Bosonic superconducting cosmic strings

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We study the classical solutions of bosonic superconducting strings for quartic and Coleman-Weinberg effective potentials. We map the parameter space of solutions, and discuss and quantify back reaction of the charged condensate upon the vortex. We address the issue of the critical current and the quench transition. We consider static loop configurations in which electromagnetic stresses balance the loop string tension. Such static loops are shown to exist only in a very small region of the parameter space. We also give accurate results for the energy per length of a nonsuperconducting gauged string for arbitrary ratio of Higgs-boson to vector mass.

I. INTRODUCTION

Recently there has been considerable interest in superconducting cosmic strings. First proposed by Witten,[1] they are cosmic strings endowed with dynamical properties which allow an effective Higgs-boson mechanism for electromagnetism to occur on the string. There are several novel ways in which such objects might be detected,[2] or have substantial effects on the formation of structure in the early Universe.[3] Superconducting strings are of two varieties: fermionic or bosonic. In the fermionic case, superconductivity arises because of the occurrence of charged Jackiw-Rossi zero modes which effectively behave as Nambu-Goldstone bosons in $1+1$ dimensions and give a longitudinal component to the photon field on the string. In some sense, this is a "natural phenomenon" in that it relies only upon certain systematic conditions being met, e.g., the presence of charged fermions with particular couplings to the vortex Higgs field. The rest is guaranteed by topology, anomalies, index theorems, and the like.

Bosonic superconductivity requires that some charged field develop a VEV (vacuum expectation value) in a region transversely localized on the string. This is a dynamical effect and must be engineered (in the scalar potential) to occur. It then becomes of interest to inquire how natural the phenomenon is: i.e., does one have to fine-tune the parameters to have such a condensate form, or is the parameter space where it occurs "large"? Furthermore, interesting dynamical questions arise: e.g., what determines the saturation (or critical) current and does the string undergo a first- or second-order phase transition when the critical current is exceeded and the superconductivity quenches? Are there solutions with and without significant back reaction of the charged condensate upon the vortex itself? Can a superconducting cosmic-string loop with a sufficiently large current and attendant electromagnetic field energy become stabilized against its string tension; i.e., is there a stable "floating solution"?[4,5] The latter question is a very delicate one because it involves the various parameters of the theory in a nontrivial way, and it is the question which led us to initiate the present study.

The aim of this paper is to give a comprehensive analysis of the microphysical phenomenon of bosonic superconducting strings. We do this by the use of accurate variational solutions for the various scalar and vector fields, an approach which significantly reduces the number of degrees of freedom and makes the analysis tractable. In the cases where we have direct comparison with analytic or very accurate numerical results there is excellent agreement between those results and ours. In short, we trust the results of our variational calculation.

The paper is organized as follows: In Sec. II, by using a variational analysis, we accurately calculate the energy per length of an ordinary (i.e., nonsuperconducting) cosmic string for arbitrary ratio of Higgs-boson to vector mass, and address the question of which regions of parameter space (for the scalar potential) permit bosonic superconducting cosmic strings; in Sec. III we study the dynamics of bosonic superconductivity; in Secs. IV and V we address the questions of critical currents and the possibility of floating or static solutions; in Sec. VI we translate our analysis from the natural space of dimensionless parameters we introduce to the parameters of the scalar potential; in Sec. VII we summarize our work and make some concluding remarks. In Table I we summarize the dimensionless parameters we introduce to simplify our analysis, and in Table II we summarize the constraints which must be satisfied for a bosonic superconducting vortex solution to exist. A summary of this work is published elsewhere.[6]

II. VARIATIONAL ANALYSIS OF BOSONIC SUPERCONDUCTING COSMIC STRINGS

Vortices arise when the first homotopy class $\Pi_1(G/H)$ associated with a symmetry breaking $G \rightarrow H$ is nontrivial. Typically $\Pi_1(G/H)$ is the set of integers corresponding to the winding numbers of scalar field configurations. The simplest realization of this is the breaking of a U(1)′ symmetry by a complex scalar field. If the U(1)′ symmetry is gauged we have a Nielsen-Olesen flux tube;[7] if not, we have a global or "axion" string.[8] For even the simplest quartic potential admitting symmetry breaking and flux tubes the classical profile of the solution is not completely known, nor would its knowledge be expected to be of great utility. There have been previous studies which obtain exact[9] or very accurate[10,11] results, and we will compare our results to these to test our variational-*Ansatz* approach. However, in the application to bosonic superconducting strings[1] no such results exist, and we must test our variational *Ansatz* by checking the stability of our results when additional terms are added to the *Ansatz*.

We begin with a variational study of the usual nonsuperconducting vortices (of both varieties) for their own sake; then we will use our approximate vortex solutions to study bosonic superconductivity. The Lagrangian density that describes the interactions of a U(1)′ charged scalar field Φ in a general potential $V_\Phi(\Phi)$ takes the form

$$L_\Phi = -\tfrac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + (D_\mu\Phi)^*(D^\mu\Phi) - V_\Phi(\Phi) \ , \qquad (2.1)$$

where $D_\mu = \partial_\mu - iqA'_\mu$, $F'_{\mu\nu} = \partial_\nu A'_\mu - \partial_\mu A'_\nu$, A'_μ is the U(1)′ gauge field, and q is the charge of the Φ field. With cylindrical symmetry the Hamiltonian (per unit length in the z direction) is

$$\tilde{H}_\Phi = \int r\,dr\,d\theta \left[\left| \frac{\partial\Phi}{\partial r} \right|^2 + \left| \frac{1}{r}\frac{\partial\Phi}{\partial\theta} - iqA'_\theta\Phi \right|^2 + V_\Phi(\Phi) + \frac{B'^2}{2} \right] . \qquad (2.2)$$

Throughout we use units where $\hbar = c = k_B = 1$ and $e^2 = 4\pi\alpha_{\rm em}$, and henceforth we use $\tilde{H}$ to designate a Hamiltonian per unit length. Here B' is the magnetic field associated with the vortex. To study the global case we set $q = 0$ (whence $B' = 0$). Following the standard conventions we write Φ (and other complex scalar fields) as

$$\Phi = (\phi_1 + i\phi_2)/\sqrt{2} \ , \qquad (2.3)$$

where ϕ_1 and ϕ_2 are real fields. Note that this convention (the factor of $1/\sqrt{2}$) is used so that the quantum theory has the usual equal-time commutation relations. Although our analysis will be classical, it is important to track the $1/\sqrt{2}$ factor in order to compare with other results.

The general vortex solution has the form

$$\Phi = \frac{\bar{v}}{\sqrt{2}} P(r) e^{i\eta} \ , \qquad (2.4)$$

where $P(r)$ is real, and $\bar{v}$ minimizes the potential $V(v) = V_\Phi(ve^{i\eta}/\sqrt{2})$ [we distinguish between the potential $V_\Phi(\Phi)$ for the complex field and the potential $V(v)$ for the real component to preserve consistency with the standard normalization conventions]. Requiring Φ to be single valued upon traversing a circular path restricts the possible changes in phase, i.e., $\Delta\eta = 2\pi N$, where N is an integer. We may then take the phase of Φ to wind uniformly over the path: $\eta = N\theta$. At the boundaries we have $P(r \rightarrow \infty) \rightarrow 1$ and $P(r \rightarrow 0) \rightarrow O(r^{|N|})$. Also, in the gauged case we have $A'_\theta(r \rightarrow \infty) \rightarrow N/qr$ and $A'_\theta(r \rightarrow 0) \rightarrow O(r)$.

