psi,\tau\rangle$, which describes the "state" of the system at that time. The state vector evolves in two ways. In the absence of observation it evolves by the Schrödinger equation

$$\left[-i\frac{\partial}{\partial\tau}+H\right]|\psi,\tau\rangle=0\ . \tag{3.1}$$

If an observation is made at one instant of time, then immediately afterward the state is the normalized projection of the state before the observation onto the subspace of the Hilbert space appropriate to the results of the observation

$$|\psi\rangle\rightarrow(\langle\psi|P|\psi\rangle)^{-1/2}(P|\psi\rangle)\ . \tag{3.2}$$

Examples will be given below.

An experiment may consist of observations made at a number of different times. The probabilities for the possible outcomes of the experiment are the probabilities of the results at the separate times multiplied together. The probability of an outcome at a single time is given by $\langle\psi|P|\psi\rangle$ where P is the projection onto the subspace consistent with the outcome of the observation.

An analyis of the experiment described in Sec. III in these terms will serve both to clarify the procedure and also to demonstrate its equivalence with the sum-over-histories formulation. Recall that this experiment (Fig. 1) consisted of a localization of the particle's position at X_1 at time τ_1, at X_4 at time τ_4, and in intervals Δ_2 and Δ_3 at intermediate times τ_2 and τ_3. Let $|X\tau\rangle$ be the states in which particle is localized at time τ normalized so that

$$\langle X'\tau \mid X\tau\rangle = \delta(X-X') \ , \tag{3.3}$$

The initial state in this experiment is $|X_1\tau_1\rangle$. After the localization at τ_2 it is

$$N_2^{-1}\int_{\Delta_2} dX_2 \, |X_2\tau_2\rangle\langle X_2\tau_2 \mid X_1\tau_1\rangle \ , \tag{3.4}$$

where N_2 is fixed so the state has unit norm

$$N_2^2 = \int_{\Delta_2} dX_2 \, |\langle X_2\tau_2 \mid X_1\tau_1\rangle|^2 \ . \tag{3.5}$$

In this state, the probability that a measurement of position at τ_3 yields a value in the range Δ_3 is

$$N_2^{-2}\int_{\Delta_3} dX_3 \, \left| \int_{\Delta_2} dX_2 \langle X_3\tau_3 \mid X_2\tau_2\rangle\langle X_2\tau_2 \mid X_1\tau_1\rangle \right|^2 . \tag{3.6}$$

The state vector after the observation at τ_3 is

$$N_3^{-1}\int_{\Delta_3} dX_3 \int_{\Delta_2} dX_2 \, |X_3\tau_3\rangle\langle X_3\tau_3 \mid X_2\tau_2\rangle\langle X_2\tau_2 \mid X_1\tau_1\rangle \tag{3.7}$$

with N_3 determined so it has unit norm. The probability that, in this state, a measurement of position at time τ_4 yields a value X_4 in a small range Δ_4 is

$$N_3^{-2}\,|\Delta_4|\, \left| \int_{\Delta_3} dX_3 \int_{\Delta_2} dX_2 \langle X_4\tau_4 \mid X_3\tau_3\rangle\langle X_3\tau_3 \mid X_2\tau_2\rangle\langle X_2\tau_2 \mid X_1\tau_1\rangle \right|^2 . \tag{3.8}$$

The probability of *both* observations is the product of (3.6) and (3.8). In fact, N_3^2 is exactly equal to (3.6) so the combined probability is just

$$p(\Delta_4\tau_4,\Delta_3\tau_3 \mid \Delta_2\tau_2 X_1\tau_1) = N_2^{-2}\,|\Delta_4|\, \left| \int_{\Delta_3} dX_3 \int_{\Delta_2} dX_2 \langle X_4\tau_4 \mid X_3\tau_3\rangle\langle X_3\tau_3 \mid X_2\tau_2\rangle\langle X_2\tau_2 \mid X_1\tau_1\rangle \right|^2 . \tag{3.9}$$

It is not difficult using (3.3) and the definition of N_2 to verify that this probability is correctly normalized.

The probability (3.9) is identical with that computed from the sum-over-histories rules (2.6). This is because, first, the paths of nonrelativistic particles move forward in time. Each path entering in (2.6) intersects the constant time surfaces τ_3 and τ_4 at a single position X_3 and X_4, respectively. Therefore

$$\Phi(X_4\tau_4,\Delta_3\tau_3,\Delta_2\tau_2,X_1\tau_1) = \int_{\Delta_3} dX_3 \int_{\Delta_2} dX_2 \Phi(X_4\tau_4,X_3\tau_3)\Phi(X_3\tau_3,X_2\tau_2)\Phi(X_2\tau_2,X_1\tau_1) \ , \tag{3.10}$$

where $\Phi(X''\tau'',X'\tau')$ is the "propagator"—the sum of $\exp(iS)$ over all paths which start at X',τ', and end at X'',τ''. As Feynman showed

$$\Phi(X''\tau'',X'\tau') = \langle X''\tau'' \mid X'\tau'\rangle \tag{3.11}$$

with an appropriate choice of the measure for the sum over histories. The normalizing factor N in (2.7) is then exactly N_2 defined by (3.5). Thus, the sum-over-histories formulation and the Schrödinger formulation of quantum mechanics are equivalent for this simple example. This sample example, however, has all the essential features of the demonstration in the general case.[5]

The description of multitime measurements can be transcribed into the Heisenberg formulation of quantum mechanics. There (see, e.g., Refs. 12–15)

$$p(\Delta_4\tau_4,\Delta_3\tau_3 \mid \Delta_2\tau_2,X_1\tau_1) = \frac{\mathrm{Tr}[P_{\Delta_4}(\tau_4)P_{\Delta_3}(\tau_3)P_{\Delta_2}(\tau_2)\rho(\tau_1)P_{\Delta_2}(\tau_2)P_{\Delta_3}(\tau_3)P_{\Delta_4}(\tau_4)]}{\mathrm{Tr}[P_{\Delta_2}(\tau_2)\rho(\tau_1)]} \ , \tag{3.12}$$

where ρ is the time-independent density matrix referred to the initial time τ_1

$$\rho(\tau_1) = |X_1\rangle\langle X_1| \ , \tag{3.13}$$

and the $P_\Delta(\tau)$ are time-dependent projections onto the interval Δ:

$$P_\Delta(\tau) = e^{-iH(\tau-\tau_1)}\left[\int_\Delta dX \, |X\rangle\langle X| \right] e^{iH(\tau-\tau_1)} \ . \tag{3.14}$$

Tr denotes the trace operation. This Heisenberg formulation is, of course, completely equivalent to the Schrödinger formulation and from the above analysis it is also fully equivalent (for the restricted class of basic observables) to the sum-over-histories formulation. Indeed, the equivalence of the sum-over-histories and the Heisenberg formulations is more easily demonstrated directly.

