Colliding almost-plane gravitational waves: Colliding plane waves and general properties of almost-plane-wave spacetimes

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It is well known that when two *precisely* plane-symmetric gravitational waves propagating in an otherwise flat background collide, they focus each other so strongly as to produce a curvature singularity. This paper is the first of several devoted to *almost*-plane gravitational waves and their collisions. Such waves are more realistic than plane waves in having a finite but very large transverse size. In this paper we review some crucial features of the well-known exact solutions for colliding plane waves and we argue that one of these features, the breakdown of "local inextendibility" can be regarded as nongeneric. We then introduce a new framework for analyzing general colliding plane-wave spacetimes; we give an alternative proof of a theorem due to Tipler implying the existence of singularities in all generic colliding plane-wave solutions; and we discuss the fact that the recently constructed Chandrasekhar-Xanthopoulos colliding plane-wave solutions are not strictly plane symmetric and thus do not satisfy the conditions and the conclusion of Tipler's theorem. Our alternative proof of Tipler's theorem emphasizes the role and the necessity of strict plane symmetry in establishing the existence of singularities in colliding plane-wave spacetimes. However, we argue on the basis of previous work that the breakdown of strict plane symmetry as exhibited in the Chandrasekhar-Xanthopoulos solutions is a nongeneric phenomenon. We then propose a definition of general *gravitational-wave spacetimes,* of which almost-plane waves are a special case; and we develop some mathematical tools for studying them. An old result of Dautcourt implies that the only gravitational-wave spacetimes with a Killing propagation direction are plane fronted waves with parallel rays (PP waves); and we prove a new, related result, that only the gravitational-wave spacetimes with a precisely sandwiched curvature distribution are PP waves. These properties imply that almost-plane waves cannot propagate without diffraction, and that as opposed to the case for precisely plane waves, the curvature in an almost-plane-wave spacetime cannot be precisely sandwiched between two null surfaces (i.e., the wave must have tails). We also prove a "peeling theorem" for components of the Weyl curvature in general gravitational-wave spacetimes.

I. INTRODUCTION AND OVERVIEW

This is the first of a series of papers describing work aimed at understanding the nonlinear interaction of colliding gravitational waves in general relativity. It has been known since the early 1970s, from work on exact solutions of the Einstein field equations, that when two gravitational plane waves propagating in an otherwise flat spacetime collide, they interact so strongly as to eventually cause a curvature singularity to develop in the future of the collision plane. It is natural to ask whether this singularity is an artifact of the unphysical idealization that the waves are precisely planar and thus extend infinitely far in the "transverse" directions, or whether a singularity would still be produced if the waves were transversely finite but had arbitrarily large "size"— i.e., if they were "almost-plane waves." And if there is a regime in which spacetime singularities are guaranteed to be produced as a result of almost-plane-wave collisions, what are the conditions on the colliding almost-plane waves which characterize this regime? This paper is the first in a series whose ultimate goal is to answer these and related questions.

This first paper in the series lays foundations for the subsequent papers by reviewing (briefly) old results and presenting some new ones on colliding exact plane-wave spacetimes, and by introducing the concept of a gravitational-wave (GW) spacetime—of which almost-plane waves are a special case—and proving some theorems about GW spacetimes which imply several important properties of almost-plane waves. More specifically, we note the following.

In Sec. II we briefly review some global properties of the exact solutions for the so-called plane fronted waves with parallel rays ("PP waves"), and for plane waves. The principal purpose of this section is to introduce the reader to our terminology and viewpoint on issues that will be central to the rest of this paper and to future papers in the series.

In Sec. III we turn attention to colliding exact plane waves. We begin, in Sec. III A, by briefly reviewing the properties of some exact solutions to Einstein's equations representing plane-wave collisions, and we discuss a peculiar property of these colliding plane-wave spacetimes: the fact that, even after one has maximally extended them in a global sense, they are not "locally inextendible" if one uses the standard notion of local inextendibility (Sec. 3.1 of Ref. 1). We elucidate this peculiarity by introducing a new notion of "generic local inextendibility," which these spacetimes do turn out to satisfy. In Sec.

III B we give an alternative proof of a theorem due to Tipler[2] which predicts that collisions of exact plane waves must produce singularities. Our alternative proof of Tipler's theorem emphasizes the role and the necessity of strict plane symmetry (a concept we shall define with care) in establishing the existence of singularities in colliding plane-wave solutions and in more general plane-symmetric spacetimes. The importance of strict plane symmetry becomes clear when, following Chandrasekhar and Xanthopoulos,[3] one notices that in contrast with the usual case where they produce spacelike spacetime singularities, some colliding plane waves can produce Killing-Cauchy horizons on which strict plane symmetry breaks down and thereby can avoid the conclusion of Tipler's theorem. (In a previous paper[4] we have shown that Killing-Cauchy horizons in plane-symmetric spacetimes are unstable against plane-symmetric perturbations, and thence that any generic colliding plane-wave solution will be devoid of such horizons. In accordance with this result but independently of it, Chandrasekhar and Xanthopoulos[5] have recently discovered that the Killing-Cauchy horizons in their colliding gravitational plane-wave spacetimes are destroyed and are replaced by spacelike singularities, when the colliding plane waves are coupled with plane symmetric null fluids propagating along with the waves. Thus, the assumption of strict plane symmetry required in the proof of Tipler's theorem is probably satisfied by all but a set of measure zero of colliding plane-wave spacetimes.)

In Sec. IV we introduce the concept of a "gravitational-wave (GW) spacetime," and we use the Newman-Penrose[6] and the characteristic initial-value formalisms[7-9] to prove several theorems about GW spacetimes. These theorems have important implications for almost-plane waves (which are special cases of GW spacetimes).

Section IV A is devoted to a careful definition of GW spacetimes and associated discussion. Roughly speaking a GW spacetime is a solution to the vacuum Einstein field equations which is flat prior to the arrival of a curvature disturbance (gravitational wave), but may or may not settle back down into flatness afterward. This section also introduces a class of "standard" coordinate systems and "standard" null tetrads to be used in studying GW spacetimes.

In Sec. IV B we discuss a previous theorem of Dautcourt[10] which directly implies that any GW spacetime possessing a null Killing vector field pointing along the propagation direction—i.e., a spacetime which represents a gravitational wave propagating in a perfectly diffraction-free manner, with no change in its wave form—must be a PP-wave spacetime. Since PP waves are always infinitely large in transverse extent, this result implies that almost-plane waves (which have finite transverse "size") must always exhibit diffraction.

In Sec. IV C we present a "peeling-off"-type theorem about the behavior of the Weyl curvature quantities associated to a standard tetrad on a general GW spacetime. A discussion of this theorem is given preceding its proof.

In Sec. IV D we introduce the characteristic initial-value formalism of Penrose,[7] Muller zum Hagen and Seifert,[8] and Friedrich[9] which we will need to prove the theorem of Sec. IV D. We give a brief review of this formalism in a form that is appropriate to our conventions and notation and we emphasize those aspects relevant to our purposes.

Section IV E is devoted to another theorem about GW spacetimes: a proof that any GW spacetime that not only begins flat before the wave arrives but also returns to perfect flatness after the wave passes (i.e., any precisely "sandwich" GW spacetime), must actually be a PP-wave spacetime. Since all PP waves are infinitely large in transverse extent, this theorem implies that almost-plane waves must always leave "tails" behind, in any region of space through which they have propagated.

In Sec. V we briefly recapitulate the principal conclusions of this paper.

Throughout this paper (with the exception of Sec. III B) we will deal with purely gravitational (vacuum) waves; for Einstein-Maxwell plane waves and for plane waves coupled with fluid motions, results similar to those of Secs. II–III hold with appropriate modifications.[5]

Throughout this paper we use, without explanation, terminology and concepts from Hawking and Ellis.[1] Our mathematical conventions and notation are those of Hawking and Ellis,[1] and Misner, Thorne, and Wheeler.[11] In particular we adopt the metric signature $(-,+,+,+)$ and use the "rationalized" Newman-Penrose equations appropriate to this signature. These equations are listed in the Appendix. Our terminology and general usage of the Newman-Penrose formalism are in accordance with those of Chandrasekhar[12] after the proper conversion from his $(+,-,-,-)$ metric signature to ours.

II. EXACT PLANE-WAVE AND PP-WAVE SPACETIMES: A REVIEW INTRODUCING OUR TERMINOLOGY AND NOTATION

A plane fronted (PP) wave with parallel rays[13] $(\mathcal{M},g)$ is a spacetime where one can introduce a global coordinate chart $(U,V,X,Y)\colon \mathcal{M}\to R^4$ in which the metric takes the form

$$g=dX^2+dY^2+h(U,X,Y)dU^2-dU\,dV\ , \tag{2.1}$$

where $h(U,X,Y)$ is C^2 and satisfies

$$\frac{\partial^2 h}{\partial X^2}+\frac{\partial^2 h}{\partial Y^2}=0\ . \tag{2.2}$$

In such a spacetime, $\partial/\partial V$ is parallel [i.e., $\nabla(\partial/\partial V)\equiv 0$] and is in general the only Killing vector field on $(\mathcal{M},g)$. The special case

$$h(U,X_i)=h_{ij}(U)X_iX_j\quad (i,j=1,2)\ , \tag{2.3}$$

where $h_{ij}(U)$ is a symmetric matrix with $h_{ii}(U)=0$, defines the plane-wave[13] spacetimes with their five-dimensional group of isometries.

Note that, when $h(U,X,Y)=0$, except for $0<U<a$, the PP-wave spacetime represents an exact "sandwich wave" for which spacetime is flat for $U\le 0$ and $U\ge a$. Note also that whatever may be h, the propagation direction $\partial/\partial V$ is Killing, so the PP wave propagates without

diffraction. The PP waves must be of infinite extent in the spacelike X,Y directions because of Eq. (2.2), even though they are not in general plane symmetric. In fact, we will show in Sec. IV that neither of the above soliton-like properties of PP waves (flatness after the passage of the wave, and diffraction-free propagation) can hold true for almost-plane gravitational waves of finite transverse extent.

For a plane polarized plane wave in the "Kerr-Schild"-type chart[13] (U,V,X,Y), the function h takes the form

$$h(U,X_i)=h(U)(X^2-Y^2)\ .$$

When $h(U)=0$ for $U\geq a$ and for $U\leq 0$ (i.e., for a sandwich plane wave), it is also useful to introduce the "Rosen-type" chart[13] (u,v,x,y), which is defined on the open domain $\{F(U)G(U)\neq 0\}$ of $\mathcal{M}$ by

$$\begin{aligned} X&=xF(u)\ ,\quad Y=yG(u)\ ,\quad U=u\ ,\\ V&=v+x^2FF'+y^2GG'\ , \end{aligned} \tag{2.4}$$

where F and G are the unique C^4 solutions to the equations

$$\frac{F''}{F}=h\ ,\quad \frac{G''}{G}=-h\ , \tag{2.5}$$

with initial conditions $F(0)=G(0)=1$, $F'(0)=G'(0)=0$, and $F(U)=G(U)=1$ for $U\leq 0$. In this local coordinate system the metric is

$$g=F^2(u)dx^2+G^2(u)dy^2-du\,dv\ , \tag{2.6}$$

and the plane symmetry generators are given by $\xi_i=\partial/\partial x^i$ on the domain of the (u,v,x,y) chart, with $(i=1,2)$ and $x^1=x$, $x^2=y$.

The field equations (2.5) imply that, for a sandwich wave (2.6), in the domain $\{u\geq a\}$ where $h=0$ and where the spacetime is flat,

$$\begin{aligned} F(u)&=\frac{F(a)}{a-f_1}(u-f_1)\ ,\\ G(u)&=\frac{G(a)}{a-f_2}(u-f_2)\ , \end{aligned} \tag{2.7}$$

where, because of the field equations (2.5), $f_1>a$, and $f_2\in[-\infty,a]\cup[f_1,+\infty]$. These metric functions display, as we shall see, the focusing effect of the plane sandwich wave (2.6) on the null geodesics propagating in the u direction. We call the case $f_1=f_2$ the anastigmatic case and the generic case $f_1\neq f_2$ the astigmatic case. We also denote the null surfaces (wave fronts) $\{u=0\}$ and $\{u=a\}$ by $\mathcal{N}$ and $\mathcal{N}'$, respectively.

