Membrane viewpoint on black holes: Gravitational perturbations of the horizon

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This paper is part of a series of papers which develop and illustrate the "membrane" formalism for black holes. In this formalism, described in earlier papers, the role of the absolute event horizon is played by a two-dimensional surface endowed with electrical, mechanical, and thermodynamic properties. The present paper deals with gravitational perturbations of holes and presents and discusses model problems that illustrate the effects of tidal gravitational fields on stationary holes. The first of these problems demonstrates the use of the Rindler approximation to Schwarzschild spacetime in the case of a static perturbing point mass and clarifies the contribution to the horizon distortion due to forces constraining the motion of the mass point. Subsequent model problems use the Rindler approximation to compute the evolution of distortions of the Schwarzschild or Kerr horizon due to mass points in motion near the horizon.

I. INTRODUCTION

For general theorems governing black holes and for the study of highly dynamical black holes the covariant four-dimensional treatment provides the best viewpoint on black-hole spacetimes. Recently, however, interest has intensified in the role played by black holes in astrophysical processes. In most of these processes the black holes are not highly dynamical; they change on a time scale long compared to the black-hole characteristic response time. In the study of such astrophysical problems the full four-dimensional spacetime viewpoint is neither necessary nor efficient. What is often needed is a viewpoint on black holes to which intuition and experience with less exotic astrophysical objects can easily be applied.

A major step towards such a viewpoint was the work by Damour[1–3] which reformulated conditions on the horizon in terms of electrical, mechanical, and thermodynamic properties ascribed to the horizon. (Znajek,[4] independently, has arrived at many of the same results in a somewhat different formulation.) These insights, however, deal only with the horizon and do not provide a formalism for connecting the physics of the horizon to that outside the horizon. Moreover, for the application of intuition, the use of the horizon as the surface of the black hole has disadvantages, partly because it is a null hypersurface and partly because of its very nature as a horizon.

This paper is part of a series[5–9] in which a complete and coherent formalism is developed for dealing with black holes. This formalism is mathematically equivalent to a fully covariant four-dimensional treatment of black-hole horizons and exteriors, but provides a viewpoint which should be much more useful for dealing with astrophysical problems involving slowly changing black holes. The starting point of this formalism is a $3+1$ split of spacetime into a "universal time" and a three-dimensional curved "absolute space." Matter and fields outside the hole are viewed as existing in absolute space and evolving in universal time. The effects of the hole are represented by a two-dimensional surface which evolves in universal time and is viewed as endowed with the physical properties ascribed to the horizon by Damour. The world tube of this surface is not taken to be the horizon but rather to be a timelike surface, the "stretched horizon," just outside (in a well-defined sense) the horizon. It is appropriate, and turns out to be useful, to think of this surface as a real physical membrane whose properties can account for the interaction of the hole with its environment.

This membrane formulation of black-hole physics is being presented in a number of works. In the first of these, Thorne and Macdonald[5] developed the $3+1$ split for electromagnetism and in a subsequent paper[6] used it to analyze the magnetospheres of black holes and the Blandford-Znajek[10] process in active galactic nuclei. In the third paper Macdonald and Suen[7] presented idealized model problems that illustrate the usefulness of the membrane viewpoint for dynamic electromagnetic fields near the horizon. In the fourth paper Price and Thorne[8] (PT) extended the membrane formalism to gravitational interactions, and developed the general theory underlying the formalism for dynamical horizons. A pedagogical review of the formalism is presented in a book-length treatise by Thorne *et al.*[9]

The goal of the present paper is to illustrate with model problems the use of this formalism in dealing with perturbing tidal gravitational fields near the horizon and, with these model problems, to provide a basis for intuition that can be applied to more complicated, but more astrophysically relevant, tidal interactions with the horizon. This paper then is meant to serve, for gravitational interactions, much the same purpose as the paper by Macdonald and Suen[7] serves for electromagnetic interactions. The model problems, presented in some detail below, also provide a basis for the more wide-ranging and more pedagogical review of horizon dynamics presented in the treatise by Thorne *et al.*[9]

Several major complications arise in dealing with gravitational, as compared with electromagnetic, perturbations. One problem, familiar in any approach to gravitational perturbations, is that of gauge arbitrariness; it will be shown that in the membrane formalism the effects of gauge choice on horizon dynamics are minimal, and its meaning is well understood. (For an exhaustive discussion of gauge effects in the formalism see PT.)

A second problem is that of solving for the fields outside the horizon. For electromagnetic perturbations of a stationary black hole the task of describing exterior fields is simplified intuitively and calculationally by the fact that many of the familiar flat-spacetime concepts (e.g., lines of force) and equations (e.g., Gauss's law, Faraday's law, Ampere's law) remain valid with only minor modification in the $3+1$ split.[5–7] For tidal perturbations no analogous simplification applies. In the case of perturbations of the Kerr horizon it turns out, however, that the only exterior tidal field that must be known is that component of the Weyl tensor described by the Teukolsky equation.[8,11] The technical difficulties of dealing with that equation would nonetheless detract from the intuitive insights which are the goal of the present paper. Instead, we arrive at tractable model problems by confining our problems to sources which are very near the horizon. This choice allows us to exploit the simplicity of the Rindler geometry[12] as an approximation to the near-horizon geometry, an approximation also used extensively by Macdonald and Suen.[7]

A third new complication that arises in dealing with gravitational perturbations is that of constraining forces. A compact source of perturbations, most conveniently a point source, allows the clearest illustrations of the relationship of a source and its effects on the horizon. For an electromagnetic point source (i.e., a point charge) the motion of the source presents no difficulty in principle; the motion is chosen arbitrarily and one simply assumes that the constraining forces driving the motion are nonelectromagnetic, and hence make no contribution to the electromagnetic perturbations of the horizon. For gravitational perturbations this simplification does not apply. Unless we limit ourselves to particles in free-fall, the forces of constraint must themselves in general be sources of tidal distortions of the horizon. We find, however, that by taking our constraining forces to be supplied by highly idealized "ropes" or "struts" the effects of constraints on the horizon are minimized, and the horizon distortions may be thought of for most purposes as due to the point particles themselves.

This paper is organized as follows: In Sec. II the basic formalism is introduced. The formalism for dealing with tidal interactions is developed in detail in PT, but in that paper no explicit prescription is given for analyzing perturbative sources of tidal distortion. The aim of Sec. II is to present such a prescription for perturbations of the Kerr geometry. Details of the formalism are presented only to the extent that they are necessary for this purpose; for other details and for proofs the relevant sections of PT are cited. Section II A gives a description of horizon kinematic quantities and the equations governing their evolution. Section II B then specializes and simplifies these equations for the case of the perturbations of the Kerr geometry. In Sec. II C the $3+1$ split is outlined and the kinematics of the horizon are shown to be equivalent to the kinematics of particles ("fiducions") in the stretched horizon. Section II D very briefly sketches the fluid-mechanical paradigm in which the equations governing horizon distortions are reinterpreted in terms of the motions of a two-dimensional fluid membrane located at the stretched horizon. The Rindler approximation[12] for the near-horizon spacetime geometry is introduced in Sec. II E and the prescription for analyzing perturbations is summarized.

In Sec. III we analyze the deformation of the Schwarzschild horizon by a point particle statically suspended near the horizon. This analysis is performed in two independent ways: (i) with the Weyl formalism[13] for static axisymmetric spacetimes, and (ii) with the Rindler[12] approximation to Schwarzschild spacetime. Section III (since it deals with a static perturbation) is not meant primarily as an illustration of the formalism for dealing with the dynamics of horizon distortions but, rather, establishes the range of validity of the Rindler approximation, and demonstrates that idealized constraining "ropes" can be considered as having a minimal tidal effect on the horizon. With these points established, Sec. IV presents models of horizon distortions arising from three different types of near-horizon particle motion: (i) a particle radially accelerating away from the horizon; (ii) a particle moving slowly parallel to the horizon; (iii) a particle in free-fall through the horizon.

For the most part in this paper we shall use the notational conventions of Misner, Thorne, and Wheeler[14] (MTW); in particular we take c and G to be unity and we use the metric signature $-+++$. Spacetime indices $\alpha, \beta, \mu, \nu, \ldots$ will run through 0–3. Indices from the middle of the lower-case latin alphabet $i, j, k, \ldots$ will run through 1–3. Indices $a, b, c, \ldots$ from the early part of the lower-case latin alphabet take the values 2, 3 and denote a two-dimensional spatial section of the stretched horizon at fixed universal time. Dyadic notation (e.g., $\overleftrightarrow{\sigma}$, $\overleftrightarrow{\mathcal{E}}$) will be used for symmetric second-rank tensors both in two and three dimensions. As in PT we denote a horizon quantity with an index H, used as a superscript or subscript with no difference in meaning.

II. TIDAL DISTORTIONS OF THE HORIZON

A. Horizon kinematics and dynamics

To describe a horizon we start by choosing a time coordinate $\bar{t}$ which is well behaved on the horizon and increases to the future but (for the moment) is otherwise arbitrary. (The simplest example is ingoing Eddington-Finkelstein time on the Schwarzschild horizon; see Box 31.2 of MTW.) We define $\vec{l}$ to be the (null) tangent to the generators of the horizon and normalize $\vec{l}$ such that

$$\langle d\bar{t}, \vec{l}\,\rangle = 1 \ . \tag{2.1}$$

On the two-dimensional spacelike section of the horizon at constant $\bar{t}$, we choose, for convenience, spatial coordinates $x^a(=x^2,x^3)$ which are comoving, i.e., which are constant on the horizon generators, implying $\vec{l}=(\partial/\partial\bar{t})_{x^a}$. We denote by $\vec{e}_a$ the basis vectors in the horizon section and, again for convenience, choose them to be coordinate basis vectors $\vec{e}_a=(\partial/\partial x^a)_{\bar{t}}$. We denote by γ_{ab} the metric in the two-dimensional horizon section:

$$\gamma_{ab}=\vec{e}_a\cdot\vec{g}\cdot\vec{e}_b$$

(where $\vec{g}$ is the spacetime metric confined to the horizon). To define all quantities entering into the description below we shall need, in addition, an outward-directed null vector $\vec{n}$ which is completely fixed (on the horizon) by the conditions that it is normal to horizon sections ($\vec{n}\cdot\vec{e}_a=0$) and satisfies

$$\vec{l}\cdot\vec{n}=-1 \ . \tag{2.2}$$

The description of tidal distortions involves the horizon surface gravity g_H, shear σ^H_{ab}, expansion θ^H, and "Hajicek field."[15] The first of these is defined as the proportionality constant in

$$\nabla_{\vec{l}}\,\vec{l}=g_H\vec{l} \ , \tag{2.3}$$

where the form of the equation follows from the fact that $\vec{l}$ is tangent to a geodesic. For stationary or nearly stationary holes g_H gives a measure of the "pull of gravity" near the horizon (see Sec. II C below). Horizon expansion and shear are defined as the trace and the trace-free part of the gradient of $\vec{l}$ in the horizon section:

$$\theta^H\equiv\gamma^{ab}\theta^H_{ab} \ , \tag{2.4a}$$

$$\sigma^H_{ab}\equiv\theta^H_{ab}-\tfrac{1}{2}\gamma_{ab}\theta^H \ , \tag{2.4b}$$

with

$$\theta^H_{ab}\equiv(\nabla_a\vec{l})\cdot\vec{e}_b \ . \tag{2.4c}$$

(Since the generator congruence is hypersurface orthogonal, its rotation vanishes and θ^H_{ab}, hence σ^H_{ab}, is symmetric.) In comoving coordinates horizon shear and expansion are simply related to the time rate of change of γ_{ab} and its determinant γ:

$$\partial\gamma_{ab}/\partial\bar{t}=2\sigma^H_{ab}+\theta^H\gamma_{ab} \tag{2.5a}$$

or

$$\theta_H=\frac{1}{2}\frac{\partial}{\partial\bar{t}}\ln\gamma \ , \qquad \sigma^H_{ab}=\frac{1}{2}\left[\frac{\partial\gamma_{ab}}{\partial\bar{t}}-\theta^H\gamma_{ab}\right] . \tag{2.5b}$$

The horizon expansion and shear have simple geometrical meaning.[16] Consider a set of generators which occupy area $\Delta\mathcal{A}^H$ on an initial horizon section at time $\bar{t}$. A differential time $d\bar{t}$ later (i.e., on the section at $\bar{t}+d\bar{t}$) the generators will cover an area which is greater by

$$d(\Delta\mathcal{A}^H)=\theta^H\Delta\mathcal{A}^H d\bar{t} \ . \tag{2.6a}$$

To understand the geometric meaning of $\vec{\sigma}^H$ consider generators which form a small circle of diameter D_0 on an initial section and introduce orthonormal basis vectors $\vec{e}_{\hat{2}}$, $\vec{e}_{\hat{3}}$ into the section such that $\sigma^H_{\hat{2}\hat{3}}$ vanishes. After a time $d\bar{t}$ the diameter $D_{\hat{2}}$ in the $\hat{2}$ direction, and the diameter $D_{\hat{3}}$ in the $\hat{3}$ direction will change according to

$$\frac{1}{D_0}\frac{d(D_{\hat{2}}-D_{\hat{3}})}{d\bar{t}}=\sigma^H_{\hat{2}\hat{2}}-\sigma^H_{\hat{3}\hat{3}}=2\sigma^H_{\hat{2}\hat{2}}=-2\sigma^H_{\hat{3}\hat{3}} \ . \tag{2.6b}$$

The Hajicek field is defined by

$$\Omega^H_a\equiv-\vec{n}\cdot\nabla_a\vec{l} \ . \tag{2.7}$$

This horizon quantity lacks the simple interpretations of g_H, θ^H, and $\vec{\sigma}^H$ and is most easily understood geometrically in terms of extrinsic curvature of the horizon. (See Sec. II of PT.) Though less familiar than the other horizon quantities it plays an important role in the membrane paradigm described in Sec. II D below.

The quantities θ^H, $\vec{\sigma}^H$, and $\vec{\Omega}^H$ are to be viewed as tensors in the horizon section, with indices raised and lowered by γ_{ab} and its inverse γ^{ab}. To discuss their evolution we introduce the time derivative $D_{\bar{t}}$ with the following meaning: covariant differentiation along $\vec{l}$ (i.e., $\nabla_{\vec{l}}$) followed by projection into the horizon section. In terms of this operator, the evolutions of θ^H, $\vec{\sigma}^H$, and $\vec{\Omega}^H$ are given by the "tidal force equation"

$$D_{\bar{t}}\sigma^H_{ab}+(\theta_H-g_H)\sigma^H_{ab}=-C_{a\mu b\nu}\,l^\mu l^\nu\equiv-\mathcal{E}^H_{ab} \ , \tag{2.8}$$

the "focusing equation"

$$D_{\bar{t}}\theta_H=g_H\theta^H-\tfrac{1}{2}\theta^2_H-\sigma^H_{ab}\sigma^{ab}_H-8\pi\mathcal{F}^H \ , \tag{2.9}$$

and the "Hajicek equation"

$$D_{\bar{t}}\Omega^H_a+(\sigma^{Hc}_{a}+\tfrac{1}{2}\delta^c_a\theta_H)\Omega^H_c+\theta_H\Omega^H_a =(g_H+\tfrac{1}{2}\theta_H)_{,a}-\sigma^{Hb}_{a}{}_{\|b}-8\pi\mathcal{G}^H_a \ . \tag{2.10}$$

In these equations $C_{a\mu b\nu}$ represents the Weyl curvature tensor; $\mathcal{F}^H$ and $\mathcal{G}^H_a$ are defined in terms of the stress-energy $T_{\mu\nu}$ at the horizon by

$$\mathcal{F}^H\equiv T_{\mu\nu}\,l^\mu\,l^\nu \ , \tag{2.11a}$$

$$\mathcal{G}^H_a\equiv-T_{a\mu}\,l^\mu \ ; \tag{2.11b}$$

the double vertical bars in the Hajicek equation indicate covariant differentiation with respect to the metric γ_{ab} of the horizon sections. Equations (2.8)–(2.10) and their equivalents in the membrane paradigm form, along with (2.5), the basis for the description of the evolution of tidal distortions.

B. Perturbations of Kerr horizons

The background for our perturbations will be a Kerr (or, as a special case, Schwarzschild) black hole of mass M and angular momentum $J=aM$. This black-hole spacetime, in standard Boyer-Lindquist coordinates ($x^0=t$, $x^i=r$, $\theta^\dagger$, $\phi^\dagger$), is given by

$$ds^2=-(\rho^2\Delta/\Sigma^2)dt^2+g_{jk}(dx^j+\beta^j dt)(dx^k+\beta^k dt) \ , \tag{2.12a}$$

where

$$g_{jk}dx^j dx^k = (\rho^2/\Delta)dr^2 + \rho^2 d\theta^{\dagger 2} + (\Sigma \sin\theta^\dagger/\rho)^2 d\phi^{\dagger 2} \,, \tag{2.12b}$$

$$\Delta \equiv r^2 + a^2 - 2Mr \,, \qquad \rho^2 \equiv r^2 + a^2\cos^2\theta^\dagger \,,$$
$$\Sigma^2 \equiv (r^2+a^2)^2 - a^2\Delta\sin^2\theta^\dagger \,, \tag{2.12c}$$

and

$$\beta^{\phi\dagger} = -2Mar/\Sigma^2 \,, \qquad \beta^r = \beta^{\theta\dagger} = 0 \,. \tag{2.12d}$$

The Kerr horizon is at

$$r = r_H \equiv M + (M^2 - a^2)^{1/2} \,, \tag{2.12e}$$

the larger root of $\Delta = 0$.

Although Boyer-Lindquist coordinates provide a useful description of the Kerr spacetime outside the horizon, they are not well suited to the study of perturbations of spacetime very near the horizon. We can arrive at a more convenient set of coordinates by first introducing as a new radial coordinate

$$\alpha \equiv \rho\Delta^{1/2}/\Sigma$$
$$= \left[\frac{1 - a^2\sin^2\theta^\dagger/2Mr_H}{Mr_H/(r_H - M)}\right]^{1/2} (r - r_H)^{1/2}$$
$$+ O((r - r_H)^{3/2}) \,, \tag{2.13}$$

which vanishes at the horizon and increases outward. We next introduce new angular coordinates θ', ϕ' by

$$\theta' = \theta^\dagger - \frac{\rho^2_{H,\theta^\dagger}}{4g_H^2\rho_H^4}\alpha^2 \,,$$
$$\phi' = \phi^\dagger - \Omega_H t \,. \tag{2.14}$$

Here Ω_H (not to be confused with the Hajicek field $\vec{\Omega}_H$) is the angular velocity of the horizon, with

$$\Omega_H \equiv a/2Mr_H \,, \tag{2.15}$$

and $\rho_H^2 = r_H^2 + a^2\cos^2\theta^\dagger$ is the value of ρ^2 at the horizon. In terms of these coordinates the spacetime metric takes the form

$$ds^2 = -\alpha^2 dt^2 + \frac{d\alpha^2}{g_H^2} + \rho_H^2 d\theta'^2 + \varpi_H^2(d\phi' + \beta^{\phi'} dt)^2 \,,$$
$$g_H \equiv \frac{r_H - M}{2Mr_H} \,, \qquad \rho_H^2 \equiv r_H^2 + a^2\cos^2\theta' \,,$$
$$\varpi^2 \equiv \frac{(2Mr_H)^2}{\rho_H^2}\sin^2\theta' \,, \tag{2.16}$$
$$\beta^{\phi'} \equiv \frac{\alpha^2 a}{g_H(2Mr_H)^2\rho_H^2}[\rho_H^2 r_H + M(r_H^2 - a^2\cos^2\theta')]$$

near the horizon. [For a detailed discussion of the order of the fractional errors in (2.16) see Appendix C of PT, where the notation agrees with that used here.] One advantage of the new angular variables is that, in the unperturbed spacetime, they comove with the horizon, i.e., θ' and ϕ' are constant on horizon generators.

The values of horizon quantities g_H, θ^H, $\overleftrightarrow{\sigma}^H$, and $\vec{\Omega}^H$ depend, in general, on the choice made for $\bar{t}$; i.e., they depend on the precise manner in which the horizon is sliced into $\bar{t}$=const sections. For a highly dynamical horizon there is no preferred choice of slicing, but for a static or stationary hole there exists a strongly preferred choice which uniquely fixes g_H, θ^H, $\overleftrightarrow{\sigma}^H$, and $\vec{\Omega}^H$. (See Appendix B of PT for details.) In particular, for the Kerr horizon the preferred slicing is

$$\bar{t} = t + \left[\frac{Mr_H}{r_H - M}\right]\ln\alpha^2 \,, \tag{2.17}$$

and leads to the value of g_H given in Eq. (2.16) and to the Hajicek field

$$\vec{\Omega}_H = -\frac{a}{(2Mr_H)^2\rho_H^2}[\rho_H^2 r_H + M(r_H^2 - a^2\cos^2\theta')]\vec{e}_{\phi'} \,. \tag{2.18}$$

As in the case of any stationary horizon, θ^H and σ^H_{ab} vanish. The metric $\overleftrightarrow{\gamma}$ for horizon sections $d\bar{t} = 0$ can be read directly from (2.16) as

$$\gamma_{\theta'\theta'} = \rho_H^2 \,, \qquad \gamma_{\phi'\phi'} = \varpi_H^2 \,, \qquad \gamma_{\theta'\phi'} = 0 \,. \tag{2.19}$$

In this paper we shall be interested in gravitationally perturbed Kerr spacetime and we shall have to deal with the fact that our coordinates acquire an arbitrariness of the same order as the spacetime perturbation. On the horizon we reduce this arbitrariness by choosing, throughout the remainder of this paper, spatial coordinates θ and ϕ which comove with the perturbed generators and which agree to zeroth order with the comoving θ' and ϕ' Kerr coordinates.