There are advantages and disadvantages to each of these formulations of quantum mechanics, and each suggests different perspectives on interperative issues. The Schrödinger formulation has the advantage of historical familiarity. Its basic notion of time-dependent "state" obeying two laws of evolution, however, breeds seeming paradox if too close an analogy with the classical notion of "state" is imagined. In fact the notions are very different.[16] In neither the Heisenberg nor the sum-over-histories formulation is a notion of "state" obeying a "reduction of the wave packet" law of evolution needed. Probabilities are computed directly. The advantages of this point of view have been expressed very clearly by

Wigner,[12] more recently by Gell-Mann,[14] Unruh,[15] and by many others.

The most important advantage of the sum-over-histories formulation for our subsequent considerations concerns the role played by time. In Schrödinger formulation a preferred time enters centrally into the formulation of the notion of state and in the evolutionary laws for that state. In the Heisenberg formulation a preferred time is needed to define the ordering of the projections in (3.12). By contrast, in the sum-over-histories formulation, although a preferred time is singled out by the nonrelativistic histories and action, it does not enter the formalism for computing probabilities in such a central way. For this reason we distinguish the Schrödinger-Heisenberg formulation on the one hand and the sum-over-histories formulation on the other.

IV. RECOVERY OF A SCHRÖDINGER-HEISENBERG FORMULATION

The discussion of the preceding two sections shows that the Schrödinger-Heisenberg formulation of nonrelativistic quantum mechanics is equivalent to the sum-over-histories formulation provided that the Hilbert space of states is constructed on surfaces of constant preferred Newtonian time. In this section we shall investigate the possibility of recovering a Schrödinger-Heisenberg formulation from the sum-over-histories formulation on more general surfaces. A brief sketch of the derivation on the surfaces of preferred Newtonian time will indicate how such a derivation might go.

Suppose the particle is prepared in a certain way prior to some particular value of the preferred time. In sum-over-histories language this preparation corresponds to a set of conditions $\mathcal{C}$ all of which occur prior to this time. The joint amplitude to find the particle at a position X on a surface of constant later time defines the wave function of the system at that time:

$$\psi_{\mathcal{C}}(X\tau)=\Phi(X\tau,\mathcal{C}) \ . \tag{4.1}$$

In a similar way we can define a wave function corresponding to a set of observations $\mathcal{O}$ which take place after time τ:

$$\psi^*_{\mathcal{O}}(X\tau)=\Phi(\mathcal{O},X\tau) \ . \tag{4.2}$$

For example, an analysis[7] of the observations corresponding to a time-of-flight measurement of momentum show $\psi_P(X\tau)\propto\exp(iPX)$. Since a particle path crosses the surface at one and only one X, one can write, for the joint amplitude to observe $\mathcal{O}$ given $\mathcal{C}$,

$$\begin{aligned}\Phi(\mathcal{O},\mathcal{C})&=\int_R dX\,\Phi(\mathcal{O},X\tau)\Phi(X\tau,\mathcal{C})\\ &=\int_R dX\,\psi^*_{\mathcal{O}}(X\tau)\psi_{\mathcal{C}}(X\tau) \ .\end{aligned} \tag{4.3}$$

This composition law across the surface of constant τ defines the Hilbert space inner product.

Because the paths of a nonrelativistic particle move only forward in time, the values of X on a surface of constant time are a set of exhaustive and exclusive observations. The wave function thus acquires a probability interpretation through (2.4) and (4.1). The probability to be in an interval Δ about a position X is

$$p(\Delta\tau\,|\,\mathcal{C})=\Delta\,|\,\psi_{\mathcal{C}}(X\tau)\,|^2\left[\int_R dX\,|\,\psi_{\mathcal{C}}(X\tau)\,|^2\right]^{-1} \ . \tag{4.4}$$

This is the familiar result.

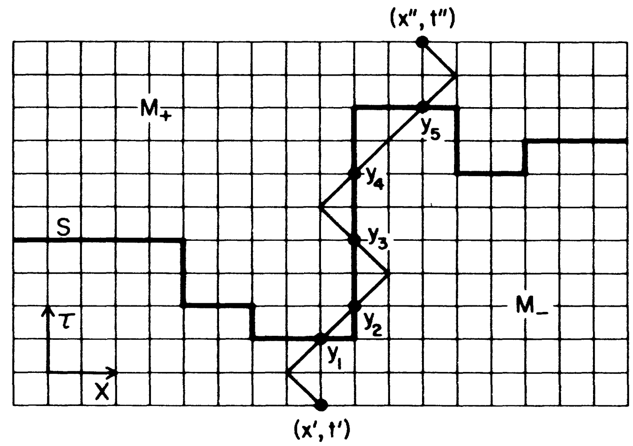

FIG. 2. A spacetime lattice with a general surface S. Points in the spacetime lattice are labeled by integer paths (x,t). Points are spaced by a distance η in X and a distance ϵ in τ. The surface S divides the spacetime lattice into a region M_- to its past and a region M_+ to its future. Lattice points on S may be numbered by an integer y starting at $-\infty$ on the far left. A typical path contributing to the propagator from $(x't')$ to $(x''t'')$ will intersect the surface S in many positions $y_1\cdots y_n$.

