# New family of exact solutions for colliding plane gravitational waves

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We construct an infinite-parameter family of exact solutions to the vacuum Einstein field equations describing colliding gravitational plane waves with parallel polarizations. The interaction regions of the solutions in this family are locally isometric to the interiors of those static axisymmetric (Weyl) black-hole solutions which admit both a nonsingular horizon, and an analytic extension of the exterior metric to the interior of the horizon. As a member of this family of solutions we also obtain, for the first time, a colliding plane-wave solution where both of the two incoming plane waves are purely anastigmatic, i.e., where both incoming waves have equal focal lengths.

## I. INTRODUCTION

As a result of the revolutionary new techniques introduced in the last decade, there now exists an extensive collection of powerful tools to generate exact solutions for the Einstein field equations in the stationary axisymmetric case.[1] More recently, there have been succesful attempts to employ these same techniques in the study of solutions with two commuting spacelike Killing vectors, i.e., in the study of plane-symmetric solutions to Einstein equations. These recent investigations have produced a rich arsenal of new exact solutions for plane-symmetric spacetimes; among these are many new solutions describing both colliding purely gravitational plane waves and colliding plane waves coupled with matter or radiation.[2]

Historically, the work on exact solutions for colliding plane waves has followed two distinct paths of development: On the one hand, the problem can be formulated as a characteristic initial-value problem for a system of nonlinear hyperbolic partial differential equations in two variables. This system involves the metric coefficients (and in the nonvacuum case the components of the matter fields) in a coordinate system where the two plane-symmetry-generating Killing vectors are equal to two members of the coordinate basis frame, so that the unknown variables are functions of the retarded and advanced time coordinates $u$ and $v$ only. The initial data for the metric coefficients (and the matter fields) are posed on the initial null boundary consisting of intersecting null surfaces $\mathcal{N}_1$ and $\mathcal{N}_2$, the past wavefronts of the two incoming waves (Fig. 1). The integration of this initial-value system to obtain the metric coefficients in the interaction region bounded by $\mathcal{N}_1 \cup \mathcal{N}_2$ is very difficult in general, in fact no general expression has been found for the solution in the generic case of colliding plane waves with nonparallel polarizations. However, in a paper of great ingenuity, Szekeres[3] was able to reduce the integration of arbitrary initial data for incoming gravitational plane waves with *parallel polarizations*, to the evaluation of a one-dimensional integral followed by two quadratures (see also Ref. 4 for another viewpoint). Despite this feat, however, the functions to be subjected to these elementary operations of integration and quadrature turned out to be very complex for general initial data. Consequently, exact solutions which were expressible in closed analytic form could only be obtained using this approach for a few very special incoming wave forms.

A very different and innovative alternative to the above approach for obtaining exact solutions of colliding plane waves was pioneered by the work of Khan and Penrose.[5] The idea is simply to work backward in time: (i) look for solutions to the field equations which have two commuting spacelike Killing vectors $\xi_1$, $\xi_2$, (ii) express the solutions in a coordinate system $(u,v,x,y)$ where $u,v$ are null coordinates and $\xi_i$ are given by $\partial/\partial x^i$, and (iii) see whether it is possible to extend these solutions across the null surfaces $\mathcal{N}_1=\{u=0\}$ and $\mathcal{N}_2=\{v=0\}$ in such a way that the extension still satisfies the field equations, and that the extended metric in regions II and III (Fig. 1) describes single plane waves

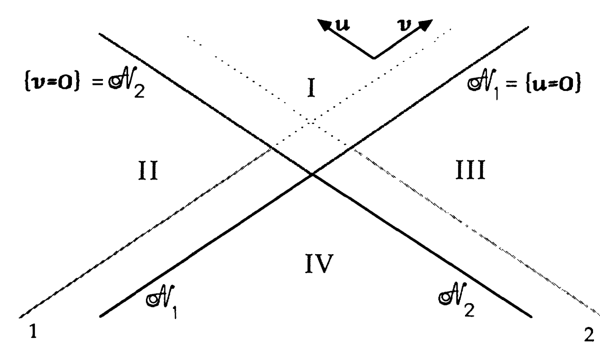


FIG. 1. The two-dimensional geometry of the characteristic initial-value problem for colliding plane waves. The null surfaces $\mathcal{N}_1=\{u=0\}$ and $\mathcal{N}_2=\{v=0\}$ are the past wave fronts of the incoming plane waves 1 and 2. Initial data corresponding to waves 1 and 2 are posed, respectively, on the upper portions of the surfaces $\mathcal{N}_2$ and $\mathcal{N}_1$ that are adjacent to the interaction region I. The geometry in region IV is flat, and the geometry in regions II and III is given by the metric describing the incoming waves 1 and 2, respectively. The geometry of the interaction region I is uniquely determined by the solution of the above initial-value problem.

 

propagating in the appropriate directions. This technique of generating exact solutions for colliding plane waves was elevated into an art form over the recent years by the work of Chandrasekhar and Xanthopoulos,[2] who have obtained not only many new solutions describing colliding plane waves with parallel polarizations coupled with matter sources, but have also obtained new exact solutions for colliding plane waves with nonparallel polarizations, which display several unexpected novel features.[6,7] It is this technique which we use in the present paper to construct our solutions; consequently we shall describe it in more detail in the subsequent sections. Here we just remark that, as it is possible in principle to use different prescriptions for extending the metric beyond the interaction region, the alternative approach we just described will in general yield several different colliding plane-wave solutions which all have the same geometry in the interaction region I, but which for each different extension describe different incoming wave forms in the regions II and III (Fig. 1). This is in contrast with the direct method where one integrates the initial data posed by the incoming plane waves and obtains a unique colliding plane-wave spacetime. The reason for this behavior is that the same solution in the interaction region may evolve from several inequivalent sets of initial data, whereas the outcome from the direct method of integrating given initial data is constrained to be unique by the well-known uniqueness results for hyperbolic systems.

For the solutions constructed in this paper, the metric in the interaction region of the colliding plane-wave spacetimes is obtained from the interiors of the static, axisymmetric "distorted black hole" (Weyl) solutions which possess an interior. Every Weyl solution of this kind has a pair of commuting spacelike Killing vectors defined throughout its interior region. The simplest example of such Weyl solutions is the Schwarzschild spacetime. The construction by which we build our colliding plane-wave spacetimes is described in detail in the next section (Sec. II) for the Schwarzschild metric, along with a discussion of the properties of the resulting colliding plane-wave solution. Then in Sec. III we discuss the generalization of this construction to the infinite-parameter family of Weyl solutions which satisfy our regularity requirements; this generalization yields a corresponding infinite-parameter family of colliding plane-wave spacetimes. In Sec. IV two specific examples of spacetimes in this family are described briefly. The first of these examples is generated from one of the simplest nonspherical Weyl solutions in our family; this Weyl solution can be interpreted as the interior metric of a Schwarzschild black hole distorted by a static, quadrupolar matter distribution outside the horizon. The second example describes a colliding plane-wave spacetime where both of the two incoming plane waves are purely anastigmatic, i.e., where both incoming waves have equal focal lengths.[8,9] In Sec. V we recapitulate our conclusions by briefly listing both the new features and the drawbacks of the solutions that we have constructed. We also discuss some open questions and suggestions for future research on the issues raised by the present work.

It is not the purpose of this paper to discuss either the physical interpretation of colliding plane wave solutions or the significance of these solutions for general relativity in a wider context. The reader is referred to Refs. 3, 5, 10, 7, 9, and 4 and the extensive literature cited therein for a detailed exposition of these issues.

## II. THE SOLUTION OBTAINED FROM THE SCHWARZSCHILD METRIC

We first write the Schwarzschild metric inside the horizon (i.e., for $r<2M$) as

$$g=-\left[\frac{1}{\frac{2M}{r}-1}\right]dr^2+\left[\frac{2M}{r}-1\right]dt^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\ . \tag{2.1}$$

Clearly, in this interior region where $r<2M$ the commuting Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$ are both spacelike. We therefore introduce new coordinates $(x,y,u,v)$ tuned to the plane symmetry generated by these Killing vectors, by the following transformation (again for $r<2M$):

$$t=x\ ,\quad \phi=(1+y/M)\ ,\qquad \theta=\frac{\pi}{2}+(v-u)\ ,\quad r=M[1-\sin(u+v)]\ . \tag{2.2}$$