No specific choice exists for the slicing function $\bar{t}$ in the perturbed spacetime. Rather we can only choose $\bar{t}$ to agree to zeroth order with the preferred choice (2.17) for the Kerr horizon. Since $\bar{t}$ has an arbitrariness of the order of the perturbation of the spacetime we must consider the effect of this arbitrariness on our horizon quantities. (For a full description of the transformation properties of horizon quantities under a change in slicing see Sec. II C of PT.) Because θ^H and $\overleftrightarrow{\sigma}^H$ vanish for the Kerr background and are of the order of the spacetime perturbation, the uncertainties in these quantities are second order in the perturbation and can be ignored. The surface gravity g_H, finite in the background, becomes uncertain to the order of the perturbation. For most purposes this arbitrariness is irrelevant since only the zeroth-order part of g_H will be needed. It is convenient, however, to fix the choice of the slicing function $\bar{t}$ such that g_H always has its constant, unperturbed value $(r_H - M)/2Mr_H$. We henceforth make this choice. (See, however, Sec. III C of PT for a discussion of slicings appropriate for a black hole undergoing sizable, but slow, changes in mass and angular momentum.) Note that a change of slicing adds a multiple of $\vec{l}$ to the spatial basis vectors $\vec{e}_a$, hence the metric $\gamma_{ab} = \vec{e}_a \cdot \vec{e}_b$ is independent of the choice of slicing. The most troublesome of the horizon quantities under the slicing transformations is the Hajicek field $\vec{\Omega}^H$. If we change from a slicing function $\bar{t}$ to a new function $\bar{t}'$, the new and the old Hajicek vectors are related by

$$\Omega_a'^H = \Omega_a^H - g_H(\partial \bar{t}'/\partial x^a)_{\bar{t}} \ . \tag{2.20}$$

(This is actually not the most general form for the transformation of $\vec{\Omega}^H$ for constant g_H. It assumes that all generators attached to the horizon long in the past, at which point $\bar{t}$ on that generator was set to $-\infty$. For details see Appendix D of PT.) We must conclude then that the perturbation of the Hajicek field, for a dynamically perturbed horizon, lacks invariant meaning; meaning can be attributed only to its curl $\Omega^H_{[a,b]}$.

For the description of small perturbations of the horizon Eqs. (2.8) and (2.9) simplify considerably. In (2.8) we treat the Weyl term $\mathcal{E}^H_{ab}$ as the perturbative driving force. The equation then requires σ^H_{ab} to be first order in the perturbation and, to first order, the left-hand side reduces to $D_{\bar{t}}\sigma^H_{ab} - g_H\sigma^H_{ab}$. We shall deal only with model problems in which no stress energy crosses the horizon, so that the $\mathcal{F}^H$ term is absent in Eq. (2.9) and this equation shows that $\theta^H \sim |\sigma^H_{ab}|^2$ is second order in the perturbation. If we confine ourselves, as we do, to comoving coordinates then the $D_{\bar{t}}$ operator may be approximated by $\partial/\partial\bar{t}$ since the difference $D_{\bar{t}} - (\partial/\partial\bar{t})$ is of the order of θ^H and σ^H_{ab} and entails only higher-order corrections in (2.8) and (2.9).

With these simplifications and with the specialization to vanishing stress energy at the horizon, the tidal force equation and focusing equation become, to lowest order,

$$\partial\sigma^H_{ab}/\partial\bar{t} - g_H\sigma^H_{ab} = -\mathcal{E}^H_{ab} \ , \tag{2.21}$$

$$\partial\theta^H/\partial\bar{t} - g_H\theta^H = -\sigma^H_{ab}\sigma^{ab}_H \ . \tag{2.22}$$

It should be noted that in the simplification of the focusing equation we have assumed

$$\theta^H \ll g_H \ . \tag{2.23}$$

This condition for the consistency of the perturbative approach is related to an important kinematic condition: no caustics have jumped onto the horizon. (See Ref. 9, Sec. VI C 7.) The remaining evolution equation we shall need is Eq. (2.5a) for the components of the metric in the comoving coordinate basis. Correct to second order in the perturbation this equation is

$$\frac{\partial\gamma_{ab}}{\partial\bar{t}} = 2\sigma^H_{ab} + \theta^H\gamma^0_{ab} \ , \tag{2.24}$$

where γ^0_{ab} is the unperturbed (e.g., Kerr) metric.

Because the background Hajicek field does not vanish, the Hajicek equation (2.10) simplifies only slightly when applied to perturbed horizons with vanishing stress energy and we shall not explicitly use the perturbation form of this equation. The exact Hajicek equation (2.10) on the other hand is of central importance in the conceptual foundations of the membrane paradigm, and provides a useful approach to the calculation of the evolution of black-hole angular momentum. (See Sec. II D.)

In using Eqs. (2.21)–(2.24) we start with a known form for $\mathcal{E}^H_{ab}$ and solve the tidal force equation (2.21) for $\overleftrightarrow{\sigma}^H$. The resulting $\overleftrightarrow{\sigma}^H$ is then used in the focusing equation (2.22) which is solved for θ^H. Finally, with the known values of $\overleftrightarrow{\sigma}^H$ and θ^H, we solve (2.24) for the time evolution of the metric.

The boundary conditions to be used with these equations deserve comment. The homogenous solution to Eq. (2.21) has the form $\overleftrightarrow{\sigma}^H \sim e^{g_H\bar{t}}$. If we take $\overleftrightarrow{\sigma}^H$ to be zero as $\bar{t}\to-\infty$ in accordance with the usual choice for causal systems, then Eq. (2.21) tells us that $\overleftrightarrow{\sigma}_H$ remains zero until the horizon feels a tidal forcing term $\mathcal{E}^H_{ab}$, after which $\overleftrightarrow{\sigma}_H$ diverges exponentially. The horizon, however, is not a causal system. It is the boundary between the regions of spacetime from which light-speed signals can and cannot ever escape to spatial infinity. Whether a light-speed signal can escape from a point in spacetime depends on the region of spacetime to the future, not the past, of that point. The appropriate boundary condition on the horizon shear is at $\bar{t}=+\infty$ where it must be taken to vanish. Similarly, the boundary condition for Eq. (2.22) is the vanishing of θ^H at $\bar{t}=+\infty$. For these boundary conditions the solutions of Eqs. (2.21) and (2.22) may immediately be written as

$$\sigma^H_{ab}(\bar{t},\theta',\phi') = \int_{\bar{t}}^{\infty} \mathcal{E}^H_{ab}(\bar{t}',\theta',\phi')\exp[g_H(\bar{t}-\bar{t}')]d\bar{t}' \ , \tag{2.25a}$$

$$\theta^H(\bar{t},\theta',\phi') = \int_{\bar{t}}^{\infty} \sigma^{ab}_H\sigma^H_{ab}(\bar{t}',\theta',\phi')\exp[g_H(\bar{t}-\bar{t}')]d\bar{t}' \ . \tag{2.25b}$$

For the metric-evolution equation (2.24) the situation is more familiar. Suppose that at time $\bar{t}_0$ the form of the metric $\gamma_{ab}(\bar{t}_0,\theta',\phi')$ is known. [It might, for example, be the standard Kerr form (2.19) for $\bar{t}_0$ preceding any horizon perturbations.] At any other time the solution to Eq. (2.24) (with θ',ϕ' dependence suppressed) is obviously

$$\gamma_{ab}(\bar{t}) = \gamma_{ab}(\bar{t}_0) + 2\Sigma^H_{ab} + \Theta^H\gamma^0_{ab} \ , \tag{2.26a}$$

where we have introduced the time-integrated shear

$$\Sigma^H_{ab} \equiv \int_{\bar{t}_0}^{\bar{t}} \sigma^H_{ab}(\bar{t}')d\bar{t}' \tag{2.26b}$$

and the time-integrated expansion

$$\Theta^H \equiv \int_{\bar{t}_0}^{\bar{t}} \theta^H(\bar{t}')d\bar{t}' \ . \tag{2.26c}$$

These quantities describe the elliptical distortion and the increase in area at time $\bar{t}$ of a set of generators that was circular at $\bar{t}_0$.

C. The 3+1 split and the stretched horizon

The 3+1 split of spacetime outside the horizon is accomplished with a special time coordinate t, which we call "universal time." The constant-t spacelike sections of spacetime outside the horizon are called "absolute space." In the membrane paradigm all such spatial sections are considered as a single three-dimensional space in which physical fields reside and evolve according to universal time. The congruence of timelike lines orthogonal to absolute space is taken as the collection of world lines of a preferred set of fiducial observers (FIDO's) at rest in absolute space. On FIDO world lines the relationship of FIDO proper time τ and universal time t defines the lapse function α according to

$$\alpha \equiv d\tau/dt \; . \tag{2.27}$$

In general a universal time can be chosen with the following properties (for proof and details see Sec. II of PT): (i) the lapse function α vanishes at the horizon and increases outward; (ii) near the horizon α is constant on a FIDO world line or, more precisely,

$$d\alpha/d\tau = O(\alpha^2) \; ; \tag{2.28}$$

(iii) in the horizon limit, $\alpha \to 0$, the FIDO world lines approach horizon generators; (iv) the equation

$$\bar{t} = t + \frac{1}{2g_H} \ln\alpha^2 + \text{const} \tag{2.29}$$

defines a well-behaved time coordinate $\bar{t}$ on the horizon, and for this $\bar{t}$ the horizon has (constant) surface gravity g_H. We always use Eq. (2.29) as the relationship between the slicing function $\bar{t}$ that determines horizon quantities and universal time t that determines FIDO world lines. With this relationship fixed, the only arbitrariness in universal time is that in the choice of the slicing function, i.e., once $\bar{t}$ is chosen (such that g_H is constant) universal time t is fixed. (See Appendix D of PT for a detailed discussion.)

The surface $\alpha=0$, the horizon, is null but the surface at $\alpha=\alpha_H \ll 1$, though very close to the horizon, is timelike. In the membrane paradigm the surface at $\alpha=\alpha_H \ll 1$ is called the "stretched horizon" and this timelike surface is used in place of the horizon as the inner boundary of the hole's external spacetime; field quantities and relations in the membrane paradigm are formulated in such a way that the precise (small) value of α_H is unimportant. By Eq. (2.28) FIDO's in the stretched horizon remain in the stretched horizon [aside from $O(\alpha_H^2)$ errors which we ignore] and, roughly speaking, the FIDO four-velocity $\vec{U}$ can be thought of as representing the generators according to the replacement

$$\vec{l} \to \alpha_H \vec{U} \; . \tag{2.30}$$

In the stretched horizon a second set of world lines, that of "fiducions," must also be invoked. These world lines are locked even more strongly than those of the FIDO's to the horizon generators. Unfortunately the mathematical definition of the fiducions is unavoidably technical and will not be given here. (For details see Sec. II B of PT.) For the concerns of this paper it suffices to state that the four-velocities of FIDO's and fiducions differ only by $O(\alpha_H^2)$ and *almost* everywhere we can treat the fiducions and FIDO's as equivalent. (The single exception is in the fluid-mechanical interpretation of horizon dynamics where the difference of FIDO and fiducion motions gives rise to fluid momentum density.)

Since the stretched horizon is a two-parameter congruence of fiducion (or, equivalently, FIDO) world lines we can define from this congruence and from the fiducions' four-velocity $\vec{U}$ an acceleration $\vec{g} = \nabla_{\vec{U}} \vec{U}$. Similarly from the projection into a constant-t section of the stretched horizon (i.e., a projection into absolute space) of $\nabla \vec{U}$ we can define fiducion expansion θ and shear $\overleftrightarrow{\sigma}$ in the manner used for three-dimensional timelike congruences. In the horizon limit $\alpha_H \to 0$, the kinematic quantities $\vec{g}$, θ, and $\overleftrightarrow{\sigma}$ diverge as α_H^{-1}.

This divergence has a simple origin. At constant α well-behaved quantities near the horizon should have a finite rate of change per unit $\bar{t}$ time, hence per unit universal time. [See Eq. (2.29).] In terms of fiducion (or FIDO) proper time, rates in the stretched horizon will then depend on α_H as α_H^{-1}. In order to arrive at kinematical quantities which are independent of the precise (small) choice of α_H we must renormalize $\vec{g}$, θ, and $\overleftrightarrow{\sigma}$ by converting them to a per-unit-universal-time basis. We therefore define

$$g_H = \alpha_H |\vec{g}| \; , \quad \theta^H \equiv \alpha_H \theta \; , \quad \overleftrightarrow{\sigma}_H \equiv \alpha_H \overleftrightarrow{\sigma} \; , \tag{2.31}$$

quantities which are independent of α_H [aside from $O(\alpha_H^2)$ fractional errors which we ignore].

The usefulness of the stretched horizon as a timelike surrogate for the true horizon is apparent in the following fact: At a point on a fiducion (or FIDO) world line g_H, θ^H, and $\overleftrightarrow{\sigma}^H$ as defined by Eq. (2.31) are identical to the horizon surface gravity, expansion and shear at the corresponding point of the horizon. (See Sec. III of PT for a proof, and for the precise meaning of "corresponding point.") The kinematic meanings of the stretched-horizon θ^H and $\overleftrightarrow{\sigma}^H$, furthermore, are those given by Eq. (2.6). The Hajicek field of the horizon, in a similar way, corresponds to a projection of the extrinsic curvature of the stretched horizon. Moreover, the equations describing the evolution of θ^H, $\overleftrightarrow{\sigma}^H$, and $\vec{\Omega}^H$ in the stretched horizon are Eqs. (2.8)–(2.10) with the following modifications: (i) $D_{\bar{t}}$ must be replaced by D_t with the meaning: $\alpha_H \nabla_{\vec{U}}$ followed by projection into the constant-t section of the stretched horizon; (ii) the Weyl term $\mathcal{E}_{ab}^H$ and the stress-energy terms $\mathcal{T}^H$ and $\mathcal{G}_a^H$ must be reinterpreted as resulting from a 3+1 decomposition of the Weyl and stress-energy tensors. [See below, especially Eqs. (2.32)–(2.34) and (2.36).]

In the use of the 3+1 split of spacetime, fields are decomposed into time components (projection with FIDO four-velocity $\vec{U}$) and space components (projection into absolute space). On the stretched horizon a further distinction is made: We introduce $\vec{N}$, a unit vector which is spatial (i.e., lying in absolute space), outwardly directed, and normal to the stretched horizon; and we introduce basis vectors $\vec{e}_a$ ($a=2,3$) lying in the constant-t sections of the stretched horizon. In the horizon limit these constant-t sections approach the constant-$\bar{t}$ sections of the horizon so we can, and do, choose to have the basis vectors $\vec{e}_a$ for the stretched horizon approach the horizon basis vectors $\vec{e}_a$ as $\alpha_H \to 0$. With this choice we have, for example, that the components σ_{ab}^H of shear in the stretched horizon are the same [aside from negligible $O(\alpha_H^2)$ errors] as the components σ^H_{ab} of horizon shear.

When the 3+1 split is used to understand electromagnetic fields[5-7] near a horizon the field tensor $F_{\mu\nu}$ is replaced by electric $E^i \equiv F^{i\nu}U_\nu$ and magnetic $B^i \equiv \frac{1}{2}\epsilon^{\nu i\lambda\beta}F_{\lambda\beta}U_\nu$ fields as measured by FIDO's. (Here $\vec{U}$ is the FIDO four-velocity and $\epsilon^{\nu\mu\lambda\beta}$ is the Levi-Civita alternating tensor in spacetime.) At the stretched horizon these spatial fields are further decomposed into components E_N, B_N in the direction $\vec{N}$ of the normal to the stretched horizon, and components E_a, B_a tangential to the stretched horizon.

For gravitational interactions the analog of $F_{\mu\nu}$ is the Weyl tensor $C_{\alpha\beta\mu\nu}$ and in our formalism it is replaced by two tensors in absolute space, the gravitoelectric field $\mathcal{E}_{ij}$ and the gravitomagnetic field $\mathcal{B}_{ij}$, given by

$$\begin{aligned} \mathcal{E}_{ij} &\equiv C_{i\mu j\nu}U^\mu U^\nu , \\ \mathcal{B}_{ij} &\equiv \tfrac{1}{2}\epsilon_{\mu i\nu\lambda}C^{\nu\lambda}{}_{j\beta}U^\mu U^\beta . \end{aligned} \tag{2.32}$$

The fact that these tensors are spatial (i.e., $\overleftrightarrow{\mathcal{E}}\cdot\vec{U}=\vec{U}\cdot\overleftrightarrow{\mathcal{E}}=\overleftrightarrow{\mathcal{B}}\cdot\vec{U}=\vec{U}\cdot\overleftrightarrow{\mathcal{B}}=0$) follows from the symmetries of the Weyl tensor.

The following facts about $\overleftrightarrow{\mathcal{E}}$ and $\overleftrightarrow{\mathcal{B}}$ are important to the formalism and are derived and discussed in PT. (i) $\overleftrightarrow{\mathcal{E}}$ and $\overleftrightarrow{\mathcal{B}}$ are symmetric and traceless and contain five independent components each; they therefore contain all the information in the ten independent components of the Weyl tensor. (ii) Because the FIDO four-velocity is pathological at the horizon some components of $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$ diverge on the stretched horizon as $\alpha_H \to 0$. Of particular importance are the "transverse-traceless" components $\mathcal{E}^{TT}_{ab}$ and $\mathcal{B}^{TT}_{ab}$ formed from the components tangential to the stretched horizon according to

$$\begin{aligned} \mathcal{E}^{TT}_{ab} &= \mathcal{E}_{ab} - \tfrac{1}{2}\gamma_{ab}(\gamma^{cd}\mathcal{E}_{cd}) , \\ \mathcal{B}^{TT}_{ab} &= \mathcal{B}_{ab} - \tfrac{1}{2}\gamma_{ab}(\gamma^{cd}\mathcal{B}_{cd}) . \end{aligned} \tag{2.33}$$

These components diverge as $O(\alpha_H^2)$ and motivate the definition of the horizon fields

$$\overleftrightarrow{\mathcal{E}}^H \equiv \alpha_H^2 \overleftrightarrow{\mathcal{E}}^{TT} \quad \text{and} \quad \overleftrightarrow{\mathcal{B}}^H \equiv \alpha_H^2 \overleftrightarrow{\mathcal{B}}^{TT} , \tag{2.34}$$

which are well behaved in the limit $\alpha_H \to 0$ and [aside from negligible $O(\alpha_H^2)$ fractional corrections] are independent of α_H near the horizon. (iii) Near the horizon $\mathcal{E}^H_{ab}$ is the same [aside from $O(\alpha_H^2)$ fractional errors] as the driving term $\mathcal{E}^H_{ab}$ on the right of Eq. (2.8). (iv) The condition that the Weyl field be nonsingular at the horizon implies, aside from $O(\alpha_H^4)$ errors, $\mathcal{E}^H_{ab} = \epsilon_{ai}{}^d\mathcal{B}^H_{db}N^i$ (where ϵ_{ijk} is the Levi-Civita tensor for absolute space), or

$$\overleftrightarrow{\mathcal{E}}^H = \vec{N} \times \overleftrightarrow{\mathcal{B}}^H . \tag{2.35}$$

This is analogous to the electromagnetic horizon condition $\vec{E} = \vec{N} \times \vec{B}$ and has an analogous interpretation: To FIDO's near the horizon $\overleftrightarrow{\mathcal{E}}$ and $\overleftrightarrow{\mathcal{B}}$ have the form of an ingoing plane gravitational wave.

The stress-energy tensor $T_{\mu\nu}$, like $F_{\mu\nu}$ and $C_{\alpha\beta\mu\nu}$, must undergo a 3+1 decomposition and renormalization with factors of α_H. Of primary concern here are the terms occurring in the focusing equation (2.9) and the Hajicek equation (2.10). In the 3+1 split these arise as the red-shifted energy crossing the stretched horizon per unit universal time,

$$\mathcal{T}^H \equiv \alpha_H^2 T_{\mu\nu}U^\mu U^\nu , \tag{2.36a}$$

and the momentum crossing the stretched horizon per unit universal time,

$$\mathcal{G}^H{}_a \equiv -\alpha_H T_{a\mu}U^\mu . \tag{2.36b}$$

In the $\alpha_H \to 0$ horizon limit these agree with the definitions given in (2.11) [cf. Eq. (2.30)].

For the Kerr geometry [Eq. (2.12)] universal time is simply Boyer-Lindquist coordinate time; the lapse function is given by Eq. (2.13) and the fiducion world lines are those of constant θ', ϕ', and $\alpha=\alpha_H$. For a perturbed Kerr geometry the spatial coordinates θ', ϕ' become uncertain to the order of the perturbation. This arbitrariness has no effect, to lowest order, on the stretched-horizon values of θ^H, σ^H, $\overleftrightarrow{\mathcal{E}}^H$, $\overleftrightarrow{\mathcal{B}}^H$, or on Eqs. (2.21), (2.22), or (2.25), which apply on the stretched horizon if $\bar{t}$ is replaced by t. We choose, however, to fix the spatial coordinates θ', ϕ' by requiring that in the perturbed geometry, as well as the unperturbed, they are constant on the fiducion world lines. With this choice the metric in the constant-t section of the stretched horizon $\gamma_{ab}=(\partial/\partial x^a)\cdot\overleftrightarrow{g}\cdot(\partial/\partial x^b)$ agrees in the horizon limit with the horizon-section metric γ_{ab}, and Eqs. (2.24) and (2.26) are valid if $\bar{t}$ is replaced by t. The mathematical description we shall use for the perturbations of the horizon may be viewed equally well as describing the evolution of the two-dimensional "fluid" of fiducions. In the discussions below we shall use "on the horizon" and "on the stretched horizon" almost always interchangeably.

D. The fluid interpretation

In the application of the membrane paradigm to the study of the interactions of a hole with electromagnetic fields[5-7] it is very useful for purposes of intuition to think of a physical membrane located at the stretched horizon. With the assignment of specific two-dimensional electrical properties (surface charge density, surface current density, surface resistivity) to this membrane, the interaction of the electromagnetic environment with the hole is understood via a more-or-less natural picture of the interactions of a physical membrane. In particular, in this picture: (i) charge cannot penetrate the membrane; charge flowing onto the membrane from the exterior resides as membrane surface charge, and moves as membrane surface current, until and unless it is cancelled by the accretion of opposite charge; (ii) the radiative boundary condition, that to FIDO's electromagnetic fields near the horizon have the form of ingoing plane waves, emerges as a consequence of the properties of the membrane.

As with electromagnetic interactions, gravitational interactions of the hole can be understood by assigning specific physical properties to the membrane, but in this case the paradigm is much richer and the details more complex. The general picture will be only briefly sketched here; for details and proofs see Sec. V of PT.