Now, $P(r)$ can in principle be determined from the equations of motion associated with Eq. (2.1) (Refs. 11 and 12), but it is simpler to adopt a variational approach. Using a combination of powers of exponentials of the form $e^{-\mu r}$ one can always engineer a function with the above short- and long-distance limits required for $P(r)$. For example, for $|N| = 1$ we choose

$$P(r) = (1 - e^{-\mu r}) \ . \qquad (2.5)$$

We shall consider the case $|N| = 1$ throughout the rest of this paper. This is the simplest choice for N, and in the gauged case there is a large portion of parameter space (corresponding to vortices that exhibit type-II superconductivity in the Ginzburg-Landau theory:[13] $b < 2$, see next section) in which a vortex with $|N| \geq 2$ is unstable and decays into $|N| = 1$ vortices.[11] Later we argue that global strings with $|N| \geq 2$ are unstable, which further motivates restricting our analysis to $|N| = 1$ vortices. The case where $|N| \geq 2$ gauged vortices are stable is currently under study,[14] and may have interesting cosmological consequences.

A. The global case

Adopting expression (2.5) as a variational *Ansatz* we find the expectation of the Hamiltonian in the ungauged U(1)′ case, expanding in powers of $e^{-\mu r}$, to be

$$\langle \tilde{H}_\Phi \rangle = \tfrac{1}{4}\pi v^2 + \pi v^2 I_\theta(\mu,\lambda) + V(v)(\pi R_\infty^2) + 2\pi \int_0^\infty r\,dr\, V'(v)(-ve^{-\mu r}) + 2\pi \int_0^\infty r\,dr \tfrac{1}{2} V''(v)(v^2 e^{-2\mu r}) + \cdots \qquad (2.6)$$

or, upon performing the integrations in the potential terms,

$$\langle \tilde{H}_\Phi \rangle = \tfrac{1}{4}\pi v^2 + \pi v^2 I_\theta(\mu,\lambda) + V(v)(\pi R_\infty^2) + 2\pi \sum_{n=1}^{\infty} \frac{(-1)^n v^n V^{[n]}(v)}{n^2 n! \mu^2} \ . \qquad (2.7)$$

Here πR_∞^2 is the infinite area normal to the z axis and this term, which acts as a cosmological constant is the dominant contribution to the energy. Hence the variational calculation for v requires that $v=\bar{v}$, where $V'(\bar{v})=0$, and the vanishing of the effective cosmological term requires as usual that $V(\bar{v})=0$. The term $I_\theta(\mu,\lambda)$ is logarithmically divergent in the global case with λ representing a large-scale cutoff. Upon varying with respect to μ the λ dependence disappears and one has

$$\frac{\partial I_\theta(\mu,\lambda)}{\partial\mu}=\frac{1}{\mu}\bigg|_{\lambda\to\infty} . \tag{2.8}$$

So upon variation of Eq. (2.7) we obtain the extremal solution for μ:

$$\mu^2=4\sum_{n=2}\frac{(-1)^n\bar{v}^{\,n-2}V^{[n]}(\bar{v})}{n^2 n!} , \tag{2.9}$$

where $\bar{v}$ solves $V'(\bar{v})=0$ (note that the $n=1$ term in the series is then zero).

The dominant contribution to the energy per length is typically the angular contribution I_θ, which is easy to calculate for arbitrary N. We take

$$\Phi=\frac{\bar{v}}{\sqrt{2}}(1-e^{-\mu r})^N e^{iN\theta} \tag{2.10}$$

as our *Ansatz*. For $\mu\lambda\gg1$ we have

$$I_\theta=N^2\int_0^{\mu\lambda}\frac{(1-e^{-y})^{2N}}{y}dy\approx N^2\ln(\mu\lambda) . \tag{2.11}$$

We then see, in our variational approximation, that the ratio of the energy per length of a vortex with vorticity $|N|$ to that of $|N|$ vortices with unit vorticity (vorticity is conserved) is $\simeq|N|$, and the decay into $|N|$ vortices each of unit vorticity is energetically favorable for vortices with $|N|\geq2$.

Therefore, upon substituting Eq. (2.9) into Eq. (2.7) the mass per unit length of the $|N|=1$ vortex takes the form

$$\begin{aligned}\langle\tilde{H}_\Phi\rangle&=[\tfrac{3}{4}+I_\theta(\mu,\lambda)]\pi\bar{v}^{\,2}\\ &\simeq[\tfrac{3}{4}+\ln(\mu\lambda)]\pi\bar{v}^{\,2}\end{aligned} \tag{2.12}$$

[note the normalization conventions chosen here are the standard ones; if one normalizes $\langle\Phi\rangle=v'$ then one obtains, for the logarithmic contribution to the energy per unit length, $2\ln(\mu\lambda)\pi v'^2$].

Typically μ is of order the mass of the Higgs boson at the minimum of the potential. For example, choosing the usual form for the scalar potential,

$$V_\Phi(\Phi)=-m_\Phi^2|\Phi|^2+\frac{\lambda_\Phi|\Phi|^4}{3!}+\frac{3m_\Phi^4}{2\lambda_\Phi} , \tag{2.13}$$

whence

$$V(v)=-\frac{m_\Phi^2v^2}{2}+\frac{\lambda_\Phi v^4}{4!}+\frac{3m_\Phi^4}{2\lambda_\Phi} \tag{2.14}$$

yields $\bar{v}^{\,2}=3!m_\Phi^2/\lambda_\Phi$ and we find

$$\mu^2=\frac{V''(\bar{v})}{2}-\frac{2\bar{v}V'''(\bar{v})}{27}+\frac{\bar{v}^{\,2}V''''(\bar{v})}{96}\approx0.62m_\Phi^2 . \tag{2.15}$$

B. The gauged case

In the case where U(1)$'$ is gauged we obtain the expectation value of the Hamiltonian:

$$\begin{aligned}\langle\tilde{H}_\Phi\rangle=&\tfrac{1}{4}\pi v^2+\pi v^2 I_\theta(\mu,v)+\langle B'^2/2\rangle+V(v)(\pi R_\infty^2)\\ &+2\pi\int_0^\infty r\,dr\,V'(v)(-ve^{-\mu r})\\ &+2\pi\int_0^\infty r\,dr\tfrac{1}{2}V''(v)(v^2e^{-2\mu r})+\cdots ,\end{aligned} \tag{2.16}$$

where the difference with the global case is the nontrivial dependence in $I_\theta(\mu,v)+\langle B'^2/2\rangle$ upon v/μ, and it is not possible to write a systematic solution for μ. Here $\langle B'^2/2\rangle$ represents the magnetic field contribution.