To see the focusing effect of the plane wave on null geodesics (discussed in greater detail, e.g., in Ref. 13), consider, for an arbitrary value of v_0, the null surface $\{v=v_0\}$ generated by null geodesics on which u is an affine parameter and along which x, y, and v are constant. In the Minkowskian region $I^-(\mathcal{N})$, these null geodesics generate a standard, flat, Minkowskian null surface; namely they generate the null surface $\{v=V=v_0\}$. On the other hand, assuming for simplicity that the plane wave is anastigmatic and using Eqs. (2.4) and (2.7), it is easily seen that in the other Minkowskian region $I^+(\mathcal{N}')$ lying to the future of the wave, the null surface $\{v=v_0\}$ is a Minkowskian null cone C_Q centered at the point Q which in the (U,V,X,Y) coordinates is given by $Q=(V=v_0,\ U=f_1,\ X=Y=0)$. In other words, after they pass through the spacetime curvature sandwiched between the wave fronts $\mathcal{N}$ and $\mathcal{N}'$ of the plane wave, the initially parallel (shear-free and convergence-free) null geodesics generating the surface $\{v=v_0\}$ are focused along the null generators of the Minkowskian null cone C_Q, to the point Q in $I^+(\mathcal{N}')$. Moreover, it is easy to see that the null generators of the surface $\{v=v_0\}$ constitute one-half of the null generators of the achronal boundary[1] $\dot{J}^+(Z_p)$ which have their past end points on Z_p. Here p is any point in $I^-(\mathcal{N})$ with $v(p)=v_0$, and Z_p is the spacelike two-surface generated by p under the action of plane symmetry. The single null generator of the null cone C_Q which runs parallel to (and thus does not intersect) the plane wave is the single past endless generator of $\dot{J}^+(Z_p)$. Similarly, in the general *astigmatic* case, one-half of the generators of $\dot{J}^+(Z_p)$ which have their past end points on Z_p generate the null surface $\{v=v_0\}$, and after passing through the plane sandwich wave these generators are focused onto a spacelike curve lying in the null plane $\{U=f_1\}$. This spacelike curve is given by $\{U=f_1\ ,\ X=0,\ V=v_0+Y^2/(f_1-f_2)\}$ in the (U,V,X,Y) coordinate system. Along the null plane $\{U=f_1\}$, which we will henceforth denote by $\mathcal{S}$, there is a one-parameter family of null generators of $\dot{J}^+(Z_p)$ which do not have past end points and which all run parallel to the plane wave.

Similar conclusions apply for the null generators of the achronal boundaries $\dot{J}^+(p)$ where $p\in I^-(\mathcal{N})$ is a point sufficiently far away from the wave (before the wave's arrival). However, in this case the null generators are focused to a point (or a spacelike curve) lying beyond the surface $\mathcal{S}$, i.e., at $U>f_1$.[13]

The plane symmetry generated by the Killing vectors ξ_i breaks down on the null surface $\mathcal{S}$; that is, in the tangent space at any point on this surface $\mathcal{S}$, the Killing vectors ξ_i generate a subspace which is *not* a two-dimensional spacelike plane (see Sec. III B of this paper). This breakdown of "strict" plane symmetry on $\mathcal{S}$ (Sec. III B) allows the null generators of the achronal boundary $\dot{J}^+(Z_p)$ to intersect each other at points in $\mathcal{S}$. In the *anastigmatic* case, the ξ_i degenerate on $\mathcal{S}$ to null Killing vectors that are proportional to $\partial/\partial V$ and that vanish on the line $X=Y=0$; hence, in this case the ξ_i span a one-dimensional null line at each point on $\mathcal{S}-\{X=Y=0\}$. In the *astigmatic case*, ξ_1 degenerates on $\mathcal{S}$ to a null Killing vector that is proportional to $\partial/\partial V$ and that vanishes along the two-surface $X=0$, while ξ_2 is still spacelike on $\mathcal{S}$, generating symmetries along the spacelike line to which null generators of $\dot{J}^+(Z_p)$ are focused. In this case, the ξ_i span a two-dimensional *null* plane at each point on $\mathcal{S}-\{X=0\}$.

A further consequence of these focusing properties is the fact that plane-wave spacetimes are not globally hyperbolic,[13] even though they are geodesically complete and satisfy stable causality. Any partial Cauchy surface

Σ in $I^-(\mathcal{S})$ cannot intersect $\mathcal{S}$; hence $\mathcal{S}$ is a future Cauchy horizon for Σ. This was to be expected, since strict plane symmetry on Σ will be preserved throughout the domain of dependence of Σ, which therefore cannot include $\mathcal{S}$. [The past Cauchy horizon for a partial Cauchy surface that intersects the wave will be the time reversed analogue of $\mathcal{S}$ lying in $I^-(\mathcal{N})$.]

III. COLLIDING EXACT PLANE WAVES

A. Review of exact solutions and a new viewpoint on the breakdown of local inextendibility

The first results on exact solutions of the vacuum Einstein equations representing colliding plane impulsive gravitational waves with parallel, linear polarizations were obtained by Khan and Penrose,[14] and Szekeres.[15] Later Nutku and Halil[16] obtained the generalization of these solutions to arbitrary relative linear polarizations for the two incoming impulsive waves. Szekeres,[17] and Chandrasekhar and Xanthopoulos[17] have obtained similar results for Einstein-Maxwell waves. The main result of these exact solutions (see Fig. 1) is that the future of the collision surface (region IV) is bounded by a curvature singularity in future directions. Surprisingly, the singularity extends over to the past endless null generators of the surfaces $\mathcal{N}_1$ and $\mathcal{N}_2$ which would lie in the respective Cauchy horizons of the single plane-wave spacetimes if the collision were not taking place.[18,14] A curious result, therefore, is that the spacetime pictured in Fig. 1 is maximally extended, i.e., the points of $\mathcal{S}_2$ and $\mathcal{S}_1$ touching the surfaces $\mathcal{N}_1$ and $\mathcal{N}_2$ cannot be added to the spacetime even though these points do not represent local curvature singularities—i.e., even though there are timelike curves running into and terminating on these singularities which are completely contained within the flat regions II and III. In other words the maximally extended colliding plane-wave spacetime is not C^k-locally inextendible although it is globally C^k-inextendible (maximal) for any $k \geq 1$ (for definitions, see Sec. 3.1 of Ref. 1).

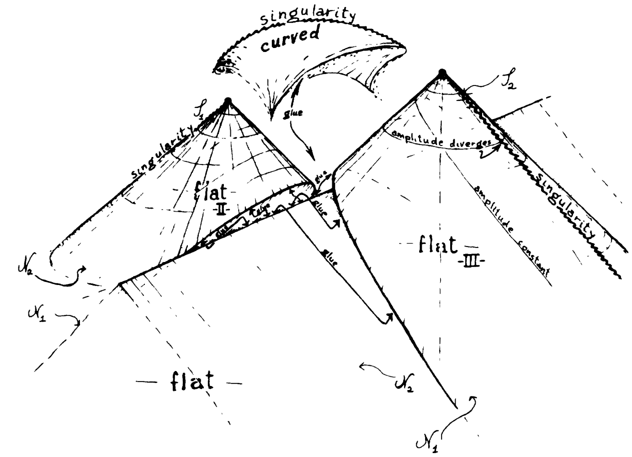

FIG. 1. Geometry of the maximally extended colliding plane-wave spacetime in the anastigmatic case. The null surfaces $\mathcal{N}_1$ and $\mathcal{N}_2$ represent the wave front(s) ($\mathcal{N}_1$, $\mathcal{N}_1'$ and $\mathcal{N}_2$, $\mathcal{N}_2'$) of the incoming colliding plane waves. Since the plane waves depicted in the above figure are impulsive, for both waves the past and future wave fronts are identified ($\mathcal{N}_1 = \mathcal{N}_1'$, $\mathcal{N}_2 = \mathcal{N}_2'$) and hence are indistinguishable. Because of the focusing of each wave by the other wave, the wave fronts $\mathcal{N}_1$ and $\mathcal{N}_2$ are represented by null cones in the future of the collision plane. The flat regions II and III lie under the cones $\mathcal{N}_2$, $\mathcal{N}_1$, respectively. The respective Cauchy horizons $\mathcal{S}_1$ and $\mathcal{S}_2$ of the incoming plane waves are completely cut off from the spacetime, except for those points which lie on the common generators of $\mathcal{S}_1$ and $\mathcal{S}_2$ with the null cones $\mathcal{N}_2$ and $\mathcal{N}_1$, respectively. However, these points are also singular since they do not possess a regular spacetime neighborhood which is isolated from the curvature singularity. Note that the coordinate system can always be arranged, by a Lorentz transformation, so that the collision is head on as in the above figure. This figure, which was drawn by R. Penrose and which was published in Ref. 18 by R. Matzner and F. Tipler, is reproduced here by the kind permissions of Penrose, Matzner, and Tipler.

The first new result of this paper is to point out that the above failure of local inextendibility is not generic in the following sense.

We define a spacetime $(\mathcal{M}, g)$ to be *generically* C^k-locally inextendible if there exists *no* open set $\mathcal{U}$ in $\mathcal{M}$ with the following properties.

(i) $\mathcal{U}$ has noncompact closure in $\mathcal{M}$.

(ii) $(\mathcal{U}, g\mid_{\mathcal{U}})$ has a C^k-extension $(\mathcal{U}', g')$ in which the image of $\mathcal{U}$ has compact closure.

(iii) There is a point $q \in \mathcal{U}$ and an open subset $\mathcal{O}$ of $T_q\mathcal{M}$ consisting of timelike vectors, such that for any vector $X \in \mathcal{O}$ the geodesic $\{\gamma_X(t) = \exp_q(tX),\ 0 \leq t < t_1\}$ from q is inextendible in $\mathcal{M}$, and t_1 is finite. Here t_1 is the smallest upper bound of all $t' \in R$ such that $\gamma_X(t) = \exp_q(tX)$ is defined and contained in $\mathcal{U}$ for all $t \in [0, t')$.

With this definition, the maximally extended colliding plane-wave spacetime of Fig. 1 is not only globally inextendible; it is also generically C^k-locally inextendible for any $k \geq 1$.

Note that the key idea behind the usual definition of local inextendibility[1] is this: One identifies as locally *extendible*, among others, those (maximal) spacetimes that possess purely topological singularities, i.e., singularities which do not involve unbounded curvature but which nevertheless cannot be removed without destroying the topological manifold structure of the spacetime. Such a singularity may be modeling the global influence of some essentially nonsingular matter distribution (e.g., a cosmic string), or may be the result of some unusual "cutting and pasting" employed in the construction of the spacetime (such as those that appear in the covering space of the two-dimensional Minkowski space with the origin removed). In all such cases our definition would identify these spacetimes as generically locally inextendible; i.e., the presence of their "noncurvature" singularities will be *directly* felt only by those freely falling observers which constitute a set of measure zero. Only the spacetimes in which such topological singularities are unavoidable for a "finite fraction" of all freely falling observers (e.g., because of the focusing of causal geodesics onto these singularities) will fail to satisfy generic local inextendibility.

For example, if the topological singularities of the colliding plane-wave spacetimes which we have described above were to lie *beyond* the respective Cauchy horizons of the colliding waves instead of lying *on* them, then these spacetimes would fail to be generically locally inextendible. However, as our discussion of Tipler's theorem in the next section will make clear, this is not a possible outcome of generic plane-wave collisions. Thus, except possibly for a set of measure zero, all colliding plane-wave spacetimes will satisfy generic local inextendibility.

Another important property of the above examples of colliding plane-wave solutions is that they are globally hyperbolic, since neither of the Cauchy horizons $\mathcal{S}_1$, $\mathcal{S}_2$ is contained in $\mathcal{M}$. In particular, the singularities present in these spacetimes are "not timelike" in the sense of Penrose;[19] that is the singular points are part of an achronal future c boundary for $(\mathcal{M},g)$.

We should also remark that, recently Chandrasekhar and Xanthopoulos[3] have obtained new exact solutions describing colliding gravitational impulsive-shock waves with nonparallel polarizations, in which the interaction region is bounded by a Killing-Cauchy horizon instead of by a spacelike singularity, and in which a timelike singularity appears when the solution is analytically extended beyond this horizon. However, as we will also discuss in the next section, it is shown in Ref. 4 that such Killing-Cauchy horizons in any plane-symmetric spacetime are unstable against purely plane-symmetric perturbations. Therefore, it is reasonable to expect that the spacetimes resulting from "generic" plane-wave collisions will always involve spacelike curvature singularities with the same global structure as the solutions we have discussed above, regardless of the relative configuration of the incoming polarizations and wave forms.

B. A general framework for studying colliding plane-wave spacetimes and an alternative proof of Tipler's theorem on their singularities

The global structure of plane-symmetric spacetimes (e.g., plane waves and colliding plane waves) is nontrivial when they possess Killing-Cauchy horizons on which their plane symmetry breaks down. When discussing such spacetimes from a general standpoint some care is needed. In this section we introduce a brief framework for analyzing general plane-wave and colliding plane-wave spacetimes. This framework is based on some intuitively plausible definitions and constructions which make precise the basic notions that one needs in such a general discussion. We conclude the section with an important application of this framework: a discussion and an alternative proof of Tipler's theorem[12] on singularities of colliding plane-wave spacetimes.