In this new coordinate system the metric (2.1) takes the form

$$g=-4M^2[1-\sin(u+v)]^2du\,dv+\left[\frac{1+\sin(u+v)}{1-\sin(u+v)}\right]dx^2+[1-\sin(u+v)]^2\cos^2(u-v)dy^2\ , \tag{2.3}$$

which explicitly displays the plane-symmetry generating, commuting, spacelike Killing vectors $\xi_1=\partial/\partial x$ and $\xi_2=\partial/\partial y$. We take the spacetime region $\{u\ge 0, v\ge 0, -\infty<x<+\infty, -\infty<y<+\infty\}$ with the metric (2.3) on it as the interaction region I of our colliding plane-wave solution. Note that, even though this interaction region is locally isometric to the region

$$J=\{r<\min\,[\,M(1+\cos\theta)\,,\,M(1-\cos\theta)\,]\,,\quad -\infty<t<+\infty,\ 0\le\phi<2\pi\}$$

of the Schwarzschild spacetime (this region $J$ is depicted in Fig. 2), we will in effect have changed the topology of the underlying manifold from $S^2\times R^2$ to $R^4$ by means of (i) extending the metric (2.3) across the surfaces $\{u=0\}$, $\{v=0\}$ (nonanalytically) in the manner described below, and (ii) by applying the coordinate transformation (2.2) in which $y$ and $v-u$ are not regarded as periodic whereas $\phi$ and $\theta$ are. More specifically, by our nonanalytic extension we shall eliminate the (coordinate) singularities of the $(u,v,x,y)$ chart at $v-u=2\pi k\pm\pi/2$ (where $k$ is any integer) that would show up in the maximal *analytic* extension, and thereby we shall transform the topology from $S^2\times R^2$ to $S^1\times R^3$. Subsequently, since $\partial/\partial\phi=M\partial/\partial y$ is Killing, the resulting metric on $S^1\times R^3$ can be lifted to the covering space $R^4$ as described by the coordinate change (2.2), and this yields us the metric (2.3) defined on $R^4$.

We extend the metric (2.3) across the wave fronts $\{u=0\}$ and $\{v=0\}$ by the Penrose prescription[5,2] $u/a \to (u/a)H(u/a)$, $v/b \to (v/b)H(v/b)$ where $H(x)$ denotes the Heaviside step function and we have introduced two length scales $a$ and $b$ into the problem by putting $u\equiv u'/a$, $v\equiv v'/b$ where $ab=4M^2$, and we have redefined $u'$ as $u$ and $v'$ as $v$. Thereby we obtain the following final metric for our colliding plane-wave spacetime:

$$g=-\left\{1-\sin\left[\frac{u}{a}H\left[\frac{u}{a}\right]+\frac{v}{b}H\left[\frac{v}{b}\right]\right]\right\}^2 du\,dv+\frac{1+\sin\left[\frac{u}{a}H\left[\frac{u}{a}\right]+\frac{v}{b}H\left[\frac{v}{b}\right]\right]}{1-\sin\left[\frac{u}{a}H\left[\frac{u}{a}\right]+\frac{v}{b}H\left[\frac{v}{b}\right]\right]}dx^2$$

$$+\left\{1-\sin\left[\frac{u}{a}H\left[\frac{u}{a}\right]+\frac{v}{b}H\left[\frac{v}{b}\right]\right]\right\}^2\cos^2\left[\frac{u}{a}H\left[\frac{u}{a}\right]-\frac{v}{b}H\left[\frac{v}{b}\right]\right]dy^2\,. \quad (2.4)$$

The geometry of this spacetime is depicted in Fig. 3, which describes a two-dimensional subspace given by $\{x=\text{const},\ y=\text{const}\}$. (Actually the geometry is more subtle than this two-dimensional projection indicates; see Refs. 11 and 9.) A curvature singularity is present at $(u/a)+(v/b)=\pi/2$; it corresponds to the curvature singularity of the interior Schwarzschild spacetime at $r=0$. The extended spacetime consists of four regions where the metric is analytic: region I, where $u>0$, $v>0$, is the interaction region in which the metric is given by Eq. (2.3); regions II and III, where $u>0$, $v<0$ and $u<0$, $v>0$ respectively, represent the two incoming plane waves; region IV, where $u<0$, $v<0$, is the flat Minkowskian region representing the spacetime before the arrival of either wave. The only vector fields that are Killing vectors on the whole spacetime are $\partial/\partial x$ and $\partial/\partial y$ (and their constant linear combinations), whereas there exist two more $R$-linearly independent (i.e., linearly independent over the reals) spacelike Killing vectors in the interaction region I [Eq. (2.3)]; these extra Killing vectors correspond to the generators of spherical symmetry for the interior Schwarzschild metric (2.1). These vector fields cannot be extended as Killing vectors to the rest of the spacetime [Eq. (2.4)]. For the generalized solutions that we describe in the next section, $\partial/\partial x$ and $\partial/\partial y$ (and their constant linear combinations) are the only Killing vectors in

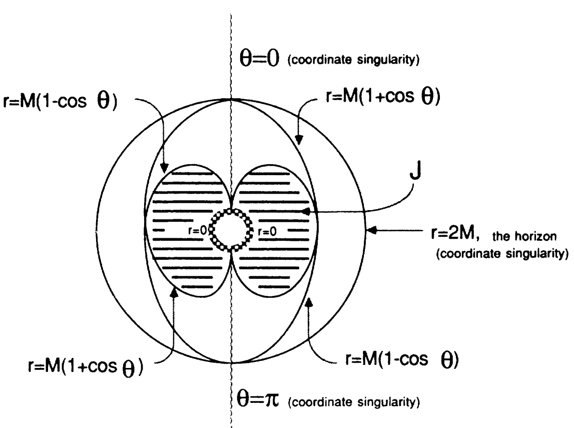


FIG. 2. The region $J$ in Schwarzschild spacetime to which the interaction region of the colliding plane-wave solution (2.4) is locally isometric. This region $J$ is shown shaded in this figure which is drawn in a $\{t=\text{const}\}$, $\{\phi=0,\pi\}$ plane. As explained in the text, the geometry in region $J$ is extended nonanalytically beyond the null surfaces $r=M(1+\cos\theta)$ and $r=M(1-\cos\theta)$, which correspond to the wave fronts $\{u=0\}$ and $\{v=0\}$, respectively. Consequently, all coordinate singularities are avoided and the Schwarzschild metric on the shaded region $J$ is lifted from $S^2\times R^2$ to $R^4$, on which the final metric (2.4) is defined.

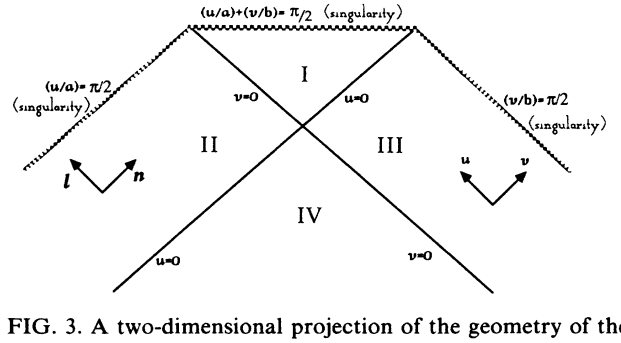


FIG. 3. A two-dimensional projection of the geometry of the colliding plane-wave solution (2.4). The metric is analytic throughout each of the regions I, II, III and IV, but it suffers discontinuities in its derivatives across the boundaries between the adjacent regions. The interaction region I is locally isometric to region $J$ (Fig. 2) of the interior Schwarzschild solution (2.1). The curvature singularity at $(u/a)+(v/b)=\pi/2$ corresponds, under this isometry, to the Schwarzschild singularity at $r=0$.

the interaction region I, since the isometry group of the distorted, static, axisymmetric Weyl solutions is in general two dimensional. The solution (2.4) and also its generalizations described in Sec. III represent colliding plane waves with parallel polarizations, since the $x-y$ part of the metric [Eq. (2.3)] in the interaction region I is in diagonal form at all points; or equivalently since the Killing vectors $\partial/\partial x$ and $\partial/\partial y$ are hypersurface orthogonal throughout the spacetime.