We shall now investigate whether a composition law can be achieved across more general surfaces. For simplicity we shall restrict attention to surfaces built up of constant X and constant τ surfaces (Fig. 2). Such surfaces are certainly possible and any continuous surface can be arbitrarily closely approximated by ones of this type. It is immediately clear that for a general surface the composition law analogous to (4.3) will involve sums over products of multicomponent wave functions because a particle path can cross a general surface an arbitrary number of times (Fig. 2). The possibility of recovering a Hilbert space formulation on a general surface depends on the existence of such sums and therefore on a more concrete definition of a sum over paths. It is to this definition that we now turn.

A. Lattice sums over histories

For simplicity and definiteness we shall consider only the case of a free nonrelativistic particle. Since we are interested in kinematical rather than dynamical issues, this will suffice for our purpose. In this theory let us consider, in particular, the propagator—the joint amplitude for the particle to be found at $X'\tau'$ and again at $X''\tau''$. The sum over histories for the propagator in the nonrelativistic quantum mechanics of a free particle may be given a concrete meaning in two steps. First, continue the time to imaginary values $\tau\to -i\tau$. The formal sum over paths becomes Euclidean:

$$\Phi(X'',-i\tau'';X',-i\tau')\equiv\Phi_E(X''\tau'',X'\tau')$$
$$=\sum_{\text{paths}}\exp\{-I[X(\tau)]\}\ , \tag{4.5}$$

where I is the Euclidean action

$$I[X(\tau)]=\frac{m}{2}\int_{\tau'}^{\tau''}d\tau\dot{X}^2\ . \tag{4.6}$$

Second, make such sums concrete by defining them as continuum limits of discrete sums on a spacetime lattice. The way to do this can be found in the close connection between sums over histories and stochastic processes.[17] In particular, Euclidean path integrals may be viewed as integrals of a probability measure on the space of paths. Discrete models of such stochastic processes can give approximations to path integrals which lead to computational algorithms and which, through the continuum limit, can be regarded as the definition of such integrals. As one might expect from the central limit theorem, or even from the numerical evaluation of ordinary integrals, there may be many discrete processes approximating a given continuum path integral.

In the case of a free particle there is a particularly simple discrete approximation process: the random walk on the line. We consider a rectangular spacetime lattice with temporal spacing ϵ and spatial spacing η. We take the lattice to coincide with the segments defining the general surface. The shape of the surface, however, is to remain fixed as the continuum limit is taken. A point on the lattice will be located by the integer variables x and t. The particle moves forward one step in time on the lattice with a "probability" of $\frac{1}{2}$ to walk one site to the left and a "probability" of $\frac{1}{2}$ to walk one site to the right. (These are "probabilities" of the random process not quantum-mechanical probabilities. We shall distinguish them by quotation marks.) The "probability" to start at site $(0,0)$ and arrive at site (x,t) is the sum over all paths P which connect the two points and which move one step in space for each step in time. If $|P|$ is the length (in steps) of P then this "probability" is

$$u(xt,00)=\sum_P\frac{1}{2^{|P|}}=\frac{1}{2^t}\begin{pmatrix}t\\ \frac{t+x}{2}\end{pmatrix}\ . \tag{4.7}$$

Here, by convention, we interpret the binomial coefficient as zero if $t+x$ is not even.

To define the continuum amplitude $\Phi_E(X\tau,00)$ we consider approximating lattices with smaller and smaller values of the spacings ϵ and η such that (X,τ) and $(0,0)$ always lie on sites. We evaluate the "probability" u at $x=X/\eta$ and $t=\tau/\epsilon$ and sum the result over the neighboring x site to eliminate the odd-even asymmetry in (4.7). We then calculate the "probability" density $u/(2\eta)$ in the limit that ϵ and η approach zero keeping X, τ, and the ratio $\epsilon/\eta^2=m$ fixed. The result is

$$(2\eta)^{-1}u(xt,00)\rightarrow\frac{e^{-mX^2/2\tau}}{\sqrt{2\pi\tau/m}}=\Phi_E(X\tau,00)\ . \tag{4.8}$$

This is the standard "diffusion limit" of the random-walk process.[18] The result (4.8) shows that, as a method of evaluating sums over paths, it reproduces the correct propagator of the free particle.

The composition law corresponding to (4.3) for the Euclidean amplitudes $\Phi_E(X''\tau'',X'\tau')$ across a surface of *constant* τ follows directly from this lattice definition of sum over paths. Since each path intersects such a surface once and only once, we have, from (4.7),

$$(2\eta)^{-1}u(x''t'',x't')$$
$$=\sum_x(2\eta)^{-1}u(x''t'',xt)u(xt,x't')\ . \tag{4.9}$$

In the continuum limit, using (4.8), this reads

$$\Phi_E(X''\tau'',X'\tau')=\sum_x(2\eta)\Phi_E(X''\tau'';\eta x,\tau)$$
$$\times\Phi_E(\eta x,\tau;X'\tau')\ . \tag{4.10}$$

$\sum_x(2\eta)(\cdots)$ becomes $\int dX(\cdots)$ and we derive the Euclidean transcription of (4.3). The important property for the existence of the continuum limit was that $u\rightarrow\eta\hat{u}$ where $\hat{u}$ is finite.

A concrete prescription for defining and calculating sums over histories in hand, we may now ask whether we can construct a Hilbert space inner product on a general surface in the manner that it was constructed for a surface of constant preferred time in (4.10). Consider for definiteness the surface S shown in Fig. 2. This surface divides spacetime into two parts: a region M_- to the "past" of S and a region M_+ to its "future." The lattice points on S may be labeled by an integer y which ranges from $-\infty$ at large negative x to $+\infty$ at large positive x. A given path P may intersect the surface S many times. We may define

$$u(y_1\cdots y_nS',x't')=\sum_{P\in M_-}(\tfrac{1}{2})^{|P|}\ . \tag{4.11}$$

The sum is over paths, connected and disconnected, which lie in M_- and have (x',t') and $y_1\cdots y_n$ as end points. A similar expression defines $u_+(x''t'',y_1\cdots y_nS)$. Next, define the composition of u_+ and u_- across the surface by

$$\sum_n\frac{1}{n!}\sum_{y_1\cdots y_n}u_+(x''t'',y_1\cdots y_nS)u_-(y_1\cdots y_nS,x't')\ . \tag{4.12}$$