We now extend our variational analysis by making an *Ansatz* for the gauge field:

$$A'_\theta=\frac{(1-e^{-hr})^2}{qr} , \tag{2.17}$$

where h is another variational parameter. This *Ansatz* for A'_θ corresponds to a magnetic flux tube of width $\sim h^{-1}$ and total flux $2\pi/q$. It is now convenient to define the dimensionless parameters:

$$a=\frac{m_\Phi^2}{\mu^2},\quad b=\frac{(q\bar{v})^2}{m_\Phi^2}=\frac{6q^2}{\lambda_\Phi}=\frac{2m_V^2}{m_H^2},\quad s=\frac{\mu}{h} , \tag{2.18}$$

where m_H is the physical mass of the Higgs particle ($m_H=\sqrt{2}m_\Phi$), and $m_V=q\bar{v}$ is the vector-boson mass. For $b>2$ ($b<2$) the vortices correspond to type-I (type-II) superconductivity in the Ginzburg-Landau theory. Physically, the width of the vortex is $\sim\mu^{-1}\sim\sqrt{a}\,m_\Phi^{-1}$, and s is $\sim$(width of the magnetic flux tube)/(width of the vortex).

Using our previous *Ansatz* for Φ and our *Ansatz* for the gauge field we readily find the angular integral contribution to the energy per length:

$$I_\theta=G(s)=\ln\left[\frac{3^4(s+4)^2(2s+3)^4(s+2)^8}{2^{11}(s+2)(s+1)^4(s+3)^8}\right] . \tag{2.19}$$

Of course, there is no large-distance logarithmic divergence here since the gauge field cancels the contribution of the Higgs field at large distances. The contribution to the energy from the scalar potential is found to be $(89/288)\pi\bar{v}^{\,2}a$, and the energy in the magnetic field is given by

$$\left\langle\frac{B'^2}{2}\right\rangle=\frac{4\pi\bar{v}^{\,2}\ln(\frac{9}{8})}{abs^2} . \tag{2.20}$$

Collecting terms, the energy per length of the gauge vortex is

$$\langle\tilde{H}_\Phi\rangle=\pi\bar{v}^{\,2}\left[\tfrac{1}{4}+G(s)+\frac{4\ln(\frac{9}{8})}{abs^2}+\frac{89}{288}a\right] , \tag{2.21}$$

where $\bar{v}$ is determined by $V'(\bar{v})=0$, and as before $V(\bar{v})=0$. From Eqs. (2.21) and (2.18) we obtain the variational equations

$$\frac{\partial G(s)}{\partial s}-\frac{89a}{144s}=0 , \tag{2.22}$$

$$\frac{\partial G(s)}{\partial s}-(8/abs^3)\ln(\tfrac{9}{8})=0\ , \qquad (2.23)$$

by varying with respect to the parameters μ and h, respectively. Subtracting these equations gives the simple relation

$$a^2s^2b=\tfrac{1152}{89}\ln(\tfrac{9}{8})\ . \qquad (2.24)$$

These equations can be solved by selecting a value for s, solving for a in Eq. (2.22), and using the above relation to obtain b. The energy per length of the vortex is easily computed in this manner, and is plotted as a function of b in Fig. 1. In addition we plot the parameters a and s, as a function of b, in Figs. 2 and 3.

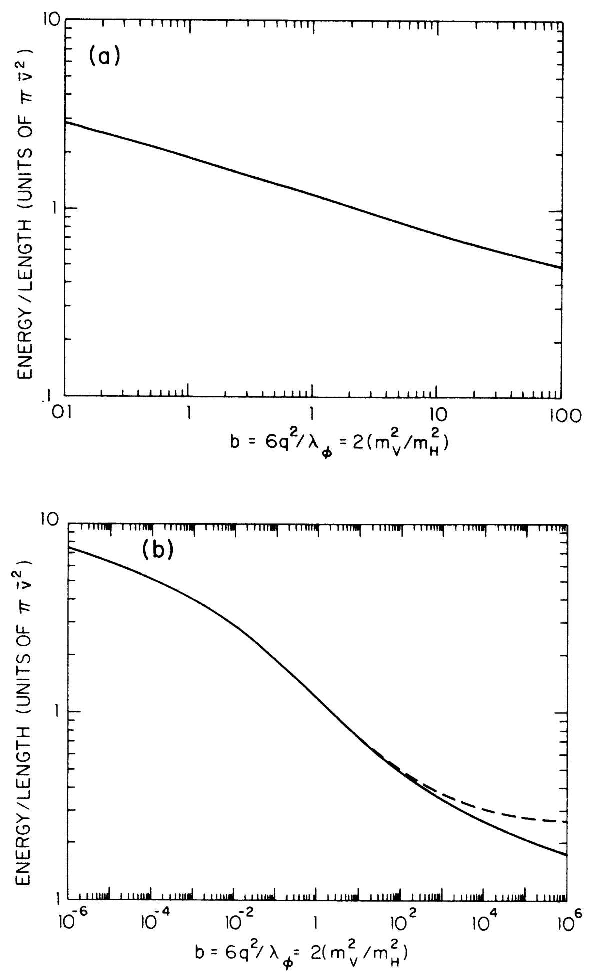

FIG. 1. In (a) we show the energy per length of the gauged vortex over the natural range of $b=6q^2/\lambda_\Phi=2m_V^2/m_H^2$. Our numerical results are well fit (to better than 5%) by energy per length $\epsilon=1.19\pi\bar{v}^2b^{-0.195}$; here $\bar{v}=\sqrt{6m_\Phi^2/\lambda_\Phi}$ is the VEV of the real part of the Φ field. For completeness, in (b) we show the energy per length over an extended range in b. In (b), the dashed curve indicates the energy per length computed with our original *Ansätze*, cf. Eqs. (2.5) and (2.17), while the continuation of the solid curve is the energy per length computed with our modified, "large b" *Ansätze*. In the limit $b\to\infty$, $\epsilon\to 2.4\pi\bar{v}^2/\ln b$ (modified *Ansätze*), $\to\pi\bar{v}^2/4$ (original *Ansätze*). Our modified *Ansätze* demonstrate that $\epsilon\to 0$ as $b\to\infty$ (albeit logarithmically).

For $b\to 0$ (equivalent to $q\to 0$), the energy per length is dominated by the angular kinetic energy and varies as $\ln b^{-1}$ (as one would expect, since $b,q\to 0$ is equivalent to the global string). We also see from Fig. 2 that $a=m_\Phi^2/\mu^2$ approaches the value we obtained in the global case ($a\simeq 1.6$), which is reassuring. In general, a is of the order of unity. Near $b\sim 1$, where all the terms in the Hamiltonian are important, we find agreement of our results with those of others[9,10] to better than 2%. Over the entire natural range of b ($0.01\lesssim b\lesssim 100$, to be discussed later) we find agreement of our results with the semiquantitative results of Bogomol'nyi and Vainshtein,[11] to the accuracy that comparison is possible ($\sim 10\%$), giving us confidence that our energy per length, and hence our variational analysis, is accurate. As a further check we also added another term to the scalar and vector field *Ansätze;* the energy decreased by $\approx 1\%$ for $b\sim 1$, and smaller changes were observed for all other values of b.

The $b\to\infty$ limit requires more careful consideration. Our *Ansatz* for Φ does not allow for a variation in the contribution of the radial kinetic energy per length; it is a constant $\pi\bar{v}^2/4$ per unit length. That the energy per length $\epsilon\to\pi\bar{v}^2/4$ for $b\to\infty$ derives from this fact. This fact gave us pause and led us to consider more general forms for $P(r)$. (Adding additional terms to the *Ansatz* of the form $r^ne^{-\mu r}$ does not change the qualitative behavior in the $b\to\infty$ limit, only the coefficient of $\pi\bar{v}^2$ for the limiting value.)