According to Eq. (2.4), the metric in region II is

$$g_{\rm II}=-[1-\sin(u/a)]^2 du\,dv+\left[\frac{1+\sin(u/a)}{1-\sin(u/a)}\right]dx^2 +[1-\sin(u/a)]^2\cos^2(u/a)dy^2\;, \tag{2.5}$$

which entails a curvature singularity at the null surface $\{u=\pi a/2\}$. The metric $g_{\rm III}$ in region III is obtained by replacing $u/a$ with $v/b$ in Eq. (2.5) and similarly displays a curvature singularity at the null surface $\{v=\pi b/2\}$. Note that, in the most famous of the solutions for colliding plane waves[5,2,3] the corresponding null surfaces are also singular, but they do not represent curvature singularities. Instead, in those solutions, these surfaces correspond to the (nonsingular) focal planes (or Killing-Cauchy horizons[7]) of the respective incoming plane waves, and they become singular in the colliding plane-wave spacetime only because of the topological effect caused by the focusing of the plane wave moving in the opposite direction.[5,11,9] In the present case, however, these null surfaces are contained *within* the incoming plane sandwich waves, by contrast with the famous solutions where they are located in the flat regions lying to the future of the curvature disturbances associated with the incoming waves. Hence, for the solution (2.4) [see Eqs. (2.13)–(2.14) below], the curvature quantity $\Psi_0$ or $\Psi_4$ representing the radiative part of the Weyl tensor diverges on these surfaces. Physically, this could be considered a serious drawback of the solution (2.4), we expect a realistic spacetime representing a single gravitational wave propagating in empty space to be free of singularities of the above kind. However, it is possible to circumvent this difficulty by cutting off the gravitational radiation in each incoming plane wave along two null surfaces $\{u=u_c\}$ and $\{v=v_c\}$, where we can choose $u_c$ and $v_c$ to be arbitrarily close to $\pi a/2$ and $\pi b/2$, respectively. This results in the colliding plane-wave spacetime depicted in Fig. 4, where the metric in the regions denoted by I, II, III, and IV is exactly the same as the metric in the regions denoted by the same symbols in the original solution (2.4). Across the surfaces $\{u=u_c\}$ and $\{v=v_c\}$ the metric is $C^1$ but not $C^2$, making these surfaces shock fronts across which the curvature quantities $\Psi_0$ or $\Psi_4$ suffer jump discontinuities without delta-function contributions. (The structure of the field equations for a plane wave makes it possible to introduce such shocks at any desired null surface $\{u=\text{const}\}$; see, for example, Ref. 9.) The geometry in the regions denoted by IIa and IIIb in Fig. 4 is flat, and the surfaces $\{u=u_f\}$ and $\{v=v_f\}$ (where $u_f$ and $v_f$ are slightly larger than $\pi a/2$ and $\pi b/2$) correspond to the focal planes of the respective plane waves. These planes would be nonsingular if the collision were not taking place; the singularities at these focal planes are solely due to the topological effect of the focusing of the wave moving in the opposite direction.[11,9] The physics of this new solution in the interaction region is determined to an arbitrarily large extent by the metric (2.3) in the region I; even though the metric in regions Ia and Ib is not determined by Eqs. (2.3) or (2.4), by choosing $u_c$ and $v_c$ arbitrarily close to $\pi a/2$ and $\pi b/2$ it is possible to make the regions Ia and Ib arbitrarily small. Hence the colliding plane-wave solution (2.4) describes arbitrarily well the collision of the more "realistic" plane waves illustrated in Fig. 4.

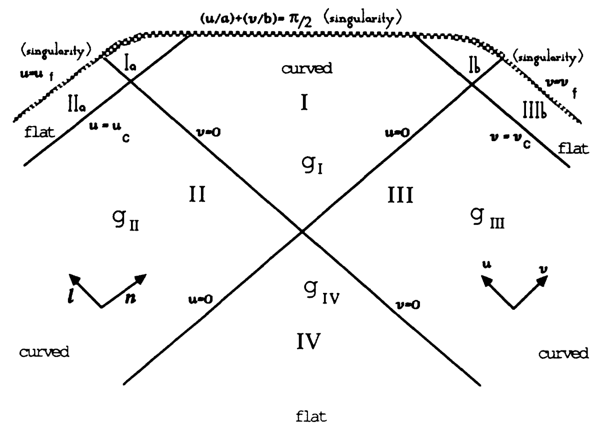


FIG. 4. Geometry of the colliding plane-wave solution that results from "cutting off" the incoming, colliding plane waves described by the solution (2.4). As explained in the text, the introduction of the secondary shocks along the surfaces $\{u=u_c\}$ and $\{v=v_c\}$ removes the curvature singularities on the focal planes $\{u=u_f\}$ and $\{v=v_f\}$. However, the geometry in the interaction region of this new solution is not everywhere described by the metric (2.4); the regions Ia and Ib are described by a different metric.

We now turn to the proof of our implicit assertion that the metric (2.4) is indeed a genuine solution (in the sense of distributions) to the vacuum Einstein field equations. For this purpose, and also for spelling out the geometric structure of the solution (2.4) more clearly, we will find it useful to introduce the following null tetrad on our colliding plane-wave spacetime:

$$\begin{aligned} l&=2e^{M}\frac{\partial}{\partial u}\;,\quad \mathbf{n}=\frac{\partial}{\partial v}\;,\\ \mathbf{m}&=N_1\frac{\partial}{\partial x}+N_2\frac{\partial}{\partial y}\;, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} N_1&=\frac{1+i}{2}e^{(U-V)/2}\;,\\ N_2&=\frac{1-i}{2}e^{(U+V)/2}\;, \end{aligned} \tag{2.7}$$

and $M$, $U$ and $V$ are functions of $u$ and $v$ only. The tetrad (2.6)—(2.7) gives rise to the metric

$$g=-e^{-M}du\,dv+e^{V-U}dx^2+e^{-(U+V)}dy^2 \; . \qquad (2.8)$$

Thus, the tetrad coefficients $M$, $U$, $V$ for the colliding plane-wave solution (2.4) are given by

$$M=-2\ln\left\{1-\sin\left[\frac{u}{a}H\left[\frac{u}{a}\right]+\frac{v}{b}H\left[\frac{v}{b}\right]\right]\right\} , \qquad (2.9a)$$

$$U=-\ln\frac{1}{2}\left\{\cos\left[\frac{2u}{a}H\left[\frac{u}{a}\right]\right]+\cos\left[\frac{2v}{b}H\left[\frac{v}{b}\right]\right]\right\} , \qquad (2.9b)$$

$$V=\ln\cos\left[\frac{u}{a}H\left[\frac{u}{a}\right]+\frac{v}{b}H\left[\frac{v}{b}\right]\right] -\ln\cos\left[\frac{u}{a}H\left[\frac{u}{a}\right]-\frac{v}{b}H\left[\frac{v}{b}\right]\right] -2\ln\left\{1-\sin\left[\frac{u}{a}H\left[\frac{u}{a}\right]+\frac{v}{b}H\left[\frac{v}{b}\right]\right]\right\} . \qquad (2.9c)$$

The vacuum field equations for the metric (2.8) are[3,4]

$$2(U_{,uu}+M_{,u}U_{,u})-U_{,u}{}^2-V_{,u}{}^2=0 \; , \qquad (2.10a)$$

$$2(U_{,vv}+M_{,v}U_{,v})-U_{,v}{}^2-V_{,v}{}^2=0 \; , \qquad (2.10b)$$

$$U_{,uv}-U_{,u}U_{,v}=0 \; , \qquad (2.10c)$$

$$V_{,uv}-\tfrac{1}{2}(U_{,u}V_{,v}+U_{,v}V_{,u})=0 \; , \qquad (2.10d)$$

where the integrability condition for the first two equations is satisfied by virtue of the last two and yields the remaining field equation

$$M_{,uv}-\tfrac{1}{2}(V_{,u}V_{,v}-U_{,u}U_{,v})=0 \; . \qquad (2.11)$$

Therefore it is sufficient to solve Eqs. (2.10c) and (2.10d) first and to obtain $M$ by quadrature from the first two equations (2.10a) and (2.10b) later, since Eq. (2.11) as well as the integrability condition for Eqs. (2.10a) and (2.10b) are automatically satisfied as a result of Eqs. (2.10c) and (2.10d).

We now proceed to verify that the field equations (2.10) and (2.11) are satisfied (in the sense of distributions) by our colliding plane-wave solution (2.4).

The field equations hold in the interaction region I (Fig. 3), since in this region (2.4) reduces to the metric (2.3), which is locally isometric to the interior Schwarzschild metric and thus is obviously vacuum.

In order to show that the field equations are satisfied in regions II and III, it is clearly sufficient to verify Eqs. (2.10) and (2.11) for the metric $g_{\rm II}$ given by Eq. (2.5), since the metric $g_{\rm III}$ in region III is locally isometric to $g_{\rm II}$ under the interchange $u \leftrightarrow v$ [which incidentally is also a discrete isometry for the metric (2.3) in the interaction region]. This can be verified directly by substituting $U, V$ and $M$ for $u>0$, $v<0$ from Eqs. (2.9) into the left side of Eqs. (2.10) and (2.11); the result is easily shown to vanish. A more elegant approach, however, is to note that (i) verifying the field equations in regions II or III is equivalent to verifying the field equations for the *analytically extended* interaction region metric (2.3) *at* the null surfaces $\{u=0\}$, $\{v=0\}$; and (ii) the field equations for the metric (2.3) clearly hold at these null surfaces, because these equations hold *throughout* the analytically extended spacetime region covered by the $(u,v,x,y)$ chart, and because this region contains the null surfaces $\{u=0\}$ and $\{v=0\}$ as nonsingular hypersurfaces.

The field equations hold in the region IV since the metric in this region is flat.