The new physical properties assigned to the membrane are those of a two-dimensional viscous fluid. The fluid elements comove with the fiducions, and as measured by FIDO's this fluid has a surface density Σ of mass-energy, a surface pressure P, a surface stress $\overleftrightarrow{S}$, and a momentum density $\vec{\Pi}$. The fluid also has fixed coefficients η and ζ, respectively, of two-dimensional shear and bulk viscosity, with the specific numerical values

$$\eta=\frac{1}{16\pi} \quad \text{and} \quad \zeta=-\frac{1}{16\pi} . \tag{2.37}$$

In terms of these, and of the shear and expansion of the (fiducion) fluid elements, the surface stress tensor has the familiar form

$$S_{ab}=(P-\zeta\theta)\gamma_{ab}-2\eta\sigma_{ab} . \tag{2.38}$$

As with other FIDO-measured quantities, some of the above fluid fields on the stretched horizon diverge in the $\alpha_H \rightarrow 0$ limit. As with other quantities these divergences can be understood as originating in the use of proper time in FIDO measurements. These divergences are remedied by a switch to measurements on a per-unit-universal-time basis or equivalently by the application to each quantity of the appropriate number of factors of α_H. In this way we arrive at the following stretched-horizon fluid quantities which [aside from negligible $O(\alpha_H^2)$ errors] are independent of α_H:

$$\Sigma^H=\alpha_H\Sigma , \quad P^H=\alpha_H P ,$$
$$\eta_H=\eta=\frac{1}{16\pi} , \quad \zeta_H=\zeta=-\frac{1}{16\pi} , \quad \vec{\Pi}^H=\vec{\Pi} , \tag{2.39}$$
$$S_{ab}^H=\alpha_H S_{ab}=(P^H-\zeta_H\theta^H)\gamma_{ab}-2\eta_H\sigma_{ab}^H .$$

We require that the shear of the two-dimensional fluid be driven by the stretched-horizon gravitoelectric field according to the tidal force equation

$$D_t\sigma_{ab}^H+(\theta_H-g_H)\sigma_{ab}^H=-\mathcal{E}_{ab}^H \tag{2.40}$$

[cf. Eq. (2.8)]. In the fluid viewpoint this is understood as the natural response of the two-dimensional fluid to a gravitational tidal field. [For a detailed discussion, and an explanation of why pressure-gradient and viscous forces do not appear in Eq. (2.40) see Sec. V B 2 of PT.]

The remaining dynamical equations of the membrane paradigm follow from the equation of motion of the fluid

$$\mathcal{S}_\mu{}^\nu{}_{;\nu}+[T_{\mu\nu}N^\nu]=0 . \tag{2.41}$$

Here $\mathcal{S}_\mu^\nu$ is the stress-energy tensor of the two-dimensional fluid ($\vec{U}\cdot\overleftrightarrow{\mathcal{S}}\cdot\vec{U}=\Sigma$, $\vec{U}\cdot\overleftrightarrow{\mathcal{S}}\cdot\vec{e}_a=-\Pi_a$, $\mathcal{S}_{ab}=S_{ab}$), $[T_{\mu\nu}N^\nu]$ denotes the discontinuity in the normal component of stress energy across the stretched horizon, and Eq. (2.41) is understood to be projected on the $(2+1)$-dimensional world tube of the stretched horizon. We further require that all stress-energy flows from the exterior are terminated at the stretched horizon, i.e.,

$$[T_{\mu\nu}N^\nu]=T_{\mu\nu}N^\nu|_{\text{just outside stretched horizon}} \tag{2.42}$$

so that all stress-energy flowing into the stretched horizon is absorbed by the two-dimensional fluid membrane.

With these requirements the fluid properties of the membrane serve, for gravitational interactions, a role similar to that served by the electrical properties in explaining the interaction of the hole with electromagnetic fields. In particular, the fluid not only terminates all stress-energy flows but, as a consequence of its equations of motion (2.40) and (2.41) and its viscous coefficients η and ζ, it "explains" the horizon boundary condition (2.35). (See Sec. V B and Appendix E of PT.) For gravitational interactions, however, the membrane paradigm goes considerably beyond this parallel with the electromagnetic paradigm. A sufficient mathematical consistency condition for the requirements (2.40) and (2.42) on the membrane fluid is the vanishing of the extrinsic curvature on the inner side of the stretched horizon's $(2+1)$-dimensional world tube. This in turn implies that our fluid dynamical variables Σ, P, and $\vec{\Pi}$ are related to the kinematical variables of the stretched horizon by

$$\Sigma_H=-\frac{1}{8\pi}\theta_H , \quad P_H=\frac{1}{8\pi}g_H , \quad \vec{\Pi}^H=-\frac{1}{8\pi}\vec{\Omega}^H . \tag{2.43}$$

With this interpretation, moreover, the membrane fluid equations are equivalent to the dynamical equations of the stretched horizon as follows: (i) Equation (2.40) with no further interpretation is, of course, identical to Eq. (2.8). (ii) The projection of Eq. (2.41) onto the FIDO four-velocity $\vec{U}$, the equation of energy conservation for the fluid, is identical to the focusing equation (2.9). (iii) The projection of Eq. (2.41) into the constant-t sections of the stretched horizon, the equation of fluid momentum conservation, is identical to the Hajicek equation (2.10). In this way the analysis of the evolution of horizon distortions may be viewed entirely in terms of the mechanics of the two-dimensional fluid membrane at the stretched horizon.

For a complete understanding of the horizon dynamics the electrical and fluid-mechanical properties of the membrane must be supplemented by the thermodynamic properties of the membrane. These properties, which follow from the work of Hawking,[17] assign to the membrane a red-shifted temperature T_H given by

$$T_H=(\hbar/2\pi k)g_H , \tag{2.44}$$

where $\hbar$ and k are, respectively, Planck's constant and Boltzman's constant; and to a cross-sectional area $\Delta\mathcal{A}^H$ of a bundle of fiducions is assigned an entropy

$$\Delta S^H=(k/4\hbar)\Delta\mathcal{A}^H . \tag{2.45}$$

From Eq. (2.6a), rewritten for the stretched horizon in terms of entropy, we can compute the rate of "heating" for a small region of the stretched horizon of area $\Delta\mathcal{A}^H$, bounded by fiducions:

$$T_H\frac{d\Delta S^H}{dt}=(g_H/8\pi)\theta^H\Delta\mathcal{A}^H . \tag{2.46}$$

We now apply the exact focusing equation for the stretched horizon [Eq. (2.22) with the θ_H^2 term and the stress-energy term of Eq. (2.9) restored and reinterpreted as stretched-horizon quantites]. With this and the fact that $\theta^H=(\Delta\mathcal{A}^H)^{-1}d\Delta\mathcal{A}^H/dt$, the heating equation can be written as

$$T_H\left[\frac{d\Delta S^H}{dt}-\frac{1}{g_H}\frac{d^2\Delta S^H}{dt^2}\right]=\frac{\Delta\mathcal{A}^H}{8\pi}(\sigma^H_{ab}\sigma^{ab}_H-\tfrac{1}{2}\theta_H^2+8\pi\mathcal{F}^H)\ . \qquad (2.47)$$

(We have used here the fact that g_H is constant for our choice of slicing; see the discussion in Sec. II B.) If this equation is applied to find the heating that occurs during a time long compared to g_H^{-1} it can be shown that the second derivative term on the left may be ignored,[18] and the equation may be written as

$$T_H\frac{d\Delta S^H}{dt}=(2\eta_H\sigma^H_{ab}\sigma^{ab}_H+\zeta_H\theta_H^2+\mathcal{F}^H)\Delta\mathcal{A}^H\ . \qquad (2.48)$$

This equation has precisely the form of the heating equation for an ordinary fluid: the first two terms on the right represent dissipative heating and the third term represents heating due to the addition of red-shifted nongravitational energy from the black-hole exterior. In our model problems there will be no nongravitational stress-energy crossing the horizon and for perturbative models we have $\theta_H^2 \ll \sigma^H_{ab}\sigma^{ab}_H$ (see Sec. II B), so that over a time interval much greater than g_H^{-1} we may use

$$T_H\frac{d\Delta S^H}{dt}=2\eta_H\sigma^H_{ab}\sigma^{ab}_H\Delta\mathcal{A}^H\ . \qquad (2.49)$$

The rate of heating for the hole may then be computed by integrating this equation over the entire horizon.

The membrane momentum density $\vec{\Pi}_H$ provides an approach to computing the rate of change of angular momentum J of a hole. For a stationary hole

$$J=\int \Pi^H_{\phi'}\sqrt{\gamma}\,d\theta' d\phi'=\int \Pi^H_{\phi'}d\mathcal{A}^H\ . \qquad (2.50)$$

where $\gamma=\det\|\gamma_{ab}\|$ and Π^H_ϕ is the projection of $\vec{\Pi}^H$ on $\partial/\partial\phi'$, the generator of rotations about the symmetry axis. (See PT Sec. II B and Hajicek.[19]) Since the integration in (2.50) is over the entire horizon, J is unaffected by the fact that an arbitrary gradient can be added to $\vec{\Pi}^H$. [See Eq. (2.20).]

We now consider a slowly evolving hole, i.e., one changing on a time scale much greater than g_H^{-1}, which begins and ends in a stationary state; the change in J will be given by the difference in integral (2.50) between the final and initial states. In order to arrive at a useful definition of J during the epoch of change we now make a further restriction: We assume that the perturbation acting on the Kerr black hole is axisymmetric about the rotation axis of the hole in a time-averaged sense, e.g., a particle orbiting the hole in its equatorial plane. (This is the model problem to be studied in Sec. IV B.) For such a situation, all generators having the same value of $\theta'(=\theta_i')$ before the perturbation will end up with their θ' values again equal to each other (though not necessarily equal to θ_i') after the perturbation. We can take these generators (or fiducions) to define $\partial/\partial\phi'$ during the epoch of evolution. It is then clear that the quantity

$$\frac{dJ}{dt}=\int\frac{\partial}{\partial t}(\sqrt{\gamma}\,\Pi^H_{\phi'})d\theta' d\phi' \qquad (2.51)$$

when integrated over time throughout the perturbation period will give the difference of the angular momenta of the Kerr black hole before and after the perturbation.

Equation (2.51) can be put into a more suggestive form. For comoving coordinates we have

$$D_{\bar t}\Omega^H_a+(\sigma^{Hc}_a+\tfrac{1}{2}\delta^c_a\theta_H)\Omega^H_c=\frac{\partial}{\partial\bar t}\Omega^H_a=\frac{\partial}{\partial t}\Omega^H_a \qquad (2.52)$$

and, from (2.5), $\partial\sqrt{\gamma}/\partial t=\theta_H\sqrt{\gamma}$. We may then use the Hajicek equation (2.10) to write (2.51) as

$$\frac{dJ}{dt}=\int(2\eta_H\sigma_{\phi'}^{Hb}{}_{\|b}+\mathcal{G}^H_{\phi'})\sqrt{\gamma}\,d\theta' d\phi'\ . \qquad (2.53)$$

The divergence of the shear can now be explicitly computed:

$$\sigma_{\phi'}^{Hb}{}_{\|b}=\frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}\,\sigma^{Hb}_{\phi'})_{,\,b}-\tfrac{1}{2}\sigma^{ab}_H\gamma_{ab,\,\phi'}\ . \qquad (2.54)$$

The first term on the right disappears under integration in (2.53) and we are left with

$$\frac{dJ}{dt}=\int(-\eta_H\sigma^{ab}_H\gamma_{ab,\,\phi'}+\mathcal{G}^H_{\phi'})d\mathcal{A}^H\ . \qquad (2.55)$$

The second term in the integral represents the rate of angular momentum falling across the horizon. The first term also has an intuitively appealing interpretation: the viscous torque due to the interaction of the tidally induced shear with the nonaxisymmetric distortion of the horizon.

It should also be noted that in the derivation of Eq. (2.55) from (2.51) the assumption of weak perturbations has nowhere been used; this result then is exact and can in principle be used to find the change in angular momentum for strong disturbances of a hole.

From the heating rate for the hole and the rate of change of the hole's angular momentum, the rate of change of the hole's mass is given by the first law of black-hole thermodynamics:[20]

$$\frac{dM}{dt}=T_H\frac{dS^H}{dt}+\Omega_H\frac{dJ}{dt}\ . \qquad (2.56)$$

where $\Omega_H\equiv a/2Mr_H$ is the hole's angular velocity given in Eq. (2.15).

E. The Rindler approximation

If we restrict attention to the region ($\alpha \ll 1$) of spacetime sufficiently close to the horizon then in Eq. (2.16) the $g_{t\phi'}$ term, which is $O(a\alpha^2\sin^2\theta')$, may be ignored and the Kerr geometry may be approximated as

$$ds^2=-\alpha^2dt^2+g_H^{-2}d\alpha^2+\rho_H^2d\theta'^2+\tilde{\omega}_H^2d\phi'^2\ . \qquad (2.57)$$

In this geometry the three-surfaces of constant t are curved with radius of curvature of order r_H. We may approximate these three-surfaces as flat by further restricting attention to regions of spacetime above a patch of the horizon small compared to r_H. More specifically we define, near $\theta'=\theta_0'$, Cartesian-type coordinates

$$x \equiv \tilde{\omega}_H(\theta'_0)\phi' \quad , \quad y \equiv \rho_H(\theta'_0)[\theta' - \theta'_0] \quad , \quad z \equiv \alpha / g_H \tag{2.58}$$

The line element in Eq. (2.57) then becomes

$$ds^2 = -(g_H z)^2 dt^2 + dx^2 + dy^2 + dz^2 \ , \tag{2.59}$$

if we ignore the $O(y^2/r_H^2)$ corrections (and assume θ'_0 is not too near 0 or π).

We shall call Eq. (2.59) the Rindler[12] approximation to Kerr. With the further transformation

$$T = z \sinh g_H t \quad , \quad Z = z \cosh g_H t \ , \tag{2.60}$$

the Rindler geometry takes the Minkowski form

$$ds^2 = -dT^2 + dx^2 + dy^2 + dZ^2 \ , \tag{2.61}$$

showing that the Rindler geometry corresponds to a region of flat Minkowski spacetime. (In general it actually contains two equivalent regions of Minkowski spacetime, $Z \geq |T|$ and $Z \leq -|T|$; here we confine our attention to the first region.)

We shall exploit the simplicity of the Rindler approximation by using flat-spacetime techniques to solve for the distortion of the horizon due to sources near the horizon. Specifically, in the model problems we start by adopting the Lorentz gauge and use the Liénard-Wiechert-like solution for the metric perturbations. From this we compute the (gauge-invariant) gravitoelectric field $\mathcal{E}^H_{ab}$ which is used in (2.25a) to find the horizon shear σ^H_{ab}. The shear in turn is used in (2.25b) to give the expansion. Comoving coordinates are then assumed; the evolution of γ_{ab} is computed and the distortion of the horizon is viewed in terms of the distortion of an initial circle of (coordinate-fixed) fiducions. (Note that σ^H_{ab} and θ^H are gauge-invariant and do not depend on the Lorentz gauge used at the outset; the introduction of comoving coordinates here is therefore not inconsistent.) Lastly, the expansion is used for an investigation of the rate of heating of the horizon [Eq. (2.46)] and of the mass added to the hole [Eq. (2.56)].

III. STATIC PARTICLE NEAR HORIZON

A. The Weyl approach

Axisymmetric, static, vacuum solutions of the Einstein field equations can be described by the Weyl formalism.[13] This formalism uses coordinates (t,ρ,z,ϕ) in which the metric takes the form

$$ds^2 = -e^{2\psi(\rho,z)}dt^2 + e^{2[\gamma(\rho,z)-\psi(\rho,z)]}(d\rho^2 + dz^2) + \rho^2 e^{-2\psi(\rho,z)}d\phi^2 \ . \tag{3.1}$$

(The Weyl coordinate z is, of course, not to be confused with the Rindler coordinate z of the previous section.) The vacuum Einstein equations in these coordinates reduce to

$$\partial^2\psi/\partial\rho^2 + (1/\rho)\partial\psi/\partial\rho + \partial^2\psi/\partial z^2 = 0 \ , \tag{3.2}$$

$$\partial\gamma/\partial\rho = \rho[(\partial\psi/\partial\rho)^2 - (\partial\psi/\partial z)^2] \ , \tag{3.3a}$$

$$\partial\gamma/\partial z = 2\rho(\partial\psi/\partial\rho)(\partial\psi/\partial z) \ . \tag{3.3b}$$

Equation (3.2) implies that ψ is a harmonic function in a fictitious Euclidean "background" space with cylindrical coordinates (ρ,z,ϕ). For asymptotically flat solutions with bounded sources, ψ approaches the Newtonian gravitational potential at spatial infinity:

$$\lim_{R\to\infty} \psi = -m/R + O(m^3/R^3) \ , \tag{3.4}$$

with $R \equiv (\rho^2+z^2)^{1/2}$; here m is the total active gravitational mass of the system, as measured at infinity. Once ψ is chosen, corresponding to some source of the field, γ is found from the integration of Eqs. (3.3).

The form of ψ for Schwarzschild spacetime is not spherical,[21] but rather is that of a uniform line mass M on the symmetry axis extending from $z=-M$ to $z=+M$. It is best described in prolate spheroidal coordinates (u, v, ϕ) related to Weyl coordinates by

$$\rho = M \sinh u \sin v \ , \qquad z = M \cosh u \cos v \ . \tag{3.5}$$

The Schwarzschild solution (denoted here with subscript 0) in these coordinates is

$$\psi_0 = \ln[\tanh(u/2)] \ , \qquad \gamma_0 = -\tfrac{1}{2}\ln\left[1 + \frac{\sin^2 v}{\sinh^2 u}\right] \ . \tag{3.6}$$

The Schwarzschild line element in standard form follows from Eqs. (3.1), (3.5), and (3.6) with the further coordinate transformation

$$r = 2M\cosh^2(u/2) \ , \qquad \theta = v \ . \tag{3.7}$$

Our approach will be to modify the Schwarzschild geometry with a harmonic weak perturbation ψ_1 corresponding to a particle suspended above the horizon on the symmetry axis. The Weyl functions are then written as

$$\psi = \psi_0 + \psi_1 \quad \text{and} \quad \gamma = \gamma_0 + \gamma_1$$

and the perturbation γ_1 is found from Eqs. (3.3) in u,v coordinates:

$$\partial\gamma_1/\partial v = 2\rho^2 \cot v[(\partial\psi_0/\partial\rho)(\partial\psi_1/\partial\rho)-(\partial\psi_0/\partial z)(\partial\psi_1/\partial z)] - 2\rho z \tan v[(\partial\psi_0/\partial z)(\partial\psi_1/\partial\rho)+(\partial\psi_0/\partial\rho)(\partial\psi_1/\partial z)] \ . \quad (3.8)$$

Here and throughout this analysis we keep terms only up to linear order in ψ_1.

To represent a perturbing particle on the symmetry axis we take ψ_1 to be

$$\psi_1 = -\mu[\rho^2+(z-b)^2]^{-1/2} = -\mu[M^2\sinh^2 u \sin^2 v + (b-M\cosh u \cos v)^2]^{-1/2} \ . \quad (3.9)$$

Though this appears to describe a spherically symmetric point particle, of apparent mass μ, at $z=b$ its meaning is actually rather more complicated due to the nonlinearities associated with γ. The field near $\rho=0$, $z=b$ is in fact the Curzon[22] solution with a highly nonspherical naked singularity at $\rho=0$, $z=b$. For $[\rho^2+(z-b)^2]^{1/2} \gg \mu$, however, the monopole part of the field dominates and the nonsphericity of the singularity can be ignored. For $\psi_1 \ll 1$, which is a requirement for our perturbation approach, the perturbing source can therefore be considered to be a point particle.

When ψ_1 from Eq. (3.9) is used in Eq. (3.8) the result is

$$\frac{\partial\gamma_1}{\partial v} = \frac{2\mu M}{[M^2\sinh^2 u \sin^2 v + (b-M\cosh u\cos v)^2]^{3/2}}(b-M\cosh u \cos v)\sin v \ . \quad (3.10)$$

Although this equation could be integrated exactly, we are only interested in the solution near the horizon. For static geometries the horizon is the surface at which $g_{tt}=0$ occurs, which for the spacetime here is the surface $u=0$ (since $\psi_0+\psi_1$ diverges to $-\infty$ at $u=0$). We can therefore approximate Eq. (3.10) as

$$\partial\gamma_1/\partial v = 2\mu M \sin v (b-M\cos v)^{-2}[1+O(u^2)] \quad \text{for } u \ll 1 \ , \quad (3.11)$$

which has the solution

$$\gamma_1 = 2\mu[1/(b-M) - 1/(b-M\cos v)][1+O(u^2)] + C \quad \text{for } u \ll 1 \ , \quad (3.12)$$

where C is an integration constant.

The condition for elementary flatness is $\gamma=0$ on the symmetry axis. A nonvanishing γ at $v=0$ or π indicates that the geometry has a conical singularity corresponding to an infinitesimal "rope" or "strut" on the axis. It is clear from Eq. (3.12) that γ cannot vanish both at $v=0$ and $v=\pi$. Furthermore, integration of $(\partial\gamma/\partial\rho)d\rho+(\partial\gamma/\partial z)dz$ from just below the point mass to just above it reveals that γ_1 increases by $4\mu M/(b^2-M^2)$, so γ cannot vanish on axis both above and below the particle. The physical origin of these mathematical results is the fact that constraints are necessary to keep the hole and the point mass static against the force of their mutual gravitational attraction. The mathematical requirements can be satisfied in a number of ways, as shown in Fig. 1, each of which corresponds to an intuitively correct system of constraints. The simplest, for example, is to choose $C=-4\mu M/(b^2-M^2)$ so that $\gamma=0$ holds on the axis except for a "strut" along $M<z<b$ holding the hole and the point mass apart.