We will call a maximal (see Sec. 3.1 of Ref. 1) spacetime $(\mathcal{M},g)$ *plane symmetric* if there exists a pair of commuting Killing vectors ξ_1,ξ_2 on $\mathcal{M}$, and an open dense subset of $\mathcal{M}$ at every point of which ξ_1,ξ_2 span a spacelike two-dimensional subspace in the tangent space. So as to exclude cylindrical symmetry, we assume that the orbits of ξ_i $(i=1,\ 2)$ are homeomorphic to R^1. If the open dense subset is all of $\mathcal{M}$, i.e., if no breakdowns of plane symmetry occur, then we say $(\mathcal{M},g)$ is *strictly plane symmetric.*

In the strictly plane-symmetric region of any plane-symmetric spacetime there exist *standard null tetrads* constructed as follows: Since ξ_i are Killing and span a spacelike two-plane at each point, there exist precisely two null geodesic congruences everywhere orthogonal to ξ_i. Let $\mathbf{l},\mathbf{n}$ be tangent vector fields to these congruences normalized so that $g(\mathbf{l},\mathbf{n})=-1$. Let $\mathbf{m}$, $\mathbf{m}^*$ be two linearly independent complex null linear combinations of ξ_i, which are complex conjugate, satisfy $g(\mathbf{m},\mathbf{m}^*)=1$, and vary smoothly over the region of strict plane symmetry. Then $(\mathbf{l},\mathbf{n},\mathbf{m},\mathbf{m}^*)$ is a null tetrad which is locally regular although it will not in general cover all of $\mathcal{M}$. We will call the tetrad $(\mathbf{l},\mathbf{n},\mathbf{m},\mathbf{m}^*)$, together with the additional requirement that $\mathbf{l},\mathbf{n}$ are Lie parallel along ξ_i (which we can obviously impose since ξ_i are Killing and commute), a *standard tetrad.*

We will say that a plane-symmetric nonflat spacetime is a *plane wave* if in a standard tetrad either $\Psi_0=\Psi_2\equiv 0$ or $\Psi_4=\Psi_2\equiv 0$. Note that this property is independent of the choice of the standard tetrad which is unique up to tetrad rotations of type III (Sec. 7.3 of Ref. 12).

We have seen in the last section that single plane sandwich wave solutions are not strictly plane symmetric, as focusing causes the breakdown of plane symmetry along a null hypersurface $\mathcal{S}$ in $I^+(\mathcal{N}\,')$, where $\mathcal{N}$, $\mathcal{N}\,'$ are the past and future wave fronts. Now consider a spacetime representing the collision between two plane waves moving in opposite directions. A plane-symmetric spacetime $(\mathcal{M},g)$ will be said to model *colliding plane waves* if there exist two null surfaces $\mathcal{N}_1$, $\mathcal{N}_2$ in $\mathcal{M}$, intersecting in a spacelike two-surface Z, such that in any standard tetrad $\Psi_4=\Psi_2\equiv 0$ but $\Psi_0\neq 0$ on $I^-(\mathcal{N}_1)$, $\Psi_0=\Psi_2\equiv 0$ but $\Psi_4\neq 0$ on $I^-(\mathcal{N}_2)$, and $\Psi_0,\Psi_4\neq 0$ on $I^+(Z)$. Figure 2 depicts such a spacetime.

In the specific colliding plane-wave solutions reviewed above,[15,16,18] the collision produces a spacetime singularity. That this is a rather general outcome of plane-wave collisions is shown by a theorem of Tipler.[2] However, a key requirement for the proof of Tipler's theorem is that strict plane symmetry holds *throughout* the colliding plane-wave spacetime. Since this notion of strict plane symmetry is crucial to the discussion that we will give in the next few paragraphs of this section, we first present a restatement and an alternative proof of Tipler's theorem

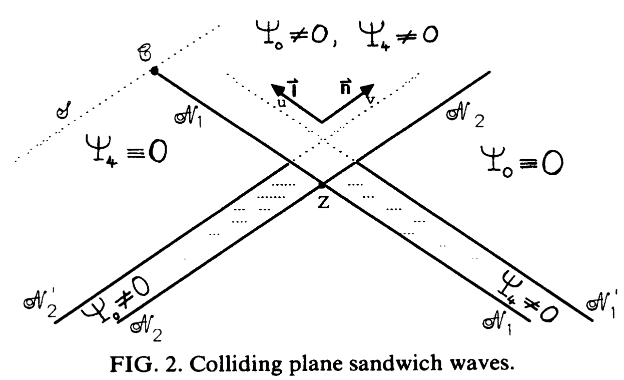

FIG. 2. Colliding plane sandwich waves.

which emphasize the requirement of strict plane symmetry explicitly.

Theorem 1 ("Tipler's theorem"[2]). Let $(\mathcal{M},g)$ be a strictly plane-symmetric spacetime with a C^2 metric g, with the following properties.

(i) Null convergence[1] holds on $\mathcal{M}$: $R_{ab}K^aK^b \geq 0$ for any null vector K.

(ii) There exists a point p at which either at least one of $(\Psi_0,\sigma,\Phi_{00})$ is nonzero or at least one of $(\Psi_4,\lambda,\Phi_{22})$ is nonzero, in some standard tetrad on $\mathcal{M}$.

(iii) Through the above point $p \in \mathcal{M}$, there exists a partial Cauchy surface Σ which intersects each null geodesic generator of $\dot{J}^{\pm}(p)$ and which is noncompact in the spacelike direction orthogonal to ξ_i.

Then, $(\mathcal{M},g)$ is *not* null geodesically complete.

Proof. Fix the standard tetrad mentioned in property (ii). Then since l,$\mathbf{n}$ are geodesic and hypersurface orthogonal, we can arrange that the following Newman-Penrose spin coefficients vanish (cf. the Appendix)

$$\kappa=\nu=\epsilon+\epsilon^*=\rho-\rho^*=\mu-\mu^*=0 \ .$$

Now assume, without loss of generality, that it is one of $(\Psi_0,\sigma,\Phi_{00})$ that is nonzero at p (otherwise interchange the role of l and $\mathbf{n}$ in the argument, and accordingly interchange the roles of the spin quantities).

Let Z_p denote the orbit of p under the action of the Killing symmetry group generated by ξ_1,ξ_2. Then by plane symmetry, properties (ii) and (iii) hold at every point $q \in Z_p$.

Now assume, in contradiction of the theorem's conclusion, that $(\mathcal{M},g)$ is null geodesically complete. Consider $\dot{J}^{\pm}(Z_p)$. These achronal boundaries are generated by null geodesics which by (iii) and because of strict plane symmetry all have their past (future) end points on Z_p, and which are everywhere orthogonal to Z_p and hence (since ξ_i are Killing) to ξ_1,ξ_2. Thus $\dot{J}^{\pm}(Z_p)$ are generated by integral curves of l and $\mathbf{n}$ that start off from Z_p.

It is shown by Tipler in Ref. 2 that as a result of the assumptions (i) and (ii) above [and of the Ricci identities (A5) and (A6)], any null geodesic γ_q parallel to l and passing through any point q in Z_p will have a conjugate point to Z_p along itself at some affine distance f from q. If we now fix our time orientation so that the conjugate point lies to the future of q, then every null generator of $\dot{J}^+(Z_p)$ parallel to l has a conjugate point to Z_p along itself at an affine distance $u=f>0$; and f is independent of the null generator. The noncompactness of the partial Cauchy surface in property (iii) guarantees that the null geodesic generators of $\dot{J}^+(Z_p)$ parallel to l cannot intersect (except on Z_p) those parallel to $\mathbf{n}$, and consequently since $\dot{J}^+(Z_p)$ has no boundary (proposition 6.3.1 of Ref. 1), the submanifold $\dot{J}_l^+(Z_p)-Z_p$ of $\dot{J}^+(Z_p)$ generated by null geodesics parallel to l has no boundary.

We construct the map

$$\phi : Z_p\times(0,f] \rightarrow \dot{J}_l^+(Z_p) - Z_p \ ,$$

given by

$$\phi : (q,u) \rightarrow \gamma_q(u) \in \dot{J}_l^+(Z_p) - Z_p \ .$$

Claim: ϕ is a diffeomorphism.

That ϕ is onto is obvious since all points on $\dot{J}_l^+(Z_p)-Z_p$ are on null geodesics γ from Z_p and for $u>f$ $\gamma_q(u)$ does not belong to $\dot{J}_l^+(Z_p)-Z_p$ for any $q\in Z_p$ (see Chap. 4 of Ref. 1). That ϕ is one to one is an immediate consequence of the strict plane symmetry holding at every point of $\mathcal{M}$, which strict plane symmetry prevents different null generators γ_q and $\gamma_{q'}$ $(q\neq q')$ from intersecting each other. That ϕ and ϕ^{-1} are smooth is clear by construction.

Thus,

$$[\dot{J}_l^+(Z_p)-Z_p]\cong Z_p\times(0,f] \cong R^2\times(0,f] \ .$$

Here, the symbol $\cong$ denotes "is diffeomorphic to." But $R^2\times(0,f]$ has a boundary which is diffeomorphic to R^2, and therefore we obtain a contradiction to the proposition 6.3.1 of Ref. 1.

Therefore, the assumption that $(\mathcal{M},g)$ is null geodesically complete must be false—a conclusion that proves the theorem. □

Tipler's theorem implies, as a specific application, that if the spacetime produced by the collision of two plane waves is strictly plane symmetric—as is the case in the classic examples (Refs. 14–16), then the collision must produce a singularity (null geodesic incompleteness). We have argued at length in Sec. III of Ref. 4 that in a spacetime which represents the collision between an exact plane gravitational wave and a plane wave of *any* physical field belonging to some restricted class, the breakdown of strict plane symmetry is incompatible with global causality. Therefore, strict plane symmetry is a *natural* restriction to impose on colliding plane-wave spacetimes. However, the fully nonlinear gravitational field does *not* belong to the class of fields for which the arguments of Ref. 4 are valid; consequently these arguments do not *prove* that colliding plane-wave spacetimes are strictly plane symmetric. In fact, just as spacetimes containing a single plane wave fail (beyond the Cauchy horizon $\mathcal{S}$) to be strictly plane symmetric, so also *some* colliding plane-wave spacetimes possess (Killing-)Cauchy horizons at which strict plane symmetry breaks down. Examples are the Chandrasekhar-Xanthopoulos[3] solutions. Tipler's theorem cannot be applied to such spacetimes.

On the other hand, as is suggested by calculations of Chandrasekhar and Xanthopoulos (Ref. 5) and proved by the author (Ref. 4), all such Killing-Cauchy horizons which break strict plane symmetry are unstable against plane-symmetric perturbations. Moreover, as was shown by Chandrasekhar and Xanthopoulos[5] for special cases, it is plausible (though not yet proved in general) that the (full nonlinear) growth of these instabilities always destroys the Killing-Cauchy horizon, thereby making the spacetime strictly plane symmetric. If this is the case, then *all colliding plane-wave spacetimes whose causal structures are stable against plane-symmetric perturbations are strictly plane symmetric, and Tipler's theorem implies that they also are all singular.*

It is interesting in revealing the depth of Tipler's theorem to note that for a single plane-wave spacetime the only conditions of the theorem that do not hold are strict plane symmetry and the existence of the partial Cauchy surface satisfying the requirements in (iii). As we argued in Sec. II these conditions are intimately related and presumably imply each other in the generic case.[4] The partial Cauchy surface condition is used to guarantee that all generators of $\dot{J}^+(Z_p)$ have past end points on Z_p; whereas the strict plane symmetry is used to show that the map ϕ is a diffeomorphism, which is the vital step in our proof of Tipler's theorem. In fact, we will use this aspect of Tipler's theorem in a future paper to produce a qualitative argument for the existence of singularities in colliding almost-plane-wave spacetimes when the relevant parameters belong to a certain regime.