To show that the field equations hold (in the distribution sense) on the boundaries between regions I and II and between regions I and III (Fig. 3), it is again sufficient to consider only the I-II boundary because of the $u \leftrightarrow v$ symmetry of the problem. Now, since all field equations hold identically throughout region I and region II, they can only fail to hold on the boundary I-II if there are contributions to the left-hand side of Eqs. (2.10) and (2.11) which are nonzero only on this boundary and which are zero everywhere else. It is seen easily from the structure of the functions $U$, $V$, $M$ displayed in Eqs. (2.9) that such contributions must involve $\delta$-functions supported on the I-II boundary. However, as $M$, $U$, and $V$ are functions of the arguments $(u/a)H(u/a)$ and $(v/b)H(v/b)$, the only way $\delta$-function contributions can arise is by a two-times differentiation of $U$, $M$, or $V$ with respect to either $u$ or $v$, but not by a differentiation of the form $\partial_u \partial_v$. Therefore the last two field equations (2.10c) and (2.10d) as well as the integrability condition Eq. (2.11) automatically hold on any of the boundaries. On the boundary I-II, the first field equation (2.10a) holds trivially since this boundary is given by $\{v=0\}$ and Eq. (2.10a) contains only double $u$ derivatives and thus cannot introduce $\delta(v)$ terms. The second equation (2.10b), however, can introduce $\delta(v)$ terms on the I-II boundary through the derivative $2U_{,vv}$. But a short calculation reveals that all $\delta(v)$ terms introduced by the differentiation $U_{,vv}$ are proportional to $\sin(2v/b)$, and thus they vanish on the I-II boundary on which $v=0$. This completes the proof that the field equations hold, in the sense of distributions, on the I-II boundary as well as on the boundary I-III between regions I and III.

The boundaries between regions II and IV and between regions III and IV (Fig. 3) are treated similarly. By the same arguments as above, and since on the II-IV boundary we have $u=0$, it is enough to show the nonexistence of $\delta(u)$ terms on the II-IV boundary. Such terms could only be introduced by the first field equation (2.10a), and the second field equation (2.10b) holds trivially on the II-IV boundary. Equation (2.10a) can introduce $\delta(u)$ terms only through the second derivative $U_{,uu}$; however, all the terms in this derivative which involve delta functions turn out to be proportional to $\sin(2u/a)$ and thus they vanish on the II-IV boundary on which $u=0$.

By either of the above boundary arguments, the field equations hold on the two-plane $\{u=v=0\}$. Morover, since the coordinate system $(u,v,x,y)$ regularly covers a neighborhood of the two-plane $\{u=v=0\}$, and since the metric coefficients in the $(u,v,x,y)$ chart are continuous on the whole spacetime including this plane, no "conical-type" singularity can be present on the spacelike two-plane $\{u=v=0\}$.

In order to elucidate further the physics of our colliding plane-wave solution, we conclude this section with a brief discussion of the behavior of the spacetime curvature associated with the metric (2.4). The Newman-Penrose curvature quantities in the null tetrad (2.6) and (2.7) are given by[3,4]

$$\Psi_0=2ie^{2M}(M_{,u}V_{,u}+V_{,uu}-V_{,u}U_{,u}) , \tag{2.12a}$$

$$\Psi_1=0 , \tag{2.12b}$$

$$\Psi_2=-e^{M}M_{,uv} , \tag{2.12c}$$

$$\Psi_3=0 , \tag{2.12d}$$

$$\Psi_4=\frac{i}{2}[(U_{,v}-M_{,v})V_{,v}-V_{,vv}] . \tag{2.12e}$$

Substituting $M$, $U$ and $V$ from Eqs. (2.9) in the above equations, we straightforwardly obtain the following information about the behavior of the curvature quantities on our colliding plane-wave spacetime (2.4).

All nonzero curvature quantities in the interaction region I (Fig. 3) diverge towards the singularity $[(u/a)+(v/b)=\pi/2]$ . The asymptotic behaviors of $\Psi_0$ , $\Psi_2$, and $\Psi_4$ near the singularity are all of the form $[(\pi/2)-(u/a)-(v/b)]^{-n}$ where $n=10$ for $\Psi_0$ , $n=6$ for $\Psi_2$, and $n=2$ for $\Psi_4$.

In region II (Fig. 3) the only nonzero curvature quantity is

$$\Psi_0=\frac{12i}{a^2}\,\frac{1}{[1-\sin(u/a)]^5} \tag{2.13}$$

whereas in region III the only nonzero curvature quantity is

$$\Psi_4=-\frac{3i}{b^2}\,\frac{1}{[1-\sin(v/b)]} . \tag{2.14}$$

In region IV all curvature quantities vanish.

On the I-II and I-III boundaries (Fig. 3) $\Psi_2$ has jump discontinuities which are finite but which diverge towards the singularity:

$$[\Psi_2]_{\text{I-II}}=-\frac{2}{ab}\,\frac{1}{[1-\sin(u/a)]^3} , \tag{2.15a}$$

$$[\Psi_2]_{\text{I-III}}=-\frac{2}{ab}\,\frac{1}{[1-\sin(v/b)]^3} . \tag{2.15b}$$

There are no δ-function contributions to the discontinuity of $\Psi_2$ along these boundaries. Along the I-II boundary $\Psi_0$ is continuous, whereas $\Psi_4$ has a jump

$$[\Psi_4]_{\text{I-II}}=-\frac{3i}{b^2}\,\frac{1}{[1-\sin(u/a)]} \tag{2.16a}$$

and also has a δ-function singularity of the form

$$-\frac{i}{b^2}\,\frac{1}{\cos(u/a)}\,\delta(v/b) . \tag{2.16b}$$

Along the I-III boundary $\Psi_4$ is continuous, whereas $\Psi_0$ has a jump

$$[\Psi_0]_{\text{I-III}}=\frac{12i}{a^2}\,\frac{1}{[1-\sin(v/b)]^5} \tag{2.17a}$$

and also has a δ-function singularity of the form

$$\frac{4i}{a^2}\,\frac{1}{\cos(v/b)[1-\sin(v/b)]^4}\,\delta(u/a) . \tag{2.17b}$$

Along the II-IV and III-IV boundaries (Fig. 3) $\Psi_2$ (being identically zero across these boundaries) is continuous. Across the II-IV boundary $\Psi_4$ (being zero) is continuous, whereas $\Psi_0$ has a jump

$$[\Psi_0]_{\text{II-IV}}=\frac{12i}{a^2} , \tag{2.18a}$$

and also has a δ-function singularity of the form

$$\frac{4i}{a^2}\delta\left[\frac{u}{a}\right] . \tag{2.18b}$$

Along the III-IV boundary $\Psi_0$ (being zero) is continuous, whereas $\Psi_4$ has a jump

$$[\Psi_4]_{\text{III-IV}}=-\frac{3i}{b^2} , \tag{2.19a}$$

and also has a δ-function singularity of the form

$$-\frac{i}{b^2}\delta\left[\frac{v}{b}\right] . \tag{2.19b}$$

## III. THE SOLUTIONS OBTAINED FROM THE WEYL METRICS

The most general static axisymmetric spacetime with a regular axis has the metric[12]

$$g=-e^{2\psi}dt^2+e^{-2\psi}\rho^2d\phi^2+e^{2(\gamma-\psi)}(d\rho^2+dz^2) , \tag{3.1}$$

where $(t,z,\rho,\phi)$ are the cylindrical (Weyl) coordinates, and $\psi$ and $\gamma$ are functions of $\rho$ and $z$ only. The vacuum Einstein field equations for the metric (3.1) are

$$\psi_{,\rho\rho}+\frac{1}{\rho}\psi_{,\rho}+\psi_{,zz}=0 , \tag{3.2a}$$

$$\gamma_{,\rho}=\rho(\psi_{,\rho}{}^2-\psi_{,z}{}^2) , \tag{3.2b}$$

$$\gamma_{,z}=2\rho\psi_{,\rho}\psi_{,z} , \tag{3.2c}$$

where Eq. (3.2a) is the integrability condition for the last two equations (3.2b) and (3.2c). The regularity of the axis $\rho=0$ requires that $\gamma=0$ at $\rho=0$. Thus, any solution $\psi(\rho,z)$ to Eq. (3.2a) uniquely determines a solution of the form (3.1) to the vacuum Einstein equations. For the Schwarzschild solution, $\psi$ and $\gamma$ are given by

$$\psi^S(\rho,z)=\tfrac{1}{2}\ln\left[\frac{\alpha-1}{\alpha+1}\right] , \tag{3.3a}$$

$$\gamma^S(\rho,z)=\tfrac{1}{2}\ln\left[\frac{\alpha^2-1}{\alpha^2-\mu^2}\right] , \tag{3.3b}$$

where

$$\mu=\frac{1}{2M}\{[\rho^2+(z-M)^2]^{1/2}-[\rho^2+(z+M)^2]^{1/2}\}\ , \qquad (3.4)$$

and

$$\alpha=\frac{1}{2M}\{[\rho^2+(z-M)^2]^{1/2}+[\rho^2+(z+M)^2]^{1/2}\}\ . \qquad (3.5)$$

The Schwarzschild coordinates $(t,r,\theta,\phi)$ are related to the Weyl coordinates by

$$\rho=r\left[1-\frac{2M}{r}\right]^{1/2}\sin\theta\ ,\quad z=(r-M)\cos\theta\ ,$$
$$\phi=\phi\ ,\quad t=t\ . \qquad (3.6)$$

The horizon $\{r=2M\}$ of the Schwarzschild spacetime corresponds to the surface $\{\rho=0,-M\leq z\leq M\}$ in Weyl coordinates. Note, however, that neither the horizon $\{r=2M\}$ nor the interior region where $r<2M$ is covered smoothly by the Weyl coordinate system.