We shall make a choice which avoids the necessity of connecting a strut or a rope to the hole: In addition to the particle at $z=b$, an identical particle is held static at $z=-b$. The total perturbation ψ_1 is then

$$\psi_1 = -\mu[\rho^2+(z-b)^2]^{-1/2} - \mu[\rho^2+(z+b)^2]^{-1/2} \ , \quad (3.13)$$

and γ_1 can be found from Eq. (3.12) by adding a similar expression with $b \to -b$ and opposite overall sign; thus, for $u \ll 1$,

$$\gamma_1 = 2\mu\left[\frac{1}{b-M} - \frac{1}{b-M\cos v} + \frac{1}{b+M} - \frac{1}{b+M\cos v}\right] . \quad (3.14)$$

(The overall sign change originates in the square roots in Eq. (3.13) where for $|z|<b$ on axis $[(z-b)^2]^{1/2}=b-z$ but $[(z+b)^2]^{1/2}=b+z$.) The choice of integration constant in Eq. (3.14) is that implying, on the axis,

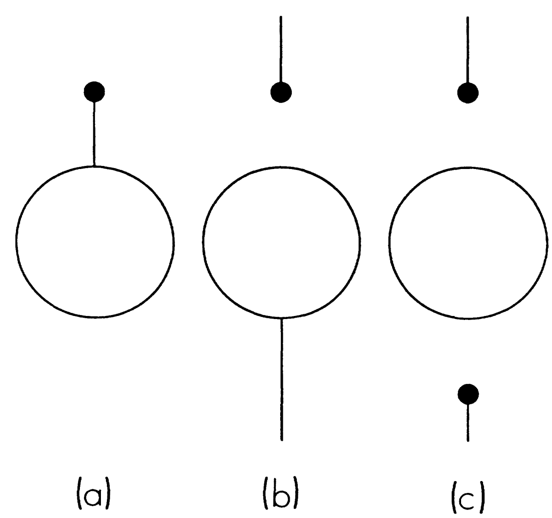

FIG. 1. Physical constraints for producing a hole-and-particle Weyl solution. In (a) a conical singularity corresponding to a strut in compresson holds the hole and particle apart; in (b) ropes to infinity provide the constraints; in (c), the solution described in the text, symmetrically placed particles supported by ropes to infinity are used.

$$\gamma=0 \quad \text{for } |z|<b\,,$$
$$\gamma=\gamma_A\equiv 4\mu M/(b^2-M^2) \quad \text{for } |z|>b\,. \tag{3.15}$$

The physical interpretation of the conical singularities is shown in Fig. 1(c); the symmetrically located particles are suspended from "ropes" anchored at infinity.

The nature of these ropes will be important in subsequent sections and warrants further discussion here. If we replace the symmetry-axis singularities with line sources of stress energy, those sources are characterized by a mass-per-unit length λ and a tension $\mathcal{T}$ related to γ_A, the value of γ on the axis, by[23]

$$\lambda=\mathcal{T}=\tfrac{1}{4}(1-e^{-\gamma_A})=\text{ const }\approx\tfrac{1}{4}\gamma_A\,. \tag{3.16}$$

The $\lambda=\mathcal{T}$ equation of state of the rope means that the passive gravitational mass of the rope vanishes. This explains why the tension $\mathcal{T}$ is constant: the rope has no weight; hence the tension at any z is only that necessary to support the particle. The lack of gravitational mass also explains why the ropes do not enter into the Newtonian-like potential ψ. These ropes then provide constraints without themselves making any apparent contribution to the gravitational field. The last (approximate) equality in (3.16) follows from (3.15), which indicates $\gamma_A\sim\psi_1\ll 1$.

A physical interpretation is also needed of the mass parameters M and μ in our solution. The surface gravity g_H can be evaluated[24] as the horizon limit of $(-g_{tt})^{1/2}|\vec{a}|$, where $\vec{a}$ is the four-acceleration of a stationary particle, with four-velocity $\vec{U}=(-g_{tt})^{-1/2}\partial/\partial t$. A straightforward calculation gives

$$g_H=(4M)^{-1}\exp(2\psi_1|_{u=0}-\gamma_1|_{u=0})\,. \tag{3.17}$$

Equations (3.9) and (3.12) show that the exponent is constant on the horizon, as it must be by the "zeroth law of black-hole mechanics."[25] A $t=$const, $u=0$ horizon section has the geometry

$$ds_H^2=4M^2e^{-2\psi_1}(e^{2\gamma_1}dv^2+\sin^2 v\,d\phi^2)\,, \tag{3.18}$$

hence the surface area

$$\mathcal{A}^H=16\pi M^2\exp(\gamma_1|_{u=0}-2\psi_1|_{u=0})\,. \tag{3.19}$$

The Smarr formula[26] for the mass M_H of a static hole,

$$M_H=g_H\mathcal{A}^H/4\pi=M\,, \tag{3.20}$$

then shows that for any ψ_1 and γ_1 nonsingular at the horizon, the hole mass is simply M.

To understand the dynamical meaning of the parameter μ we confine ourselves to the weak-field region close to the upper particle, i.e., we take

$$\mu\ll[\rho^2+(z-b)^2]^{1/2}\ll b-M\,. \tag{3.21}$$

The background potentials ψ_0 and γ_0 change on a length scale $b-M$ so we may approximate them as constant with the values at the particle location $v=0$, $u=u_p=\text{arccosh}(b/M)$:

$$\psi_0\approx\ln[\tanh(u_p/2)] \quad \text{and} \quad \gamma_0\approx 0\,. \tag{3.22}$$

From Eq. (3.10), and the fact that γ_1 vanishes on axis, we can also approximate $\gamma_1\approx 0$ away from the particle and the rope, e.g., in a neighborhood of the axis below the particle, so that the metric becomes

$$ds^2=-\tanh^2(u_p/2)e^{2\psi_1}dt^2+\coth^2(u_p/2)e^{-2\psi_1}(d\rho^2+dz^2+\rho^2d\phi^2)\,. \tag{3.23}$$

When reexpressed in local time and distance coordinates for the background,

$$\hat{t}=\tanh(u_p/2)t\,, \quad \hat{\rho}=\coth(u_p/2)\rho\,,$$
$$\hat{z}=\coth(u_p/2)z\,, \quad \text{and} \quad \hat{\phi}=\phi\,, \tag{3.24}$$

the metric takes the simple form

$$ds^2=-e^{2\psi_1}d\hat{t}^2+e^{-2\psi_1}(d\hat{\rho}^2+d\hat{z}^2+\hat{\rho}^2d\hat{\phi}^2)\,. \tag{3.25}$$

This is just the form of the weak-field metric with Newtonian potential ψ_1 (see, e.g., MTW, Sec. 18.4). In the careted coordinates ψ_1 is given by

$$\psi_1=-\mu\coth(u_p/2)/[\hat{\rho}^2+(\hat{z}-\hat{b})^2]^{1/2}\,, \tag{3.26}$$

with $\hat{b}=b\coth(u_p/2)$ and where the negligible contribution from the particle at $z=-b$ has been omitted. It follows that the active gravitational mass of each particle is

$$m=\mu\coth(u_p/2)=\mu\left[\frac{b+M}{b-M}\right]^{1/2}\,. \tag{3.27}$$

It should be noted that in the Schwarzschild geometry the acceleration of gravity at the position of the upper particle is

$$|\vec{g}|=\left[1-\frac{2M}{r}\right]^{-1/2}M/r^2$$
$$=(4M)^{-1}\coth(u_p/2)\cosh^{-4}(u_p/2) \tag{3.28}$$

implying

$$m|\vec{g}|=\frac{\mu}{M\sinh^2 u_p}=\frac{\mu M}{b^2-M^2}=\frac{1}{4}\gamma_A\,. \tag{3.29}$$

From Eq. (3.16) it follows that the tension in the ropes supporting the particles satisfies

$$\mathcal{T}=m|\vec{g}| \tag{3.30}$$

as, of course, it must.

The computation of Riemann components for the diagonal static metric of Eq. (3.1) is straightforward though tedious. We present the results in the orthonormal basis for a static observer:

$$\vec{e}_{\hat{t}}=(-g_{tt})^{-1/2}\partial/\partial t=\coth(u/2)e^{-\psi_1}\partial/\partial t,\quad \vec{e}_{\hat{u}}=(g_{uu})^{-1/2}\partial/\partial u=(2M)^{-1}\mathrm{sech}^2(u/2)e^{\psi_1-\gamma_1}\partial/\partial u\ ,$$
$$\vec{e}_{\hat{v}}=(g_{vv})^{-1/2}\partial/\partial v=(2M)^{-1}\mathrm{sech}^2(u/2)e^{\psi_1-\gamma_1}\partial/\partial v,\quad \vec{e}_{\hat{\phi}}=(g_{\phi\phi})^{-1/2}\partial/\partial\phi=(2M)^{-1}\mathrm{sech}^2(u/2)\csc v e^{\psi_1}\partial/\partial\phi\ . \tag{3.31}$$

Since we are interested only in the fields near the horizon, the computations of the Riemann components are carried out only to lowest order in u. A typical result is

$$R_{\hat{t}\hat{u}\hat{t}\hat{u}}=\{-[1/(4M^2)]+(2\mu b/M^2)/(b^2-M^2)-(\mu b/M^2)(b^2+3M^2\cos^2 v)(b^2-M^2)/(b^2-M^2\cos^2 v)^3\}[1+O(\psi_1^2)+O(u^2)]\ . \tag{3.32}$$

The $1/(4M^2)$ term in the above expression is the Riemann component due to the unperturbed Schwarzschild hole; the remaining terms are the perturbations, linear in ψ_1, due to the particle. The characteristic size of ψ_1 on the horizon is $\mu/(b-M)$ so in Eq. (3.32) we see that the second term is smaller than the first by a factor of order ψ_1. At or near $v=0$ or π, however, the ratio of the third term to the first is of order $\psi_1 M/(b-M)$. We are specifically interested in configurations in which the spacetime curvature on the horizon, near the particles, is dominated by the tidal fields of the particles, so we take as a further constraint on our parameters $\psi_1 M/(b-M)\gg 1$ or

$$\frac{m}{b-M}\left[\frac{M}{b-M}\right]^{1/2}\gg 1\ . \tag{3.33}$$

It is useful now to express the Riemann components near $v=0$ in notation appropriate for later comparisons. We therefore introduce $\tilde{\omega}$ as the proper distance along the horizon from the symmetry axis at $v=0$, and we note that $\vec{e}_{\tilde{\omega}}=\vec{e}_{\hat{v}}$. We also label the spatial normal to the horizon section $\vec{e}_n=\vec{e}_{\hat{u}}$, and we use s to represent proper distance normal to the horizon, i.e.,

$$s=2Mu[1+O(u^2)+O(\psi_1)]\ ,$$

with s_0 the distance to the particles:

$$s_0=2Mu_p\simeq 2M\,\mathrm{arccosh}(b/M)\simeq 2M[2(b/M-1)]^{1/2}\ .$$

With this notation the Riemann components on the horizon (i.e., at $s=u=0$) to first order in ψ_1 are

$$R_{\hat{t}\tilde{\omega}\hat{t}\tilde{\omega}}=R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}}=-R_{\hat{n}\tilde{\omega}\hat{n}\tilde{\omega}}=-R_{\hat{n}\hat{\phi}\hat{n}\hat{\phi}}=\tfrac{1}{2}R_{\tilde{\omega}\hat{\phi}\tilde{\omega}\hat{\phi}}=-\tfrac{1}{2}R_{\hat{t}\hat{n}\hat{t}\hat{n}}=\frac{8ms_0^3}{(s_0^2+\tilde{\omega}^2)^3}\left[1+O\left[\frac{s_0^2}{M^2}\right]+O\left[\frac{\tilde{\omega}^2}{M^2}\right]\right]\ . \tag{3.34}$$

Components not related to these by index symmetry vanish on the horizon. Components $R_{\hat{t}\hat{n}\hat{t}\tilde{\omega}}$ and $R_{\hat{n}\hat{\phi}\tilde{\omega}\hat{\phi}}$ vanish on the horizon linearly in u and are computed to be

$$R_{\hat{t}\hat{n}\hat{t}\tilde{\omega}}=R_{\hat{n}\hat{\phi}\tilde{\omega}\hat{\phi}}=\frac{48ms_0^3\tilde{\omega}s}{(s_0^2+\tilde{\omega}^2)^4}\left[1+O\left[\frac{s_0^2}{M^2}\right]+O\left[\frac{\tilde{\omega}^2}{M^2}\right]\right]\ . \tag{3.35}$$

The above results apply of course only if our restrictions on the geometry and on mass parameters are satisfied. In the present notation these restrictions are (i) the condition $\psi_1\ll 1$ requires that m be small enough, i.e.,

$$\frac{m}{s_0}\ll 1\ . \tag{3.36a}$$

(ii) The condition [Eq. (3.33)] that the particle dominate the tidal field at the horizon requires that (s_0/M) be small enough to give

$$\frac{m}{s_0}\left[\frac{M}{s_0}\right]^2\gg 1\ . \tag{3.36b}$$

It should also be noted that the tidal field described by Eqs. (3.34) and (3.35) are those due to the particle at $z=b$. The particle at $z=-b$ has negligible influence in this region of validity. If, for example, the constraints in Fig. 1(b) were used the results would be unchanged.

B. The Rindler approach

We now use the Rindler geometry of Eq. (2.59) to find the tidal fields on the horizon for a model problem analogous to that analyzed above in the Weyl formalism. The analogous configuration consists of a particle of mass m at $x=0$, $y=0$, and $z=s_0$. To constrain the particle we invoke a rope of infinitesimal cross section, with mass-per-unit-length λ and tension $\mathcal{T}$, extending upward (i.e., to larger z) from $z=s_0$, along the $x=y=0$ axis. As the equation of state of the rope we use $\lambda=\mathcal{T}$ so that the rope is weightless. [See the discussion following Eq. (3.16).]

The stress energy of the particle source $T^{\mu\nu}_{\text{particle}}$ has, as its only nonvanishing component in these coordinates,

$$T^{tt}_{\rm particle}=(m/g_H^2z^2)\delta(x)\delta(y)\delta(z-s_0)\ . \qquad (3.37)$$

For the stress energy of the rope the nonvanishing components are

$$T^{tt}_{\rm rope}=[\lambda(z)/g_H^2z^2]\delta(x)\delta(y)H(z-s_0)\ ,$$
$$T^{zz}_{\rm rope}=-\mathcal{T}(z)\delta(x)\delta(y)H(z-s_0)\ , \qquad (3.38)$$

where H is the unit step function. The force-balance equation $(T^{z\mu}_{\rm particle}+T^{z\mu}_{\rm rope})_{;\mu}=0$ requires

$$\frac{d\mathcal{T}}{dz}=\frac{\lambda-\mathcal{T}}{z} \quad \text{and} \quad \mathcal{T}|_{z=z_0}=\frac{m}{s_0} \qquad (3.39)$$

so that, for the $\lambda=\mathcal{T}$ equation of state, $\mathcal{T}$ is constant and is given by

$$\lambda=\mathcal{T}=m\,|\vec{g}_0|=mg_H/\alpha_0\ , \qquad (3.40)$$

where $\vec{g}_0$ is the acceleration of gravity at $z=s_0=\alpha_0/g_H$ (i.e., the acceleration of a particle stationary in Rindler coordinates at $z=s_0$).

It will be useful in the calculations below to write the stress-energy tensor in a coordinate-independent form. To do this we let $\vec{U}(\tau)$ be the particle four-velocity, as a function of its proper time τ, and we take $\vec{U}(s,\tau)$ to be the four-velocity of the segment of rope at proper length s. The unit vector tangent to the rope in the rope rest frame is denoted $\vec{t}$. At any spacetime point $\vec{x}$ the total stress energy can be written

$$T^{\mu\nu}(\vec{x})=m\int U^{\mu}(\tau)U^{\nu}(\tau)\delta^{4}[\vec{x}-\vec{\xi}(\tau)]d\tau+\lambda\int\int[U^{\mu}(s,\tau)U^{\nu}(s,\tau)-t^{\mu}(s,\tau)t^{\nu}(s,\tau)]\delta^{4}[\vec{x}-\vec{\xi}(s,\tau)]d\tau\, ds\ , \qquad (3.41)$$

where $\vec{\xi}(\tau)$ specifies the world line of the particle, and $\vec{\xi}(s,\tau)$ gives the world line of the rope segment at s.

The perturbations of Rindler spacetime due to the particle and rope are easily found if Minkowski coordinates (T, x, y, Z) are used for the Rindler geometry [see Eqs. (2.60) and (2.61)]. We can then follow the standard approach (see, e.g., MTW, Chap. 18) of defining the metric perturbations $h_{\mu\nu}$ of flat spacetime (with metric $g^F_{\mu\nu}$) by

$$h_{\mu\nu}\equiv g_{\mu\nu}-g^F_{\mu\nu} \qquad (3.42)$$

and the trace-reversed perturbations $\bar{h}_{\mu\nu}$ by

$$\bar{h}_{\mu\nu}\equiv h_{\mu\nu}-\tfrac{1}{2}g^F_{\mu\nu}h_{\beta}{}^{\beta}\ . \qquad (3.43)$$

In all calculations only perturbations of first order in $h_{\mu\nu}$ are kept, implying, e.g., $h_{\beta}{}^{\beta}=h_{\mu\nu}g^{F\mu\nu}$. When the Lorentz gauge condition

$$\bar{h}^{\mu\nu}{}_{;\nu}=0 \qquad (3.44)$$

is invoked, the solution for $h_{\mu\nu}$ at spacetime point $\vec{x}=(\mathbf{x},t)$, due to perturbing stress energy $T_{\mu\nu}$, takes the familiar form

$$\bar{h}_{\mu\nu}(\vec{x})=\int\frac{4T_{\mu\nu}(\mathbf{x}',t')\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)d^3x'dt'}{|\mathbf{x}-\mathbf{x}'|}\ , \qquad (3.45)$$

but only in a Minkowski coordinate basis system.

When the particle stress energy, the first integral in Eq. (3.41), is used in Eq. (3.45) the result is the linearized-gravity analog of the Liénard-Wiechert potentials

$$\bar{h}_{\mu\nu}(\vec{x})|_{\rm particle}=\left.\frac{4mU_{\mu}U_{\nu}}{\vec{U}\cdot\vec{k}}\right|_{\rm ret}\ , \qquad (3.46)$$

where $\vec{k}$ is the null vector from the field point $\vec{x}$ to the retarded event $\vec{x}_{\rm ret}$ at the intersection of the particle world line and the past light cone of $\vec{x}$:

$$\vec{k}=\vec{x}_{\rm ret}-\vec{x}\ . \qquad (3.47)$$

The "ret" subscript on the right-hand side of Eq. (3.46) indicates that $\vec{U}$ is to be evaluated at $\vec{x}_{\rm ret}$. The rope contribution to $\bar{h}_{\mu\nu}$ is

$$\bar{h}_{\mu\nu}(\vec{x})|_{\rm rope}=4\mathcal{T}\int\frac{U_{\mu}(s)U_{\nu}(s)-t_{\mu}(s)t_{\nu}(s)}{\vec{U}(s)\cdot\vec{k}(s)}ds\ , \qquad (3.48)$$

where $\vec{k}(s)$ is the null vector from $\vec{x}$ to the retarded event for the rope segment at s.

To evaluate the expressions in Eqs. (3.46) and (3.48) we start by considering a segment of the rope at $z=z'=\alpha'/g_H$, and for a field point at t, x, y, $z=\alpha/g_H$ we denote by $t'_{\rm ret}$ the retarded time for the contribution from the segment at α'. The four-velocity and tangent for the segment are $\vec{U}=(1/\alpha')\vec{e}_t$ and $\vec{t}=\vec{e}_z$ in the Rindler basis. But Eq. (3.48) is valid only in a Minkowski basis, so we use Eq. (2.59) at time $t'_{\rm ret}$ to arrive at

$$U^T=\cosh g_H t'_{\rm ret} \quad \text{and} \quad U^Z=\sinh g_H t'_{\rm ret}\ ;$$
$$t^T=\sinh g_H t'_{\rm ret} \quad \text{and} \quad t^Z=\cosh g_H t'_{\rm ret}\ . \qquad (3.49)$$

Proper length along the rope can be parametrized with α', using $ds=dz'=g_H^{-1}d\alpha'$, with the rope extending from $\alpha_0=g_Hs_0$ to $\alpha=\infty$. Equation (3.48) then gives, as the only components of the rope contribution,

$$\bar{h}_{TT}(\vec{x})|_{\rm rope}=-\bar{h}_{ZZ}(\vec{x})|_{\rm rope}=\frac{4\mathcal{T}}{g_H}\int_{\alpha_0}^{\infty}\frac{d\alpha'}{\vec{U}(\alpha')\cdot\vec{k}(\alpha')}\ . \qquad (3.50)$$

At the field point $\vec{x}$ the Rindler and Minkowski bases for the flat background spacetime are related by

$$\vec{e}_t = \alpha(\cosh g_H t \; \vec{e}_T + \sinh g_H t \; \vec{e}_Z) \; ,$$
$$\vec{e}_z = \sinh g_H t \; \vec{e}_T + \cosh g_H t \; \vec{e}_Z \; . \tag{3.51}$$

With these we can compute the Rindler components of $\bar{h}_{\mu\nu}$. The only nonvanishing components are

$$\alpha^{-2}\bar{h}_{tt} \mid_{\rm rope} = -\bar{h}_{zz} \mid_{\rm rope} = \frac{4\mathcal{T}}{g_H}\int_{\alpha_0}^{\infty} \frac{d\alpha'}{\vec{U}(\alpha')\cdot\vec{k}(\alpha')} \; . \tag{3.52}$$

If Eq. (2.60) is used to evaluate the Rindler components of $\vec{k}$, the denominator in Eq. (3.52) can be written

$$\vec{U}(\alpha')\cdot\vec{k}(\alpha') = -U^T(T'_{\rm ret}-T) + U^Z(Z'_{\rm ret}-Z)$$
$$= \alpha g_H^{-1} \sinh[g_H(t-t'_{\rm ret})] \; . \tag{3.53}$$

If we are to evaluate the integral in Eq. (3.52) we must solve for $t-t'_{\rm ret}$ in terms of the field-point coordinates and the source-point parameter α'. This relationship follows from the retardation equation $\vec{k}\cdot\vec{k}=0$. To simplify notation we shall henceforth use

$$S' \equiv \sinh[g_H(t-t'_{\rm ret})] \; , \quad C' \equiv \cosh[g_H(t-t'_{\rm ret})] \; , \tag{3.54a}$$

$$\varpi^2 = x^2 + y^2 \; . \tag{3.54b}$$

The retardation equation then gives

$$\vec{k}\cdot\vec{k} = 0 = -(T'_{\rm ret}-T)^2 + \varpi^2 + (Z'_{\rm ret}-Z)^2$$
$$= \varpi^2 + g_H^{-2}(\alpha'^2 + \alpha^2 - 2\alpha\alpha' C') \; . \tag{3.55}$$

This implies

$$C' = \frac{\alpha^2 + \alpha'^2 + g_H^2 \varpi^2}{2\alpha\alpha'} \tag{3.56}$$

and

$$\vec{U}(\alpha')\cdot\vec{k}(\alpha') = \alpha g_H^{-1} S' = \alpha g_H^{-1}(C'^2-1)^{1/2}$$
$$= [(g_H^2\varpi^2 + \alpha^2 + \alpha'^2)^2 - 4\alpha^2\alpha'^2]^{1/2}/2g_H\alpha' \; . \tag{3.57}$$

When Eq. (3.57) is used in the integral in Eq. (3.52) a difficulty emerges: $\vec{U}\cdot\vec{k} \to \alpha'/2g_H$ as $\alpha' \to \infty$, so the integral is logarithmically divergent. This is a pure gauge effect; an explicit gauge transformation can be performed to produce finite expressions for $\bar{h}_{\mu\nu}\mid_{\rm rope}$. There is, however, a simpler and more instructive way of eliminating the divergent integral: The rope can be terminated at some arbitrary $\alpha=\alpha_f=g_H s_f$. The force necessary to support the upper end of the rope can be supplied conveniently, if somewhat unphysically, by a negative-mass particle. Intuition suggests that the magnitude m_f of the mass must satisfy

$$m \mid \vec{g} \mid_{{\rm at}\ \alpha_0} = \mathcal{T} = m_f \mid \vec{g} \mid_{{\rm at}\ \alpha_f}$$

or

$$\frac{m g_H}{\alpha_0} = \mathcal{T} = \frac{m_f g_H}{\alpha_f} \; . \tag{3.58}$$

It is straightforward to add negative mass contributions to Eq. (3.37) [an additional term with $m\delta(z-s_0)$ replaced by $-m_f\delta(z-s_f)$] and to Eqs. (3.38) [an additional factor of $H(s_f-z)$] and to verify that the force balance equation $T^{z\mu}{}_{;\mu}=0$ does indeed reduce to Eq. (3.58).