We note that the form $P(r)=A_1\ln r+A_2$ over the interval $r=[r_1,r_2]$ [with $P(r_1)=0$, $P(r_2)=1$] leads to a radial kinetic energy per length $E_r=\pi\bar{v}^2/\ln(r_2/r_1)$; for $r_2 \ggg 1$ this contribution becomes logarithmically small. (Of course, such an *Ansatz* does not satisfy the boundary conditions at $r=0,\infty$, nor is its first derivative continuous at $r=r_1,r_2$.) This suggests that for $b\gg 1$, where we have two disparate scales (m_H^{-1}, the size of the vortex, and $m_V^{-1}\sim b^{-1/2}m_H^{-1}$, the size of the flux tube), we should consider different *Ansätze*. To study the behavior

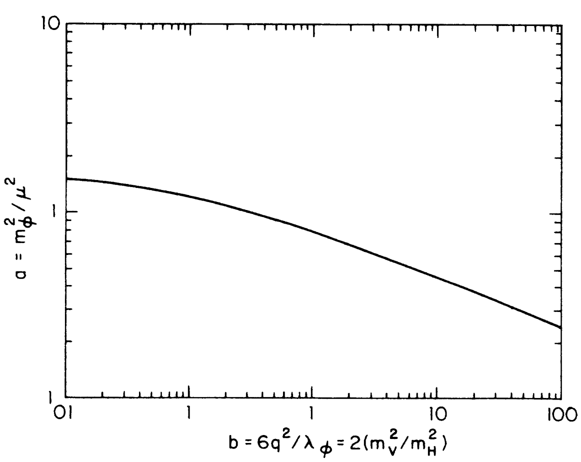

FIG. 2. The parameter $a=m_\Phi^2/\mu^2$ vs b for the gauged vortex. The width of the vortex $\mu^{-1}\sim\sqrt{a}\,m_\Phi^{-1}$.

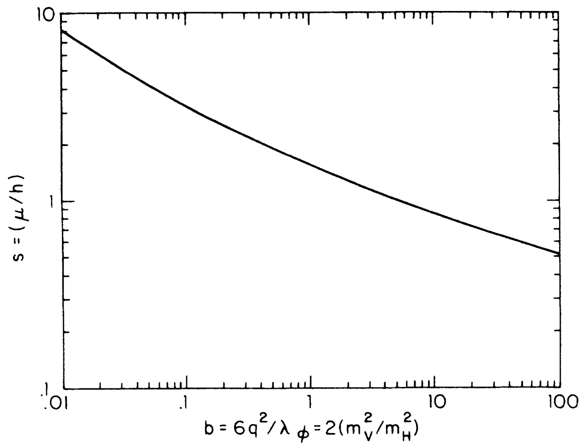

FIG. 3. The variational parameter s ($\approx$the ratio of the size of the magnetic flux tube to that of the vortex) is shown vs b.

of ϵ for $b \ggg 1$ we take, as *Ansätze,*

$$P(y)=\begin{cases} Q_1 y/a_1, & 0<y<a_1, \\ Q_1[1+\ln(y/a_1)], & a_1<y<a_2, \\ 1-Q_2e^{-cy}, & y>a_2, \end{cases}$$

$$A'_\theta=(qr)^{-1}\times\begin{cases} 3(y/a_3)^2-2(y/a_3)^3, & 0<y<a_3, \\ 1, & y>a_3, \end{cases}$$

where

$$y \equiv rm_\Phi \ ,$$

$$Q_1 \equiv [1+\ln(a_2/a_1)+(ca_2)^{-1}]^{-1} \ ,$$

$$Q_2 \equiv Q_1 e^{ca_2}/ca_2 \ .$$

These *Ansätze* and their first derivatives are continuous and satisfy the appropriate boundary conditions at $r=0,\infty$. In addition, all the relevant integrations can be performed analytically. The new variational parameters are a_1, a_2, a_3, and c.

Using these *Ansätze,* we find for $b \gg 1$ that the energy per length is minimized for $a_1 \sim a_3 \sim b^{-1/2}$, and $a_2 \sim c = O(1)$. Thus for these *Ansätze* the characteristic size of the vortex is $\sim a_2 m_H^{-1} \sim m_H^{-1}$, and that of the flux tube is $\sim a_3 m_H^{-1} \sim m_V^{-1}$, just as expected. Moreover, the radial kinetic energy per length

$$E_r \simeq \pi\bar{v}^2/\ln(a_2/a_1) \simeq 2\pi\bar{v}^2/\ln b \ ,$$

which $\to 0$ as $b \to \infty$. The total energy per length is (for $b \gg 1$)

$$\epsilon' \simeq 2.4\pi\bar{v}^2/\ln b$$

(an analytic fit to the range $10^6 \lesssim b \lesssim 10^{100}$).

Several points are to be noted. First, $\epsilon' \to 0$ as $b \to \infty$, albeit logarithmically. Second, the energy per length for these *Ansätze* is only lower than that of our previous *Ansätze* ($\epsilon \simeq \pi\bar{v}^2/4$ for $b \gg 1$) when $\ln b \gtrsim 4\times 2.4$, or $b > 1.5\times 10^4$. (For $b \lesssim 10^4$, the energy per length for the new *Ansätze* is essentially identical to that of the old *Ansätze.*) A comparison of the energy per length computed by the two different *Ansätze* for $b \gg 1$ is shown in Fig. 1(b). Although the energy per length $\to 0$ as $b \to \infty$, the logarithmic approach to the limit makes this a moot point for any physically reasonable, finite value of b. For example, even for $b=10^{12}$, the energy per length ϵ' is still $\simeq 0.09\pi\bar{v}^2$ vs $0.25\pi\bar{v}^2$ for our original *Ansätze.* (The fact that the energy per length $\to 0$ as $b \to \infty$ has also recently been noticed independently by Bennett and Turok.[15])

C. The superconducting condensate

The discussion of this subsection is independent of the choice of a global or gauged vortex. We now consider the bosonic superconducting cosmic strings which arise in a $U(1)\otimes U(1)'$ gauge theory with the general scalar potential[1] for which the Lagrangian density takes the form

$$L=L_\Phi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+(D_\mu\sigma)^*(D^\mu\sigma) - U_\sigma(\sigma)-f\,|\sigma|^2|\Phi|^2 \ , \tag{2.25}$$

where $D_\mu=\partial_\mu-ieA_\mu$, $F_{\mu\nu}=\partial_\nu A_\mu-\partial_\mu A_\nu$, and A_μ is the U(1) gauge field. The field σ carries U(1) (ordinary electromagnetic) charge e and no U(1)′ charge, the field Φ carries no U(1) charge and U(1)′ charge q (we have not written the photon-vector potential explicitly). We now obtain the Hamiltonian per unit length:

$$\tilde{H}=\tilde{H}_\Phi+\tilde{H}_\sigma+f\int_0^{2\pi}\int_0^\infty r\,dr\,d\theta\,|\Phi|^2|\sigma|^2 \ , \tag{2.26}$$

where

$$\tilde{H}_\sigma=\int_0^{2\pi}\int_0^\infty r\,dr\,d\theta\left[\left|\frac{\partial\sigma}{\partial r}\right|^2+\left|\frac{1}{r}\frac{\partial\sigma}{\partial\theta}-ieA_\theta\sigma\right|^2 +U_\sigma(\sigma)+\frac{B^2}{2}\right] . \tag{2.27}$$

For the moment we take the superconducting current to be zero.