Also note that (Fig. 2) Tipler's theorem [simply by taking the point p as an arbitrary point in $I^+(\mathcal{N}_2) \cap I^-(\mathcal{N}_2')$] implies that the points on the past endless generators of $\dot{J}^+(Z_p)$ which would lie in the Cauchy horizon $\mathcal{S}$ will become singular, and consequently $\mathcal{S}$ will be cut off completely from the colliding plane-wave spacetime, a result that is not obvious from the analytical structure of the known exact solutions.[18]

IV. GRAVITATIONAL-WAVE (GW) SPACETIMES

In this section we turn attention to general solutions to the vacuum Einstein equations which represent a single "gravitational wave" propagating in an otherwise flat space. Plane-wave and PP-wave spacetimes are simple examples of such solutions; and we frequently will refer to them for comparison and motivation while discussing more general gravitational-wave (GW) spacetimes. Our primary interest in studying GW spacetimes is to learn about "almost-plane waves"—GW spacetimes that in some suitable sense are of "finite spatial extent," representing a transversely bounded curvature disturbance carrying finite "energy" and propagating in an otherwise flat spacetime. (We will define almost-plane waves more precisely in paper 2 of this series.) Clearly almost-plane waves cannot be plane symmetric, since they have an amplitude that must satisfy suitable falloff conditions at large "transverse" distances.[20] We will see in Sec. IV B, by a theorem of Dautcourt,[10] that relaxing the assumption of plane symmetry on such a spacetime forces it to have no Killing vectors in general and hence leaves little hope for an exact solution. Indeed, one can already guess that for a nonplanar gravitational wave the linear and nonlinear effects of diffraction and backscattering might cause the wave to evolve as it propagates, thereby preventing the existence of a Killing propagation vector. However, it is by no means clear whether the nonlinearity of the field equations can make possible the existence of localized, nondispersive, solitonlike solutions. Dautcourt's result shows that it cannot.

The plan of this section is as follows.

Section IV A is devoted to a careful definition of GW spacetimes and associated discussion. Roughly speaking a GW spacetime is a solution to the vacuum Einstein field equations which is flat prior to the arrival of a curvature disturbance (gravitational wave), but may or may not settle back down into flatness afterward. This section also introduces a class of "standard" coordinate systems and "standard" null tetrads to be used in studying GW spacetimes.

In Sec. IV B we discuss a previous theorem of Dautcourt[10] which directly implies that any GW spacetime possessing a null Killing vector field which points along the propagation direction—i.e., possessing a gravitational wave which propagates in a perfectly diffraction-free manner, indefinitely preserving its wave form—must be a PP-wave spacetime. Since PP waves are always infinitely large in transverse extent (Sec. II), this result implies that almost-plane waves (which have finite transverse "size") must always exhibit diffraction.

In Sec. IV C we discuss, present, and prove a "peeling-off"-type theorem about the behavior of the Weyl curvature quantities associated with a standard tetrad on a general GW spacetime.

In Sec. IV D we introduce the characteristic initial-value formalism of Penrose,[7] Muller zum Hagen and Seifert,[8] and Friedrich[9] which we will need to prove the theorem of Sec. IV D. We give a brief review of this formalism in a form that is appropriate to our conventions and notation, and we emphasize those aspects relevant to our purposes.

Section IV E is devoted to another theorem about GW spacetimes: a proof that any GW spacetime that not only begins flat before the wave arrives but also returns to perfect flatness after the wave passes (i.e., any precisely "sandwich" GW spacetime), must actually be a PP-wave spacetime. Since all PP waves are infinitely large in transverse extent (Sec. II), this theorem implies that almost-plane waves must always leave "tails" behind in any region of space through which they have propagated.

A. Definition of a GW spacetime

Definition. A gravitational-wave (GW) spacetime is a geodesically complete (hence maximal), vacuum spacetime $(\mathcal{M},g)$ with a C^2 metric g, satisfying the following conditions.

(i) $\mathcal{M}$ is diffeomorphic to R^4.

(ii) There exist two nonintersecting, null, achronal three-dimensional C^2 submanifolds (without edge) $\mathcal{N}$ and $\mathcal{N}'$, whose null geodesic generators have no past or future end points in $\mathcal{M}$, and which satisfy $\mathcal{N} \subset I^-(\mathcal{N}')$.

(iii) $(\mathcal{M},g)$ is flat on $I^-(\mathcal{N})$.

(iv) There exists a noncompact partial Cauchy surface through every point $p \in \mathcal{M}$.

(v) g is C^∞ outside $\mathcal{N}$ and $\mathcal{N}'$.

Remark. The differentiability class of $\mathcal{M}$ is assumed C^∞. It can be shown, using the characteristic initial-value formalism which we will outline in Sec. IV D, that there exist spacetimes satisfying all the above conditions except geodesic completeness. Completeness cannot be proved for these spacetimes because of the local nature of the existence theorems; nevertheless, in view of its mathematical naturalness and the relatively unimportant role it will play in what follows, we retain the assumption of completeness. We also remark that, by appealing to Christodoulou's recent theorems[21] proving the global existence of solutions to the initial-value problem for the vacuum Einstein equations with "small" initial data, it seems physically plausible (and in fact extremely likely) that for sufficiently "weak" gravitational waves—not a serious restriction for our purposes—the completeness condition will indeed be satisfied.

On any GW spacetime there exist local coordinate systems (u,v,x,y) for which $\mathcal{N}$, $\mathcal{N}'=\{u=0\},\{u=a\}$ and in which we can find a local null tetrad with the form

$$l=R\left[\frac{\partial}{\partial u}+A\frac{\partial}{\partial v}\right], \quad \mathbf{n}=\frac{\partial}{\partial v}, \qquad m=\hat{M}\frac{\partial}{\partial x}+\hat{N}\frac{\partial}{\partial y}+\omega\frac{\partial}{\partial v}, \tag{4.1}$$

where $R(u,v,x,y)$, $A(u,v,x,y)$, and $\omega(u,v,x,y)$ are real and $\hat{M}(u,v,x,y)$, $\hat{N}(u,v,x,y)$ are complex functions. (A proof and detailed discussion will be given in a future paper.[20]) We call such a local chart and tetrad a "standard coordinate system" and its "associated standard tetrad." If the GW spacetime has a (null) Killing propagation direction, we also require a standard coordinate system (u,v,x,y) to satisfy $\partial/\partial v$ = Killing vector, but we drop the requirement that $\mathcal{N}'=\{u=a\}$. Note that for both the general case and for a GW spacetime with a Killing propagation direction, neither the standard charts nor the standard tetrads are uniquely defined; in both cases a large amount of coordinate and tetrad transformation freedom remains in the choice of these charts and tetrads. For example, for a sandwich plane-wave spacetime (Sec. II), the "Kerr-Schild"-type chart and the "Rosen"-type chart are both standard coordinate systems.

B. The only diffraction-free GW spacetimes are PP waves

In a short paper[10] published in 1964, Dautcourt classified all vacuum spacetimes possessing a null Killing vector. According to his classification, such spacetimes either are PP waves or are certain solutions of Petrov type II or I. Furthermore, his solutions of Petrov type II or I have the property that their curvature-invariant $R_{abcd}R^{abcd}$ is nonzero on a region that extends into $I^-(\mathcal{N})$ for any null surface $\mathcal{N}$ satisfying property (ii) above, and diverges on a three-dimensional timelike hypersurface.[10] Obviously, these type-I or type-II solutions cannot be gravitational-wave spacetimes according to our definition above. Therefore, as we have stated earlier, the only GW spacetimes with a Killing propagation direction (the only diffraction-free GW spacetimes) are PP waves; and this in turn implies that almost-plane waves must always exhibit diffraction.

C. A "peeling"-type property of general GW spacetimes

We now prove a "peeling"-type result about the behavior of the curvature tensor in a general GW spacetime.

Theorem 2. Let $(\mathcal{M},g)$ be a gravitational-wave spacetime with wave fronts $\mathcal{N}$, $\mathcal{N}'$; thus $(\mathcal{M},g)$ is flat on $I^-(\mathcal{N})$ where $\mathcal{N}$ is the past wave front. Then, there exists a collection of open sets $\{U_\alpha\}$, $U_\alpha\subset I^+(\mathcal{N})$, such that $\overline{\cup_\alpha U_\alpha}\supset\mathcal{N}$, and on each U_α, $\Psi_1=\Psi_2=\Psi_3=\Psi_4=0$ in any standard chart and tetrad.

Remarks. This result implies that any generic observer through whom the curvature disturbance of the GW spacetime passes will first feel only the Ψ_0 component of the Weyl tensor in any standard tetrad. Only later, and in a "sudden" (i.e., nonanalytic, shocklike) fashion, the other components Ψ_1, Ψ_2, Ψ_3, and Ψ_4 (which represent backscattered curvature) will appear in the measured gravitational field. Hence, if we trace the history of the observer's measurements backwards in time, the quantities Ψ_1, Ψ_2, Ψ_3, Ψ_4 will "peel off" (not necessarily in that order) before the quantity Ψ_0 vanishes and the disturbance is turned off.

Proof. In any standard chart (u,v,x,y), the surface $\mathcal{N}$ is given by $\mathcal{N}=\{u=0\}$ and the standard tetrad is of the form of Eq. (4.1):

$$l=R\left[\frac{\partial}{\partial u}+A\frac{\partial}{\partial v}\right] \quad (R\neq 0)$$

$$\mathbf{n}=\frac{\partial}{\partial v}, \quad \mathbf{m}=\hat{M}\frac{\partial}{\partial x}+\hat{N}\frac{\partial}{\partial y}+\omega\frac{\partial}{\partial v}.$$

Since the metric is C^2, and the spacetime is flat on $I^-(\mathcal{N})$, all curvature quantities vanish on $\mathcal{N}=\{u=0\}$. Now assume, in contradiction to the theorem's conclusion, that there is no set of neighborhoods $\{U_\alpha\}$, $U_\alpha\subset I^+(\mathcal{N})$ satisfying $\overline{\cup_\alpha U_\alpha}\supset\mathcal{N}$ such that on each U_α, $\Psi_1=\Psi_2=\Psi_3=\Psi_4\equiv 0$. Let $\{V_\alpha\}$ be the collection of all open sets in $I^+(\mathcal{N})$ on which $\Psi_i\equiv 0$, $i=1,2,3,4$. Then the complement $\mathcal{O}$ of $\overline{\cup_\alpha V_\alpha}$ in $\mathcal{M}$ is an open set intersecting $\mathcal{N}$, and $\mathcal{O}\cap I^+(\mathcal{N})$ does not contain any open neighborhoods on which $\Psi_i\equiv 0$. Thus $\mathcal{O}\cap I^+(\mathcal{N})\subset\operatorname{supp}(\Psi_1,\Psi_2,\Psi_3,\Psi_4)$; in fact, $\operatorname{Int}\operatorname{supp}(\Psi_1,\Psi_2,\Psi_3,\Psi_4)=\mathcal{O}\cap I^+(\mathcal{N})$, and therefore $[\mathcal{O}\cap I^+(\mathcal{N})]\cap\{p\in\mathcal{M},(\Psi_1,\Psi_2,\Psi_3,\Psi_4)(p)\neq 0\}$ is a nonempty open set whose closure intersects $\mathcal{N}$ in the closure of an open set $\mathcal{W}$ in $\mathcal{N}$. Then, it follows from the repeated application of the argument below to noncharacteristic surfaces in a neighborhood of $\mathcal{N}$, that there exists at least one open subset $\mathcal{W}$ of $\mathcal{N}$ and an open neighborhood $\mathcal{U}'$ around it, for which at least one of $\Psi_1,\Psi_2,\Psi_3,\Psi_4$ is nonzero at any point in $\mathcal{U}'\cap I^+(\mathcal{N})$.

Now in general Ψ_0, will be nonzero on $\mathcal{U}'$. Then, perform a type-II tetrad rotation[12] with a local function b to make $\Psi_0'\equiv 0$ on $\mathcal{U}'$. [This can be done since $(\Psi_1,\Psi_2,\Psi_3,\Psi_4)\neq 0$ at any point.] The rotated tetrad will be of the form

$$l'=R\frac{\partial}{\partial u}+RA'\frac{\partial}{\partial v}+P'\frac{\partial}{\partial x}+Q'\frac{\partial}{\partial y},$$

$$\mathbf{n}'=\mathbf{n}=\frac{\partial}{\partial v}, \quad \mathbf{m}'=\hat{M}\frac{\partial}{\partial x}+\hat{N}\frac{\partial}{\partial y}+\omega'\frac{\partial}{\partial v}.$$

Henceforth we will omit primes over the quantities belonging to the new tetrad.

Now in this new tetrad $\Psi_0\equiv 0$ on $\mathcal{U}'$. But then, the Bianchi identities give us

$$-D\Psi_1=-3\kappa\Psi_2+2(\epsilon+2\rho)\Psi_1\ ,$$
$$-D\Psi_2=-\delta^*\Psi_1-2\kappa\Psi_3+3\rho\Psi_2+2(\pi-\alpha)\Psi_1\ ,$$
$$-D\Psi_3=-\kappa\Psi_4-\delta^*\Psi_2-2(\epsilon-\rho)\Psi_3+3\pi\Psi_2-2\lambda\Psi_1\ ,$$
$$-D\Psi_4=-\delta^*\Psi_3-(4\epsilon-\rho)\Psi_4+(4\pi+2\alpha)\Psi_3-3\lambda\Psi_2\ ,$$

which, when written in terms of partial derivatives with respect to the coordinates according to the tetrad above, and when the spin coefficients and metric components are regarded as known functions, yields us a system of first-order linear partial differential equations for Ψ_1, Ψ_2, Ψ_3, Ψ_4.