This sum includes not only the connected paths which define $u(x''t'',x't')$ but also "closed loop" paths (Fig. 3) which are not connected to (x',t') or (x'',t''). With a familiar[19] manipulation (4.12) can be rewritten as

$$\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{y_1 \cdots y_k} (\text{connected paths from } x't' \text{ to } x''t'' \text{ with } y_1 \cdots y_k \text{ fixed on } S)$$

$$\times \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{y_1 \cdots y_p} (\text{closed loop paths with } y_1 \cdots y_p \text{ fixed on } S) . \quad (4.13)$$

The first factor in (4.13) is $u(x''t'',x't')$. Defining $D(S)$ to be the second factor we have

$$D(S)u(x''t'',x't') = \sum_{n \text{ odd}} \frac{1}{n!} \sum_{y_1 \cdots y_n \in S} u_+(x''t'',y_1 \cdots y_n S)u_-(y_1 \cdots y_n S,x't') . \quad (4.14)$$

The sums occurring on the right-hand side of (4.14) and those defining D are all finite for a fixed lattice and surface S because it is easily seen that there is a largest value of $|y|$ which can contribute and a largest number of vertices which will make either a closed loop or connected path fitting in the given interval t' to t''.

Equation (4.14) shows that a composition law for amplitudes exists on the spacetime lattice for an arbitrary surface S which divides spacetime into two regions. We shall now investigate whether the existence of a composition law on the lattice implies the existence of such a law in the continuum. By dividing spacetime by constant τ surfaces, we can reduce the study of the continuum limit on a general surface S to its study in cases of the type shown in Fig. 4. Figure 4(b) is the familiar one of composition across a constant τ surface and has already been discussed.

The special case in Fig. 4(c) will illuminate the general situation. Consider the surface S shown in Fig. 5 with (x'',t'') and (x',t') arranged as shown. A typical term in the connected part of (4.14) represents the contribution of paths which intersect the origin n times. The question of the existence of the continuum limit is the question of whether the sums over the y_i organize themselves into a multiple integral over the surface. To put the question more precisely rewrite (4.14) as

$$\begin{aligned} D(S)\hat{u}(x''t'',x't') &= (2\eta)^{-1}u(x''t'',x't')D(S) \\ &= \sum_{n \text{ odd}} \frac{1}{n!} \sum_{t_1 \cdots t_n} (2\epsilon)^n \hat{u}_+(x'',t'',t_1 \cdots t_n S) \\ &\quad \times \hat{u}_-(t_1 \cdots t_n S,x't') , \end{aligned} \quad (4.15)$$

where

$$\hat{u}_-(t_1 \cdots t_n S,x't') = (2\epsilon)^{-n/2}(2\eta)^{-1/2}u_-(t_1 \cdots t_n S,x't') \quad (4.16)$$

and a similar expression for $\hat{u}_+$. If the various $\hat{u}$ remain finite in the continuum limit then the sum over the t_i can be replaced by a multiple time integral over the interval $[\tau',\tau'']$ and a continuum composition law will exist. We now test the finiteness of the $\hat{u}$'s in this limit by evaluating them explicitly.

The amplitude $u_-(t_1 \cdots t_n S,x't')$ is

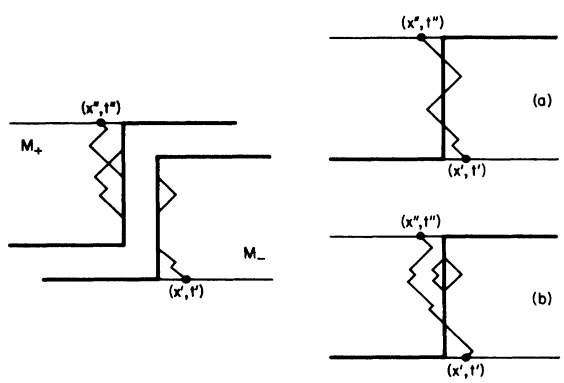

FIG. 3. The composition of two lattice amplitudes across a general surface S will typically give rise to closed loops. Illustrated on the left are two amplitudes. One is constructed from a sum over paths in M_-, both connected and disconnected, which link $(x't')$ and three points y_1,y_2,y_3 on S. A similar sum in M_+ defines the second amplitude. The composition will contain paths linking $(x't')$ to $(x''t'')$ which are connected as in (a) but also paths which are disconnected as in (b).

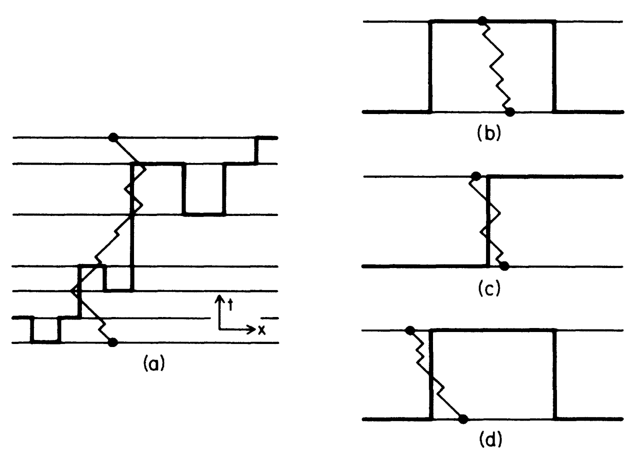

FIG. 4. An investigation of the continuum limit of the lattice composition law across a general surface such as that shown in (a) may be broken up into a series of elementary cases, examples of which are shown in (b), (c), and (d).