We presently assume that $U_\sigma(\sigma)$ is an unstable, quartic scalar potential,

$$U_\sigma(\sigma)=-m_\sigma^2\,|\sigma|^2+\frac{\lambda_\sigma}{3!}\,|\sigma|^4 \ , \tag{2.28}$$

and the overall stability of the theory against the breaking of electromagnetism (far from the flux tube) is controlled by this potential and the $f\,|\sigma|^2|\Phi|^2$ term in Eq. (2.25). The condition that U(1) remains unbroken outside the flux tube is that $f\bar{v}^2\sigma^2/2+U_\sigma(\sigma)$ has no global minimum for nonzero σ (shortly, we will be more specific). Nonetheless, the basis of bosonic superconductivity is that in the core region in which $\langle\Phi\rangle\to 0$ the f term no longer stabilizes the σ field, and it may be energetically favorable for a condensate to form. Or course, since $\langle\sigma\rangle\to 0$ as $r\to\infty$, it costs kinetic energy to allow the σ field to develop a nonzero condensate at $r\to 0$, and *a priori* it is not clear whether the gain in potential energy wins out over the cost in kinetic energy.

To begin, we assume that the terms involving the σ field are sufficiently weak that they do not back react upon the Φ field, and the σ condensate can be studied in the fixed background of the Φ vortex solution just discussed. We refer to this as the *concrete-vortex* approximation. Later we study the validity of this approximation.

With standard normalization we introduce the real part of σ as $\mathrm{Re}\sigma = u/\sqrt{2}$ and $U(u) = U_\sigma(u/\sqrt{2})$. The standard potential becomes

$$U(u) = -\frac{m_\sigma^2}{2}u^2 + \frac{\lambda_\sigma u^4}{4!} . \tag{2.29}$$

We examine the properties of the equation of motion in the absence of superconducting currents:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + m_\sigma^2 u - \frac{\lambda_\sigma u^3}{3!} - f\,|\Phi|^2 u = 0 . \tag{2.30}$$

At this point it is very useful to introduce dimensionless parameters which rescale the various dimensionful parameters relative to the size of the vortex:

$$\begin{aligned} &\alpha = m_\sigma^2/\mu^2 = a m_\sigma^2/m_\Phi^2 ,\\ &\beta = f\bar{v}^2/2\mu^2 = 3af/\lambda_\Phi ,\\ &\tilde{\sigma}^2 = \frac{u^2\lambda_\sigma}{\mu^2},\quad y = \mu r , \end{aligned} \tag{2.31}$$

where the additional relations follow from substituting in $\mu^2 = m_\Phi^2/a$. The equation of motion now becomes

$$\frac{d^2\tilde{\sigma}}{dy^2} + \frac{1}{y}\frac{d\tilde{\sigma}}{dy} - \tilde{\sigma}[\beta P(y)^2 - \alpha] - \frac{\tilde{\sigma}^3}{3!} = 0 . \tag{2.32}$$

As $y \to 0$ we find $\tilde{\sigma} \to \tilde{\sigma}_0 + O(y^2)$. For $y \gtrsim 1$ and $\tilde{\sigma}^2 \ll 6(\beta-\alpha)$ we have $\tilde{\sigma}^2 \propto K_0(y\sqrt{\beta-\alpha})$, and

$$\tilde{\sigma}^2 \propto \frac{e^{-2y(\beta-\alpha)^{1/2}}}{y\sqrt{\beta-\alpha}} \tag{2.33}$$

in the limit $y\sqrt{\beta-\alpha} \gg 1$. Here K_0 is a modified Bessel function of zeroth order. We also note that for $\alpha = \beta$ and $y \gtrsim 1$, $\tilde{\sigma}^2 \to 6/y^2$.

To investigate the dynamics of the superconducting condensate we make a variational *Ansatz* of the form

$$\sigma(t,z,r,\theta) = \frac{\sigma_0}{\sqrt{2}} e^{-\kappa r}(1 + \kappa r + \kappa' r^2 + \kappa'' r^3) e^{i\phi(z,t)} \tag{2.34}$$

with four variational parameters σ_0, κ, κ', and κ'' (note, we do not endow σ with vorticity, i.e., $\partial\sigma/\partial\theta = 0$). We choose four variational parameters so that convergence can be checked with the second- and third-order *Ansatz*. This *Ansatz* has the correct short-distance limit and κ^{-1} represents the size of the σ condensate. The fact that the charged field σ acquires a vacuum expectation value in the core of the string signals that the string is superconducting and $\phi(z,t)$ is the massless mode which supplies the longitudinal degree of freedom for the photon on the string.

To describe the essential physics we find it useful to consider a simple, truncated *Ansatz*

$$\sigma(t,z,r,\theta) = \frac{\sigma_0}{\sqrt{2}} e^{-\kappa r} e^{i\phi(z,t)} , \tag{2.35}$$

with which we can discuss many results analytically. We presently use this *Ansatz* and also make use of the previously obtained profile for the vortex field Φ to obtain the expectation value for the Hamiltonian:

$$\langle \tilde{H} \rangle = \langle \tilde{H}_\Phi \rangle + \tfrac{1}{4}\pi\sigma_0^2 + \frac{\pi \tilde{U}(\sigma_0)}{\kappa^2} + \frac{\pi f v^2 \sigma_0^2}{8\kappa^2} F(\mu/\kappa) . \tag{2.36}$$

Here $\tilde{U}$ is derived from U upon performing the cylindrical integration normal to the string axis. If, for example, $U(u)$ has the polynomial expansion

$$U(u) = \sum_{n\ \mathrm{even}} U_n u^n , \tag{2.37}$$

then we have

$$\tilde{U}(\sigma_0) = 2 \sum_{n\ \mathrm{even}} \frac{U_n}{n^2}\sigma_0^n . \tag{2.38}$$

The function $F(x)$ represents the overlap of the Φ profile turning on to its asymptotic value $\bar{v}$ over a distance scale μ^{-1} from the flux tube and the σ profile turning off over a distance scale κ^{-1}. It takes the form

$$\begin{aligned} F(x) &= 1 - \frac{2}{(1+x/2)^2} + \frac{1}{(1+x)^2} \\ &= \frac{x^4 + 6x^3 + 6x^2}{x^4 + 6x^3 + 13x^2 + 12x + 4} , \end{aligned} \tag{2.39}$$

where $x \equiv \mu/\kappa \sim$(width of the σ condensate)/(width of the vortex). We see that $F(x)$ is positive over its range and we further note the limits

$$\begin{aligned} &F(x) \approx 3x^2/2 \quad (0 \le x \le 0.5) ,\\ &F(x) \approx x/3 \quad (0.5 \le x \le 2) ,\\ &F(x) \to 1 \quad (x \to \infty) . \end{aligned} \tag{2.40}$$