By the C^2-ness of g, $\Psi_1=\Psi_2=\Psi_3=\Psi_4=0$ on $\mathcal{N}$, and hence also on $\mathcal{W}\subset\mathcal{N}$. Since $R=R'\neq 0$ at any point, the surface $\mathcal{W}$ is a noncharacteristic surface for the above system of equations, given locally by $\{u=0\}$. Since the coefficients are smooth [at least C^3 since g is C^4 on $I^+(\mathcal{N})\cap I^-(\mathcal{N}')$], any C^1 solution $(\Psi_1,\Psi_2,\Psi_3,\Psi_4)$ of the system above is uniquely determined in some neighborhood $\mathcal{V}'$ of $\mathcal{W}$ by Holmgren's uniqueness theorem extended to nonanalytic equations (John[22] and Smoller[23]). (In fact, one can safely assume g to be piecewise analytic thus eliminating the need for such an extended uniqueness theorem: that piecewise analyticity, by the Cauchy-Kovalewski theorem,[22] implies that the unique C^1 solution of the above system is also analytic on its domain of uniqueness.) But as we clearly see from the above system of equations, $\Psi_1=\Psi_2=\Psi_3=\Psi_4=0$ is a C^1 solution in any neighborhood of $\mathcal{W}$, of the above initial-value problem. Therefore it is the unique solution in some neighborhood $\mathcal{V}'$ of $\mathcal{W}$, and $\Psi_1=\Psi_2=\Psi_3=\Psi_4\equiv 0$ in that neighborhood $\mathcal{V}'$.

But since $\Psi_0\equiv 0$ on $\mathcal{U}'$, $\mathcal{U}'\cap\mathcal{V}'$ is a flat neighborhood of $\mathcal{W}$ and hence in the original tetrad, on $\mathcal{U}'\cap\mathcal{V}'$, $\Psi_1=\Psi_2=\Psi_3=\Psi_4\equiv 0$. This contradicts our assumption about $\mathcal{W}$ and $\mathcal{U}'$, since $(\mathcal{U}'\cap\mathcal{V}')\cap I^+(\mathcal{N})$ is nonempty and is contained in $\mathcal{U}'\cap I^+(\mathcal{N})$. This contradiction proves theorem 2. □

D. Review of the characteristic initial-value formalism

Our next result about GW spacetimes is a uniqueness theorem similar to that of Dautcourt: Whereas Dautcourt's theorem (Sec. IV B above) says that the only diffraction-free GW spacetimes are PP waves, our next theorem (Sec. IV E below) says that the only sandwich GW spacetimes are PP waves. Since the proof of the theorem makes extensive use of the characteristic initial-value formalism as developed by Penrose,[7] Muller zum Hagen and Seifert,[8] Friedrich,[9] and others, we first give in this subsection a brief review of this formalism, emphasizing those aspects that are relevant to our purposes. We follow Friedrich[9] quite closely, though with entirely different conventions.

We assume that we are given a "spacetime $\mathcal{M}$ with boundary," where the boundary $\partial\mathcal{M}\equiv S$ consists of two null surfaces $\mathcal{N}_1$, $\mathcal{N}_2$ intersecting and terminating in the past directions on a spacelike two-dimensional submanifold $Z=\mathcal{N}_1\cap\mathcal{N}_2$; $\partial\mathcal{M}\equiv S=\mathcal{N}_1\cup\mathcal{N}_2\cup Z$. Here the geometry on the boundary $\partial\mathcal{M}$ is to be understood as the geometry given by the limit of the metric g which lives in the open interior of $\mathcal{M}$; this limiting metric defines smooth tensor fields on the manifolds without boundary: $\mathrm{Int}\mathcal{N}_1$, $\mathrm{Int}\mathcal{N}_2$, and Z. We describe the situation in Fig. 3.

We will now outline the construction of a local coordinate system and tetrad on $\mathcal{M}$, which are particularly well suited for the discussion of the initial-value problem. We will call them Friedrich's tetrad and coordinate system.[9] They are constructed as follows.

On Z choose coordinates $x^3,x^4\equiv x^A(\equiv x,y)$.

On $\mathcal{N}_1$ choose a function $u\geq 0$ which vanishes on Z and which is the affine parameter along integral curves of $\mathbf{e}_1\equiv\mathbf{l}$, the null geodesic generators of $\mathcal{N}_1$. Let Z_{u_0} be the two-dimensional submanifold $\{u=u_0\}$ in $\mathcal{N}_1$. Choose on Z complex vector fields $\mathbf{e}_3$, $\mathbf{e}_4=\mathbf{e}_3^*$ with $g(\mathbf{e}_3,\mathbf{e}_4)=1$, $g(\mathbf{e}_3,\mathbf{e}_3)=0$ which are tangent to Z. Propagate $\mathbf{e}_3,\mathbf{e}_4$ onto $\mathcal{N}_1$ in the following manner: Construct $\mathbf{e}_3'$, $\mathbf{e}_4'$ as the parallel transports of $\mathbf{e}_3,\mathbf{e}_4$ along $\mathbf{e}_1$. At any point in $\mathcal{N}_1$, $\mathbf{e}_3$ ($\mathbf{e}_4$) lies in the intersection of the $\mathbf{e}_3'\wedge\mathbf{e}_1$ ($\mathbf{e}_4'\wedge\mathbf{e}_1$) plane with the two-surface Z_{u_0} through that point, and $g(\mathbf{e}_3,\mathbf{e}_4)=1$, $g(\mathbf{e}_3,\mathbf{e}_3)=g(\mathbf{e}_4,\mathbf{e}_4)=0$.

Choose a coordinate $u\geq 0$ on $\mathcal{M}$ coinciding with u on $\mathcal{N}_1$, such that $\forall\ u_0$, $u=u_0$ is a null hypersurface in $\mathcal{M}$. Put $\mathbf{e}_2\equiv-\nabla u$ on $\mathcal{M}$. Parallel transport $\mathbf{e}_1$, $\mathbf{e}_3$, $\mathbf{e}_4$ from $\mathcal{N}_1$ to all of $\mathcal{M}$ along integral curves of $\mathbf{e}_2$. Choose a function $v\geq 0$ on $\mathcal{M}$ and functions x^A on $\mathcal{M}$ ($A=3,4$) such that (i) x^A coincide with x^A on Z, (ii) x^A are constant along null generators of $\mathcal{N}_1$ and null generators of the $\{u=u_0\}$ hypersurfaces [and hence $\mathbf{e}_2(x^A)=0$], (iii) $v=0$ on $\mathcal{N}_1$, and (iv) $\{u,v,x^A\}$ form a coordinate system such that $\mathbf{e}_2\equiv\partial/\partial v$.

As a result of these constructions, we have

$$\mathbf{l}\equiv\mathbf{e}_1=\frac{\partial}{\partial u}+U\frac{\partial}{\partial v}+X^A\frac{\partial}{\partial x^A}\ ,$$
$$\mathbf{n}\equiv\mathbf{e}_2=\frac{\partial}{\partial v}\ ,\quad \mathbf{m}\equiv\mathbf{e}_3=\omega\frac{\partial}{\partial v}+\xi^A\frac{\partial}{\partial x^A}\ , \tag{4.2}$$
$$\mathbf{m}^*\equiv\mathbf{e}_4=\omega^*\frac{\partial}{\partial v}+\xi^{*A}\frac{\partial}{\partial x^A}$$

as a null tetrad on $\mathcal{M}$. We also have $\mathcal{N}_1=\{v=0\}$, $\mathcal{N}_2=\{u=0\}$, and $Z=\{u=v=0\}$. Moreover,

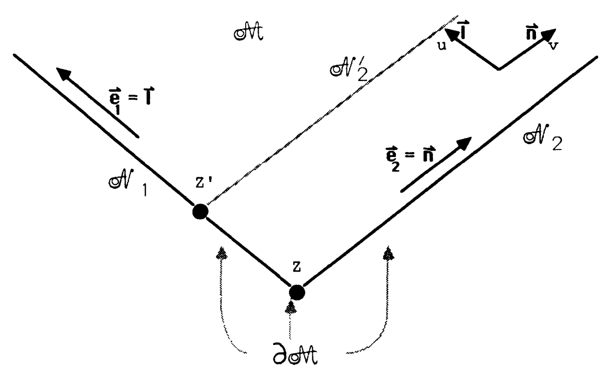

FIG. 3. Characteristic initial-value problem.

$$U = X^A = \omega = 0 \quad \text{on } \mathcal{N}_1 \quad (v=0) ,$$
$$\kappa = \epsilon = 0 \quad \text{on } \mathcal{N}_1 \quad (v=0) , \tag{4.3}$$

while on the whole spacetime $\mathcal{M}$

$$\nu = \gamma = \tau = \pi - (\alpha + \beta^*) = \mu - \mu^* = 0 \quad \text{on } \mathcal{M} . \tag{4.4}$$

We now formulate the fundamental theorems of the characteristic initial-value formalism.

An initial data set is a set of complex- and real-valued functions

$$U, X^A, \omega, \xi^A, \mu, \beta, \alpha, \lambda, \rho, \epsilon, \sigma, K, \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$$

on $S \equiv \partial\mathcal{M} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup Z$. A reduced initial data set is a set of complex- and real-valued functions μ, ρ, σ, λ, π, ξ^A on Z such that $g^{AB} = \xi^A \xi^{*B} + \xi^B \xi^{*A}$ is a positive definite metric on Z, and complex-valued functions Ψ_4 on $\mathcal{N}_2$ and Ψ_0 on $\mathcal{N}_1$. It is assumed that the initial and reduced initial data sets satisfy Eqs. (4.3) and (4.4).

Theorem 3. Let an initial data set on S satisfy all the constraint equations obtained by restricting the vacuum Einstein field equations onto the initial surface S. Then this initial data set uniquely determines, in some neighborhood of S, a vacuum spacetime $(\mathcal{M}, g)$ with boundary $\partial\mathcal{M} = S$ and with the data on S coinciding with the restrictions to S of the spin quantities on $\mathcal{M}$ in some suitable null tetrad and coordinate system on $\mathcal{M}$.

Theorem 4. A *reduced initial data set* on S uniquely determines, by using the constraint equations, an *initial data set* on S which satisfies the constraints.

For the proof of theorem 3, see Refs. 9 and 8. In our proof of theorem 5 in the next section, we will need the intermediate steps of the proof of theorem 4. Therefore, we sketch here an outline of this proof, following Friedrich.[9]

Proof of theorem 4 .