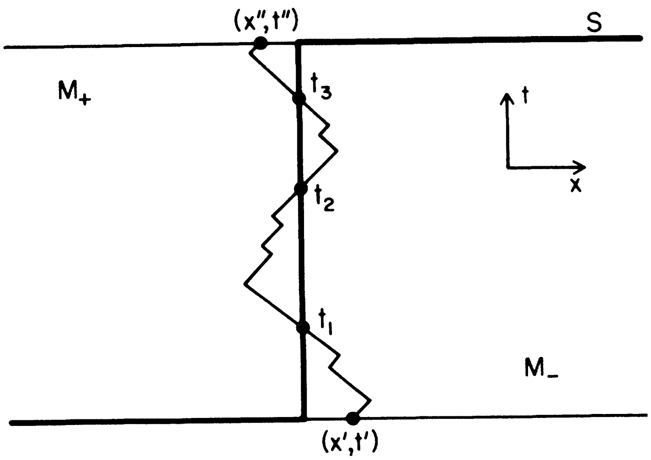

FIG. 5. An elementary case of the lattice composition law. The amplitude to go from $(x't')$ to $(x''t'')$ is a sum over paths such as the one illustrated which intersect the vertical part of the surface S many times. On the lattice this sum may be decomposed into a sum over paths to the right of S which intersect it at n positions t_i, a similar sum to the left, a sum over the values of $t_i, i=1,\ldots,n$, and finally a sum over the number of crossings n. On the lattice the sums over the paths to the left and right of S may be thought of as defining "wave functions" on S and the remaining sum over y_i and n as defining their inner product. However, in the continuum limit these functions vanish for any finite n because the sum is dominated by n's of the order $(t''-t')^{1/2}$ which goes to infinity.

$$u_-(t_1\cdots t_nS,x't') = \sum_{\text{perm}} u_-(0t_n,0t_{n-1})u_-(0t_{n-2},0t_{n-3})\cdots \times u_-(0t_3,0t_2)u_-(0t_1,x't')\ , \tag{4.17}$$

where the sum is over all permutations of the t_i and

$$u_-(x't',xt) = \sum_{P\in M_-} (\tfrac{1}{2})^{|P|}\ , \tag{4.18}$$

the sum being over paths which connect the two points and which lie entirely in M_-. It is not difficult to evaluate (4.18); it is the "probability" in a random walk with one "absorbing barrier" at $x=-1$. This is a standard problem.[18] Briefly, $u_-(x't',xt)$ satisfies the difference equation

$$u_-(x',t'+1;xt)=\tfrac{1}{2}[u_-(x'-1,t';xt) + u_-(x'+1,t';xt)] \tag{4.19}$$

with the boundary conditions

$$u_-(x't,xt)=\delta_{xx'}\ , \tag{4.20a}$$

$$u_-(-1t',xt)=0,\quad t'>t\ , \tag{4.20b}$$

$$u_-(x't',xt)\to 0,\quad x'\to\infty\ . \tag{4.20c}$$

This difference equation can be solved by the "method of images" in terms of the unrestricted random walk $u(x't',xt)$:

$$u_-(x't',xt)=u(x't';xt)-u(x't';-2-x,t)\ . \tag{4.21}$$

This is a solution because $u(x't';xt)=u(x'-x,t'-t;00)$ and $u(-x,t;00)=u(x,t;00)$. Since $u(xt,00)$ is given explicitly by (4.7), we can explicitly evaluate the terms entering (4.18). Since $u_-(0t',xt)=u_-(00,x,t-t')$ it is suffucient to evaluate $u_-(00,xt)$. This is

$$u_-(00,xt)=\frac{1}{2^t}\frac{x+1}{t+1}\begin{pmatrix} t+1 \\ \dfrac{t+x}{2}+1\end{pmatrix}\ . \tag{4.22}$$

To take the continuum limit of (4.22) put $x=X/\eta$, $t=\tau/\epsilon$, and take the limit $\eta\to 0$ while X and τ remain constant and $\epsilon=m\eta^2$. There are two cases. For $X\neq 0$ one finds

$$u_-(00,xt)\to\frac{2\epsilon X}{\sqrt{2\pi\tau^3/m}}e^{-mX^2/2\tau}\ , \tag{4.23}$$

while for $X=0$ one has

$$u_-(00,0t)\to\frac{2\eta\epsilon}{\sqrt{2\pi\tau^3/m}}\ . \tag{4.24}$$

Thus, a typical term in (4.16) is

$$\hat{u}_-(t_1\cdots t_nS,x't')\sim(2\epsilon)^{-n/2}(2\eta)^{-1/2}(2\eta\epsilon)^{(n-1)/2}(2\epsilon) \times\left[\frac{m}{2\pi(t_n-t_{n-1})^3}\right]^{1/2}\cdots\left[\frac{m}{2\pi(t_1-t')^3}\right]^{1/2}X'\exp\left[\frac{-mX'^2}{2(t_1-t')}\right]\ . \tag{4.25}$$

The leading power in η, as η becomes small keeping $\epsilon=m\eta^2$, is therefore $(n/2)$. This is always positive and $\hat{u}_-$ vanishes in the continuum limit. A continuum composition law does not exist, and we do not recover from the sum over histories of Hilbert space of states on the general surface.

It is not difficult to convince oneself that this result will hold for the other cases such as that shown in Fig. 4(d). There, the evaluation of the lattice amplitudes is equivalent to calculating "probabilities" in random walks with *two* absorbing barriers. This too is a standard problem[18] and the additional complication in no way changes the behavior (4.23) and (4.24) which are essentially local. There are a few other amplitudes to discuss but the main result remains unchanged: there is a composition law across a general surface for Euclidean amplitudes on the lattice, but its continuum limit does not exist.

The above discussion has been for amplitudes continued to Euclidean time. Some such continuation is needed just to *define* the sums over $\exp(iS)$. However, one could worry that there might exist a composition law for sums over $\exp(iS)$ despite the nonexistence of the analogous Euclidean law. This is not the case. The question of the existence of a composition law across a general surface may be decomposed into a series of elementary problems as in Fig. 4. Consider, for example, the problem in Fig. 5. The amplitudes defined as sums of $\exp(iS)$ have the same value when computed by a method which does not involve continuing the time (say, solving the Schrödinger equation) as when computed as the continuum limit of the Euclidean time lattice sums above—namely, zero. The Euclidean time computation is thus correct as judged by these alternative standards. From vanishing amplitudes it not possible to assemble a composition law which gives a nonvanishing result.