In order to isolate those Weyl solutions which, like Schwarzschild spacetime, possess a nonsingular horizon and an interior region, we will find it convenient to define new metric functions $\hat\psi$ and $\hat\gamma$ by

$$\hat\psi\equiv\psi-\psi^S\ ,\quad \hat\gamma\equiv\gamma-\gamma^S\ . \qquad (3.7)$$

Since the field equation (3.2a) for $\psi$ is linear, it is satisfied in exactly the same form by the function $\hat\psi$. On the other hand, the field equations satisfied by $\hat\gamma$ as obtained from Eqs. (3.2b), (3.2c), and (3.3) are given by

$$\hat\gamma_{,\rho}=\rho\left[\hat\psi_{,\rho}{}^2-\hat\psi_{,z}{}^2+\frac{2}{\alpha^2-1}(\alpha_{,\rho}\hat\psi_{,\rho}-\alpha_{,z}\hat\psi_{,z})\right]\ , \qquad (3.8a)$$

$$\hat\gamma_{,z}=2\rho\left[\hat\psi_{,\rho}\hat\psi_{,z}+\frac{1}{\alpha^2-1}(\alpha_{,z}\hat\psi_{,\rho}+\alpha_{,\rho}\hat\psi_{,z})\right]\ , \qquad (3.8b)$$

where $\alpha(\rho,z)$ is defined by Eq. (3.5). (The mass $M$ that enters into the definition of $\gamma^S$ and $\psi^S$ can be chosen arbitrarily, and in particular can be set equal to 1. However, we retain this free parameter $M$ since it will be helpful when introducing length scales into the colliding plane-wave solutions that we are going to build shortly.) To solve the field equation (3.2a) satisfied by $\hat\psi$, we introduce spherical polar coordinates $\nu$ and $\eta$ defined by

$$\rho=\nu\sin\eta\ ,\quad z=\nu\cos\eta\ .$$

The general solution of Eq. (3.2a) can now be written in terms of Legendre polynomials $P_k(x)$ (Ref. 13):

$$\hat\psi(\nu,\eta)=\sum_{k=0}^{\infty}(d_k\nu^k+c_k\nu^{-k-1})P_k(\cos\eta)\ . \qquad (3.9)$$

For the time being, the coefficients $d_k$ and $c_k$ are simultaneously included in the above expression, because both asymptotic flatness and regularity of the horizon are irrelevant restrictions for our purposes. However, the terms involving the Legendre functions of the second kind, $Q_k(\cos\eta)$, are left out of the sum (3.9), since we assume that the axis on which $\cos\eta=\pm1$ is nonsingular throughout spacetime. This assumption, together with the regularity condition that we impose below, will guarantee that the spacetime admits a nonsingular horizon which is located at $r=2M$ in the Schwarzschild-type coordinate system (3.6). Combining Eq. (3.1) with Eqs. (3.7), (3.3)—(3.5), and (3.6), we obtain the following general Weyl metric written in the Schwarzschild-type coordinates $(t,r,\theta,\phi)$:

$$g=-\left[1-\frac{2M}{r}\right]e^{2\hat\psi(r,\theta)}dt^2+e^{-2\hat\psi(r,\theta)}r^2\sin^2\theta\,d\phi^2$$
$$+e^{2[\hat\gamma(r,\theta)-\hat\psi(r,\theta)]}\left[\frac{dr^2}{1-\frac{2M}{r}}+r^2d\theta^2\right]\ . \qquad (3.10)$$

The functions $\hat\psi(r,\theta)$ and $\hat\gamma(r,\theta)$ in the above metric are calculated from the formulas [cf. Eqs. (3.8) and (3.9)]

$$\hat\psi(\rho,z)=\sum_{k=0}^{\infty}[d_k(\rho^2+z^2)^{k/2}+c_k(\rho^2+z^2)^{-(k+1)/2}]P_k\left[\frac{z}{\sqrt{\rho^2+z^2}}\right]\ , \qquad (3.11)$$

$$\hat\gamma(\rho,z)=\int_C\rho\left[\left[\hat\psi_{,\rho}{}^2-\hat\psi_{,z}{}^2+\frac{2}{\alpha^2-1}(\alpha_{,\rho}\hat\psi_{,\rho}-\alpha_{,z}\hat\psi_{,z})\right]d\rho+2\left[\hat\psi_{,\rho}\hat\psi_{,z}+\frac{1}{\alpha^2-1}(\alpha_{,z}\hat\psi_{,\rho}+\alpha_{,\rho}\hat\psi_{,z})\right]dz\right]\ , \qquad (3.12)$$

by substituting for $\rho$ and $z$ their expressions (3.6) as functions of $r$ and $\theta$. In Eq. (3.12) the function $\alpha(\rho,z)$ is defined by Eq. (3.5), and the line integral is evaluated on *any* contour $C$ that starts on the axis $\rho=0$ (where $\hat\gamma$ vanishes), and that ends at the point $(\rho,z)$ where $\hat\gamma$ is to be computed.[13]

Since we are interested in solutions with two spacelike Killing vectors, we now turn to the characterization of those Weyl solutions in the family (3.10)—(3.12) which possess an "interior region" in which $\partial/\partial t$ is spacelike. As we have noted before, the Weyl coordinates $(t,\rho,z,\phi)$ cannot cover the interior region regularly even if such a region exists. However, as the form of the metric (3.10) indicates clearly, the Schwarzschild-type coordinates $(t,r,\theta,\phi)$ [which are defined formally by Eqs. (3.6)] will cover the interior region $r<2M$, whenever this region exists as a spacetime region with a well-defined metric (3.10). Morover, these coordinates will cover the interior region $r<2M$ regularly, apart from the usual singularities associated with spherical coordinates. It is also clear that the interior region $r<2M$ will have a well-defined metric (3.10) if and only if the functions $\hat{\psi}(r,\theta)$ and $\hat{\gamma}(r,\theta)$ given by Eqs. (3.11) and (3.12) are well defined for $r<2M$. We now claim that in order for these functions $\hat{\psi}$ and $\hat{\gamma}$ be well defined for $r<2M$, it is necessary and sufficient that in Eqs. (3.9) and (3.11) all of the coefficients $c_k$ vanish. This restriction on the general solution (3.9) is necessary for the existence of the interior region $r<2M$, because the expression

$$\rho^2+z^2 \equiv r^2-2Mr+M^2\cos^2\theta$$

assumes negative values at some points in the region $r<2M$; therefore we have to eliminate any term involving the product of an half-odd-integer (integer) power of $\rho^2+z^2$ with an even-indexed (odd-indexed) Legendre polynomial $P_k$ from Eqs. (3.9) and (3.11). The sufficiency of the above condition for the existence of a well-defined metric (3.10) on the region $r<2M$ will become clear after the following discussion. We also note that the above restriction on the general solution (3.9) guarantees not only the existence of the interior region $\{r<2M\}$, but also the existence and regularity of the "horizon" $\{r=2M\}$.

We now have the following infinite-parameter family of interior Weyl solutions, defined on the region $\{r<2M\}$ where both of the two commuting Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$ are spacelike:

$$g=\left[\frac{2M}{r}-1\right]e^{2\hat{\psi}(r,\theta)}dt^2+e^{-2\hat{\psi}(r,\theta)}r^2\sin^2\theta\, d\phi^2 +e^{2[\hat{\gamma}(r,\theta)-\hat{\psi}(r,\theta)]}\left[r^2d\theta^2-\frac{dr^2}{\frac{2M}{r}-1}\right] \quad \text{for } r<2M\ , \tag{3.13}$$

where

$$\hat{\psi}(r,\theta)=\sum_{k=0}^{\infty} d_k(r^2-2Mr+M^2\cos^2\theta)^{k/2} \times P_k\left[\frac{(r-M)\cos\theta}{(r^2-2Mr+M^2\cos^2\theta)^{1/2}}\right], \tag{3.14}$$

$$\hat{\psi}(\rho,z)=\sum_{k=0}^{\infty} d_k(\rho^2+z^2)^{k/2}P_k\left[\frac{z}{\sqrt{\rho^2+z^2}}\right], \tag{3.15}$$

and where $\hat{\gamma}(r,\theta)$ is computed by inserting Eq. (3.15) into Eq. (3.12), using Eq. (3.5), and substituting for $\rho$ and $z$ their expressions (3.6) as functions of $r$ and $\theta$. In Eqs. (3.13)—(3.15) we have combined Eq. (3.11) with Eq. (3.6) to obtain Eq. (3.14).