(1) To find the initial data on Z from the reduced initial data, first use the commutation relations

$$(\alpha - \beta^*)\xi^A + (\beta - \alpha^*)\xi^{*A} = (\xi^B \xi^{*A}{}_{,B} - \xi^{*B} \xi^A{}_{,B}) \quad \text{on } Z \tag{4.5}$$

and $\pi = \alpha + \beta^*$ [Eqs. (4.4)] to find α, β, π on Z. All other initial data on Z are known from the reduced initial data and Eqs. (4.3) and (4.4) (since $Z \subset \mathcal{N}_1$), except Ψ_1, Ψ_2, and Ψ_3 which are found from the following Ricci identities restricted to Z:

$$\delta^*\sigma - \delta\rho = \rho(\alpha^* + \beta) - \sigma(3\alpha - \beta^*) + \Psi_1 , \tag{4.6}$$

$$\delta^*\beta - \delta\alpha = \mu\rho - \lambda\sigma + \alpha\alpha^* + \beta\beta^* - 2\alpha\beta + \Psi_2 , \tag{4.7}$$

$$\delta^*\mu - \delta\lambda = \mu(\alpha + \beta^*) + \lambda(\alpha^* - 3\beta) + \Psi_3 . \tag{4.8}$$

(2) To find the initial data on $\mathcal{N}_1$, proceed as follows: First use Ψ_0 on $\mathcal{N}_1$ given by the reduced initial data, and the following commutation relations and Ricci identities, restricted to $\mathcal{N}_1$,

$$\xi^A{}_{,u} = -\rho^* \xi^A - \sigma \xi^{*A} , \tag{4.9}$$

$$-D\rho = -\rho_{,u} = \rho^2 + \sigma\sigma^* ,$$
$$-D\sigma = -\sigma_{,u} = \sigma(\rho + \rho^*) - \Psi_0 . \tag{4.10}$$

Use these to integrate, onto $\mathcal{N}_1$, by ordinary differential equations along null generators of $\mathcal{N}_1$, the reduced initial data $\{ \xi^A, \rho, \sigma \text{ on } Z \}$. Then use the Ricci identities on $\mathcal{N}_1$,

$$\Psi_1 = \delta^*\sigma - \delta\rho - \rho(\alpha^* + \beta) + \sigma(3\alpha - \beta^*) , \tag{4.11a}$$

$$-D\alpha = -\alpha_{,u} = \rho\alpha + \beta\sigma^* + \rho(\alpha + \beta^*) ,$$
$$-D\beta = -\beta_{,u} = (2\alpha + \beta^*)\sigma + \rho^*\beta - \delta^*\sigma + \delta\rho + \rho(\alpha^* + \beta) - \sigma(3\alpha - \beta_*) , \tag{4.11b}$$

to determine α, β, Ψ_1 on $\mathcal{N}_1$ from $\omega, \xi^A, \rho, \sigma$ on $\mathcal{N}_1$ which are known from the preceding step and from Eqs. (4.3). Similarly, use the Ricci identities on $\mathcal{N}_1$,

$$\Psi_2 = \delta^*\beta - \delta\alpha - (\rho\mu - \sigma\lambda) - \alpha\alpha^* - \beta\beta^* + 2\alpha\beta , \tag{4.12a}$$

$$-D\lambda = -\lambda_{,u} = -\delta^*(\alpha + \beta^*) + \rho\lambda + \sigma^*\mu + (\alpha + \beta^*)^2 + \alpha^2 - \beta^{*2} ,$$

$$-D\mu = -\mu_{,u} = -\delta(\alpha + \beta^*) + \rho^*\mu + \sigma\lambda + |\alpha + \beta^*|^2 - (\alpha + \beta^*)(\alpha^* - \beta) - \delta^*\beta + \delta\alpha + (\rho\mu - \sigma\lambda) + \alpha\alpha^* + \beta\beta^* - 2\alpha\beta , \tag{4.12b}$$

to determine μ, λ, and Ψ_2 on $\mathcal{N}_1$ by integrating, by ordinary differential equations (ODE's) along $\mathcal{N}_1$, the reduced initial data on Z. Finally, use the Ricci and Bianchi identities on $\mathcal{N}_1$,

$$\Psi_3 = \delta^*\mu - \delta\lambda - \mu(\alpha + \beta^*) - \lambda(\alpha^* - 3\beta) , \tag{4.13}$$

$$-D\Psi_4 = -\Psi_{4,u} = -\delta^*\Psi_3 + \rho\Psi_4 + (6\alpha + 4\beta^*)\Psi_3 - 3\lambda\Psi_2 , \tag{4.14}$$

to determine Ψ_3 and Ψ_4 on $\mathcal{N}_1$.

(3) To find the initial data on $\mathcal{N}_2$, proceed as follows: Use the commutation relations and the Ricci identities on $\mathcal{N}_2$,

$$U_{,v} = -(\epsilon + \epsilon^*) + \pi\omega + \pi^*\omega^* ,$$
$$X^A{}_{,v} = \pi\xi^A + \pi^*\xi^{*A} , \tag{4.15a}$$

$$\omega_{,v} = -\pi^* + \mu\omega + \lambda^*\omega^* ,$$
$$\xi^A{}_{,v} = \mu\xi^A + \lambda^*\xi^{*A} , \tag{4.15b}$$

$$\Delta\beta = \beta_{,v} = \beta\mu + \alpha\lambda^* ,$$
$$-\Delta\alpha = -\alpha_{,v} = -\beta\lambda - \mu\alpha + \delta^*\mu - \delta\lambda - \mu(\alpha + \beta^*) - \lambda(\alpha^* - 3\beta) , \tag{4.15c}$$

to determine X^A, ξ^A, β, α, ω, and π on $\mathcal{N}_2$. This should be done *after* finding λ and μ on $\mathcal{N}_2$ from Ψ_4 on $\mathcal{N}_2$, by using the following Ricci identities on $\mathcal{N}_2$:

$$\Delta\mu=\mu_{,v}=\mu^2+\lambda\lambda^* ,$$
$$\Delta\lambda=\lambda_{,v}=2\mu\lambda-\Psi_4 . \quad (4.16)$$

(Note that Ψ_4 on $\mathcal{N}_2$ is given by reduced initial data.) Next find Ψ_3, Ψ_2, ρ, and σ on $\mathcal{N}_2$ as follows: First find ρ and σ on $\mathcal{N}_2$ by integrating the ODE's on $\mathcal{N}_2$ which follow from the Ricci identities

$$\Delta\sigma=\sigma_{,v}=\mu\sigma+\lambda^*\rho ,$$
$$-\Delta\rho=-\rho_{,v}=-(\rho\mu+\sigma\lambda)+\delta^*\beta-\delta\alpha-(\mu\rho-\lambda\sigma) -\alpha\alpha^*-\beta\beta^*+2\alpha\beta . \quad (4.17)$$

Then use the Ricci identities

$$\Psi_3=\delta^*\mu-\delta\lambda-\mu(\alpha+\beta^*)-\lambda(\alpha^*-3\beta) , \quad (4.18a)$$
$$\Psi_2=\delta^*\beta-\delta\alpha-(\mu\rho-\lambda\sigma)-\alpha\alpha^*-\beta\beta^*+2\alpha\beta , \quad (4.18b)$$

on $\mathcal{N}_2$ to compute Ψ_3 and Ψ_2 on $\mathcal{N}_2$. Finally, use Eqs. (4.15a) and the Ricci and Bianchi identities on $\mathcal{N}_2$,

$$\Delta\epsilon=\epsilon_{,v}=\pi^*\alpha+\pi\beta-\Psi_2 , \quad (4.19)$$
$$\Delta\kappa=\kappa_{,v}=\pi^*\rho+\pi\sigma-\Psi_1 , \quad (4.20)$$
$$\Psi_1=\delta^*\sigma-\delta\rho-\rho(\alpha^*+\beta)+\sigma(3\alpha-\beta^*) , \quad (4.21)$$
$$-\Delta\Psi_0=-\Psi_{0,v}=-\delta\Psi_1+3\sigma\Psi_2-\mu\Psi_0-2\beta\Psi_1 , \quad (4.22)$$

to determine ϵ, κ, Ψ_1, Ψ_0, and U on $\mathcal{N}_2$.

The uniqueness statement in the theorem now follows straightforwardly, since we have only integrated ordinary differential equations to determine the initial data on S from the reduced initial data on S. □

E. The only sandwich GW spacetimes are PP waves

Theorem 5. Let $(\mathcal{M},g)$ be a gravitational-wave spacetime with wave fronts $\mathcal{N}_2$, $\mathcal{N}_2'$; hence $(\mathcal{M},g)$ is flat on $I^-(\mathcal{N}_2)$. If $(\mathcal{M},g)$ is also flat on $I^+(\mathcal{N}_2')$, and if the fundamental theorems of the characteristic initial-value formalism hold globally (rather than just locally) on $\mathcal{M}$, then $(\mathcal{M},g)$ is a PP-wave spacetime.

Remark. A gravitational wave of this type, which leaves spacetime precisely flat both before and after its passage, is called a sandwich wave. This theorem then says that the only sandwich gravitational waves are PP waves.

Proof. We assume, as stated in the theorem, that $I^+(\mathcal{N}_2')$ (Fig. 3) is flat and that theorem 3 holds globally on $\mathcal{M}$; and we seek to show that $(\mathcal{M},g)$ is a PP-wave spacetime.

Choose a null surface $\mathcal{N}_1$ which intersects $\mathcal{N}_2$ transversely in a spacelike two-submanifold Z (and $\mathcal{N}_2'$ in Z') (Fig. 3). Then, that portion of the spacetime which lies to the future of the initial null boundary $S=\mathcal{N}_1\cup\mathcal{N}_2\cup Z$ is uniquely determined by the reduced initial data it induces on S. From here on, we will only be interested in this region $I^+(S)$ of the spacetime $(\mathcal{M},g)$ together with the boundary of this region $S=\partial I^+(S)$, and we will denote the spacetime region with boundary, $I^+(S)\cup S$, by the same symbol $\mathcal{M}$, where $\partial\mathcal{M}=S$. The following proof will show that if $(\mathcal{M},g)$ is a precisely sandwich GW spacetime as defined above, then the reduced initial data induced on the null boundary S is PP-wave reduced initial data. This is sufficient to prove theorem 5, since the location of the transverse null surface $\mathcal{N}_1$ is arbitrary.

Before proceeding with the proof, we observe that the flatness of $I^+(\mathcal{N}_2')$ and $I^-(\mathcal{N}_2)$ requires $\Psi_0=\Psi_1=\Psi_2=\Psi_3=\Psi_4=0$ on $\mathcal{N}_2$, on Z, and on $\mathcal{N}_1\cup I^+(\mathcal{N}_2')$. That is, all curvature quantities (in any tetrad) vanish on the null boundary S *except* on that portion of $\mathcal{N}_1$ lying between $\mathcal{N}_2$ and $\mathcal{N}_2'$ (Fig. 3). We also note that, in general there is some coordinate freedom in choosing Friedrich's coordinate system and tetrad on the spacetime $\mathcal{M}$ with (null) boundary $S=\partial\mathcal{M}=\mathcal{N}_1\cup\mathcal{N}_2\cup Z$. In the following, we will use this gauge freedom in the choice of Friedrich's chart, coupled with the freedom to choose the transverse null surface $\mathcal{N}_1$ (the choice of which is completely arbitrary), to construct *a specific* null boundary S and *a specific* Friedrich-type coordinate chart on the spacetime $\mathcal{M}$ with boundary S. These choices for S and for the Friedrich-type coordinate chart on $\mathcal{M}=I^+(S)$ will be particularly well suited for studying the exactly sandwich GW spacetime $(\mathcal{M},g)$.

The gauge freedom in the choice of a Friedrich-type coordinate system can be decomposed into two different types of coordinate transformations. The transformations of the first type are generated by the successive application of two transformations: (i) transformations of the form $u'=\alpha(x^A)u$ on $\mathcal{N}_1$ which give, upon extending u' uniquely as a null coordinate, $u'=u'(x_A,u,v)$ on $\mathcal{M}$, where α is an arbitrary function on Z which is extended to S by keeping it constant along the null geodesic generators of $\mathcal{N}_1$ and $\mathcal{N}_2$; (ii) transformations of the form $v'=v'(u,v,x^A)$ which are so adjusted that when $x^{A'}=x^{A'}(x^A,u,v)$ on $\mathcal{M}$ are obtained from the x^A on $\mathcal{N}_1$ by keeping them constant along the new integral curves of $\mathbf{e}_1'=-\nabla u'$, we have $\partial/\partial v'=-\nabla u'$ and $v'=0$ on $\mathcal{N}_1$ in the new primed coordinate system. The second type of coordinate transformations generating the gauge freedom in the choice of Friedrich's chart are given by

$$v'=v+v'(u,x^A) , \quad u'=u , \quad x^{A'}=x^{A'}(u,x^A) ,$$

where all primed quantities are arbitrary functions of their arguments. Since $x^{A'}$ (and x^A) are to be constant on null generators of $\mathcal{N}_1$ and $\mathcal{N}_2$, these transformations reduce to

$$v'=v+v'(u,x^A) , \quad x^{A'}=x^{A'}(x^A) , \quad u'=u .$$

And if $\mathcal{N}_1$ and $\mathcal{N}_2$ are fixed, since we have $v'=0$ on $\mathcal{N}_1$, these transformations further reduce to

$$v'=v , \quad x^{A'}=x^{A'}(x^A) , \quad u'=u .$$

Both types of coordinate transformations induce tetrad rotations, since the old $\mathbf{n},\boldsymbol{l},\mathbf{m},\mathbf{m}^*$ in the new coordinate system will not be of the form (4.2), and tetrad rotations are necessary to bring them back into the form (4.2) in the $(u',v',x^{A'})$ chart.

In the next two paragraphs we will use, as we have indicated before, the above coordinate and tetrad freedom coupled with the freedom in the choice of $\mathcal{N}_1$, to bring

Friedrich's coordinate system and associated tetrad into a form that meshes nicely with the structure of our sandwich GW spacetime.