The artifice of the negative mass is very convenient since it requires no additional computation. Once the particle contribution in $\bar{h}_{\mu\nu}$ and in the Riemann tensor have been found, the changes $\alpha_0 \to \alpha_f$ and $m \to -m_f = -(\alpha_f/\alpha_0)m$ give the contribution of the negative mass; in the rope integral in Eq. (3.52) the upper limit of integration is taken to be α_f. Furthermore, the negative mass can be eliminated from our final results for the Riemann components. Since the Riemann components are gauge invariant they are not plagued by the gauge-dependent singularity of $\bar{h}_{\mu\nu}$. There is therefore no difficulty in taking the $\alpha_f \to \infty$ limit. (The results have been checked by also computing the Riemann components following an explicit gauge transformation of $\bar{h}_{\mu\nu}$.)

With a cutoff at α_f the integral in Eq. (3.52) can now be evaluated:

$$\alpha^{-2}\bar{h}_{tt} \mid_{\rm rope} = -\bar{h}_{zz} \mid_{\rm rope}$$
$$= \frac{4\mathcal{T}}{g_H}\int_{\alpha_0}^{\alpha_f} \frac{d\alpha'}{\vec{U}(\alpha')\cdot\vec{k}(\alpha')}$$
$$= 4\mathcal{T} \ln\left[\frac{\delta_f}{\delta_0}\right] \tag{3.59a}$$

with

$$\delta_0 \equiv \xi_0 + (\xi_0^2 + 4\alpha^2 g_H^2 \varpi^2)^{1/2} \quad \text{and} \quad \xi_0 \equiv g_H^2\varpi^2 + \alpha_0^2 - \alpha^2 \; , \tag{3.59b}$$

and where δ_f is defined by the same expressions with α_0 replaced by α_f throughout.

We now turn to the evaluation of the contributions from the ends of the rope. It is convenient to define $t_{\rm ret\ 0}$ and $t_{\rm ret\ f}$ as the retarded times at α_0 and at α_f, and to define C_0, S_0, C_f, and S_f as in Eqs. (3.54a), for example,

$$C_0 \equiv \cosh[g_H(t-t_{\rm ret\ 0})]$$
$$= (\alpha^2 + \alpha_0^2 + g_H^2\varpi^2)/2\alpha\alpha_0 \; , \tag{3.60}$$

where the second equality follows from Eq. (3.56) with α' taken to be α_0. It is then straightforward to evaluate the Minkowski components of the particle contribution (3.46) with Eqs. (3.49) and (3.53). When the results are expressed as components in the Rindler basis at the field point we find

$$\alpha^{-2}\bar{h}_{tt} \mid_{\rm particle} = \frac{4g_H m}{\alpha}\,\frac{C_0^2}{S_0} \; ,$$
$$\alpha^{-1}\bar{h}_{tz} \mid_{\rm particle} = \frac{4g_H m}{\alpha} C_0 \; , \tag{3.61}$$
$$\bar{h}_{zz} \mid_{\rm particle} = \frac{4g_H m}{\alpha} S_0 \; .$$

The contributions due to the negative-mass particle then follow immediately:

$$\alpha^{-2}\bar{h}_{tt}\,|_{\text{neg mass}} = -\frac{4g_H m_f}{\alpha}\frac{C_f^2}{S_f}\,,$$
$$\alpha^{-1}\bar{h}_{tz}\,|_{\text{neg mass}} = -\frac{4g_H m_f}{\alpha}C_f\,, \tag{3.62}$$
$$\bar{h}_{zz}\,|_{\text{neg mass}} = -\frac{4g_H m_f}{\alpha}S_f\,.$$

We can now sum Eqs. (3.59), (3.61), and (3.62) to find $\bar{h}_{\mu\nu}$ and can invert Eq. (3.43):

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \tfrac{1}{2}g^F_{\mu\nu}\bar{h}_\beta{}^\beta\,, \tag{3.63}$$

where, of course, $g^F_{\mu\nu}$ is the Rindler metric. The results, the perturbations of Rindler spacetime in the Lorentz gauge, are

$$\alpha^{-2}h_{tt} = h_{zz} = \frac{4mg_H}{\alpha}\left[\left[S_0 + \frac{1}{2S_0}\right] - \frac{\alpha_f}{\alpha_0}\left[S_f + \frac{1}{2S_f}\right]\right],$$
$$\alpha^{-1}h_{tz} = \frac{4mg_H}{\alpha}\left[C_0 - \frac{\alpha_f}{\alpha_0}C_f\right], \tag{3.64}$$
$$h_{xx} = h_{yy} = \frac{2mg_H}{\alpha}\left[\frac{1}{S_0} - \frac{\alpha_f}{\alpha_0}\frac{1}{S_f}\right] + \frac{4mg_H}{\alpha_o}\ln\left[\frac{\delta_f}{\delta_0}\right],$$

in which Eq. (3.58) has been used to eliminate $\mathcal{T}$ and m_f.

With these results the Riemann components can be calculated from

$$R_{\mu\beta\gamma\delta} = \tfrac{1}{2}(h_{\mu\delta;\beta\gamma} + h_{\beta\gamma;\mu\delta} - h_{\beta\delta;\mu\gamma} - h_{\mu\gamma;\beta\delta})\,. \tag{3.65}$$

The perturbed spacetime is clearly symmetric about the z axis so it is convenient to introduce polar coordinates $(\tilde{\omega},\phi)$ for the xy sections and to give the Riemann components in the orthonormal basis

$$\vec{e}_{\hat{t}} = \alpha^{-1}\vec{e}_t\,,\quad \vec{e}_{\hat{\tilde{\omega}}} = \cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y\,,$$
$$\vec{e}_{\hat{z}} = \vec{e}_z\,,\quad \vec{e}_{\hat{\phi}} = -\sin\phi\,\vec{e}_x + \cos\phi\,\vec{e}_y\,. \tag{3.66}$$

When the metric in Eq. (3.64) is used in Eq. (3.65) a typical result is

$$R_{\hat{t}\hat{\tilde{\omega}}\hat{t}\hat{\tilde{\omega}}} = \frac{mg_H^3}{\alpha^3}\left\{\frac{1}{S_0^3} - \frac{3}{S_0^5}\left[\frac{\tilde{\omega}}{s_0}\right]^2 - \frac{\alpha_f}{\alpha_0}\left[\frac{1}{S_f^3} - \frac{3}{S_f^5}\left[\frac{\tilde{\omega}}{s_f}\right]^2\right]\right\}, \tag{3.67}$$

where $s_0 \equiv \alpha_0/g_H$ and $s_f \equiv \alpha_f/g_H$ are proper distances defined as were the analogous quantities in the previous subsection.

We can now take the rope to be infinitely long, that is, we assume $\alpha_f \to \infty$, hence $S_f \to \alpha_f/2\alpha \to \infty$. In this limit the Riemann components are

$$R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -R_{\hat{z}\hat{\tilde{\omega}}\hat{z}\hat{\tilde{\omega}}} = \frac{mg_H^3}{\alpha^3 S_0^3}\,,$$
$$R_{\hat{t}\hat{\tilde{\omega}}\hat{t}\hat{\tilde{\omega}}} = -R_{\hat{\phi}\hat{z}\hat{\phi}\hat{z}} = \frac{mg_H^3}{\alpha^3 S_0^3}\left[1 - \frac{3}{S_0^2}\left[\frac{\tilde{\omega}}{s_0}\right]^2\right],$$
$$R_{\hat{\phi}\hat{\tilde{\omega}}\hat{\phi}\hat{\tilde{\omega}}} = -R_{\hat{t}\hat{z}\hat{t}\hat{z}} = \frac{2mg_H^3}{\alpha^3 S_0^3}\left[1 - \frac{3}{2S_0^2}\left[\frac{\tilde{\omega}}{s_0}\right]^2\right], \tag{3.68}$$
$$R_{\hat{\tilde{\omega}}\hat{t}\hat{z}\hat{t}} = R_{\hat{\tilde{\omega}}\hat{\phi}\hat{z}\hat{\phi}} = \frac{3mg_H^4\tilde{\omega}}{\alpha^2\alpha_0 S_0^5}\left[\frac{C_0}{\alpha} - \frac{1}{\alpha_0}\right].$$

We next take the horizon limit $\alpha \to 0$, noticing

$$S_0 \to \frac{\alpha_0^2 + g_H^2\tilde{\omega}^2}{2\alpha_0\alpha} = \frac{s_0^2 + \tilde{\omega}^2}{2\,ss_0}\,,$$

where we have used $s \equiv \alpha/g_H$. In this limit ($\alpha \ll 1$, infinite rope) the Riemann components are

$$R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -R_{\hat{z}\hat{\tilde{\omega}}\hat{z}\hat{\tilde{\omega}}} = R_{\hat{t}\hat{\tilde{\omega}}\hat{t}\hat{\tilde{\omega}}} = -R_{\hat{\phi}\hat{z}\hat{\phi}\hat{z}} = \tfrac{1}{2}R_{\hat{\phi}\hat{\tilde{\omega}}\hat{\phi}\hat{\tilde{\omega}}} = -\tfrac{1}{2}R_{\hat{t}\hat{z}\hat{t}\hat{z}} = \frac{8m\,s_0^3}{(s_0^2 + \tilde{\omega}^2)^3}\,,$$
$$R_{\hat{\tilde{\omega}}\hat{t}\hat{z}\hat{t}} = R_{\hat{\tilde{\omega}}\hat{\phi}\hat{z}\hat{\phi}} = \frac{48\,m\,s_0^3\,\tilde{\omega}\,s}{(s_0^2 + \tilde{\omega}^2)^4}\,, \tag{3.69}$$

which agree perfectly with the results in Eqs. (3.34) and (3.35).

C. Interpretation

In Sec. III A we computed the near-horizon Riemann components for a static particle in the Schwarzschild geometry, under certain geometric restrictions. In Sec. III B these were found to agree with the results of the computation for the analogous configuration in the Rindler geometry. In both computations the constraining force on the mass point is provided by an idealized "rope" which is weightless and can be considered to have a minimal effect on the tidal fields at the horizon. We can therefore consider the horizon fields as being produced by the mass point alone. The justification of this interpretation is particularly clear in Sec. III A in which the stress energy of the rope never enters; further justification will appear in Sec. IV in which certain results will be seen to be consistent with the viewpoint that tidal fields emanate only from the particle.

The agreement of the results in Secs. III A and III B is more-or-less expected since the Rindler geometry approximates the near-horizon Schwarzschild geometry. It may seem that the constraining rope, extending to infinite height, does not lie within the realm of this approximation. The rope, however, need not be infinite for our results to apply. We can reconsider results for a finite rope extending from α_0 to α_f and terminated there (by a negative-mass particle, or whatever). In Sec. III B this was explicit in expressions such as (3.67). In Sec. III A we can infer the finite-rope result by adding to Eqs. (3.34) and (3.35) analogous expressions with α_0 replaced by α_f and m replaced by $-(\alpha_f/\alpha_0)m$. The condition that the contributions from the upper end of the rope can be ignored is

$$S_f^3/\alpha_f \gg S_0^3/\alpha_0 \,. \tag{3.70}$$

[See, e.g., Eq. (3.67).] This requirement is satisfied at a particular value of $\tilde{\omega}$ if α_f is chosen large enough to ensure

$$\alpha_f \gg \alpha_0 \quad \text{and} \quad \alpha_f \gg (g_H\tilde{\omega})^3/\alpha_0^2 \,. \tag{3.71}$$

For sufficiently small $\tilde{\omega}$ these conditions can be satisfied with $\alpha_f \ll 1$ so that the entire source configuration can lie within the region in which the Rindler geometry approximates the Schwarzschild geometry.

With the nature of the Rindler approximation clarified by the above example, we may now use the approximation to extend our results. For a patch of the horizon with $g_H\tilde{\omega} \ll 1$, and for a source lying in the $\alpha \ll 1$ region, the Rindler spacetime approximates Kerr spacetime as well as Schwarzschild. We may therefore interpret the results of Secs. III A and III B as the horizon fields due to a mass point and rope comoving above the Kerr horizon, i.e., due to a mass point at constant Boyer-Lindquist coordinates $r=r_0$ and $\theta^\dagger=\theta_0^\dagger$ and with $\phi^\dagger=\phi_0^\dagger+(a/2Mr_H)t$ [cf. Eq. (2.14)]. In this case the horizon fields are those given by Eq. (3.68) with the interpretation [from (2.13) and (2.14)]:

$$\begin{aligned} s_0 &= 2\left[\frac{r_0-r_H}{r_H-M}\right]^{1/2}(Mr_H-\tfrac{1}{2}a^2\sin^2\theta_0^\dagger)^{1/2}, \\ \tilde{\omega}^2 &= \rho_H^2\sin^2\theta_0^\dagger(\theta^\dagger-\theta_0^\dagger)^2 \\ &\quad +\frac{(2Mr_H)^2}{\rho_H^2}[\phi^\dagger-\phi_0^\dagger-(a/2Mr_H)t]^2 \,, \\ \rho_H^2 &= r_H^2+a^2\cos^2\theta_0^\dagger \,. \end{aligned} \tag{3.72}$$

The restriction to a limited region of spacetime near the horizon has, in the above example, allowed us to find an approximate solution of a difficult problem (a particle orbiting near a Kerr black hole) with a relatively simple computation in Rindler spacetime. Effects, such as centrifugal acceleration, absent in Rindler spacetime, are guaranteed to make negligible $O(\alpha_f^2)$ fractional corrections to the results. (For details see PT.) The use of the Rindler approximation in this way will be the basis of the model problems in Sec. IV involving mass points which are not comoving with the horizon.

The mass point comoving with the horizon, i.e., the situation considered in Secs. III A and III B, is itself a rather trivial example of the formalism outlined in Sec. II. The results in Eq. (3.34) or (3.69) show that the components $R_{\hat\rho\hat\varphi\hat\beta\hat\gamma}=C_{\hat\rho\hat\varphi\hat\beta\hat\gamma}$ as measured by FIDO's are finite on the horizon and the horizon fields $\overleftrightarrow{\mathcal{E}}^H$, $\overleftrightarrow{\mathcal{B}}^H$ vanish [see Eqs. (2.32)–(2.34)]. This is compatible with the boundary conditions in Eq. (2.35). Note in fact that $\overleftrightarrow{\mathcal{B}}$ will always vanish for a static spacetime, a consequence of $C_{ijk0}=0$ [cf. Eq. (2.32)]. It follows from the horizon boundary conditions (2.35) that for a static spacetime $\overleftrightarrow{\mathcal{E}}^H$ must vanish. The vanishing of $\overleftrightarrow{\mathcal{E}}^H$ means that there is no source in the tidal force equation (2.8) or in (2.25a). The horizon shear $\overleftrightarrow{\sigma}^H$ must then vanish and this in turn means that there is no source in the focusing equation (2.9) or in (2.25b), so that horizon expansion θ^H also must vanish. These results, $\overleftrightarrow{\sigma}^H=\theta^H=0$, are of course always true for a stationary horizon, as is obvious intuitively and is explicit in Eq. (2.5).

It is useful to consider qualitatively how our results would change if the particle were not held stationary, but rather were lowered in a quasistationary manner, on a time scale $t_D \gg g_H^{-1}$, toward the hole. Since the fiducion (or generator) positions change on this same time scale, the horizon shear must be of order $1/t_D$. From Eq. (2.21) with the time derivative ignored (using $\partial/\partial t \sim 1/t_D \ll g_H$) we infer $\mathcal{E}_{ab}^H \sim 1/t_D$, and Eq. (2.22) implies $\theta^H \sim 1/t_D^2$. The time-integrated shear $\Sigma_{ab}^H \sim \sigma_{ab}^H t_D$ is independent of t_D and the time-integrated expansion $\Theta^H \sim \theta^H t_D$ is of order $1/t_D$. The nonvanishing of Σ_{ab}^H in the limit $t_D \to \infty$ accounts for the fact that during the process of lowering the particle, no matter how slowly it is done, the intrinsic geometry of the horizon must change, since the horizon geometry in the static case is distorted. This distortion, to first order in the mass of the particle, is best described by the Ricci curvature scalar for the horizon section

$$R=\frac{32ms_0^3}{(\tilde{\omega}^2+s_0^2)^3} \,, \tag{3.73}$$

which is straightforward to compute in the Weyl or Rindler geometries. [It can, for example, be found from the component $R_{\tilde{\omega}\hat\phi\tilde{\omega}\hat\phi}$ in Eq. (3.34) with the use of the Gauss-Codazzi equations.] In Sec. IV B an explicit example will be given in which this result emerges as the static limit for a slowly moving particle. The vanishing of Θ^H has the obvious meaning that the area of the horizon remains constant; this of course is required by the fact that the quasistationary process is reversible.

We conclude this section by summarizing the conditions for the validity of the Rindler approximation to the Kerr geometry and the perturbation scheme for point particles, the approach that will be used throughout the next section. (i) The field of the particle must be weak at the horizon. If the particle mass is m, and its distance to the horizon is s_0, this condition is

$$m/s_0 \ll 1 \tag{3.74}$$

[cf. Eq. (3.36a)]. This condition must be satisfied for that portion of the particle motion that produces the horizon-distorting fields being studied. (ii) During the field-producing particle motion, the particle must be in the region $\alpha \ll 1$, i.e.,

$$s_0 \ll g_H^{-1} . \tag{3.75}$$

(iii) The horizon region considered must be small in extent in comparison with the horizon radius,

$$\tilde{\omega} \ll r_H \tag{3.76a}$$

[see Eqs. (2.57)–(2.59)]. We will confine attention to holes not too near the Kerr limit and take this condition to be

$$\tilde{\omega} \ll M . \tag{3.76b}$$

In Sec. III A, we made an additional assumption that at the horizon the Riemann curvature due to the particle dominates that due to the background. [See Eqs. (3.33) and (3.36b).] If a Kerr hole is not near its extreme Kerr limit (i.e., a is not too near M), and we are interested in the horizon not too near the polar regions, this condition remains that given in (3.36b):

$$m/s_0^3 \gg M^{-2} . \tag{3.77}$$

In Sec. IV we shall be interested in dynamical changes in the horizon, changes which are driven only by the components of Riemann curvature that constitute $\overleftrightarrow{\mathcal{E}}^H$ and $\overleftrightarrow{\mathcal{B}}^H$. These components vanish for the background (and are slicing invariant) so that the particle field dominates that of the background, without the imposition of condition (3.77). This condition is relevant only to one consideration: In using the Rindler approximation to find the perturbation to the metric of horizon sections, we may be ignoring larger static constributions unless condition (3.77) is satisfied.

IV. MODEL PROBLEMS WITH POINT MASSES

A. Radially accelerating point mass

As an example of a dynamical source of horizon distortion we now consider a particle of mass m which moves on a trajectory normal to the horizon with a constant acceleration "upward" (i.e., away from the horizon). We adjust the parameters of the motion so that the particle moves downward from infinity and reaches a minimum height $z=s_m$ above the Rindler horizon before returning to infinity. To treat the problem perturbatively we shall assume

$$m/s_m \ll 1 . \tag{4.1}$$

(Further constraints necessary for the validity of the Rindler approximation will be discussed at the end of the subsection.) The condition for the particle to return to infinity from $z=s_m$ is that the (constant) acceleration A of the particle be greater than the acceleration of gravity $1/s_m \equiv g_H/\alpha_m$ at $z=s_m$. We therefore take

$$A=(1+f)/s_m , \tag{4.2}$$

with $f>0$. This acceleration is to be provided by a $\lambda=\mathcal{T}$ weightless rope which we have seen to produce minimal tidal gravitational effects.

Our approach will be to solve the problem with the Rindler approximation to find the tidal fields at the horizon. We can, in fact, do this by a simple transformation of the tidal fields found in Sec. III. We let (t, x, y, z) be our Rindler coordinates and (T, x, y, Z) the associated Minkowski coordinates, related by

$$T=z\sinh(g_H t) \quad \text{and} \quad Z=z\cosh(g_H t) . \tag{4.3}$$

We now note that the trajectory of the particle with constant acceleration A and minimum "height" $z=s_m=\alpha_m/g_H$ is

$$\begin{aligned} &(Z-d)^2-T^2=A^{-2} , \\ &d\equiv s_m - A^{-1}=s_m f/(1+f) . \end{aligned} \tag{4.4}$$

The world line of this particle is the hyperbola pictured in Fig. 2.