The full potential for both fields is

$$V_{\Phi,\sigma}(\Phi,\sigma) = -m_\Phi^2\,|\Phi|^2 + \frac{\lambda_\Phi |\Phi|^4}{3!} - m_\sigma^2\,|\sigma|^2 + \frac{\lambda_\sigma}{3!}\,|\sigma|^4 + f\,|\Phi|^2|\sigma|^2 + \frac{3m_\Phi^4}{2\lambda_\Phi} , \tag{2.41}$$

$$V(v,u) = -m_\Phi^2 v^2/2 + \frac{\lambda_\Phi v^4}{4!} - m_\sigma^2 u^2/2 + \frac{\lambda_\sigma u^4}{4!} + f v^2 u^2/4 + \frac{3m_\Phi^4}{2\lambda_\Phi} .$$

Within our concrete-vortex approximation we will now derive the constraints necessary for the existence of superconducting vortices. First, the theory must be such that far from a vortex the U(1)′ symmetry is spontaneously broken while the U(1) symmetry remains unbroken; i.e., we desire the global minimum of Eq. (2.41) to have $\langle\sigma\rangle = 0$, $\langle\Phi\rangle \neq 0$. The minima of $V(v,u)$ are determined by $\partial V/\partial u = \partial V/\partial v = 0$, and $\partial^2 V/\partial u^2, \partial^2 V/\partial v^2 > 0$. The potential $V(v,u)$ has three minima:

$$v_1^2 = 6m_\Phi^2/\lambda_\Phi,\quad u_1^2 = 0,\quad V_1 = 0 ;$$

$v_2^2=0,\ \ u_2^2=6m_\sigma^2/\lambda_\sigma,\ \ V_2=3m_\Phi^4/2\lambda_\Phi-3m_\sigma^4/2\lambda_\sigma\ ;$

$$v_3^2=\frac{2(fm_\sigma^2-\lambda_\sigma m_\Phi^2/3)}{f^2-\lambda_\Phi\lambda_\sigma/9},\quad u_3^2=\frac{2(fm_\Phi^2-\lambda_\Phi m_\sigma^2/3)}{f^2-\lambda_\Phi\lambda_\sigma/9},$$

$$V_3=\frac{(m_\sigma^2\lambda_\Phi-3m_\Phi^2 f)^2}{6\lambda_\Phi(f^2-\lambda_\Phi\lambda_\sigma/9)}\ .$$

The constant term in Eq. (2.41) was added so that the energy of the desired global minimum [$\langle\sigma\rangle=0$, $\langle\Phi\rangle\neq 0$; unbroken U(1), broken U(1)′] is zero. Furthermore, the condition that $\partial^2 V/\partial u^2, \partial^2 V/\partial v^2>0$ at this extremum requires $3fm_\Phi^2/\lambda_\Phi>m_\sigma^2$. The second minimum [$\langle\Phi\rangle=0$, $\langle\sigma\rangle\neq 0$; broken U(1) and unbroken U(1)′] will be energetically disfavored if we require $m_\Phi^4/\lambda_\Phi>m_\sigma^4/\lambda_\sigma$. Finally, consider the third minimum [$\langle\Phi\rangle,\langle\sigma\rangle\neq 0$; broken U(1) and U(1)′]. If $f^2-\lambda_\Phi\lambda_\sigma/9$ is greater than 0, then V_3 is manifestly positive, and so this cannot be the global minimum. On the other hand, if $f^2-\lambda_\Phi\lambda_\sigma/9<0$, then V_3 is manifestly negative. In order for this to be a physical solution v_3^2 and u_3^2 must both be positive; however, if $fm_\Phi^2<\lambda_\Phi m_\sigma^2/3$ (as required above so that the desired solution is a minimum), u_3^2 is *less* than zero. In sum, we find that the necessary and sufficient constraints for the global minimum to be that with $\langle\Phi\rangle\neq 0$ and $\langle\sigma\rangle=0$ are

$$\frac{m_\Phi^4}{\lambda_\Phi}>\frac{m_\sigma^4}{\lambda_\sigma}\quad\text{(constraint 1)}\tag{2.42}$$

and

$$f\bar{v}^2/2=\frac{3fm_\Phi^2}{\lambda_\Phi}>m_\sigma^2\quad\text{or } \beta>\alpha\quad\text{(constraint 2)}\ .\tag{2.43}$$

For the potential given by Eq. (2.41), $\tilde{U}(\sigma_0)$ is given by

$$\tilde{U}(\sigma_0)=-\frac{m_\sigma^2\sigma_0^2}{4}+\frac{\lambda_\sigma\sigma_0^4}{192}\ .\tag{2.44}$$

Our problem is thus the minimization of the energy per length:

$$\begin{aligned}\tilde{H}(\sigma_0,\kappa)&=\tfrac{1}{4}\pi\sigma_0^2+\frac{\pi\tilde{U}(\sigma_0)}{\kappa^2}+\frac{\pi f v^2\sigma_0^2}{8\kappa^2}F(\mu/\kappa)\\&=-\frac{\pi}{4}[\alpha x^2-\beta x^2F(x)-1]\sigma_0^2+\frac{\pi\lambda_\sigma x^2\sigma_0^4}{192\mu^2}\ .\end{aligned}\tag{2.45}$$

Note that a nontrivial minimum will occur if the overall coefficient of σ_0^2 is negative, or

$$\alpha x^2-\beta x^2F(x)-1>0\quad\text{(constraint 3)}\ .\tag{2.46}$$

We can thus determine a lower limit to α. Using constraint (2) we have the condition

$$\alpha x^2[1-F(x)]-1>0\ .\tag{2.47}$$

It is readily verified that the

$$\max\{x^2[1-F(x)]\}=\lim_{x\to\infty}x^2[1-F(x)]=7\ ,$$

and thus the lower limit to α is

$$\alpha\geq\tfrac{1}{7}\ .\tag{2.48}$$

(It should be noted that in his discussion Witten considers the limit $\lambda_\sigma\approx 0$ and $\alpha\approx\beta$, and argues that the solution exists because the two-dimensional Schrödinger equation with negative-definite potential admits a normalizable bound state (in the $\alpha=\beta$ limit this corresponds to a negative coefficient to u^2 in the above potential). Strictly speaking, however, the absence of the λ_σ term causes the overall theory to be unstable [constraint (1) cannot be satisfied]. The region external to the vortex is a false vacuum and σ_0 grows without bound, eventually expelling the Φ field to infinity; the vortex ceases to exist. Therefore, the λ_σ term must always be present at some level and the normalization of σ_0 is always determined as above. Moreover, such a term is induced by interactions and one cannot have the strict $\alpha=\beta$ case without Coleman-Weinberg symmetry breaking by the σ field in far vacuum. Indeed, we see below that our variational calculation picks out the family of solutions with $\alpha\approx\beta$, except when $\alpha<\frac{1}{7}$. Why is our result in apparent conflict with the theorem that such solutions should always exist for any α including $\alpha\to 0$? The answer is that our variational *Ansatz* cannot probe the extreme weak potential case. We will see, however, that we obtain sufficient information about the parameter space that we can infer its structure as $\alpha\to 0$ by Witten's analytic result.)