We begin with the freedom to choose $\mathcal{N}_1$. We shall choose $\mathcal{N}_1$ so that it is given by $\hat{v}=0$, where $\hat{v}$ is a Minkowskian null coordinate in $I^+(\mathcal{N}_2')$. That is, $d\hat{v}$ is a parallel null 1-form in the flat region $I^+(\mathcal{N}_2')$, or in other words on $I^+(\mathcal{N}_2')$ there is a coordinate system $(\hat{v},u',x^{A'})$ in which the metric is

$$g=dx^{3'}\otimes dx^{3'}+dx^{4'}\otimes dx^{4'} -\tfrac{1}{2}(du'\otimes d\hat{v}+d\hat{v}\otimes du') \text{ on } I^+(\mathcal{N}_2') .$$

Note that this choice of $\mathcal{N}_1$ on $I^+(\mathcal{N}_2')$ completely fixes it everywhere in spacetime including the region between $\mathcal{N}_2$ and $\mathcal{N}_2'$, because there exist precisely two null surfaces passing through *any* spacelike two-surface. In other words, $\mathcal{N}_1$ is extended in the past directions *beyond* the spacelike two-surface Z' (Fig. 3) as that null surface, which together with $\mathcal{N}_2'$ constitutes the unique pair of null surfaces through Z'.

Now the scaling freedom in u, i.e., the freedom of coordinate transformations of the first kind, is fixed by the arrangement that the wave front $\mathcal{N}_2'$ coincides with the null surface $\{u=a\}$. (Then the coordinate v is constructed as usual from $-\nabla u$, using u and some choice of coordinates x^A on Z and thereby on $\mathcal{M}$.) We are then left with the following coordinate freedom of the second type:

$$v'=v \;, \quad x^{A'}=x^{A'}(x^A) \;, \quad u'=u \;.$$

We fix this remaining freedom totally by noting that, since Z' is a two-dimensional spacelike hypersurface in flat spacetime contained in $\{\hat{v}=0\}$, the induced metric on it is flat, and hence we can arrange $x^{A'}=x^{A'}(x^A)$ on Z in such a manner that at $\{u=a\}$ on $\mathcal{N}_1$:

$$\xi^{A'}(u=a)=\xi^B(u=a)\frac{\partial x^{A'}}{\partial x^B}=\frac{1+i}{2} \quad \text{for } A=3$$
$$=\frac{1-i}{2} \quad \text{for } A=4 \;. \tag{4.23}$$

(Note that we are leaving the tetrad vectors $\mathbf{m}$, $\mathbf{m}^*$ fixed during the above arrangement of coordinates.) Then $g^{AB}(u=a)=\delta^{AB}$. But using Eqs. (4.10), since $Z'\subset\mathcal{N}_1$, this gives

$$\beta-\alpha^*=0 \text{ on } Z' \;. \tag{4.24}$$

Now, note that we have two coordinate systems covering $I^+(\mathcal{N}_2')$: $(u',\hat{v},x^{A'})$ and (u,v,x^A), where the first one is Minkowskian. As we will argue later, $\mathcal{N}_2'$ (as well as $\mathcal{N}_2$) is a flat null surface in Minkowski spacetime, and therefore by construction one can find a Minkowskian coordinate system $(u'',v'',x^{A''})$ on $I^+(\mathcal{N}_2')$ [rotating $(u',\hat{v},x^{A'})$ by a Lorentz transformation, if necessary] such that $\partial/\partial v=\partial/\partial v''$ on Z'. But then on Z', $\partial/\partial v=\partial/\partial v''=\mathbf{n}$; and since $\mathbf{m}$ is tangent to Z', $\nabla_{\mathbf{m}}\mathbf{n}$ on Z' does not depend on the extension of $\mathbf{n}$ from Z' to $\mathcal{M}$. In particular, $\partial/\partial v''$ is an extension of $\mathbf{n}\,|_{Z'}$ to $\mathcal{M}$; and hence

$$(\nabla_{\mathbf{m}}\mathbf{n})\,|_{Z'}=\left.\left[\nabla_{\mathbf{m}}\frac{\partial}{\partial v''}\right]\right|_{Z'}=0 \;,$$

since $\partial/\partial v''$ is a parallel vector field on $I^+(\mathcal{N}_2')$ and the metric is C^2. But $\nabla_{\mathbf{m}}\mathbf{n}=(\alpha^*+\beta)\mathbf{n}-\lambda^*\mathbf{m}^*-\mu\mathbf{m}$. Thus, on Z' we have $\alpha^*+\beta=0$, which by Eq. (4.24) gives $\alpha=\beta=0$ on Z'. Therefore, in this way we fix the above remaining coordinate freedom so that on $\mathcal{N}_1$, $\alpha(u=a)=\beta(u=a)=0$, $\xi^3(u=a)=(1+i)/2$, $\xi^4(u=a)=(1-i)/2$, and $g^{AB}(u=a)=\delta^{AB}$. Note that, with this procedure we also fix the remaining freedom for tetrad transformations of type III: $\mathbf{l}\to\mathbf{l}$, $\mathbf{n}\to\mathbf{n}$, $\mathbf{m}\to e^{i\theta}\mathbf{m}$, $\mathbf{m}^*\to e^{-i\theta}\mathbf{m}^*$, where θ is a function which depends only on x^A.

This completes our specialization of Friedrich's coordinate system and tetrad. In the next paragraph we shall derive the special values of the spin coefficients associated with this tetrad.

Now, since $\mathcal{N}_1$ is a flat null surface in Minkowski space for $u\geq a$, its null geodesic generators have no shear or convergence; and hence, since on $\mathcal{N}_1$ the null generators are tangent to $\mathbf{l}$, we have $\rho=\sigma=0$ for $u\geq a$ on $\mathcal{N}_1$. But by the Ricci identities (A20) and (A21) on $\mathcal{M}$,

$$\Delta\sigma=\sigma_{,v}=\mu\sigma+\lambda^*\rho \quad \text{for } u\geq a \;,$$
$$-\Delta\rho=-\rho_{,v}=-(\rho\mu+\sigma\lambda) \quad \text{for } u\geq a \;.$$

These imply, by the uniqueness theorem for ODE,

$$\rho=\sigma=0 \quad \text{for } u\geq a$$

on all of $\mathcal{M}$. Applying the Ricci identities (A8) and (A9) on $\mathcal{N}_1$ for $u\geq a$, and using the same arguments as in the last few equations, we obtain

$$\pi=\alpha=\beta=0 \quad \text{for } u\geq a$$

on all of $\mathcal{M}$. (Here we have used the fact that, by the choice of the coordinates x^A, we have $\alpha=\beta=0$ at $u=a$.) Similarly, we obtain

$$\epsilon=\kappa=0 \quad \text{for } u\geq a \;,$$

on all of $\mathcal{M}$. Since $\mathcal{N}_2$ and $\mathcal{N}_2'$ are nonsingular null surfaces whose null geodesic generators have no end points in $\mathcal{M}$ (and $\mathcal{M}$ is complete), we have

$$\lambda=\mu=0 \quad \text{on } \mathcal{N}_2 \text{ and } \mathcal{N}_2' \;.$$

Then using Eqs. (A11) and (A12) on $\mathcal{M}$ for $u\geq a$ we obtain

$$\lambda=\mu=0 \quad \text{for } u\geq a$$

on all of $\mathcal{M}$.

Now, having completed the construction of our specific Friedrich-type coordinate system and its associated tetrad and the specific null boundary $S=\mathcal{N}_1\cup\mathcal{N}_2\cup Z$ on which our sandwich GW spacetime induces a characteristic initial data set, we return to the proof that the sandwich GW spacetime $(\mathcal{M},g)$ is actually a PP wave. Clearly, by theorems 3 and 4, there is a one to one correspondence between vacuum *sandwich* GW spacetimes, and the reduced initial data sets they induce on the null boundary S, expressed in the coordinate system and tetrad constructed above. We will call such reduced initial data, which correspond to sandwich GW spacetimes, "good reduced initial data." Note that this condition of "goodness" on a reduced initial data set is equivalent to the demand that the spacetime which develops uniquely from it according to theorem 4 is flat on $I^+(\mathcal{N}_2')$ and $I^-(\mathcal{N}_2)$.

It is not hard to prove, using Eqs. (4.5)–(4.22), that any good reduced initial data set on S is completely determined by giving Ψ_0 on $\mathcal{N}_1$, between $u=0$ and $u=a$. Therefore, the set of good initial data is in one to one correspondence with a certain subset of the set of all C^2 functions Ψ_0 on $\mathcal{N}_1$, which vanish for $u=0$ and $u\geq a$. In the following paragraphs, we will prove that any good reduced initial data set is necessarily a PP-wave reduced initial data set, which will prove the theorem.

We begin by noting that a PP-wave metric in the Kerr-Schild coordinates is associated with the null tetrad

$$l'=2\left[\frac{\partial}{\partial U}+h(U,X,Y)\frac{\partial}{\partial V}\right],$$

$$\mathbf{n}'=\frac{\partial}{\partial V},\quad \mathbf{m}'=\frac{1+i}{2}\frac{\partial}{\partial X}+\frac{1-i}{2}\frac{\partial}{\partial Y},$$

in which the only nonzero spin quantities are κ' and $\Psi_0'=-\delta'\kappa'$, and in which $\delta'^*\kappa'=0$. When we transform this coordinate system and tetrad into Friedrich's form [Eq. (4.2)], the only nonzero spin quantities are ρ, σ, and $\Psi_0=\Psi_0'$, where $\delta^*\Psi_0=0$. Therefore the PP-wave reduced initial data will consist of (i) ξ^A,ρ,σ on Z with $\mu=\lambda=\pi=0$ on Z, (ii) $\Psi_4=0$ on $\mathcal{N}_2$, and (iii) Ψ_0 on $\mathcal{N}_1$ with $\delta^*\Psi_0=0$ and $\Psi_0=0$ for $u=0$, $u\geq a$.

A necessary condition for the reduced initial data induced from Ψ_0 to be good is that, when the Eqs. (4.10), (4.9), and (4.11b) are solved with initial conditions (4.23) and ρ, σ, α, $\beta=0$ at $u=a$, and when Eqs. (4.12b) are then solved for μ, λ with initial conditions μ, $\lambda=0$ at $u=a$, one then obtains, at $u=0$ (on Z):

$$\mu(0)=\lambda(0)=0,$$

$$\Psi_2(0)=[\delta^*\beta-\delta\alpha-(\mu\rho-\lambda\sigma)-\alpha\alpha^*-\beta\beta^*+2\alpha\beta]\,|_{u=0}=0, \tag{4.25}$$

$$\Psi_1(0)=[\delta^*\sigma-\delta\rho-\rho(\alpha^*+\beta)+\sigma(3\alpha-\beta^*)]\,|_{u=0}=0.$$

We claim that these conditions can only be satisfied if $\delta^*\Psi_0\equiv 0$ on $\mathcal{N}_1$.

Since the proof of this claim is rather long, we will only outline in this paragraph the main steps. First define $A\equiv\delta\rho-\delta^*\sigma$. A satisfies, on $\mathcal{N}_1$,

$$DA=A_{,u}=-(2\rho+\rho^*)A+\sigma A^*-\delta^*\Psi_0, \qquad A=0 \text{ on } u=a \text{ in } \mathcal{N}_1. \tag{4.26}$$

Now using theorem 2 and the Bianchi identities, this gives $A=0$ in some neighborhood U_a of $\{u=a\}$ in $\mathcal{N}_1$. By theorem 2 and the Goldberg-Sachs theorem,[24] $\lambda=0$ in some neighborhood of $\{u=0\}$ and $\{u=a\}$ in $\mathcal{N}_1$, and this gives $\alpha=\beta=0$ at $u=0$. Again by the Bianchi identities this implies $\delta^*\Psi_0=0$ on some neighborhood of $\{u=0\}$ in $\mathcal{N}_1$. It then follows from Eqs. (4.26) using standard arguments for ordinary differential equations [specifically, using an energy-type inequality, which involves a positive-definite expression depending on $|DA|$ and $|\Psi_0|^2$ and which is obtained from Eqs. (4.26), (4.10), (4.9), and (4.25)] that if $\delta^*\Psi_0\neq 0$ at any point on $\mathcal{N}_1$, $|DA|$ and thence $A=\delta\rho-\delta^*\sigma$ are nonzero at $u=0$. But this contradicts Eq. (4.25). Therefore $\delta^*\Psi_0\equiv 0$ on $\mathcal{N}_1$ for any "good" Ψ_0 on $\mathcal{N}_1$ and the claim is proved. [To understand this claim more intuitively, first note that we still have some freedom left in the choice of the null surface $\mathcal{N}_1$, even though we have restricted it to be a flat Minkowskian surface on $I^+(\mathcal{N}_2')$. This freedom consists of (i) rotating the surface $\mathcal{N}_1$ by Lorentz transformations applied in the flat region $I^+(\mathcal{N}_2')$, and (ii) translating $\mathcal{N}_1$ linearly in $I^+(\mathcal{N}_2')$. Thus, even if the fact $\Psi_0\neq 0$ on $\mathcal{N}_1$ *were compatible* with the Eqs. (4.25) for a *particular* choice of the surface $\mathcal{N}_1$, we could readjust the orientation of $\mathcal{N}_1$ by using the above freeedom in such a way that with the new choice of $\mathcal{N}_1$, Eqs. (4.25) would be violated.]