We now introduce a shifted set of Minkowski coordinates

$$T'=T \quad \text{and} \quad Z'=Z-d, \tag{4.5}$$

and the associated shifted Rindler coordinates t', x, y, and $z'=\alpha'/g_H$ defined by

$$T'=z'\sinh(g_H t') \quad \text{and} \quad Z'=z'\cosh(g_H t') . \tag{4.6}$$

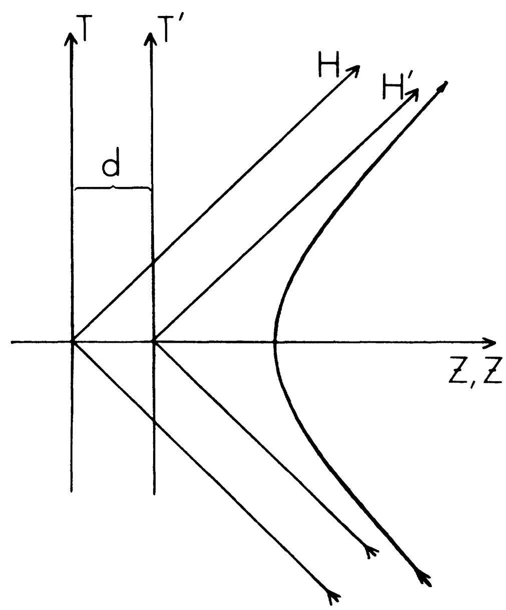

FIG. 2. Coordinate systems for the analysis of a radially accelerating particle. The hyperbola is the world line of a particle with constant acceleration A. Minkowski coordinates T,Z are those associated with the Rindler coordinates t,z in which the particle acceleraton is $A\vec{e}_z$; in these coordinates the Rindler horizon H is at $T=Z$. Minkowski coordinates T', Z' are shifted relative to T,Z, by d, as shown. In the associated Rindler coordinates t', α' the world line is described by $\alpha'=g_H/A$. See text for details.

In the shifted Rindler coordinates (cf. Fig. 2) then the particle is static at position $x=y=0$ and

$$z' \equiv z_0' \equiv \alpha_0'/g_H = A^{-1} = s_m/(1+f). \tag{4.7}$$

But the tidal fields of a static particle, supported by a weightless rope, have already been given in Sec. III. We need only change notation appropriately to infer from Eq. (3.68), for example,

$$R_{\hat{t}'\hat{\phi}\hat{t}'\hat{\phi}} = \frac{mg_H^3}{\alpha'^3 S_0'^3}, \tag{4.8}$$

with

$$(S_0')^2 \equiv (C_0')^2 - 1$$
$$= [(\alpha'^2+\alpha_0'^2+g_H^2\varpi^2)^2 - 4\alpha'^2\alpha_0'^2]/4\alpha'^2\alpha_0'^2. \tag{4.9}$$

It remains to transform from the shifted Rindler basis to the original Rindler basis using the transformations in Eqs. (4.3), (4.5), and (4.6). A typical result is

$$R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = \cosh^2[g_H(t-t')]R_{\hat{t}'\hat{\phi}\hat{t}'\hat{\phi}}$$
$$+2\sinh[g_H(t-t')]\cosh[g_H(t-t')]R_{\hat{t}'\hat{\phi}\hat{z}'\hat{\phi}}$$
$$+\sinh^2[g_H(t-t')]R_{\hat{z}'\hat{\phi}\hat{z}'\hat{\phi}}. \tag{4.10}$$

The results of this composite transformation are somewhat lengthy and we present here only the components of greatest interest in a mixed, but concise, notation:

$$R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = \frac{mg_H^3}{(\alpha' S_0')^3}\left[1+\frac{3g_H^4T^2d^2(A\varpi)^2}{(\alpha\alpha' S_0')^2}\right], \tag{4.11a}$$

$$R_{\hat{t}\hat{\varpi}\hat{t}\hat{\varpi}} = \frac{mg_H^3}{(\alpha' S_0')^3}\left[1-\frac{3g_H^4(A\varpi)^2}{(\alpha\alpha' S_0')^2}(ZZ'-TT')^2\right], \tag{4.11b}$$

$$R_{\hat{t}\hat{\varpi}\hat{z}\hat{\varpi}} = \frac{3mg_H^7(A\varpi)^2}{\alpha^2(\alpha' S_0')^5}Td(ZZ'-T^2) = -R_{\hat{t}\hat{\phi}\hat{z}\hat{\phi}}. \tag{4.11c}$$

In this notation all primed symbols (α', S_0', Z', T') are understood to represent functions of the original Rindler coordinates (α, t, ϖ). Considering $\alpha'^2 = g_H^2(Z'^2 - T'^2)$ it would seem that α' is imaginary and ambiguous (choice of branch) for $|Z'| < |T'|$ or $|Z-d| < |T|$, a region which includes points on the horizon $T=Z$. No such difficulties arise, in fact. Factors of α' occur only in the combination

$$\alpha' S_0' = [(\alpha'^2+\alpha_0'^2+g_H^2\varpi^2)^2 - 4\alpha'^2\alpha_0'^2]^{1/2}/2\alpha_0'. \tag{4.12}$$

In this expression of course we must allow α'^2 [representing $g_H^2(Z'^2-T'^2)$] to take on negative values, but the argument of the square root in Eq. (4.12) remains non-negative and all expressions in Eq. (4.11) and below are real and unambiguous.

We find the horizon fields $\overleftrightarrow{\mathcal{E}}^H$ and $\overleftrightarrow{\mathcal{B}}^H$ [see Eqs. (2.32)–(2.34)] by taking the horizon limit ($\alpha \to 0$, $Z-T \to 0$). The only nonvanishing components are

$$\mathcal{E}^H_{\hat{\phi}\hat{\phi}} = -\mathcal{E}^H_{\hat{\varpi}\hat{\varpi}} = \mathcal{B}^H_{\hat{\phi}\hat{\varpi}} = \frac{3mg_H^7 f^2\varpi^2 T^2}{(\alpha' S_0')^5} \tag{4.13}$$

(Note that the vanishing of $\mathcal{E}^H_{\hat{\phi}\hat{\varpi}}$, $\mathcal{B}^H_{\hat{\phi}\hat{\phi}}$, and $\mathcal{B}^H_{\hat{\varpi}\hat{\varpi}}$ follows from the axial symmetry.) The results in Eq. (4.13) are therefore compatible with the condition that $\overleftrightarrow{\mathcal{E}}^H$ and $\overleftrightarrow{\mathcal{B}}^H$ be traceless and with the horizon boundary condition (2.35), relating $\overleftrightarrow{\mathcal{E}}^H$ and $\overleftrightarrow{\mathcal{B}}^H$.

On the stretched horizon Minkowski and Rindler time are related by $T+Z=2T=(\alpha_H/g_H)e^{g_H t}$. It is convenient to shift the zero of universal time to $t=0$ at $T=d/2$, the time of arrival of the first signal from the downward moving particle (see Fig. 2); this choice implies

$$T = \frac{d}{2}e^{g_H t}. \tag{4.14}$$

[Note: At constant α, universal time and the horizon time coordinate $\bar{t}$ differ only by a constant, cf. Eq. (2.29). We may therefore use $\bar{t}$ rather than t in (4.14). On the horizon itself $T=d/2e^{g_H\bar{t}}$ remains meaningful, though universal time is ill defined.] We may now write out explicitly the t-dependence inherent in the factors T and $(\alpha' S_0')^5$ occurring in Eq. (4.13):

$$\mathcal{E}^H_{\hat{\phi}\hat{\phi}} = -\mathcal{E}^H_{\hat{\varpi}\hat{\varpi}} = \mathcal{B}^H_{\hat{\phi}\hat{\varpi}} = \frac{24mg_H^2A(A\varpi)^2(p+f^2)^2}{\{[p+1-(A\varpi)^2]^2+4(A\varpi)^2\}^{5/2}},$$
$$p \equiv f^2(e^{g_H t}-1). \tag{4.15}$$

It is clear from Fig. 2 that the first signal from the descending particle hits the horizon at $T=d/2$, i.e, at $t=0$, so that $\overleftrightarrow{\mathcal{E}}^H$ and $\overleftrightarrow{\mathcal{B}}^H$ vanish for $t<0$. [The curvature perturbations described by Eqs. (4.8)–(4.15), derived in the shifted Rindler coordinate patch, apply only in and to the future of that region. Elsewhere the Riemann curvature vanishes in this Rindler model.] At $t=0$ the fields jump discontinuously from zero to

$$\mathcal{E}^H_{\hat{\phi}\hat{\phi}}|_{t=0} = -\mathcal{E}^H_{\hat{\varpi}\hat{\varpi}}|_{t=0}$$
$$= \mathcal{B}^H_{\hat{\phi}\hat{\varpi}}|_{t=0} = \frac{24mg_H^2f^4A(A\varpi)^2}{[(A\varpi)^2+1]^5}. \tag{4.16}$$

For $\varpi \ll s_m/(1+f)$ these initial fields increase with radial distance as ϖ^2, reach a maximum strength of order $mg_H^2f^4(1+f)/s_m$, and for $\varpi \gg s_m/(1+f)$ they fall off as ϖ^{-8}.

It might seem that the initial signals at $t=0$ should be divergent. At Minkowski time $T<0$ the particle moves toward the horizon with velocity $\beta = dZ/dT = AT[1+(AT)^2]^{-1/2}$ and with a Lorentz factor $\gamma \equiv (1-\beta^2)^{-1/2} = [1+(AT)^2]^{1/2}$. In the distant past, $T \to -\infty$, the Lorentz factor diverges and one might expect divergent Riemann fields at the horizon due to the infinite blue shift of the gravitational radiation from the particle source. The finite fields at $t=0$ are not due to the α^2 factors built into $\overleftrightarrow{\mathcal{E}}^H$ and $\overleftrightarrow{\mathcal{B}}^H$. These correct for the fact that the FIDO's are pathological; the Riemann components in the Minkowski basis are in fact finite at $t=0$. The explanation lies, rather, in the Lorentz contraction of the distance between the particle and the horizon. In the rest frame of the particle emitting "gravitons" radially inward (i.e., in the direction $-\vec{e}_Z$) the distance between events of emission and the reception of the gravitons at the horizon is $\frac{1}{2}[A^{-1}+d(1+\beta)\gamma]$. For $\beta\to-1$ this distance is of order A^{-1}, which is comparable to the characteristic wavelength of the gravitational bremsstrahlung emitted by the particle. The horizon then is not in the radiation zone of the particle's emission from the distant past, and the idea of the "blueshift" of that emission is inapplicable. It should be noted, however, that for $T\to+\infty$, hence $\beta\to+1$, the horizon *is* in the particle's radiation zone and the concept of geometric optics should apply (see below).

For $p\gg\max(1,f^2)$, long after the initial signal arrives, the character of the horizon fields is quite different from that of the initial fields. The dependence on time and location is then given by

$$F(t,\tilde{\omega})=(A\tilde{\omega})^2p^2\{[p+1-(A\tilde{\omega})^2]^2+4(A\tilde{\omega})^2\}^{-5/2} \tag{4.17}$$

which is plotted as a function of p and $(A\tilde{\omega})$ in Fig. 3. At a given p, i.e., at a given time t, the peak of F is in an annulus at

$$\tilde{\omega}\approx p^{1/2}/A\propto e^{g_Ht/2} \tag{4.18}$$

and this annulus has a width of order

$$\Delta\tilde{\omega}\sim 1/A\ . \tag{4.19}$$

The field strength in this annulus is of order $mAg_H^2p^{1/2}$ and is much greater than the maximum field strength $mAg_H^2f^4$ at early times with $p\ll1$ [cf. Eq. (4.16)]. The most interesting horizon perturbations then occur at late times, in this narrow strong-field annulus.

The location of this strong-field annulus can be understood in terms of geometric optics. At late times the particle moves away from the horizon at near-light velocity beaming most of its "gravitational bremsstrahlung" away from the horizon due to the headlight effect. Let θ be the angle, as measured in the rest frame of the mass point, between the outgoing radial direction (i.e., the z direction) and the path of an emitted graviton (i.e., a null geodesic). The relationship among θ, the time of reception, and the location $\tilde{\omega}$ at which the graviton hits the horizon, is

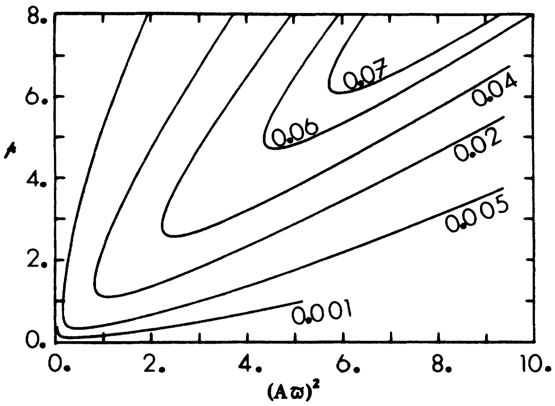

FIG. 3. The function F characterizing the strength at late times of horizon fields due to a radially accelerating particle. On the curves F is constant at the indicated value. As p increases the strong-field region is confined to a narrower annulus, with a greater peak value of F.

$$[p+1-(A\tilde{\omega})^2]^2=4(A\tilde{\omega})^2\cot^2\theta\ . \tag{4.20}$$

At late times, with $p\gg1$, most of these gravitons hit the horizon at

$$\tilde{\omega}\approx p^{1/2}/A\pm1/A$$

in agreement with Eqs. (4.18) and (4.19). Only for $|\cot\theta|\gtrsim p^{1/2}$ (i.e., only for emission very nearly in the ingoing or outgoing radial directions) do the gravitons hit the horizon outside this annulus.

For large $\tilde{\omega}$ (i.e., $A\tilde{\omega}\gg1$) the horizon fields have the nature of a pulse, with a maximum at $p=(A\tilde{\omega})^2$, or at time

$$t=t_{\max}(\tilde{\omega})=2g_H^{-1}\ln(A\tilde{\omega}/f)\ .$$

In this case we can approximate the horizon fields as

$$\mathcal{E}^H_{\hat{\phi}\hat{\phi}}=-\mathcal{E}^H_{\tilde{\omega}\tilde{\omega}}=\mathcal{B}^H_{\hat{\phi}\tilde{\omega}}\approx 2mg_HA\delta[t-t_{\max}(\tilde{\omega})]\ . \tag{4.21}$$

With this approximation in Eq. (2.25) the shear and expansion due to the distorting strong-field annulus are found to be

$$\sigma^H_{\hat{\phi}\hat{\phi}}=-\sigma^H_{\tilde{\omega}\tilde{\omega}}=\begin{cases}2mg_HAe^{g_H[t-t_{\max}(\tilde{\omega})]}=2mg_HAf^2(A\tilde{\omega})^{-2}e^{g_Ht} & \text{for } t<t_{\max}\ ,\\ 0 \ \text{for } t>t_{\max}\ ,\end{cases} \tag{4.22}$$

and

$$\theta^H=\begin{cases}8m^2g_HA^2f^2(A\tilde{\omega})^{-2}e^{g_Ht}(1-e^{g_H[t-t_{\max}(\tilde{\omega})]}) & \text{for } t<t_{\max}\ ,\\ 0\ \text{for } t_{\max}\ .\end{cases} \tag{4.23}$$

In this approximation the horizon shear and expansion vanish inside the annulus, i.e., for $(A\tilde{\omega})^2 < f^2 e^{g_H t}$. Outside the annulus, at $(A\tilde{\omega})^2 > f^2 e^{g_H t}$, the shear and expansion fall off with distance as $\tilde{\omega}^{-2}$. The shear and expansion can be found more accurately with the use of $\overleftrightarrow{\mathscr{E}}^H$ from Eq. (4.15) in Eq. (2.25):

$$\sigma^H_{\hat{\phi}\hat{\phi}} = -\sigma^H_{\tilde{\omega}\tilde{\omega}} = -\tfrac{3}{2} m f^2 g_H A (A\tilde{\omega})^{-2} e^{g_H t} F_1(\psi) , \quad (4.24a)$$

$$\theta^H = 9 m^2 f^2 g_H A^2 (A\tilde{\omega})^{-3} e^{g_H t} F_2(\psi) , \quad (4.24b)$$

with

$$F_1(\psi) \equiv \cos\psi - \tfrac{1}{3}\cos^3\psi - \tfrac{2}{3} , \quad (4.24c)$$

$$F_2(\psi) \equiv \tfrac{8}{9}\tan\tfrac{1}{2}\psi - \tfrac{5}{24}\psi - \tfrac{4}{9}\sin\psi + \tfrac{1}{9}\sin 2\psi - \tfrac{1}{288}\sin 4\psi , \quad (4.24d)$$

$$\cos\psi \equiv \frac{p+1-(A\tilde{\omega})^2}{\{[p+1-(A\tilde{\omega})^2]^2 + 4(A\tilde{\omega})^2\}^{1/2}} . \quad (4.24e)$$

For late times $(p \gg 1)$ far outside the annulus $[(A\tilde{\omega})^2 - p \gg A\tilde{\omega}]$ these expressions agree with those in Eqs. (4.22) and (4.23). For late times far inside the annulus $[p - (A\tilde{\omega})^2 \gg A\tilde{\omega}]$ they give the nonzero values of shear and expansion

$$\sigma^H_{\hat{\phi}\hat{\phi}} = -\sigma^H_{\tilde{\omega}\tilde{\omega}} = 6 m g_H A f^{-6} (A\tilde{\omega})^2 e^{-3 g_H t} , \quad (4.25a)$$

$$\theta^H = \tfrac{72}{7} m^2 g_H A^2 f^{-12} (A\tilde{\omega})^4 e^{-6 g_H t} . \quad (4.25b)$$

These values are very small compared to those outside the strong-field annulus. This justifies the approximate results in Eqs. (4.22) and (4.23) that the annulus "turns off" θ^H and $\overleftrightarrow{\sigma}^H$ at a point on the horizon as it expands outward across that point.

With the approximation in Eqs. (4.22) and (4.23) the time-integrated shear and expansion [Eqs. (2.26)] are

$$\Sigma^H_{\hat{\phi}\hat{\phi}} = -\Sigma^H_{\tilde{\omega}\tilde{\omega}} = \begin{cases} 2 m A f^2 (A\tilde{\omega})^{-2} e^{g_H t} & \text{for } t < t_{\max} , \\ 2 m A f^2 (A\tilde{\omega})^{-2} e^{g_H t_{\max}} = 2 m A & \text{for } t > t_{\max} , \end{cases} \quad (4.26a)$$

and

$$\Theta^H = \begin{cases} 4 m^2 A^2 f^2 (A\tilde{\omega})^{-2} e^{g_H t} (2 - e^{g_H (t - t_{\max})}) & \text{for } t < t_{\max} , \\ 4 m^2 A^2 f^2 (A\tilde{\omega})^{-2} e^{g_H t_{\max}} = 4 m^2 A^2 & \text{for } t > t_{\max} . \end{cases} \quad (4.26b)$$

These results indicate that a reference set of fiducions which was circular in the distant past becomes elongated in the ϕ direction and shortened in the $\tilde{\omega}$ direction, and increases in area. This deformation increases with time until the annulus passes, after which the deformation does not change.

To understand the deformation of the intrinsic geometry of the horizon section we consider the first-order perturbations of the horizon metric γ_{ab} at constant late time $(p \gg 1)$. Since only the transverse-traceless part of the metric is affected to first order we can write the line element to first order in m as

$$ds^2 = (1+\Gamma) d\tilde{\omega}^2 + \tilde{\omega}^2 (1-\Gamma) d\phi^2 . \quad (4.27)$$

The condition for such a geometry to be Riemann flat is

$$\Gamma = a + b/\tilde{\omega}^2 \quad \text{with} \quad a,b = \text{const} . \quad (4.28)$$

Inside the annulus (i.e., at smaller $\tilde{\omega}$) we have $\Gamma = \Sigma_{\tilde{\omega}\tilde{\omega}} = \text{const}$, and outside (larger $\tilde{\omega}$) $\Gamma \propto \tilde{\omega}^{-2}$, from Eq. (4.26a). Thus the intrinsic geometry is flat except in the region of the annulus. The passage of the annulus deforms initially circular rings of fiducions and produces a deformed line element (4.27) if the comoving coordinates tied to fiducions are used. The geometry remains flat, but the spatial coordinates in which the flatness is manifest are not the comoving coordinates.

The heating rate of the horizon is given by Eq. (2.49) as

$$\frac{1}{8\pi} g_H \frac{d\mathscr{A}_H}{dt} = T_H \frac{dS_H}{dt} = \int 2\eta_H \sigma^H_{ab} \sigma^{ab}_H d\mathscr{A}$$
$$= \int 2\eta_H \sigma^H_{ab} \sigma^{ab}_H 2\pi\tilde{\omega} d\tilde{\omega} . \quad (4.29)$$

In the late-time $(p \gg 1)$ limit we have been considering, the main contribution to this integral comes from the region of the horizon at larger $\tilde{\omega}$ than the strong-field annulus; the contribution at smaller $\tilde{\omega}$ is smaller by a factor p^{-4}. On the Rindler horizon this contributing region extends to infinite $\tilde{\omega}$. But to apply our results to problems involving Kerr black holes, we must consider the heating of a finite region of the horizon, cf. (3.76a). Let this finite region be a disc of some radius $\tilde{\omega}_{\rm LIM}$ (characterized by $\tilde{\omega} < \tilde{\omega}_{\rm LIM}$). Thus restricted, the integral (4.29), with σ^H_{ab} from Eq. (4.22), gives

$$\frac{1}{8\pi} g_H \frac{d\mathscr{A}_H}{dt} = \begin{cases} m^2 f^2 g_H^2 [e^{g_H t} - e^{g_H (2t - t_{\rm LIM})}] & \text{for } t < t_{\rm LIM} , \\ 0 & \text{for } t > t_{\rm LIM} , \end{cases} \quad (4.30)$$

here $t_{\rm LIM}$ denotes the time at which the strong-field annulus would pass beyond $\tilde{\omega} = \tilde{\omega}_{\rm LIM}$, given by

$$e^{g_H t_{\rm LIM}} = f^{-2} (A\tilde{\omega}_{\rm LIM})^2 . \quad (4.31)$$

For $t \geq t_{\rm LIM}$, all of the disc $\tilde{\omega} < \tilde{\omega}_{\rm LIM}$ lies inside the strong-field annulus; the heating of the region is then negligible, as Eq. (4.30) shows. The total increase in area for this portion of the horizon is

$$\Delta\mathscr{A}_H = 8\pi m^2 f^2 g_H \int_{-\infty}^{t_{\rm LIM}} (e^{g_H t} - e^{g_H (2t - t_{\rm LIM})}) dt$$
$$= 4\pi m^2 (A\tilde{\omega}_{\rm LIM})^2 . \quad (4.32)$$

(Although the integrand is a late-time approximation, the lower limit of the integral is extended to $t = -\infty$. This changes the result negligibly; as indicated by the exponential time dependence of $d\mathscr{A}_H/dt$, the integral is dominated by the contribution at late times.) The fractional increase in area, or entropy, for this region is

$$\frac{\Delta\mathscr{A}_H}{\mathscr{A}_H} = \frac{\Delta S_H}{S_H} = 4 m^2 A^2 , \quad (4.33)$$

in accord with Eq. (4.26b).