The extremal solution for σ_0^2 is

$$\sigma_0^2=\frac{24\mu^2}{\lambda_\sigma}[\alpha-\beta F(x)-1/x^2]\ ,\tag{2.49}$$

which is a valid solution provided it is positive. This is just a restatement of constraint (3). Substituting the solution for σ_0^2 into Eq. (2.45) gives the energy per length as a function of κ,

$$\tilde{H}(x)=-\frac{3\pi\mu^2}{\lambda_\sigma x^2}[\alpha x^2-\beta x^2F(x)-1]^2\ ,\tag{2.50}$$

which leads to the extremal equation for κ:

$$\alpha x^2-\beta[x^2F(x)+x^3F'(x)]+1=0\tag{2.51}$$

(note, $\tilde{H}$ is manifestly negative at the extremal value for σ_0^2, so that the existence of a condensate does indeed lower the energy per length of the vortex).

We have scanned over the parameter space defined by α, β, and x for solutions consistent with constraint (2), and constraint (3). The allowed solution space and the width of the σ condensate are shown in Figs. 4 and 5. Our procedure consisted of choosing a value of β, solving for α using Eq. (2.51), and then scanning over values of x, checking for consistency with the constraints. We can also obtain the equation for the outer boundary of solutions, parametrized by x, by setting $\sigma_0^2=0$:

$$\beta(x)=2/x^3F'(x)\ ,\tag{2.52}$$

$$\alpha(x)=\beta(x)F(x)+1/x^2\ .\tag{2.53}$$

[Of course, constraint (1) must also be satisfied; this

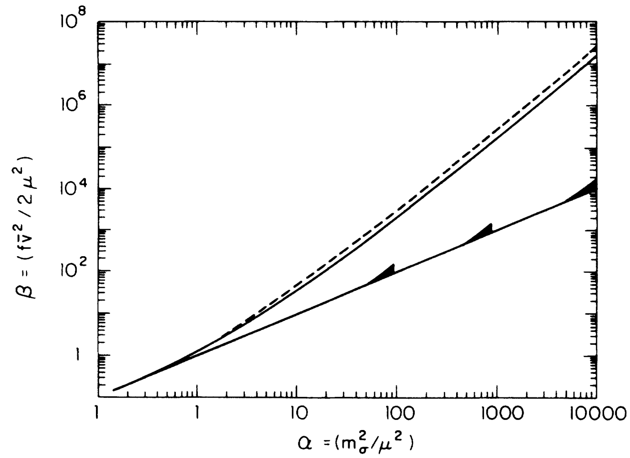

FIG. 4. The allowed α,β parameter space for superconducting solutions with the ordinary unstable scalar potential. The region between the two solid lines is the allowed region mapped out by our simple *Ansatz*; the dashed line is the upper boundary obtained from our full four-parameter *Ansatz*. For $\alpha \geq 10$ the upper boundary from the full *Ansatz* is $\beta \simeq 0.5\alpha^{1.93}$; for $\alpha \leq 10$ the upper boundary is $\beta \simeq 1.32\alpha^{1.35}$. Note that the wedge of superconducting solutions is terminated by constraint (1): $\alpha < a(b)(\lambda_\sigma/\lambda_\Phi)^{1/2} = (a/2\gamma)^{1/2}$. The solid triangles indicate regions of parameter space where the back reaction of the σ condensate on the vortex is significant (so defined by $y \geq 1.2$) for $\gamma = 10^{-4}$, 10^{-6}, and 10^{-8}.

merely truncates the wedge of solutions in α-β space at $\alpha = a(\lambda_\sigma/\lambda_\Phi)^{1/2}$.]

The parameter space of solutions from the full *Ansatz* (2.34) was also determined. This was done by looking for a global minimum in the energy with $\sigma_0^2 > 0$ and $\beta > \alpha$. The outer boundary obtained using the full *Ansatz* is also shown in Fig. 4, and it is not significantly different from the boundary determined by our truncated *Ansatz*. We also note that while the energy given by the full *Ansatz* (2.34) has converged to within a few percent, our simple *Ansatz* can give an energy per length which differs from that of the full *Ansatz* by $\sim 50\%$ for unfortunate choices of α and β. However, both *Ansätze* yield energies that agree to within a few percent near $\alpha = \beta$, which happens to be one of the most important regions for the rest of this study. At this point we abandon the more complicated *Ansatz*.

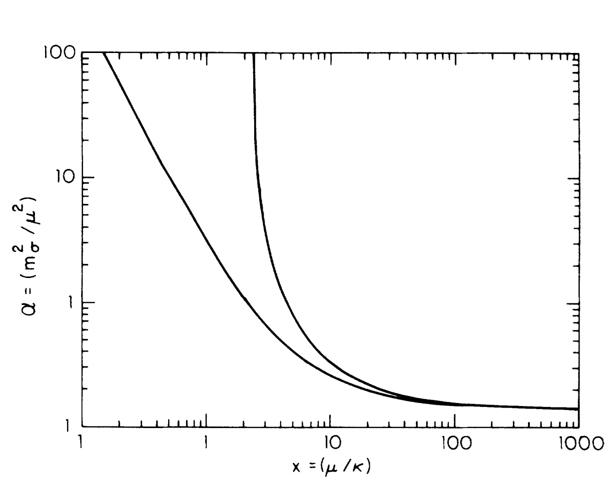

FIG. 5. The variational parameter x ($\approx$ the ratio of the size of the condensate to that of the vortex) is shown vs α. Solutions with $\alpha = \beta$ correspond to the right boundary ($x \simeq 2.5$).

The line of solutions corresponding to $\alpha = \beta$ is already known from Witten's argument and it extends in reality down to $\alpha = \beta = 0$. We see that the parameter space of solutions is restricted for small α and β and grows to the indicated wedge for large values. Our definitions of α and β have been very convenient because the allowed parameter space, in the *concrete-vortex* approximation, is independent of whether or not the string is gauged. [However, it should be mentioned that if m_σ^2 and $f\bar{v}^2/2$ were normalized by m_Φ^2, instead of by our variational parameter μ^2, global strings would have a fixed-wedge of allowable parameter space while the gauge strings would have a wedge of a size determined by the ratio of scalar and vector-boson masses. The parameter space with this normalization is obtained by the mapping $(\alpha,\beta) \rightarrow (\alpha/a, \beta/a)$, where a can be obtained from Fig. 2 in the gauged case, and $a \approx 1.6$ in the global case. We then see, with this normalization, that the gauged and global strings have the same available parameter space for $b \lesssim 0.1$, and for $b \gtrsim 0.1$ the gauged parameter space can be significantly larger than the global parameter space.]

In sum, for a superconducting solution to exist (within the *concrete-vortex* approximation), α and β must be within the allowed wedge (see Fig. 4) and constraint (1) must be satisfied: $m_\Phi^4/\lambda_\Phi > m_\sigma^4/\lambda_\sigma$, or, equivalently, $\alpha < a(\lambda_\sigma/\lambda_\Phi)^{1/2}$. That is, for a specified value of $\lambda_\sigma/\lambda_\Phi$, the wedge of solutions terminates at $\alpha = a(\lambda_\sigma/\lambda_\Phi)^{1/2}$. In Sec. VI we will discuss how one translates from the parameters α and β, which are computationally convenient, back to physical parameters in the scalar potential.