It is easily seen that the initial data set associated with a good reduced initial data set induced from a Ψ_0 with $\delta^*\Psi_0\equiv 0$ on $\mathcal{N}_1$ has the following form: (i) on $\mathcal{N}_1$

$$\Psi_0=0 \text{ for } u\geq a \text{ and } u=0,$$

$$\rho,\ \sigma=0 \text{ for } u\geq a,$$

$$\delta^*\Psi_0\equiv 0,$$

$$\epsilon=\kappa=\mu=\lambda=\alpha=\beta=\pi=0,$$

$$\Psi_1=\Psi_2=\Psi_3=\Psi_4=0,$$

$$X^A=U=\omega=0,$$

while ξ^A are found by Eq. (4.9). (ii) On Z

$$\epsilon=\kappa=\mu=\lambda=\alpha=\beta=\pi=0,$$

$$\Psi_0=\Psi_1=\Psi_2=\Psi_3=\Psi_4=0,$$

$$X^A=\omega=U=0,$$

while ξ^A,ρ,σ are nonzero. (iii) On $\mathcal{N}_2$

$$\mu=\lambda=\pi=\alpha=\beta=\epsilon=\kappa=0,$$

$$\Psi_0=\Psi_1=\Psi_2=\Psi_3=\Psi_4=0,$$

$$U=X^A=\omega=0,$$

while ξ^A,ρ,σ are nonzero but independent of v. (iv) On the whole spacetime, $\tau=\nu=\gamma=0$.

Now we are ready to show that good reduced initial data corresponding to Ψ_0 on $\mathcal{N}_1$ with $\delta^*\Psi_0\equiv 0$ are PP-wave reduced initial data. To prove this, it is enough to prove that the spacetime which uniquely develops from the initial data above (which are induced from reduced initial data with $\delta^*\Psi_0\equiv 0$) is a PP wave.

To find the spacetime that develops from these initial data, just put any of the quantities

$$\xi^A, X^A, \omega, U, \mu, \lambda, \kappa, \epsilon, \alpha, \beta, \pi, \rho, \sigma,$$
$$\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4(v,u,x^3,x^4)$$

at (v,u,x^3,x^4) equal to their values at $(v=0,u,x^3,x^4)$; in other words, just transport identically every quantity on $\mathcal{N}_1$ along integral curves of $\mathbf{n}=\partial/\partial v$, independently of v. Clearly the resulting spacetime will be vacuum (the Ricci and Bianchi identities are trivially checked) and will induce the above initial data on $S=\mathcal{N}_1 \cup \mathcal{N}_2 \cup Z$. Morover, by uniqueness (theorem 3), it will be the unique vacuum spacetime developing from the above initial data. Clearly the vector $\mathbf{n}=\partial/\partial v$ is a Killing vector for this spacetime (and is also parallel), and the resulting spacetime is flat on $I^+(\mathcal{N}_2)$ and $I^-(\mathcal{N}_2)$. Hence the spacetime is a PP wave.

This completes the proof of theorem 5. □

V. CONCLUSIONS

We have reviewed in this paper the general structure of exact colliding plane-wave solutions of the vacuum Einstein equations; and we have argued on the basis of previous work, both by the author and largely by others, that those solutions whose causality structures are stable against plane-symmetric perturbations will involve all-embracing spacelike curvature singularities bounding the spacetime in the future of the collision plane. We have given a detailed qualitative review of the well-known focusing effect of plane waves in both single and colliding plane-wave spacetimes, and by discussing and giving an alternative proof of a singularity theorem originally discovered by Tipler,[2] we have described how this focusing property makes inevitable the occurrence of singularities in *generic* plane-wave collisions. We have carefully stressed the subtle aspects of Tipler's singularity theorem and emphasized the reason for its inapplicability to single plane-wave spacetimes and to colliding plane-wave solutions which possess Killing-Cauchy horizons.[3,4]

We have defined and analyzed general gravitational-wave spacetimes and we have seen that the PP-wave solutions—a particular family of GW spacetimes—satisfy strong uniqueness theorems, much like the Kerr-Newman family which satisfies the well-known black-hole uniqueness results. We have pointed out the insight that these results give into the structures of almost-plane waves, which constitute a special case of gravitational-wave spacetimes. In particular we have seen that almost-plane waves must always exhibit diffraction, since by the classification theorem of Dautcourt[10] the only diffraction-free GW spacetimes are PP waves; and we have seen that almost-plane waves must leave behind "tails" in any region of space through which they have propagated, since the only GW spacetimes with a precisely "sandwiched" curvature distribution are the PP waves.

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APPENDIX: NEWMAN-PENROSE EQUATIONS IN RATIONALIZED FORM

The Newman-Penrose equations as originally formulated[6] were based on the metric signature $(+,-,-,-)$. When one adopts, instead, the signature $(-,+,+,+)$, they assume the following "rationalized" form.

Commutation relations:

$$\Delta D - D\Delta = -(\gamma+\gamma^*)D - (\epsilon+\epsilon^*)\Delta + (\tau^*+\pi)\delta + (\tau+\pi^*)\delta^* , \quad (A1)$$

$$\delta D - D\delta = -(\alpha^*+\beta-\pi^*)D - \kappa\Delta + (\rho^*+\epsilon-\epsilon^*)\delta + \sigma\delta^* , \quad (A2)$$

$$\delta\Delta - \Delta\delta = \nu^* D - (\tau-\alpha^*-\beta)\Delta - (\mu-\gamma+\gamma^*)\delta - \lambda^*\delta^* , \quad (A3)$$

$$\delta^*\delta - \delta\delta^* = (\mu-\mu^*)D + (\rho-\rho^*)\Delta + (\beta^*-\alpha)\delta + (\alpha^*-\beta)\delta^* . \quad (A4)$$

Ricci identities:

$$\delta^* K - D\rho = (\rho^2+\sigma\sigma^*) + \rho(\epsilon+\epsilon^*) - \kappa^*\tau - \kappa(3\alpha+\beta^*-\pi) - \Phi_{00} , \quad (A5)$$

$$\delta\kappa - D\sigma = \sigma(\rho+\rho^*+3\epsilon-\epsilon^*) - \kappa(\tau-\pi^*+\alpha^*+3\beta) - \Psi_0 , \quad (A6)$$

$$\Delta\kappa - D\tau = (\tau+\pi^*)\rho + (\tau^*\pi)\sigma + (\epsilon-\epsilon^*)\tau - (3\gamma+\gamma^*)\kappa - \Psi_1 - \Phi_{01} , \quad (A7)$$

$$\delta^*\epsilon - D\alpha = (\rho+\epsilon^*-2\epsilon)\alpha + \beta\sigma^* - \beta^*\epsilon - \kappa\lambda - \kappa^*\gamma + (\epsilon+\rho)\pi - \Phi_{10} , \quad (A8)$$

$$\delta\epsilon - D\beta = (\alpha+\pi)\sigma + (\rho^*-\epsilon^*)\beta - (\mu+\gamma)\kappa - (\alpha^*-\pi^*)\epsilon - \Psi_1 , \quad (A9)$$

$$\Delta\epsilon - D\gamma = (\tau+\pi^*)\alpha + (\tau^*+\pi)\beta - (\epsilon+\epsilon^*)\gamma - (\gamma+\gamma^*)\epsilon + \tau\pi - \nu\kappa - \Psi_2 + \Lambda - \Phi_{11} , \quad (A10)$$

$$\delta^*\pi - D\lambda = (\rho\lambda+\sigma^*\mu) + \pi^2 + (\alpha-\beta^*)\pi - \nu\kappa^* - (3\epsilon-\epsilon^*)\lambda - \Phi_{20} , \quad (A11)$$

$$\delta\pi - D\mu = (\rho^*\mu+\sigma\lambda) + \pi\pi^* - (\epsilon+\epsilon^*)\mu - \pi(\alpha^*-\beta) - \nu\kappa - \Psi_2 - 2\Lambda , \quad (A12)$$

$$\Delta\pi - D\nu = (\pi+\tau^*)\mu + (\pi^*+\tau)\lambda + (\gamma-\gamma^*)\pi - (3\epsilon+\epsilon^*)\nu - \Psi_3 - \Phi_{21} , \quad (A13)$$

$$\delta^*\nu - \Delta\lambda = -(\mu+\mu^*)\lambda - (3\gamma-\gamma^*)\lambda + (3\alpha+\beta^*+\pi-\tau^*)\nu + \Psi_4 , \quad (A14)$$

$$\delta^*\sigma - \delta\rho = \rho(\alpha^*+\beta) - \sigma(3\alpha-\beta^*) + (\rho-\rho^*)\tau + (\mu-\mu^*)\kappa + \Psi_1 - \Phi_{01} , \quad (A15)$$

$$\delta^*\beta-\delta\alpha=(\mu\rho-\lambda\sigma)+\alpha\alpha^*+\beta\beta^*-2\alpha\beta+\gamma(\rho-\rho^*)+\epsilon(\mu-\mu^*)+\Psi_2-\Lambda-\Phi_{11}\ , \tag{A16}$$

$$\delta^*\mu-\delta\lambda=(\rho-\rho^*)\nu+(\mu-\mu^*)\pi+\mu(\alpha+\beta^*)+\lambda(\alpha^*-3\beta)+\Psi_3-\Phi_{21}\ , \tag{A17}$$

$$\Delta\mu-\delta\nu=(\mu^2+\lambda\lambda^*)+(\gamma+\gamma^*)\mu-\nu^*\pi+(\tau-3\beta-\alpha^*)\nu-\Phi_{22}\ , \tag{A18}$$

$$\Delta\beta-\delta\gamma=(\tau-\alpha^*-\beta)\gamma+\mu\tau-\sigma\tau-\sigma\nu-\epsilon\nu^*-\beta(\gamma-\gamma^*-\mu)+\alpha\lambda^*-\Phi_{12}\ , \tag{A19}$$

$$\Delta\sigma-\delta\tau=(\mu\sigma+\lambda^*\rho)+(\tau+\beta-\alpha^*)\tau-(3\gamma-\gamma^*)\sigma-\kappa\nu^*-\Phi_{02}\ , \tag{A20}$$

$$\delta^*\tau-\Delta\rho=-(\rho\mu^*+\sigma\lambda)+(\beta^*-\alpha-\tau^*)\tau+(\gamma+\gamma^*)\rho+\nu\kappa+\Psi_2+2\Lambda\ , \tag{A21}$$

$$\delta^*\gamma-\Delta\alpha=(\rho+\epsilon)\nu-(\tau+\beta)\lambda+(\gamma^*-\mu^*)\alpha+(\beta^*-\tau^*)\gamma+\Psi_3\ . \tag{A22}$$

Bianchi identities (in vacuum):

$$\delta^*\Psi_0-D\Psi_1=-3\kappa\Psi_2+2(\epsilon+2\rho)\Psi_1+(\pi-4\alpha)\Psi_0\ , \tag{A23}$$

$$\delta^*\Psi_1-D\Psi_2=-2\kappa\Psi_3+3\rho\Psi_2+2(\pi-\alpha)\Psi_1-\lambda\Psi_0\ , \tag{A24}$$

$$\delta^*\Psi_2-D\Psi_3=-\kappa\Psi_4-2(\epsilon-\rho)\Psi_3+3\pi\Psi_2-2\lambda\Psi_1\ , \tag{A25}$$

$$\delta^*\Psi_3-D\Psi_4=-(4\epsilon-\rho)\Psi_4+(4\pi+2\alpha)\Psi_3-3\lambda\Psi_2\ , \tag{A26}$$

$$\delta\Psi_1-\Delta\Psi_0=(4\gamma-\mu)\Psi_0-(4\tau+2\beta)\Psi_1+3\sigma\Psi_2\ , \tag{A27}$$

$$\delta\Psi_2-\Delta\Psi_1=\nu\Psi_0+2(\gamma-\mu)\Psi_1-3\tau\Psi_2+2\sigma\Psi_3\ , \tag{A28}$$

$$\delta\Psi_3-\Delta\Psi_2=2\nu\Psi_1-3\mu\Psi_2-2(\tau-\beta)\Psi_3+\sigma\Psi_4\ , \tag{A29}$$

$$\delta\Psi_4-\Delta\Psi_3=3\nu\Psi_2-(2\gamma+4\mu)\Psi_3-(\tau-4\beta)\Psi_4\ . \tag{A30}$$

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