There is a simple physical reason for these results. The dominant contribution to the sum over histories defining the propagator in nonrelativistic quantum mechanics comes from paths which are nowhere differentiable.[6–8] For example, $\langle X''\tau'' \,|\, (dX/d\tau)^2 \,|\, X'\tau' \rangle = \infty$. The general particle path intersects a surface of constant τ only once, but it intersects a surface on which τ is not constant an arbitrarily large number of times. On the lattice the "composition law" (4.14) is valid and, indeed, the sum over n is finite since n must be less than $t''-t'$. As the lattice spacing becomes smaller the dominant contribution to the sum comes[18] from n's comparable to $[(\tau''-\tau')/\epsilon]^{1/2}$ which does not approach a continuum limit. Put differently, the joint amplitude to cross the surface *a finite* number of times [Eq. (4.25)] is zero in the continuum limit because the most likely number of crossings is infinite.

The results of this section show that starting from a sum-over-histories formulation of nonrelativistic quantum mechanics it is possible to recover a Schrödinger-Heisenberg formulation on a surface of constant preferred Newtonian time but not on a general surface. The absence of Schrödinger-Heisenberg formulation on a general surface is, of course, no surprise when viewed from the perspectives of the more familiar formulations. Positions at different times do not commute and, therefore, one cannot have a complete set of commuting positions on a general surface. Alternatively, if the Schrödinger equation is transformed so that the label of a general surface replaces Newtonian time, the equation becomes second order in that label so a positive inner product cannot be constructed without further restrictions on the states. From the sum-over-histories perspective, however, a more general reason can be given which does not, like these, presume the existence of a Schrödinger-Heisenberg formulation on *some* surface. A Schrödinger-Heisenberg formulation is possible on surfaces of constant Newtonian time because *those* are surfaces which the histories intersect *once and only once.* Given the locality of the measure, a composition law can then be constructed. Such a construction is not possible on a general surface which histories intersect arbitrarily often.

V. LESSON OF THE RELATIVISTIC PARTICLE

The discussion of the preceding section shows that the straightforward sum-over-histories derivation of a Schrödinger-Heisenberg formulation fails on spacetime surfaces which the histories cross arbitrarily often. However, the argument does not exclude a construction by a cleverer choice of variables. In particular, the same theory may have a variety of sum-over-histories formulations using different variables and a Schrödinger-Heisenberg formulation may follow from one but not the other. The quantum mechanics of a free relativistic particle is a good example of this.

A useful form of the classical action for path-integral formulations of the quantum mechanics of a free relativistic particle is[20]

$$S[x^\alpha(\tau),N(\tau),\tau'',\tau'] = \frac{m}{2}\int_{\tau'}^{\tau''} d\tau \left[\frac{(\dot{x}^\alpha)^2}{N} - N\right], \tag{5.1}$$

where, $x^\alpha(\tau)$, $\alpha=0,1,2,3$ describes the particle's world line in a particular Lorentz frame, $N(\tau)$ is a Lagrange multiplier, and $(\dot{x}^\alpha)^2$ is shorthand for the four-vector inner product of $\dot{x}^\alpha$ with itself. Variation of (5.1) with respect to x^α and N yield the classical equations of motion and a constraint which implies that N is the rate of change of proper time with respect to parameter time, τ. The action (5.1) is invariant under reparametrizations of the parameter time (including the end points of integration unless suitably restricted) in which $\bar{x}^\alpha(\tau) = x^\alpha(f(\tau))$ and $\bar{N}(\tau) = \dot{f}(\tau)N(f(\tau))$.

The action (5.1) can serve as the starting point for a sum-over-histories quantum mechanics of the relativistic particle.[21,20] When $\exp(iS)$ is summed over all paths which connect two spacetime points x' and x'', including both those moving forward in time and backward, there results the Feynman propagator of the many particle theory. When $\exp(iS)$ is summed over paths which move only forward in the time of a particular Lorentz frame, the result is the Newton-Wigner propagator of the single-particle theory. More specifically, if $|\,\mathbf{p}\rangle$ are the invariantly normalized momentum eigenstates of the single-particle theory, the Newton-Wigner states in which the particle is located at $\mathbf{x}$ at time t are

$$|\,\mathbf{x}t\,\rangle = \int \frac{d^3p}{(2p^0)^{1/2}} \exp(ip_\alpha x^\alpha)\,|\,\mathbf{p}\rangle\ , \tag{5.2}$$

where $p^0 = (\mathbf{p}^2+m^2)^{1/2}$. The propagator which results from properly summing $\exp(iS)$ over forward-moving paths is $\langle \mathbf{x}''t'' \,|\, \mathbf{x}'t' \rangle$. The reparametrization invariance of the action must be taken into account in carrying out sums over $\exp(iS)$ and suitable "gauge fixing" conditions must be used in each case so that each path is counted only once. A nontrivial measure is also required. These details needed to actually carry out the sums over paths can be found in Refs. 21 and 20. What is important for us is that a sum over histories exists for both the relativistic Feynman propagator and for the Newton-Wigner propagators of all Lorentz frames. We will now investigate the construction of a Schrödinger-Heisenberg formulation in each of these path-integral versions of the theory.

Let us begin with the relativistic propagator and attempt a construction of a Hilbert space on a surface of constant time. The paths which contribute to the sum cross and recross such a surface an arbitrarily large number of times. The experience with the nonrelativistic theory in the previous sections would suggest that it is not possible to construct a Hilbert space based on the inner product which is the continuum analog of (4.15). This is correct. It is also misleading. A Hilbert space does exist for the relativistic particle as the single-particle version of the theory shows. A Schrödinger-Heisenberg formulation for the relativistic particle can be constructed from the path integral for the Newton-Wigner propagator in a way that is completely analogous to the nonrelativistic construction at the start of Sec. IV. The forward moving paths intersect a surface of constant time at one and only one position. Single-particle wave functions can be defined and the inner product is $\int d^3x\, \psi^*(\mathbf{x})\chi(\mathbf{x})$.