The above calculations have been carried out entirely in the Rindler geometry. We must now consider what restrictions apply for these results to apply to a Kerr hole. An obvious restriction is

$$\tilde{\omega} \ll r_H \sim M \ . \tag{4.34}$$

[See Eq. (3.76).] A more subtle constraint is that the horizon fields must be generated when the particle is within the realm of the Rindler approximation, i.e., at $\alpha \ll 1$ or

$$z \ll g_H^{-1} \approx 4M \ . \tag{4.35}$$

This means that the initial horizon fields of Eq. (4.16) cannot be trusted, as they arise from particle motions far from the horizon. These initial fields, however, are not of primary concern; the interesting horizon dynamics is associated with the "late time" development of the strong-field annulus. We must therefore ask at what height z the particle generates the strong-field annulus.

The relevant equations to answer this question are (i) the relation between the height z of the particle and the Lorentz factor $\gamma \equiv (1-\beta^2)^{-1/2}$ for its outward motion through the Minkowski (T,Z) spacetime, viz.,

$$(Az)^2 = 1 + f(f+2\gamma) \ , \tag{4.36a}$$

and (ii) the relation of the motion parameters β and γ at emission of a signal and the radius $\tilde{\omega}$ at which that signal strikes the horizon, viz.,

$$A\tilde{\omega} = 1 + f\gamma(1+\beta) \ . \tag{4.36b}$$

(Here it is assumed that in the particle rest frame the emission is at 90° to the vertical direction.) From these equations it follows that of the two constraints (4.34) and (4.35) the former is more restrictive, and means that only the particle motion with $z \lesssim (1+f)^{-1/2} M (s_m/M)^{1/2}$ and $\gamma \lesssim (1+f) f^{-1} (M/s_m)$ is legitimately within the Rindler approximation.

From the above considerations it follows that Eq. (4.32) should be correct, to order of magnitude, for a patch of the horizon of radius $\tilde{\omega} \approx M$ so that the increase in area of the hole should be of order

$$\Delta \mathcal{A}_H \approx 4\pi m^2 A^2 M^2 \ , \tag{4.37}$$

and the increase in mass of the hole of order

$$\Delta M \approx \tfrac{1}{8} m^2 A^2 M = \frac{(1+f)^2}{8} M \left[\frac{m}{s_m} \right]^2 . \tag{4.38}$$

B. Point mass in uniform motion

We consider next a particle of mass m moving in the Rindler background at a fixed height $z = s_0 = \alpha_0 g_H^{-1}$ and with a constant velocity β in the positive x-direction, as measured by FIDO's. The Rindler-coordinate position of the particle is then

$$x = \alpha_0 \beta t \ , \quad y = 0, \quad z = s_0 \ . \tag{4.39}$$

The rope supporting the particle is taken to be "vertical" in the Rindler geometry, i.e., all elements of the rope obey $x = \alpha_0 \beta t$. [This implies that the velocity, as measured by a FIDO, of the elements at height z is $(s_0/z)\beta$.] In terms of the linear density λ and the tension $\mathcal{T}$ of the rope, the total (particle plus rope) stress energy is

$$\begin{aligned}
T^{tt} &= D(x,y,t)\alpha^{-2}(mf_P + \lambda f_R) \ , \\
T^{zz} &= -D(x,y,t)(1-\beta^2\alpha_0^2/\alpha^2)\mathcal{T} f_R \ , \\
T^{xx} &= D(x,y,t)\beta^2(\alpha_0/\alpha)^2(mf_P + \lambda f_R) \ , \\
T^{xt} &= D(x,y,t)\beta\alpha_0\alpha^{-2}(mf_P + \lambda f_R) \ ,
\end{aligned} \tag{4.40}$$

with

$$D(x,y,t) \equiv (1-\beta^2\alpha_0^2/\alpha^2)^{-1/2}\delta(y)\delta(x-\beta\alpha_0 t) \ ,$$

$$f_P \equiv \delta(z-s_0) \ , \quad \text{and} \quad f_R \equiv H(z-s_0) \ ,$$

where H is the unit step function. The equation of force balance $(T^{z\mu}{}_{;\mu} = 0)$ gives us

$$\frac{d\mathcal{T}}{dz} = \frac{\lambda - \mathcal{T}}{z(1-\beta^2\alpha_0^2/\alpha^2)} \ , \tag{4.41}$$

$$\mathcal{T}|_{z=s_0} = \gamma^2 m g_H/\alpha_0 \ , \quad \gamma \equiv (1-\beta^2)^{-1/2} \ . \tag{4.42}$$

We again choose the equation of state of the rope to be $\lambda = \mathcal{T}$ so that the rope is weightless and has tension constant at the value given by (4.42). As in Sec. III B we avoid infinities in $h_{\mu\nu}$ by terminating the rope at $z_f = \alpha_f g_H^{-1}$ with a particle of mass $-m_f$. This requires modifying Eq. (4.40) with the replacements

$$\begin{aligned}
m\delta(z-s_0) &\rightarrow m\delta(z-s_0) - m_f\delta(z-z_f) \ , \\
H(z-s_0) &\rightarrow H(z-s_0) - H(z-z_f) \ .
\end{aligned} \tag{4.43}$$

The force-balance equation for the amended stress-energy gives

$$m_f = m\gamma^2(\alpha_f/\alpha_0)(1-\beta^2\alpha_0^2/\alpha_f^2) \ . \tag{4.44}$$

The solution now proceeds very much as in Sec. III B. The stress energy can be written in the form of Eq. (3.41) and the Liénard-Wiechert-type solution for $\bar{h}^{\mu\nu}$ can be written

$$\begin{aligned}
\bar{h}^{\mu\nu} = {} & 4\mathcal{T} \int_{\alpha_0}^{\alpha_f} \frac{U^\mu(\alpha')U^\nu(\alpha') - t^\mu(\alpha')t^\nu(\alpha')}{\vec{U}(\alpha')\cdot\vec{k}(\alpha')} d\left[\frac{\alpha'}{g_H}\right] \\
& + 4m\frac{U^\mu(\alpha_0)U^\nu(\alpha_0)}{\vec{U}(\alpha_0)\cdot\vec{k}(\alpha^0)} - 4m_f\frac{U^\mu(\alpha_f)U^\nu(\alpha_f)}{\vec{U}(\alpha_f)\cdot\vec{k}(\alpha_f)} \ ,
\end{aligned} \tag{4.45}$$

where the notation is that of Sec. III B [cf. Eqs. (3.46)–(3.52)] and it is understood that the fields are retarded. The components of $\vec{U}$ and $\vec{t}$ at the retarded source point, when expressed in the Rindler basis at the field point, are

$$U^t(\alpha')=\frac{1}{\alpha}\Gamma'C' , \qquad t^t=-\frac{1}{\alpha}S' ,$$
$$U^z(\alpha')=-\Gamma'S' , \qquad t^z=C' , \tag{4.46}$$
$$U^x(\alpha')=\Gamma'\beta\alpha_0/\alpha' , \quad \text{and} \quad t^x=0 ,$$

with

$$\Gamma'\equiv(1-\beta^2\alpha_0^2/\alpha'^2)^{-1/2} ,$$

and the denominator occurring in Eq. (4.45) is given by

$$\vec{k}(\alpha')\cdot\vec{U}(\alpha')=\Gamma'[\alpha g_H^{-1}S'-\beta(\alpha_0/\alpha')(x-\beta\alpha_0 t'_{\rm ret})] . \tag{4.47}$$

We are using here the notation of Eq. (3.54a) for S' and C'; as in Sec. III B the components of $\vec{U}$ and the value of $\vec{k}\cdot\vec{U}$ for the particle terms in Eq. (4.45) follow by replacing α' with α_0 and α_f.

Equations (4.46) and (4.47) give the expressions needed to evaluate the right-hand side of Eq. (4.45). Those expressions, however, are given in terms of retarded time $t'_{\rm ret}$, not explicitly in terms of the field point location (t,α,x,y) and the parameters of the particle motion. As in Sec. III B the evaluation of $t'_{\rm ret}$ follows from the retardation equation $\vec{k}\cdot\vec{k}=0$ which here has the form

$$0=\vec{k}(\alpha')\cdot\vec{k}(\alpha')$$
$$=g_H^{-2}(\alpha^2+\alpha'^2-2\alpha\alpha'C')+(x-\beta\alpha_0 t'_{\rm ret})^2+y^2 . \tag{4.48}$$

The retarded time $t'_{\rm ret}$ as a function of t,α,x,y, and α' is then given implicitly by

$$\cosh[g_H(t-t'_{\rm ret})]=(2\alpha\alpha')^{-1}[g_H^2(x-\beta\alpha_0 t'_{\rm ret})^2+g_H^2y^2+\alpha^2+\alpha'^2] . \tag{4.49}$$

The problem studied in Sec. III B is the present problem with $\beta=0$. In that case the retardation equation (4.49) could be solved explicitly for $t'_{\rm ret}$. The needed expressions C' and S' were found, of course, to be independent of the field-point time t. For the present case, with $\beta\neq0$, $\vec{U},\vec{t}$, and $\vec{k}\cdot\vec{U}$ will depend on t. Since we shall be primarily interested in the fields on or near the horizon we will need a time coordinate well behaved there. We therefore introduce an ingoing time coordinate

$$\bar{t}\equiv t+g_H^{-1}\ln(\alpha/\alpha_0) \tag{4.50}$$

[cf. Eq. (2.29)]. We also define

$$\bar{x}\equiv x-\beta\alpha_0\bar{t} ,$$
$$\Xi\equiv x-\beta\alpha_0 t'_{\rm ret}=\bar{x}+\beta\alpha_0[t-t'_{\rm ret}+g_H^{-1}\ln(\alpha/\alpha_o)] . \tag{4.51}$$

Equation (4.49) cannot be solved in closed form for $\beta\neq0$. We can, however, get a reasonably tractable solution by restricting attention to the case $\beta\ll1$ and by finding a solution valid only to first order β. (In the use of this approximation of course $\bar{x}$ must be considered zero order in β.) To find $\bar{h}_{\mu\nu}$ we shall need to evaluate the $t'_{\rm ret}$-dependent quantities S', C', and Ξ, appearing in Eqs. (4.46) and (4.47), correct to first order in β. We start by expanding Ξ as

$$\Xi=\bar{x}+\beta\delta_1+\beta^2\delta_2+\cdots . \tag{4.52}$$

With this expansion the left-hand side of Eq. (4.49) is

$$C'=\cosh\left[g_H\frac{\Xi-\bar{x}}{\alpha_0\beta}-\ln(\alpha/\alpha_0)\right]$$
$$=\cosh[g_H\delta_1/\alpha_0-\ln(\alpha/\alpha_0)+O(\beta)] \tag{4.53}$$

and the right-hand side is

$$(2\alpha\alpha')^{-1}(g_H^2\Xi^2+g_H^2y^2+\alpha^2+\alpha'^2)$$
$$=(2\alpha\alpha')^{-1}(g_H^2\tilde{\omega}^2+\alpha^2+\alpha'^2+2\beta g_H^2\bar{x}\delta_1)+O(\beta^2) , \tag{4.54a}$$

where

$$\tilde{\omega}^2\equiv\bar{x}^2+y^2 . \tag{4.54b}$$

We find δ_1 by equating the parts of Eqs. (4.53) and (4.54) which are zero order in β:

$$\delta_1=\left[\frac{\alpha_0}{g_H}\right]\left[\operatorname{arccosh}\left[\frac{g_H^2\tilde{\omega}^2+\alpha^2+\alpha'^2}{2\alpha\alpha'}\right]+\ln(\alpha/\alpha_0)\right] . \tag{4.55}$$

It should be noted that δ_1 is well behaved in the horizon limit, with

$$\delta_1\underset{\alpha\to0}{\sim}\left[\frac{\alpha_0}{g_H}\right]\ln\left[\frac{g_H^2\tilde{\omega}^2+\alpha'^2}{\alpha_0\alpha'}\right] . \tag{4.56}$$

With the solution for δ_1 we have Ξ to first order in β. With δ_1 we also have Eq. (4.54) and hence C' to first order in β. The remaining function S' is given by $(C'^2-1)^{1/2}$.

With these results the right-hand side of Eq. (4.45) is known as an explicit function of $\tilde{\omega}$ and α and a straightforward calculation, similar to that in Sec. III B but rather more tedious, can be done to find the metric perturbations $h_{\mu\nu}$, and from them the Riemann components correct to first order in β. The goal of this computation, the horizon field $\overleftrightarrow{\mathcal{E}}^H$, has components

$$\mathcal{E}^H_{yy}=-\mathcal{E}^H_{xx}=4\beta m g_H^2\bar{x}\frac{2y^2+s_0^2}{(\tilde{\omega}^2+s_0^2)^2} ,$$
$$\mathcal{E}^H_{xy}=4\beta m g_H^2 y\frac{\bar{x}^2-y^2-s_0^2}{(\tilde{\omega}^2+s_0^2)^2} , \tag{4.57}$$

where we have taken the large-α_f limit, with

$$\alpha_f\gg\alpha_0 \quad\text{and}\quad \alpha_f\gg g_H\tilde{\omega} . \tag{4.58}$$

The horizon shear and expansion follow from Eqs. (2.21) and (2.22). The time derivatives in these equations are higher order in β and can be ignored, so that to lowest order in β

$$\sigma^H_{ab}=g_H^{-1}\mathcal{E}^H_{ab} \quad\text{and}\quad \theta^H=g_H^{-1}\sigma^H_{ab}\sigma^{ab}_H \tag{4.59}$$

can be used. The facts that $\overleftrightarrow{\mathcal{E}}^H$ and $\overleftrightarrow{\sigma}^H$ are proportional to β (i.e., inversely proportional to the time scale for the change in fields) and that θ^H is quadratic in β are in accord with one's intuition about how these quantities must depend on the time scale; see the discussion in Sec. III C.

The above results allow us to infer the nature of the distortion in the metric for horizon sections. To first order in β we have, from Eq. (2.24),

$$\frac{\partial \gamma_{ab}}{\partial \bar{t}} = -\beta\alpha_0 \frac{\partial \gamma_{ab}}{\partial \bar{x}} = 2\sigma^H_{ab} = 2g_H^{-1}\mathcal{E}^H_{ab} , \tag{4.60}$$

where we use the fact that the time dependence in γ_{ab} must occur only in the combination $\bar{x} = x - \alpha_0\beta\bar{t}$. Equation (4.60) can be integrated explicitly, with the initial condition that the perturbation in $\overleftrightarrow{\gamma}$ vanishes at $\bar{t} \to -\infty$:

$$\gamma_{ab} = \gamma^{(0)}_{ab} + \gamma^{(1)}_{ab} ,$$

$$\gamma^{(1)}_{xx} = -\gamma^{(1)}_{yy} = -\frac{4m}{s_0}\frac{2y^2+s_0^2}{\varpi^2+s_0^2} , \tag{4.61}$$

$$\gamma^{(1)}_{xy} = \frac{8m}{s_0}\frac{\bar{x}y}{\varpi^2+s_0^2} .$$

In the low-velocity limit we have considered, these perturbations are independent of velocity. They correspond in fact to the horizon distortions produced by a particle statically suspended at height s_0 above the horizon. A computation of the scalar curvature of a horizon section gives

$$R = \frac{32ms_0^3}{(\varpi^2+s_0^2)^3} , \tag{4.62}$$

in agreement with Eq. (3.73). At any time $\bar{t}$ then the horizon geometry is distorted as if statically. With $\beta \neq 0$, however, an additional meaning can be given to the metric perturbations in Eq. (4.61): they describe the distortion of an initially (i.e., at $\bar{t} = -\infty$) circular reference ring of fiducions. For $\bar{x}, y \ll s_0$ this distortion takes a particularly simple form,

$$\gamma^{(1)}_{xx} = -\gamma^{(1)}_{yy} \approx -4m/s_0 \quad \text{and} \quad \gamma^{(1)}_{xy} \approx 0 , \tag{4.63}$$

which indicates that, as the particle passes overhead, near $\varpi = 0$ the ring of fiducions is elongated in the direction orthogonal to the direction of particle motion. The pattern of distortion both for $\varpi \ll s_0$ and for $\varpi \gg s_0$ is depicted in Fig. 4.

The horizon expansion in Eq. (4.59) has the form

$$\theta^H = 32\beta^2 m^2 g_H \varpi^2 \frac{\varpi^2 y^2 + 2y^2 s_0^2 + s_0^4}{(\varpi^2+s_0^2)^4} \tag{4.64}$$

and governs the heating of the hole according to Eq. (2.46). The distribution of heating on the horizon, i.e., the dependence of θ^H on $\bar{x}$ and y, is shown in Fig. 5. We can find the total heating of a patch of the horizon by integrating over θ^H:

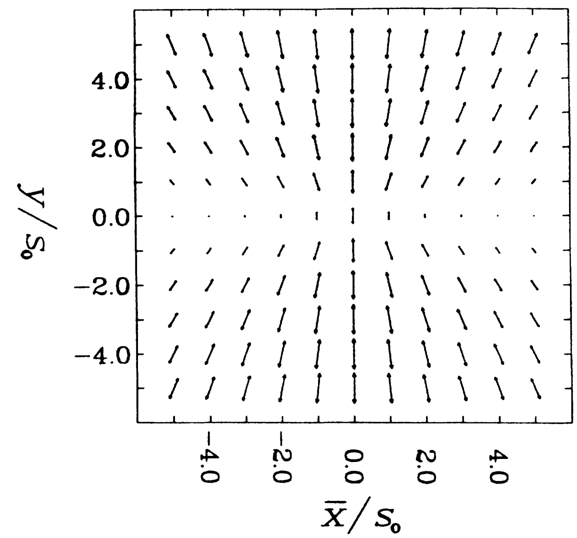

FIG. 4. Illustration of the distortion of initially circular rings of fiducions (i.e., the horizon-metric perturbation) as a function of comoving coordinates $\bar{x}$, y, when a particle passes over the horizon at a fixed height s_0 at low speed in the x direction. The length of the two-headed arrow centered at each point is proportional to the difference in length between major and minor axes of the fiducion ellipse produced at that point (i.e., to the eigenvalue of the horizon-metric-perturbation matrix) and the direction of the arrow is that of the major axis.

$$T_H \delta\Delta S^H \equiv T_H(\Delta S^H|_{\text{final}} - \Delta S^H|_{\text{initial}})$$
$$= \frac{g_H}{8\pi}\Delta\mathcal{A}^H \int_{-\infty}^{+\infty}\theta^H d\bar{t}$$
$$= \tfrac{1}{4} g_H \beta m^2 \frac{8y^4 + 8y^2 s_0^2 + s_0^4}{s_0(y^2+s_0^2)^{5/2}}\Delta\mathcal{A}^H . \tag{4.65}$$

Here the $\pm\infty$ limits of integration represent integration from negative values of $\bar{t}$ satisfying $\alpha_0\beta\bar{t} - x$

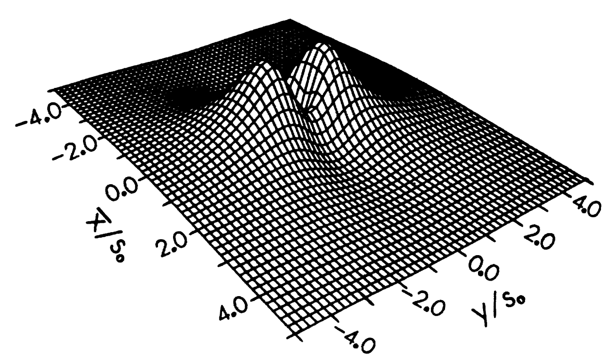

FIG. 5. Horizon expansion θ^H for a particle in uniform motion in the x direction. The magnitude of the horizon expansion is represented by the height of the surface above the $\bar{x}$-y plane. For a particle of mass m and speed β moving at $z = s_0$ the maximum value of θ^H is $8\beta^2 m^2 g_H s_0^{-2}$ at $\bar{x} = 0$, $y = \pm s_0$. Saddle points occur at $\bar{x} = \pm 3^{-1/2} s_0$, $y = 0$. For $\bar{x}^2 + y^2 \gg s_0^2$ the expansion falls off as ϖ^{-2}.

$\ll -|y|, -s_0$ to positive values satisfying $\alpha_0 \beta \bar{t} - x \gg |y|, s_0$.

Equation (4.65) clearly cannot be integrated over the horizon area to find the total heat added to the horizon. The infinite heat added corresponds to the fact that the heating proceeds at a constant rate for an infinite time. More surprising is the fact that the heating rate itself cannot be integrated over the whole horizon area. If we integrate θ^H over area out to some maximum $\tilde{\omega} = \tilde{\omega}_{\max} \gg s_0$ we find

$$T_H \frac{dS^H}{d\bar{t}} = \frac{g_H}{8\pi} \int \theta^H d\mathcal{A}^H \approx 4\beta^2 m^2 g_H^2 \ln(\tilde{\omega}_{\max}/s_0) , \quad (4.66)$$

so that the total heating rate for the entire horizon appears to be logarithmically infinite. This result is to be contrasted with that for the analogous case of a point electrical charge moving uniformly above the horizon,[7] in which the heating rate for the entire horizon is finite. The result in Eq. (4.66) is misleading. In arriving at the horizon fields in Eq. (4.57) we have used $\tilde{\omega} \ll g_H^{-1} \alpha_f$ [see Eq. (4.58)]. Integration of θ_H, as given in Eq. (4.64), to arbitrarily large $\tilde{\omega}$ is therefore not valid. If we restore to the $\overleftrightarrow{\mathcal{E}}^H$ fields the terms representing the termination of the constraint rope at α_f, the resulting θ^H falls off as $\tilde{\omega}^{-6}$ and is integrable over the entire horizon area to give

$$T_H \frac{dS^H}{d\bar{t}} \approx 2\beta^2 m^2 g_H^2 \ln(\alpha_f/\alpha_0) \quad (4.67)$$

for $\alpha_f \gg \alpha_0$.