To see that the functions $\hat{\psi}(r,\theta)$ and $\hat{\gamma}(r,\theta)$ defined by Eqs. (3.14), (3.15), (3.12), (3.5), and (3.6) are well defined and real for $r<2M$, note the following facts.

(i) The Legendre polynomials $P_{2n}(x)$ are polynomials in $x^2$ of order $n$. Hence, for all even $k$ the expression

$$(r^2-2Mr+M^2\cos^2\theta)^{k/2}P_k\left[\frac{(r-M)\cos\theta}{(r^2-2Mr+M^2\cos^2\theta)^{1/2}}\right]$$

is real, well defined, and finite for $r<2M$, even at the points where $(r^2-2Mr+M^2\cos^2\theta)$ is zero or negative. Similarly, $P_{2n+1}(x)$ is equal to the product of $x$ with a polynomial in $x^2$ of order $n$, and $x^{2n+1}=x(x^2)^n$. Hence, also for all odd $k$ the above expression is real, well defined, and finite for $r<2M$.

(ii) The integral (3.12) can be put into the form

$$\hat{\gamma}(\rho,z)=\int_C\left[\left[\hat{\psi}_{,\rho}{}^2-\hat{\psi}_{,z}{}^2+\frac{2}{\alpha^2-1}(\alpha_{,\rho}\hat{\psi}_{,\rho}-\alpha_{,z}\hat{\psi}_{,z})\right]d(\tfrac{1}{2}\rho^2)+2\left[\rho\hat{\psi}_{,\rho}\hat{\psi}_{,z}+\frac{1}{\alpha^2-1}(\alpha_{,z}\rho\hat{\psi}_{,\rho}+\rho\alpha_{,\rho}\hat{\psi}_{,z})\right]dz\right], \tag{3.16}$$

where $\alpha$ is defined in Eq. (3.5) and the contour $C$ is as in Eq. (3.12). Morover, by Eq. (3.15) and because of the relation

$$\rho^2+(z\pm M)^2=[r-M(1\mp\cos\theta)]^2\ , \tag{3.17}$$

all of the expressions $\alpha_{,z}$, $\rho\alpha_{,\rho}$, $\alpha_{,\rho}\hat{\psi}_{,\rho}$, and $\alpha_{,z}\hat{\psi}_{,z}$ as well as the expressions $\rho\hat{\psi}_{,\rho}$, $\hat{\psi}_{,z}$, $\hat{\psi}_{,\rho}{}^2$, and $\hat{\psi}_{,z}{}^2$ which appear in Eq. (3.16) are well defined and real throughout the region $r<2M$.

(iii) By Eq. (3.17) and because of the fact that $\alpha^2-1\neq 0$ at all points in the interior region $r<2M$, all improper integrals which are involved in the evaluation of $\hat{\gamma}(\rho,z)$ [Eq. (3.16)] are convergent; and thus the integral (3.16) yields $\hat{\gamma}$ as a well defined and real function of $\rho^2$ and $z$.

Now that we have an infinite-parameter family of interior solutions described by Eqs. (3.13)—(3.15), we can turn to the construction of the corresponding family of colliding plane-wave spacetimes. This construction proceeds in exact parallel to Sec. II, where we constructed the colliding plane-wave solution (2.4) starting from the interior Schwarzschild metric (2.1). In fact, the interior Schwarzschild solution is the special case of the family of solutions (3.13)—(3.15) for which all of the parameters $d_k$ are zero.

We build our infinite-parameter family of colliding plane-wave solutions by the following steps.

(i) We apply the coordinate transformation (2.2) to the generalized interior metric (3.13) whose metric coefficients are defined by Eqs. (3.14) and (3.15).

(ii) We introduce two length scales $a$ and $b$ into the resulting metric by defining $u=u'/a$ and $v=v'/b$ where $ab=4M^2$. We then redefine $u'$ as $u$ and $v'$ as $v$. We also redefine our parameters $d_k$ so that the new $d_k$ are equal to the dimensionless quantities $M^k d_k$.

(iii) We then extend the resulting interaction-region metric across the wave fronts $\{u=0\}$ and $\{v=0\}$ by the Penrose prescription; i.e., we replace $u/a$ by $(u/a)H(u/a)$, and $v/b$ by $(v/b)H(v/b)$.

(iv) The resulting metric on the interaction region is locally isometric to the generalized interior metric (3.13). However, as a result of the above extension and the coordinate transformation (2.2), we change the topology of our solution from $S^2\times R^2$ [which is the topology of the manifold on which the metric (3.13) is defined], to $R^4$ (which is the topology of our maximal colliding plane-wave spacetime, see Sec. II for details).

For each choice of the parameters $\{d_k\}$, the above construction yields a unique colliding plane-wave solution. In the following equations we describe the metric of this solution in the interaction region (Fig. 1); the complete expression for the metric on the maximally extended spacetime is obtained by replacing each $u/a$ by $(u/a)H(u/a)$ and each $v/b$ by $(v/b)H(v/b)$ in Eqs. (3.18), (3.19), and (3.22) below:

$$g_{\rm I}=-e^{2[\hat\gamma(u,v)-\hat\psi(u,v)]}\left[1-\sin\left[\frac{u}{a}+\frac{v}{b}\right]\right]^2du\,dv+e^{2\hat\psi(u,v)}\frac{1+\sin\left[\frac{u}{a}+\frac{v}{b}\right]}{1-\sin\left[\frac{u}{a}+\frac{v}{b}\right]}dx^2$$
$$+e^{-2\hat\psi(u,v)}\left[1-\sin\left[\frac{u}{a}+\frac{v}{b}\right]\right]^2\cos^2\left[\frac{u}{a}-\frac{v}{b}\right]dy^2\,, \tag{3.18}$$

where

$$\hat\psi(u,v)=\sum_{k=0}^{\infty}d_k\left[\sin^2\left[\frac{u}{a}+\frac{v}{b}\right]-\cos^2\left[\frac{u}{a}-\frac{v}{b}\right]\right]^{k/2}P_k\left[\frac{\sin\left[\frac{u}{a}+\frac{v}{b}\right]\sin\left[\frac{u}{a}-\frac{v}{b}\right]}{\left[\sin^2\left[\frac{u}{a}+\frac{v}{b}\right]-\cos^2\left[\frac{u}{a}-\frac{v}{b}\right]\right]^{1/2}}\right], \tag{3.19}$$

$$\hat\psi(\rho,z)=\sum_{k=0}^{\infty}d_k(\tfrac{1}{4}ab)^{-k/2}(\rho^2+z^2)^{k/2}\,P_k\left[\frac{z}{\sqrt{\rho^2+z^2}}\right], \tag{3.20}$$

and $\hat\gamma(u,v)$ is evaluated (i) by inserting Eq. (3.20) into the integral given in Eq. (3.16) where the contour $C$ is as in Eq. (3.12) and where the function $\alpha(\rho,z)$ is given by

$$\alpha(\rho,z)=\frac{1}{\sqrt{ab}}\{[\rho^2+(z-\tfrac{1}{2}\sqrt{ab}\,)^2]^{1/2}+[\rho^2+(z+1/2\sqrt{ab}\,)^2]^{1/2}\}\,, \tag{3.21}$$

and (ii) by formally substituting

$$\rho^2\equiv-\frac{ab}{4}\cos^2\left[\frac{u}{a}+\frac{v}{b}\right]\cos^2\left[\frac{u}{a}-\frac{v}{b}\right], \tag{3.22a}$$

$$z\equiv-\frac{\sqrt{ab}}{2}\sin\left[\frac{u}{a}+\frac{v}{b}\right]\sin\left[\frac{u}{a}-\frac{v}{b}\right], \tag{3.22b}$$

into $\hat\gamma(\rho,z)$, which is a smooth function of $\rho^2$ and $z$.

The functions $\hat\gamma(u,v)$ and $\hat\psi(u,v)$ are smooth functions throughout the interaction region I (Fig. 3), and generically, the metric (3.18) has a curvature singularity at $(u/a)+(v/b)=\pi/2$.

The proof that the colliding plane-wave spacetimes constructed above are genuine solutions (in the sense of distributions) to the vacuum Einstein equations is provided by exactly the same arguments with which we have shown the solution (2.4) of Sec. II to be a genuine vacuum solution in the distribution sense. The crucial observation to note in this regard is that the metric function $U(u,v)$ [Eq. (2.8)] for any of the solutions in the above family (3.18)—(3.22) is given by precisely the same expression [Eq. (2.9b)] as the corresponding function for the solution (2.4) of Sec. II.