The particle paths which move forward in the preferred time of a particular Lorentz frame will intersect a Lorentz-boosted slice an arbitrarily large number of times. Experience with the nonrelativistic theory would therefore suggest that it is not possible to construct a Hilbert space on this slice. This is again correct by misleading. It is correct that a Hilbert space on the boosted slice cannot be constructed using the same notions of localization and particle path as on the unboosted slice. However, since the theory is Lorentz invariant, if a Hilbert space can be constructed in one Lorentz frame, it is possible to do it in all Lorentz frames. However, different notions of localization are used in each frame. In fact, any two such Hilbert-space formulations are unitarily equivalent.

The lesson of the relativistic particle is this: The sum-over-histories formulation of quantum mechanics yields a Schrödinger-Heisenberg formulation on surfaces which a history crosses once and only once. It typically does not yield a construction on surfaces which a generic history can be expected to cross an arbitrarily large number of times. The failure of this construction, however, does not preclude *rewriting* the theory of terms of a *different set of variables* in terms of which a Schrödinger-Heisenberg formulation can be given on such surfaces. Indeed, we are familiar with another set of variables which are generally useful in this way. They are called fields.

VI. QUANTUM MECHANICS ON GENERAL SURFACES

In nonrelativistic particle quantum mechanics, the absence of a notion of state for a general spacetime surface does not mean that one cannot calculate probabilities for the outcomes of experiments whose observation and conditions lie on such surfaces. This can be done by considering "multitime" observations in the usual measurement theory or considerably more conveniently and covariantly in the Heisenberg or sum-over-histories formulation.[5,11] The meaningful experiments will, however, in general require both conditions as well as observations on the surface. We will illustrate these conclusions in this section with a few simple thought experiments. One of these will also be an example of the more general basic observables of sum-over-histories quantum mechanics discussed in Sec. II.

A simple surface, which is not a surface of constant preferred time, is the surface shown in Fig. 6. To define experiments we must specify conditions and a complete and exclusive set of observations. Two of these conditions or observations must specify the end points of a segment of the particle's world line. For definiteness let us take two of the conditions to be that the particle is localized at position X' at time τ' and passes immediately through a centered slit of width Δ'. What are some possibilities for the remaining conditions and observations?

We know that we cannot ask for observations which include specifying the number of intersections of a history with the surface. The amplitude for these is zero. An alternative is to place a finite number of particle detectors at fixed places on the surface and to ask whether or not they register. A complete and exclusive set of observations for one detector is whether it registers or does not and a complete and exclusive set for the collection are the various combinations of the individual possibilities. One of the detectors must localize the end of the particle's world line. For definiteness take this to be a detector at position X'' on the horizontal branch at τ''. This localization may be taken to be part of the conditions. We place the remaining detectors at the origin at time $\tau_1, \ldots, \tau_N$ and assume they each have a *spatial* fiducial

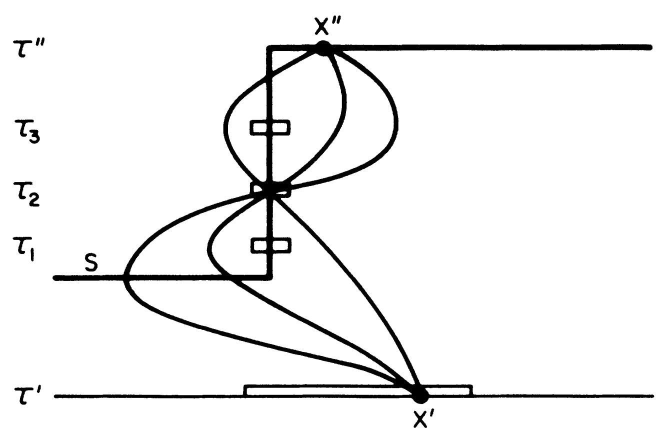

FIG. 6. One type of experiment involving outcomes ranged along a general surface S. The conditions of this experiment are that the particle start localized in an interval Δ' at τ' *and* be detected at position X'' on S. Possible outcomes may be defined by putting a detector at the origin which localizes the particle to a spatial volume Δ' but is only "on" at times τ_1, τ_2, and τ_3 along the surface. Some paths contributing to the joint amplitude for the outcome "only the detector at τ_2 registers" are shown. The probability for this outcome is correctly computed by the sum-over-histories prescription but not as the square of a state defined on S. This experiment is characteristic of meaningful experiments on a general surface in having a condition on the surface, in this case the restriction to the value X''.

volume Δ. This experiment could be realized in real time by putting a detector at the origin which is only "on" (interacting) at the times $\tau_1, \ldots, \tau_N$, "preparing" the system by localizing the particle at $X'\tau'$, and finally measuring the position at τ'' and discarding all results in which it is not X''. It might be objected that such idealizations border on the fictitious, but they do so no more than those used in the usual formulations of quantum mechanics.

The joint amplitudes for the outcomes of this experiment can now be calculated according to the sum-over-histories framework of Sec. II. Some typical paths contributing to the outcome "only detector 2 registers" are shown in Fig. 6. The calculation of such amplitudes was discussed in Secs. III and IV. For example, if there is a single detector at τ_1 the joint amplitude for it to register ("click") is

$$\Phi(\text{click},X''\tau'',\Delta'\tau') = \int_\Delta dX_1 \int_{\Delta'} dX' \Phi_0(X''\tau'',X_1\tau_1)\Phi_0(X_1\tau_1,X'\tau') , \tag{6.1}$$

where Φ_0 is the *unrestricted* propagator. The joint amplitude not to register is

$$\Phi(\text{no click},X''\tau'',\Delta'\tau') = \int_{R-\Delta} dX_1 \int_{\Delta'} dX' \Phi_0(X''\tau'',X_1\tau_1)\Phi_0(X_1\tau_1,X'\tau') . \tag{6.2}$$

The conditional probabilities for either of these two possible outcomes "click" or "no click" are the normalized squares of these two amplitudes.

As an illustration of an experiment which involves basic observables not restricted to a moment of time discussed in Sec. II, consider the following: Suppose we place a detector at the origin which registers "yes" if the particle crosses the origin, irrespective of the number of times it crosses, and registers "no" if the particle never crosses the origin. We might realize such a device by having a detector at the origin which ceases to interact after the first time it registers. For conditions, we suppose, as before, that the particle is localized at X'' at time τ'' and at X' at time τ'. These fix the end points of the history segment. The exhaustive and exclusive set of observations given these conditions are whether the detector registers "yes" or "no."