In applying the above considerations to a black hole, as opposed to a Rindler horizon, we must take into account that the Rindler approximation is in any case valid only for $\alpha_f \ll 1$ and $\tilde{\omega} \ll M$. To order of magnitude we can infer the heating rate for a hole either by using $\alpha_f \approx 1$ in Eq. (4.67) or $\tilde{\omega}_{\max} \approx M$ in Eq. (4.66) to find

$$T_H \frac{dS^H}{d\bar{t}} \approx \beta^2 (m/M)^2 \ln(M/s_0) \quad (4.68)$$

for a particle of mass m moving at a FIDO-measured uniform velocity β at a proper distance s_0 from the horizon, outside a hole of mass M.

We now consider the specific case of a particle moving in an equatorial orbit, very close to a Kerr hole, with a small FIDO-measured velocity β. In this case we may use Eq. (2.55) to write the rate of change of hole angular momentum as

$$\frac{dJ}{dt} = -\frac{1}{16\pi} \int \sigma_H^{ab} \gamma_{ab,\phi'} d\mathcal{A}^H . \quad (4.69)$$

Our Rindler approximation replaces ϕ' by $x = \tilde{\omega}_H \phi' = 2M\phi'$ for an equatorial orbit [cf. Eq. (2.58)] so that Eq. (4.69) becomes

$$\frac{dJ}{dt} = -\frac{\tilde{\omega}_H}{16\pi} \int \sigma_H^{ab} \frac{\partial \gamma_{ab}}{\partial x} d\mathcal{A}^H . \quad (4.70)$$

But to lowest order in the velocity β we have

$$\frac{\partial \gamma_{ab}}{\partial x} = \frac{\partial \gamma_{ab}}{\partial \bar{x}} = -\frac{1}{\alpha_0 \beta} \frac{\partial \gamma_{ab}}{\partial \bar{t}} = -\frac{2}{\alpha_0 \beta} \sigma_{ab}^H \quad (4.71)$$

[cf. Eq. (4.60)] and therefore

$$\frac{dJ}{dt} = \frac{\tilde{\omega}_H}{8\pi \alpha_0 \beta} \int \sigma_H^{ab} \sigma_{ab}^H d\mathcal{A}^H = \frac{\tilde{\omega}_H}{\alpha_0 \beta} T^H \frac{dS^H}{dt} . \quad (4.72)$$

It is useful to reexpress this in terms of the angular velocity

$$\Omega_P \equiv \frac{d\phi^\dagger}{dt} \quad (4.73)$$

of the particle relative to Boyer-Lindquist coordinates. With this notation the particle trajectory is given by

$$x_P = \tilde{\omega}_H \phi' = \tilde{\omega}_H (\Omega_P - \Omega_H) t \quad (4.74)$$

[cf. Eqs. (2.14) and (2.58)] so that $\alpha_0 \beta = \tilde{\omega}_H (\Omega_P - \Omega_H)$ follows from Eq . (4.38), and the rate of change of angular momentum and the rate of heating are related by

$$T^H \frac{dS^H}{dt} = (\Omega_P - \Omega_H) \frac{dJ}{dt} . \quad (4.75)$$

In arriving at this result we have nowhere used the specific form of the perturbing tidal fields. The only feature of the model which has entered is the fact $\partial \gamma_{ab}/\partial t = -\alpha_0 \beta \partial \gamma_{ab}/\partial x$. The relation in Eq. (4.75) therefore holds for any perturbing source which is reflection symmetric in the equatorial plane, and rigidly rotates around the hole.

C. Freely falling point mass

As a further example of dynamical perturbations of the horizon we consider a particle freely falling, in the Rindler background, from rest at $z = s_0$. As in previous problems we choose the mass m of the particle to be small so that its tidal influence can be treated perturbatively. Here this means that we want distances small compared to $g_H^{-1} \approx 4M$ to be in the weak-field region of the particle, hence we require

$$m/M \ll 1 . \quad (4.76)$$

Additional constraints required for validity of the perturbation approach will be discussed at the end of this section; initially we shall not impose any constraint on s_0, the height from which the particle falls.

The perturbation calculation is particularly simple since no constraining ropes are necessary and since, in Minkowski coordinates [cf. Eq. (2.61)], we can take the trajectory of the unaccelerated particle to be stationary at

$$x = y = 0 \quad \text{and} \quad Z = s_0 . \quad (4.77)$$

The FIDO's of course see the particle as moving inward with decreasing α and increasing velocity. It is easily verified that the Lorentz factor γ, as measured by FIDO's, and the α position of the particle are related by

$$\alpha \gamma = g_H s_0 \equiv \alpha_0 . \quad (4.78)$$

The Minkowski components of the metric perturbations are easily computed for a stationary particle (see MTW, Chap. 18):

$$h_{xx}=h_{yy}=h_{ZZ}=h_{TT}=2m[(Z-s_0)^2+\varpi^2]^{-1/2}\ , \qquad (4.79)$$

where, as in previous problems, we use $\varpi^2 \equiv x^2+y^2$.

From the results for $h_{\mu\nu}$ it is straightforward and simple to compute, to first order in $h_{\mu\nu}$, the components of the Weyl tensor and $\mathcal{E}_{ij}$. To describe the results we need a time coordinate well behaved near the horizon. We choose this to be the ingoing time

$$\bar{t}=t+g_H^{-1}\ln(\alpha/2\alpha_0) \qquad (4.80)$$

[cf. Eq. (2.29)] shifted so that $\bar{t}=0$ corresponds to the plunge of the particle through the horizon. On or near the horizon this is related to Minkowski time according to

$$g_H T=\alpha_0 e^{g_H\bar{t}}[1+O(\alpha^2/\alpha_0^2)]\ . \qquad (4.81)$$

In terms of this time parameter the transverse components of $\overleftrightarrow{\mathcal{E}}$ (not $\overleftrightarrow{\mathcal{E}}^H$), on the stretched horizon at α_H, are

$$\begin{aligned}\mathcal{E}_{\hat\varpi\hat\varpi}&=-\mathcal{E}_{\hat\phi\hat\phi}\\ &=-\frac{3m(g_H s_0)^2 e^{2g_H\bar{t}}\varpi^2}{\alpha_H^2[\varpi^2+s_0^2(e^{g_H\bar{t}}-1)^2]^{5/2}}[1+O(\alpha_H^2/\alpha_0^2)]\ ,\\ \mathcal{E}_{\hat\varpi\hat\phi}&=0\ .\end{aligned} \qquad (4.82)$$

We can understand this result physically if we confine our attention to times near the plunge time, i.e., with $|g_H\bar{t}|\ll 1$, if we use Eq. (4.78), and if we introduce FIDO proper time $\tau=\alpha_H\bar{t}$ on the stretched horizon, shifted to give $\tau=0$ [more precisely $O(\alpha_H^3)$] when the particle plunges through the stretched horizon. Aside from fractional corrections of order $g_H\bar{t}$ and α_H^2 the $\overleftrightarrow{\mathcal{E}}$ field components can then be written

$$\mathcal{E}_{\hat\varpi\hat\varpi}=-\mathcal{E}_{\hat\phi\hat\phi}=-\frac{3m\gamma^2\varpi^2}{[\varpi^2+(\gamma\tau)^2]^{5/2}}\ , \qquad (4.83)$$

where γ, from Eq .(4.78), is the Lorentz factor, at the stretched horizon, of the particle motion relative to the FIDO's. This is precisely the form of the Weyl field for a moving point mass; the "pancaking" of the field intensity into the transverse plane is the same as that for the electric field of a moving charge. Here we are interpreting τ as the appropriate local time in the FIDO frame. Since the FIDO frame has an acceleration of order $g_H\alpha_H^{-1}$ the locally flat approximation is valid only for $|\tau|\ll\alpha_H g_H^{-1}$ or $|g_H\bar{t}|\ll 1$.

The horizon field $\overleftrightarrow{\mathcal{E}}^H$ has components given by Eq. (4.82) multiplied by α_H^2. The "pancaking" of the field means that for $\varpi\ll s_0$, $\overleftrightarrow{\mathcal{E}}^H$ is sizable [of order $m(g_H s_0)^2/\varpi^3$] only for a time $\Delta\bar{t}\approx\varpi/(g_H s_0)\ll g_H^{-1}$. We may therefore approximate the time dependence of $\overleftrightarrow{\mathcal{E}}_H$, in the region $\varpi\ll s_0$, with a δ function; thus,

$$\mathcal{E}^H_{\hat\varpi\hat\varpi}=-\mathcal{E}^H_{\hat\phi\hat\phi}=-4m(g_H s_0)\varpi^{-2}\delta(\bar{t}) \qquad (4.84)$$

(see Fig. 6).

The horizon shear $\overleftrightarrow{\sigma}^H$ and expansion θ^H can be found from Eq. (4.82) with Eqs. (2.25). In these equations the terms have been omitted which account for stress energy passing through the horizon so we can apply them only for

$$\varpi>R_P\equiv \text{(radius of particle as it passes through horizon)}. \qquad (4.85)$$

The results are

$$\sigma^H_{\hat\varpi\hat\varpi}=-\sigma^H_{\hat\phi\hat\phi}=\frac{3m(g_H s_0)}{\varpi}{}^2\left[\frac{\xi}{\Delta}-\frac{1}{3}\frac{\xi^3}{\Delta^3}-\frac{2}{3}\right] \qquad (4.86a)$$

and

$$\theta^H=-\frac{2m^2(g_H s_0)e^{g_H\bar{t}}}{\varpi^4}\left\{8\xi-\frac{15}{8}\left[\varpi\left[\arctan\frac{\xi}{\varpi}-\frac{\pi}{2}\right]+\frac{\varpi^2\xi}{\Delta^2}\right]-\frac{\varpi^4\xi}{4\Delta^4}+\frac{4\varpi^2}{\Delta}-8\Delta\right\}\ , \qquad (4.86b)$$

with

$$\xi\equiv s_0(e^{g_H\bar{t}}-1)\quad\text{and}\quad \Delta^2\equiv\varpi^2+\xi^2\ . \qquad (4.86c)$$

The meaning of these results is clearer if we limit attention to the region $\varpi\ll s_0$ and use Eq. (4.84) to find the approximations (see Fig. 6)

$$\sigma^H_{\hat\varpi\hat\varpi}=-\sigma^H_{\hat\phi\hat\phi}\approx\begin{cases}-4mg_H s_0\varpi^{-2}e^{g_H\bar{t}} & \text{for } \bar{t}<0\ ,\\ 0 \text{ for } \bar{t}>0\ ,\end{cases} \qquad (4.87a)$$

$$\theta^H\approx\begin{cases}32m^2 s_0^2 g_H\varpi^{-4}e^{g_H\bar{t}}(1-e^{g_H\bar{t}}) & \text{for } \bar{t}<0\ ,\\ 0 \text{ for } \bar{t}>0\ .\end{cases} \qquad (4.87b)$$

These time dependences illustrate the teleological nature of the response of the horizon; the horizon shear and expansion grow on a time scale g_H^{-1} until, around $\bar{t}=0$, the tidal field pulse hits the horizon.

The time-integrated shear [Eqs. (2.26)] can be found in closed form, but again it is more instructive to consider

$\tilde{\omega} \ll s_0$ and to use the approximation of Eq. (4.84) that the tidal fields hit the horizon in a sharp pulse. From Eq. (4.87a) and the conditions $\overleftrightarrow{\Sigma}^H=0$ and $\Theta^H=0$ at $\bar{t}=-\infty$ we have

$$\Sigma^H_{\hat{\tilde{\omega}}\hat{\tilde{\omega}}}=-\Sigma^H_{\hat{\phi}\hat{\phi}}\approx\begin{cases}-4ms_0\tilde{\omega}^{-2}e^{g_H\bar{t}} & \text{for } \bar{t}<0\ ,\\ -4ms_0\tilde{\omega}^{-2} & \text{for } \bar{t}>0\end{cases} \tag{4.88a}$$

and

$$\Theta^H\approx\begin{cases}16m^2s_0^2\tilde{\omega}^{-4}e^{g_H\bar{t}}(2-e^{g_H\bar{t}}) & \text{for } \bar{t}<0\ ,\\ 16m^2s_0^2\tilde{\omega}^{-4} & \text{for } \bar{t}>0\ .\end{cases} \tag{4.88b}$$

As in the model problem of Sec. IV A, initially circular rings of fiducions are distorted, elongated in the tangential direction and shortened in the radial direction. As in that model problem also the first-order transformation $\tilde{\omega}'=(1+\Sigma^H_{\hat{\tilde{\omega}}\hat{\tilde{\omega}}})\tilde{\omega}$ reduces the line element to $ds^2=d\tilde{\omega}'^2+\tilde{\omega}'^2d\phi^2$. The tidal fields produce a deformation of the (fiducion-tied) comoving coordinates, not of the intrinsic geometry.

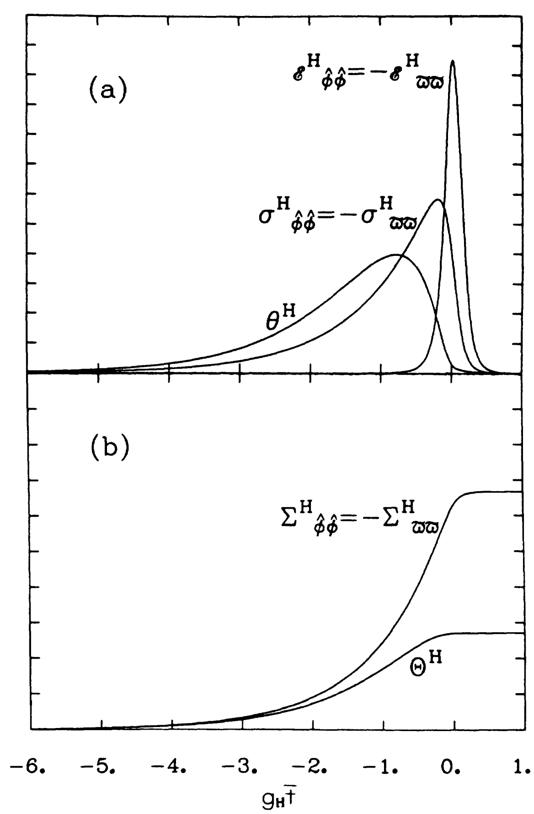

FIG. 6. The time dependence of horizon fields—instantaneous expansion, shear, and tidal curvature in (a), time-integrated shear and expansion in (b)—due to a particle freely falling near the Rindler horizion. The vertical scales of these graphs are arbitrary and different for each function. The curves shown are for $\tilde{\omega}/s_0=0.25$; for smaller values of $\tilde{\omega}/s_0$ the $\overleftrightarrow{\mathcal{E}}^H$ pulse would be narrower and would more closely correspond to the delta-function approximation used in the text.

The rate of heating of the horizon follows from Eq. (2.46):

$$T_H\frac{dS^H}{d\bar{t}}=\frac{g_H}{8\pi}\int\theta^H d\mathcal{A}=\frac{1}{4}g_H\int\theta^H\tilde{\omega}d\tilde{\omega}\ . \tag{4.89}$$

Since we are omitting the stress energy of the particle we must limit our calculation to the heating due to tidal fields in the region $\tilde{\omega}\geq R_P$. For clarity we shall also use the approximation for θ^H in (4.87b); this will give an accurate estimate of the heating (outside $\tilde{\omega}=R_P$) given the condition $s_0 \gg R_P$. The result is

$$T_H\frac{dS^H}{d\bar{t}}=4(m/R_P)^2s_0^2g_H^2e^{g_H\bar{t}}(1-e^{g_H\bar{t}}) \quad \text{for } \bar{t}<0\ . \tag{4.90}$$

From this the total mass energy deposited in the horizon by tidal heating for $\tilde{\omega}\geq R_P$ is given by

$$\Delta M=\int_{-\infty}^{0}T_H\frac{dS^H}{d\bar{t}}d\bar{t}=2(m/R_P)^2s_0^2g_H\ . \tag{4.91}$$

We now consider the application of the above results to a black hole (as contrasted with a Rindler horizon). The Rindler approximation requires that the source and the fields be confined to $\alpha \ll 1$ and $\tilde{\omega}<M$ (see Sec. III C). For results correct to order of magnitude we choose $\alpha_0=g_Hs_0=1$. According to Eq. (4.78) this corresponds to infall with $\gamma=1/\alpha$, the condition for radial infall into a Schwarzschild hole from a starting position far outside the horizon. We have several times used the simplifying assumption $\tilde{\omega}\ll s_0$, so that our results are reliable to order of magnitude only for $\tilde{\omega}\lesssim s_0$. For $\alpha_0=1$, however, this simply means $\tilde{\omega}\lesssim g_H^{-1}=4M$, a condition that in any case must be met if the Rindler approximation is to be used.

Equation (4.91), with $\alpha_0=1$ and $g_H^{-1}=4M$, says that the increase in hole mass due to tidal heating is of order

$$\Delta M\simeq 8m\left[\frac{m}{R_P}\right]\left[\frac{M}{R_P}\right]\ . \tag{4.92}$$

For our weak-field perturbation approach to be valid on the horizon at $\bar{t}=0$ and $\tilde{\omega}=R_P$ we must have $m/R_P \ll 1$. The M/R_P factor on the right-hand side of Eq. (4.92) can, on the other hand, be large, suggesting that ΔM due to tidal heating could be comparable to or larger than m. (This of course would be unphysical; the other contributions to ΔM, those of the particle's stress energy and tidal heating for $\tilde{\omega}<R_P$, will certainly be positive, and the total increase in hole mass must be less than m due to gravitational radiation losses outward to infinity.) The size of M/R_P must, in fact, be limited for our calculations to be valid. A condition for validity of the approach is $\theta^H \ll g_H$. [See Eq. (2.23).] From Eq. (4.87b), θ^H has a maximum value (at $e^{g_H\bar{t}}=\frac{1}{2}$ and $\tilde{\omega}=R_P$) corresponding to

$$\frac{\theta^H}{g_H}=8\frac{m^2\alpha_0^2}{g_H^2R_P^4}=128\left[\frac{m}{R_P}\right]^2\left[\frac{M}{R_P}\right]^2\ . \tag{4.93}$$

Our results then are valid only when ΔM for tidal heating is much smaller than the particle mass m.

V. SUMMARY

In this paper a point of view has been introduced and developed for the way to treat distortions, especially perturbative distortions, of a black-hole event horizon. The basic procedure starts with the calculation of the tidal distorting force (the horizon gravitoelectric field $\mathcal{E}^H_{ab}$). With this as a source term, the horizon shear is calculated via the tidal force equation, and the shear in turn is used as a source term in the focusing equation to find the horizon expansion. Finally, from the shear and the expansion, the time dependence of the horizon metric is inferred. It has been shown how the dynamics of the horizon can alternately be viewed as the dynamics of a two-dimensional fluid membrane.

Both this procedure and the nature of horizon dynamics have been illustrated with model problems based on a simplifying approximation: that the source of the horizon distortion is very close to the horizon. With this simplification, spacetime near the horizon can be approximated with the Rindler metric, and the horizon gravitoelectric fields can be computed fairly simply as retarded integrals. This approach was first checked by applying it to the case of a point particle statically suspended above a Schwarzschild horizon by minimal constraining forces. In this static case the approximate result could be compared with the result of a perturbation calculation done in the Schwarzschild geometry. The comparison not only confirmed the validity of the Rindler approximation, but also clarified the limits of this approximation, limits that must be observed in applying the results of a Rindler calculation to a black hole. In addition, these calculations suggested that the effects on the horizon could be ascribed to the particle alone, without significant contribution from the minimal constraining forces.

The Rindler approximation was applied to three specific model problems with point particles: a particle accelerating at a constant rate away from a horizon, a particle moving with uniform velocity parallel to a horizon, and a particle freely falling towards a horizon. These model calculations provided illustrations of horizon phenomena and showed the usefulness of elements of the membrane formalism (the $3+1$ split, the use of horizon quantities renormalized by powers of the lapse function α, etc.). Of particular interest in these model problems were concrete examples of the teleological nature of the horizon; horizon shear and expansion were seen not to react causally to perturbations, but rather to anticipate the perturbations. These models also gave examples of the "heating" of the horizon associated with time-dependent distortions. Because of the limitations of the approximation the precise results for heating of the Rindler horizon could not be applied directly to black holes. Those results, however, lead to useful estimates of heating effects for black-hole horizons, and provide insight into the perturbation-induced change in mass and angular momentum of a Kerr hole.

A major motivation for the present work is to make horizon dynamics more intuitively accessible. The model problems did in fact exhibit many effects that could be understood in terms of familiar physical phenomena. The horizon distortions in the radially accelerating particle problem were characterized by a narrow expanding annulus on the horizon; the location of that annulus could be explained by geometric optics, i.e., by considering the propagation of disturbances from the particle to the horizon along null geodesics. For particle motion parallel to the horizon, the dependence of horizon distortions (gravitoelectric field, shear, expansion, and metric) on velocity could be understood as a simple scaling law. For the particle falling freely towards the horizon, the gravitoelectric field hits the horizon as a sudden pulse; the detailed form of this pulse could be understood as the appropriately Lorentz transformed field of a static particle; the pulse hitting the horizon is the result of the "pancaking" of the fields, similar to the more familiar pancaking of the electrical field of a rapidly moving charged particle.

Along with these familiar aspects of the horizon distortion problems, there were new details that were not so familiar. Neither could these details be understood on the basis of electromagnetic interactions with the horizon.[7] Because of the intrinsically more complex nature of the horizon's gravitational interactions, the effects of gravitational perturbations are considerably more intricate than the analogous electromagnetic effects. Intuition about the electrical properties of the horizon has been shown to be useful in achieving a qualitative understanding of a complex astrophysical system.[6,9] Considerable effort will be required to achieve a comparably well developed intuition about the gravitational/fluid mechanical properties of the horizon. This paper represents some progress toward that goal.

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