For completeness, we conclude this section by describing the interaction region I of our solutions in an alternative coordinate system defined by

$$u'/a=\sin(u/a)\,,\quad v'/b=\sin(v/b)\,,$$
$$x'=x\,,\quad y'=y\,. \tag{3.23}$$

In the following, we omit the primes over the new coordinate functions. The interaction-region metric for the family of solutions (3.18)—(3.22) is expressed below in the new coordinates (3.23). The extension of the metric beyond the interaction region is again accomplished by the substitutions $u/a \to (u/a)H(u/a)$ and $v/b \to (v/b)H(v/b)$, and these substitutions result in a colliding plane-wave spacetime globally isometric to the corresponding spacetime (3.18)—(3.22) [even though the coordinate transformation (3.23) does not hold outside region I]. Thus, the following region-I expressions produce exactly the same family of colliding plane-wave solutions as above, written in the new coordinates (3.23):

$$g_{\rm I} = -e^{2(\hat\gamma-\hat\psi)} \frac{\left[1-\frac{u}{a}\left[1-\frac{v^2}{b^2}\right]^{1/2}-\frac{v}{b}\left[1-\frac{u^2}{a^2}\right]^{1/2}\right]^2}{\left[1-\frac{u^2}{a^2}\right]^{1/2}\left[1-\frac{v^2}{b^2}\right]^{1/2}} du\,dv + e^{2\hat\psi}\frac{1+\frac{u}{a}\left[1-\frac{v^2}{b^2}\right]^{1/2}+\frac{v}{b}\left[1-\frac{u^2}{a^2}\right]^{1/2}}{1-\frac{u}{a}\left[1-\frac{v^2}{b^2}\right]^{1/2}-\frac{v}{b}\left[1-\frac{u^2}{a^2}\right]^{1/2}}dx^2$$

$$+e^{-2\hat\psi}\left[1-\frac{u}{a}\left[1-\frac{v^2}{b^2}\right]^{1/2}-\frac{v}{b}\left[1-\frac{u^2}{a^2}\right]^{1/2}\right]^2\left[\left[1-\frac{u^2}{a^2}\right]^{1/2}\left[1-\frac{v^2}{b^2}\right]^{1/2}+\frac{uv}{ab}\right]^2 dy^2 , \tag{3.24}$$

where

$$\hat\psi(u,v)=\sum_{k=0}^{\infty} d_k \left\{\left[\frac{u}{a}\left[1-\frac{v^2}{b^2}\right]^{1/2}+\frac{v}{b}\left[1-\frac{u^2}{a^2}\right]^{1/2}\right]^2-\left[\left[1-\frac{u^2}{a^2}\right]^{1/2}\left[1-\frac{v^2}{b^2}\right]^{1/2}+\frac{uv}{ab}\right]^2\right\}^{k/2}$$

$$\times P_k\left[\frac{\frac{u^2}{a^2}-\frac{v^2}{b^2}}{\left\{\left[\frac{u}{a}\left[1-\frac{v^2}{b^2}\right]^{1/2}+\frac{v}{b}\left[1-\frac{u^2}{a^2}\right]^{1/2}\right]^2-\left[\left[1-\frac{u^2}{a^2}\right]^{1/2}\left[1-\frac{v^2}{b^2}\right]^{1/2}+\frac{uv}{ab}\right]^2\right\}^{1/2}}\right] , \tag{3.25}$$

and $\hat\gamma(u,v)$ is evaluated (i) by inserting Eq. (3.20) into the integral given in Eq. (3.16), where the contour $C$ is as in Eq. (3.12) and where $\alpha(\rho,z)$ is given by Eq. (3.21), and (ii) by formally substituting

$$\rho^2 \equiv -\tfrac{1}{4}ab[1-(u^2/a^2)-(v^2/b^2)]^2 , \tag{3.26a}$$

$$z \equiv -\tfrac{1}{2}\sqrt{ab}\,[(u^2/a^2)-(v^2/b^2)] , \tag{3.26b}$$

in $\hat\gamma(\rho,z)$ which is a smooth function of $\rho^2$ and $z$. The curvature singularity, which, in the generic case, constitutes an achronal future $c$-boundary in the interaction region of the solution (3.24), is located at

$$\{(u/a)[1-(v^2/b^2)]^{1/2}+(v/b)[1-(u^2/a^2)]^{1/2}=1\}$$

in the new coordinate system (3.23).

## IV. EXAMPLES

As we have noted before, when all parameters $d_k$ are zero, the general solution (3.18)—(3.22) reduces to the solution (2.4) of Sec. II which was obtained from the interior Schwarzschild metric. From now on, we will denote by the symbol $\{d_k\}$ the unique colliding plane wave solution (3.18)—(3.22) which corresponds to a given choice of the parameters $d_k$. When all $d_k$ are zero except for the parameter $d_0$, the solution $\{d_k\}=\{d_0,0,0,0,\dots\}$ is again equal to (2.4), except in this case the mass $M=\sqrt{ab}/2$ (and the boost-invariant product $\sqrt{ab}$ of the characteristic wavelengths) is rescaled by a factor $e^{-d_0}$ corresponding to a monopolar distortion of the solution (2.4).

To illustrate the evaluation of the function $\hat\gamma(u,v)$ by Eqs. (3.20)—(3.22), we write down below the functions $\hat\psi_2(u,v)$ and $\hat\gamma_2(u,v)$ corresponding to the solution $\{0,0,1,0,0,\dots\}$, where all $d_k$ are zero except for $d_2=1$. The metric in the interaction region of this solution is locally isometric to an interior Weyl metric (3.13); this Weyl solution can be interpreted as the interior metric of a Schwarzschild black hole distorted by a static, quadrupolar matter distribution outside the horizon:

$$\hat\psi_2(u,v)=\frac{1}{2}\left[3\sin^2\left[\frac{u}{a}+\frac{v}{b}\right]\sin^2\left[\frac{u}{a}-\frac{v}{b}\right]+\cos^2\left[\frac{u}{a}-\frac{v}{b}\right]-\sin^2\left[\frac{u}{a}+\frac{v}{b}\right]\right] , \tag{4.1}$$

$$\hat\gamma_2(u,v)=\frac{4}{a^2b^2}(q^2-8qz^2)-\frac{2}{\sqrt{ab}}[I_1(q;z^2-M^2,2(z^2+M^2),(z-M)^2,(z^2-M^2)^2)$$

$$+I_1(q;z^2-M^2,2(z^2+M^2),(z+M)^2,(z^2-M^2)^2)]$$

$$-\frac{4}{\sqrt{ab}}z[(z-M)I_2(q;z^2-M^2,2(z^2+M^2),(z-M)^2,(z^2-M^2)^2)$$
$$+(z+M)I_2(q;z^2-M^2,2(z^2+M^2),(z+M)^2,(z^2-M^2)^2)]\ , \tag{4.2}$$

where

$$M\equiv\tfrac{1}{2}\sqrt{ab}\ ,$$
$$q\equiv-\frac{ab}{4}\cos^2\left[\frac{u}{a}+\frac{v}{b}\right]\cos^2\left[\frac{u}{a}-\frac{v}{b}\right],$$
$$z\equiv-\frac{\sqrt{ab}}{2}\sin\left[\frac{u}{a}+\frac{v}{b}\right]\sin\left[\frac{u}{a}-\frac{v}{b}\right], \tag{4.3}$$
$$I_1(q;a,b,c,d)\equiv\int_0^q\frac{s\,ds}{\sqrt{s+c}\,(s+a+\sqrt{s^2+bs+d}\,)}\ ,$$
$$I_2(q;a,b,c,d)\equiv\int_0^q\frac{ds}{\sqrt{s+c}\,(s+a+\sqrt{s^2+bs+d}\,)}\ .$$

We now turn to our second example of a colliding plane-wave solution in the family (3.18)—(3.22): a solution which describes colliding purely anastigmatic plane waves.[7-9] According to Eqs. (3.18)—(3.22), the metric $g_{\rm II}$ on the region II (Fig. 3) of a solution $\{d_k\}$ is given by

$$g_{\rm II}=-e^{[\hat\gamma_{\rm II}(u)-\hat\psi_{\rm II}(u)]}[1-\sin(u/a)]^2du\,dv$$
$$+e^{2\hat\psi_{\rm II}(u)}\left[\frac{1+\sin(u/a)}{1-\sin(u/a)}\right]dx^2$$
$$+e^{-2\hat\psi_{\rm II}(u)}[1-\sin(u/a)]^2\cos^2(u/a)dy^2\ , \tag{4.4}$$

where

$$\hat\psi_{\rm II}(u)=\sum_{k=0}^{\infty}d_k(2t-1)^{k/2}P_k\left[\frac{t}{\sqrt{2t-1}}\right], \tag{4.5a}$$
$$t\equiv\sin^2(u/a) \tag{4.5b}$$

The metric (4.4) and (4.5) describes the geometry of one of the two incoming colliding plane waves in the solution $\{d_k\}$. As before, the metric $g_{\rm III}$ in region III (Fig. 3) (which describes the remaining incoming wave) is obtained by replacing $u/a$ by $v/b$ in the above equations.