The joint amplitude $\Phi(\text{no},X''\tau'',X'\tau')$ is the sum of $\exp(iS)$ over all paths which connect $X'\tau'$ and $X''\tau''$ and do not cross the origin. This is not difficult to evaluate using the lattice methods of Sec. IV. The amplitude is proportional to the analytic continuation to real time of the continuum limit of the "probability" of starting at $x't'$ and arriving at the $x''t'$ in a one-dimensional random walk with absorbing barrier at the origin. Indeed, we have already evaluated this "probability"; it is the $u_-(x''t''\,|\,x't')$ of Eq. (4.21). Using (4.7) taking the continuum limit as in (4.8), and returning to real time, we have

$$\Phi(\text{no},X'',\tau'',X'\tau') = \left[\frac{m}{2\pi i(\tau''-\tau)}\right]^{1/2} \times \left[\exp\left[\frac{im(X''-X')^2}{2(\tau''-\tau')}\right] - \exp\left[\frac{im(X''+X')^2}{2(\tau''-\tau')}\right]\right] . \tag{6.3}$$

To calculate the amplitude $\Phi(\text{yes},X''\tau'',X'\tau')$ we sum $\exp(iS)$ over all paths which cross the origin at least once. This sum is the continuum limit of the corresponding "probability" in a random walk on a spacetime lattice as described in Sec. IV. Let t_1 be the lattice time at which the particle *first* crosses the origin. The "probability" $u_*(x''t''\,|\,x't')$ to random walk from $x't'$ to $x''t''$ crossing the origin at least once is the composition of the "probabilities" $u_-(0t_1\,|\,x't')$ to the first crossing with the unrestricted "probability" $u(x''t''\,|\,0t_1)$ to move on to $x''t''$ summed over all possible first crossing times t_1:

$$u_*(x''t'',x't') = \sum_{t_1=t'}^{t''} u(x''t''\,|\,0t_1)u_-(0t_1\,|\,x't') . \tag{6.4}$$

The amplitude $\Phi(\text{yes},X''\tau'',X'\tau')$ is the continuum limit of $(2\eta)^{-1}u_*(x''t''\,|\,x't')$ returned to real times. Equation (4.8) gives the continuum limit of u. Equation (4.23) gives the continuum limit of u_- taking time translation invariance into account. The continuum limit exists because the sum organizes itself into the form $\Sigma_{t_1}(2\epsilon)\times$(finite factors). The result is

$$\Phi(\text{yes},X''\tau'',X'\tau') = \int_{\tau'}^{\tau''} d\tau_1 \left[\left[\frac{m}{2\pi i(\tau''-\tau_1)}\right]^{1/2}\left[\frac{m}{2\pi i(\tau_1-\tau')^3}\right]^{1/2} X'\exp\left[\frac{imX''^2}{2(\tau''-\tau_1)}+\frac{imX'^2}{2(\tau_1-\tau')}\right]\right] . \tag{6.5}$$

The conditional probabilities for the two possible outcomes of the experiment are the squares of (6.3) and (6.5) normalized so their sum is unity. While such experiments appear naturally in the sum-over-histories framework, their status in the context of the familiar measurement theory is not clear without a realization of the experimental interaction. Understanding such experiments is essential if the basic observables are not to be restricted to an instant of the preferred time.

Both of these experiments on general surfaces involve conditions *on the surface* as well as in its past. This is unusual from the point of view of familiar nonrelativistic quantum mechanics, but in quantum cosmology all experiments are of this type.

VII. CONCLUSIONS

The sum-over-histories formulation of quantum mechanics provides an approach to the problem of prediction for quantum systems which does not by itself require a special family of spacetime surfaces defining a preferred time. The two laws of evolution of the Schrödinger formulation are unified into a single prescription in the sum-over-histories framework. With it, predictions can be made about observations ranged along any general surface provided appropriate care is taken to define meaningful experimental conditions. Thus, all surfaces have the same status within the general framework. Indeed, there is no requirement that observations even be arranged on a surface. For these reasons the sum-over-histories formulation seems to be a natural framework for investigating quantum theories of spacetime, such as general relativity, which do not single out a preferred family of spacelike surfaces.

Many of the familiar elements of the Schrödinger-Heisenberg formulation of quantum mechanics—a preferred time, a Hilbert space of states on surfaces of constant time, unitary evolution of states, the transformation theory of observables, and so forth—are not primary notions in the sum-over-histories framework but derived ones. A Schrödinger-Heisenberg formulation is possible on a given surface if a composition law for amplitudes across this surface can be derived. In turn, this depends on several features of the theory such as having a measure which is sufficiently local and the precise relation between the class of histories and the surface. In the nonrelativistic quantum mechanics investigated here, there was a direct construction of a Hilbert space of states on those surfaces which each history intersects once and only once. These were surfaces of constant Newtonian time. A similar construction on other surfaces which the histories cross arbitrarily often was not possible. The possibility of a Schrödinger-Heisenberg formulation of nonrelativistic quantum mechanics may then be viewed, in part, as a consequence of a special property of its histories, namely, that particle paths move forward in Newtonian time.

What if there were a theory in whose natural sum-over-histories formulation there were *no* surfaces which the histories intersected once and only once? The example of general surfaces nonrelativistic quantum mechanics suggests that it may not be possible to construct a Schrödinger-Heisenberg in a straightforward way. However, the example of the relativistic particle shows that it may still be possible by introducing other variables. We offer no general rules for when a Schrödinger-Heisenberg formulation is possible. Our main point is that a solution to this problem, or even the existence of one, is *not* a prerequisite for making predictions in the sum-over-histories framework. Sum-over-histories quantum mechanics is predictive even in the absence of a Schrödinger-Heisenberg formulation. For theories with no preferred time the sum-over-histories formulation is a more accessible, equally predictive, and probably more general formulation of quantum mechanics. In the following two papers we shall argue that covariant theories of curved spacetime are examples of theories of this type.

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