Unfortunately, the polynomials $(2t-1)^{k/2}P_k(t/\sqrt{2t-1})$ are not orthogonal polynomials with respect to any weight function, since they fail the Darboux-Christoffel test (Ref. 14, Sec. 8.90). However, we shall construct one particular infinite sequence $(\bar d_k)$ [Eqs. (4.11) below], such that for the corresponding solution $\{\bar d_k\}$ the function $\hat\psi_{\rm II}(u)$ [Eqs. (4.5)] has the right asymptotic behavior as $u\to\pi a/2$ to make the incoming plane wave in region II [Eq. (4.4)] purely anastigmatic.[7-9] Clearly, because of the $u\leftrightarrow v$ symmetry of our solutions, with this choice of the parameters $d_k$ the other incoming plane wave (which is represented by the metric $g_{\rm III}$ on region III) will also be purely anastigmatic. Also note that Eqs. (4.11) represent only one particular solution in our family of solutions for which the incoming plane waves are purely anastigmatic; the details of the construction below will make it clear that infinitely many different solutions $\{d_k\}$ with the same property can be found in the family (3.18)–(3.22).

To proceed, consider the following function $f(t)$, defined on the interval $(-1,1)$:

$$f(t)=\ln(1-t)\quad\text{for } t\geq 0\ ,$$
$$f(t)=\ln(1+t)\quad\text{for } t\leq 0\ . \tag{4.6a}$$

Since $f(t)$ is even, there is an expansion

$$f(t)=\sum_{k=0}^{\infty}\hat d_kP_{2k}(t)\ . \tag{4.6b}$$

Since $f(t)$ is square integrable on $(-1,1)$, this expansion converges absolutely everywhere on $(-1,1)$ with the exception of the point $t=0$ at which $f(t)$ is not $C^1$. In fact,

$$\hat d_k=\frac{4k+1}{2}\int_{-1}^{1}f(t)P_{2k}(t)dt$$
$$=(4k+1)\int_0^1\ln(1-t)P_{2k}(t)dt\ . \tag{4.7}$$

Now consider the solution $\{\bar d_k\}$, where $\bar d_k$ are defined by

$$\bar d_k=\hat d_{k/2}\quad\text{for } k \text{ even}\ ,$$
$$\bar d_k=0\quad\text{for } k \text{ odd}\ . \tag{4.8}$$

Then, the function $\hat\psi_{\rm II}(u)$ [Eqs. (4.5)] for this solution $\{\bar d_k\}$ is given by

$$\hat\psi_{\rm II}(u)=\sum_{k=0}^{\infty}\hat d_k(2t-1)^kP_{2k}\left[\frac{t}{\sqrt{2t-1}}\right], \tag{4.9}$$

where $t\equiv\sin^2(u/a)$. However, for $t\in(0,1)$ and for all $k\geq1$ we have[14]

$$\left|(2t-1)^kP_{2k}\left[\frac{t}{\sqrt{2t-1}}\right]\right|\ <\ 1\ =\ \sup_{t\in(0,1)}|P_{2k}(t)|\ .$$

Therefore, the series (4.9) converges absolutely to a continuous function on the interval (0,1). We can write

$$\sum_{k=0}^{\infty}\hat d_k(2t-1)^kP_{2k}\left[\frac{t}{\sqrt{2t-1}}\right]=\sum_{k=0}^{\infty}\hat d_kP_{2k}(t)$$
$$+\sum_{k=1}^{\infty}\delta_k(1-t)^k\ ,$$

where the second series is convergent being the difference of two convergent series on (0,1). Hence, by Eq. (4.9) and Eqs. (4.6)

$$\hat\psi_{\rm II}(u)=\ln\left[1-\sin^2\frac{u}{a}\right]+S(u)\ ,$$

where

$$\lim_{u\to\pi a/2} S(u)=0 \ .$$

We thus obtain the following for the asymptotic behavior of $\hat{\psi}_{\rm II}(u)$ as $u\to\pi a/2$:

$$e^{2\hat{\psi}_{\rm II}(u)} \underset{u\to\pi a/2}{\sim} \left[1-\sin\frac{u}{a}\right]^2 \left[1+\sin\frac{u}{a}\right]^2 , \tag{4.10a}$$

$$e^{-2\hat{\psi}_{\rm II}(u)} \underset{u\to\pi a/2}{\sim} \left[1-\sin\frac{u}{a}\right]^{-2} \left[1+\sin\frac{u}{a}\right]^{-2} . \tag{4.10b}$$

The asymptotic behavior of the function $\hat{\psi}_{\rm III}(v)$ [which is the counterpart of $\hat{\psi}_{\rm II}(u)$ in region III of the solution $\{\bar{d}_k\}$] will have the analogous form near the focal plane $\{v=\pi b/2\}$. From Eqs. (4.10) we conclude, by inspecting the metric $g_{\rm II}$ (and $g_{\rm III}$) given by Eq. (4.4), that for our solution $\{\bar{d}_k\}$ both incoming plane waves are purely anastigmatic, i.e., for both incoming plane waves the metric coefficients $g_{xx}$ and $g_{yy}$ vanish simultaneously on the respective focal planes $\{u=\pi a/2\}$ and $\{v=\pi b/2\}$.

The coefficients $\bar{d}_k$ for the solution $\{\bar{d}_k\}$ can be calculated explicitly using Eqs. (4.7) and (4.8). This gives[14]

$$\bar{d}_{2k}=\hat{d}_k=-\frac{4k+1}{2^{2k}} \times\sum_{l=0}^{k}\frac{(-1)^l(4k-2l)!}{(2k-2l+1)l!(2k-l)!(2k-2l)!} \times[\psi(2k-2l+2)-\psi(1)] \ , \tag{4.11a}$$

where

$$\psi(x)=\frac{d}{dx}[\ln\Gamma(x)]$$

is Euler's psi function,[14] and

$$\bar{d}_{2k+1}=0 \tag{4.11b}$$

for any $k\geq 0$.

## V. CONCLUSIONS

The infinite-parameter family of colliding plane-wave solutions we have constructed in Sec. III have the following new features.

(i) The interaction regions of our solutions are locally isometric to interior Weyl black-hole solutions. An observer who enters the interaction region will not be able to distinguish, by local measurements that he performs completely within the interaction region, the geometry of the surrounding spacetime from the geometry in the interior of a black hole.

(ii) The metric functions of our solutions have oscillatory forms in a suitable coordinate system [Eqs. (3.18)—(3.22)].

(iii) By constructing an infinite series expansion for the function $\hat{\psi}(u,v)$ [Eq. (3.19)], we have built a colliding plane-wave solution in our family for which both of the two incoming colliding plane waves are purely anastigmatic, i.e., for which both incoming waves have equal focal lengths (Sec. IV).

On the other hand, our solutions suffer from the following drawbacks, some of which are common to all presently known exact solutions for colliding plane waves.

(i) As with the famous Khan-Penrose solution,[5] so also here, there are δ-function contributions to the Riemann curvature (gravitational shock waves) on the boundaries between the adjacent regions (Fig. 3); i.e., the metric is not $C^1$. The reason for this discontinuous behavior is the particular prescription that we use to extend the metric beyond the interaction region. It is clear, from the form of our metric as described by Eqs. (3.18)—(3.22), that no finite sum (3.19) will eliminate the δ-function shocks across the boundaries as long as we use the Penrose prescription for extending the metric beyond the interaction region. Since infinite sums of the form we have discussed in Sec. IV will in general converge only in the mean (i.e., in the $L^2$ sense), we cannot reliably employ infinite series expansions of the form (3.19) to construct smoother wave forms.

(ii) Except for the characteristic wavelengths $a$ and $b$ which can be freely adjusted by scale transformations, the two incoming colliding plane waves in our solutions have exactly the same functional form. The reasons for this are the $u\leftrightarrow v$ symmetry of the metric (3.18) in the interaction region, and the $u\leftrightarrow v$ symmetry of the Penrose prescription for extending the metric beyond the interaction region.

(iii) The incoming colliding plane waves in our solutions are not of sandwich type in general; i.e., the spacetime regions II and III (Fig. 3) representing the incoming waves are not flat near the respective focal planes of these waves. As we have discussed in detail in Sec. II, this property is responsible for the curvature singularities that are present at the focal planes of our solutions. The technique of "cutting off" the incoming waves just before their focal planes, which we have discussed in Sec. II, successfully avoids this difficulty from a physical viewpoint; however, it appears exceedingly difficult to determine whether the resulting solution (e.g., the solution depicted in Fig. 4) can be expressed in closed form as a member of the family of solutions (3.18)—(3.22) which we have constructed.

We conclude by listing some open questions which suggest directions for further research on some of the issues that we have raised in this paper.

(i) Are there different prescriptions for extending the metric (3.18) beyond the interaction region which could resolve some of the drawbacks in our solutions listed above?

(ii) Can the technique of using static axisymmetric black-hole metrics to generate parallel-polarized colliding plane-wave spacetimes be generalized to stationary axisymmetric solutions? Such a generalization presumably would yield an infinite-parameter family of solutions representing colliding plane waves with nonparallel polarizations.

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