D. Back reaction onto the vortex

In the previous analysis we have viewed the Φ background solution as fixed, i.e., the width of the vortex, μ^{-1}, is held fixed as a parameter determined from the potential for Φ alone. In this section we relax our *concrete-vortex* approximation and vary the full Hamiltonian with respect to μ to study the validity of this approximation. We also map out the regions of solution space where there are significant deviations from our *concrete-vortex* approximation; fortunately, they occupy only a small fraction of the entire parameter space of solutions. Since the full Hamiltonian depends upon whether or not the string is gauged, we consider each case separately.

1. The global case

The full Hamiltonian for the global string is

$$\langle \tilde{H} \rangle = \frac{\pi \bar{v}^2}{4} + \pi \bar{v}^2 I_\theta(\mu) + 2\pi \sum_{n=1}^{\infty} \frac{(-1)^n \bar{v}^n V^{[n]}(\bar{v})}{n^2 n! \mu^2} - \frac{3\pi\mu^2}{\lambda_\sigma x^2}[\alpha x^2 - \beta x^2 F(x) - 1]^2 . \qquad (2.54)$$

It is now useful to introduce a new variational parameter which is the ratio of the value obtained in the *concrete-vortex* approximation ($\equiv \mu_0$) to the true value of μ. We thus introduce

$$y = \mu_0/\mu \ ,$$

where (2.55)

$$\mu_0^2 = 4 \sum_{n=2}^{\infty} \frac{(-1)^n \bar{v}^{\,n-2} V^{[n]}(\bar{v})}{n^2 n!} \ .$$

Also, we define α', β', and x' to be ratios with μ_0, i.e., $x' = \mu_0/\kappa = yx$, and, correspondingly, $\alpha' = \alpha/y^2$, $\beta' = \beta/y^2$. Removing overall constant factors and additive constants we have the variational Hamiltonian in y and x':

$$\langle \tilde{H} \rangle = -\ln y + y^2/2 - \frac{\gamma}{x'^2}[\alpha' x'^2 - \beta' x'^2 F(x'/y) - 1]^2 \ . \quad (2.56)$$

We have also introduced the parameter $\gamma = 3\mu_0^2/\bar{v}^{\,2}\lambda_\sigma$; note the ratio $\mu_0/\bar{v}$ has already been determined implicitly in the above discussion for the general quartic polynomial potential: $\gamma \approx 0.31\lambda_\Phi/\lambda_\sigma$. In the *concrete-vortex* approximation, $y = 1$, the primed parameters coincide with the unprimed ones; this redefinition has the advantage of minimizing the y dependence of the additional term.

The joint extremal equations in x' and y are

$$\alpha' x'^2 - \beta'[x'^2 F(x'/y) + x'^3 F'(x'/y)/y] + 1 = 0 \ , \quad (2.57)$$

$$1 - y^2 + 2\beta' \gamma W x' F'(x'/y)/y = 0 \ , \quad (2.58)$$

where $W = \alpha' x'^2 - \beta' x'^2 F(x'/y) - 1$. We also recover our previous constraints recast in the present variables: $W > 0$, and $\beta' - \alpha' > 0$. Note that since $W > 0$ we must have $y > 1$, i.e., the vortex is always larger than the size given by our *concrete-vortex* approximation.

Upon specification of γ the allowed values of α' are restricted by constraint (1):

$$\alpha' < \frac{0.9}{\sqrt{\gamma}} \ . \quad (2.59)$$

We present in Figs. 6–9 both α' and β' as a function of y for two choices of γ, 10^{-3} and 10^{-8}. (We choose small values of γ here and elsewhere only for the convenience of having a large region of α,β parameter space to study.) These results were obtained by solving for α' and β' in the extremal equations and scanning over x' and y (always checking that the constraints are satisfied, and that the charged condensate lowers the energy per length of the original vortex). We see, as a general rule, that back reaction can only be significant if the above inequality approaches equality. In other words, a necessary condition for back reaction to be important is

$$\frac{m_\Phi^4}{\lambda_\Phi} \approx \frac{m_\sigma^4}{\lambda_\sigma} \ . \quad (2.60)$$

This should come as no surprise since in this limit the vacuum energy associated with a σ condensate is about the same as that associated with a Φ condensate; i.e., the $\langle \Phi \rangle \neq 0$, $\langle \sigma \rangle = 0$ and the $\langle \sigma \rangle \neq 0$, $\langle \Phi \rangle = 0$ minima are nearly degenerate. We further see, upon comparison of the graphs with the same γ, that large values of y (i.e., significant back reaction) only occur when condition (2.60) is satisfied and

$$\alpha' \approx \beta' \quad \text{or} \quad m_\sigma^2 \approx \frac{3 f m_\Phi^2}{\lambda_\Phi} \ . \quad (2.61)$$

For the choice $\gamma = 10^{-4}$, 10^{-6}, and 10^{-8} we also show the α',β' parameter space where $y \geq 1.2$ in Fig. 4. We see that our concrete-vortex approximation is valid over most of the parameter space, and solutions with even mild back reaction are very rare, and are restricted to be near the line $\alpha = \beta$.

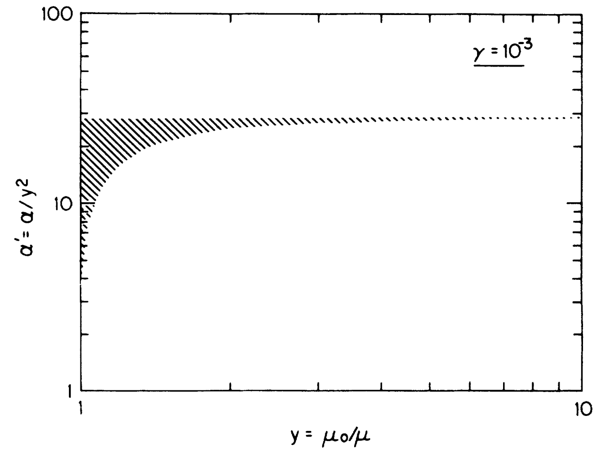

FIG. 6. The allowed α' parameter space as a function of y for a global string with $\gamma = 10^{-3}$. Significant deviations from the *concrete-vortex* approximation occur for large y.

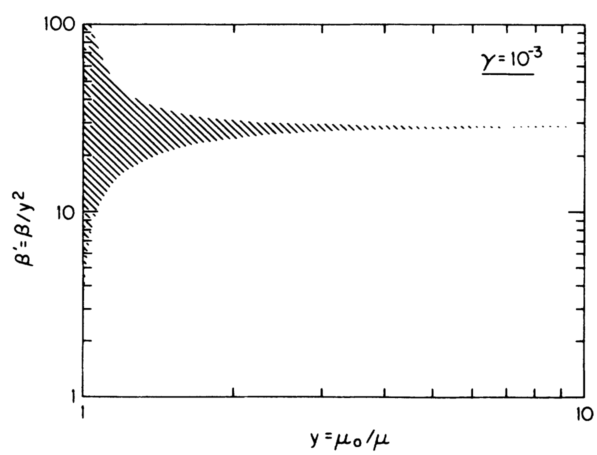

FIG. 7. The allowed β' parameter space as a function of y for a global string with $\gamma = 10^{-3}$.

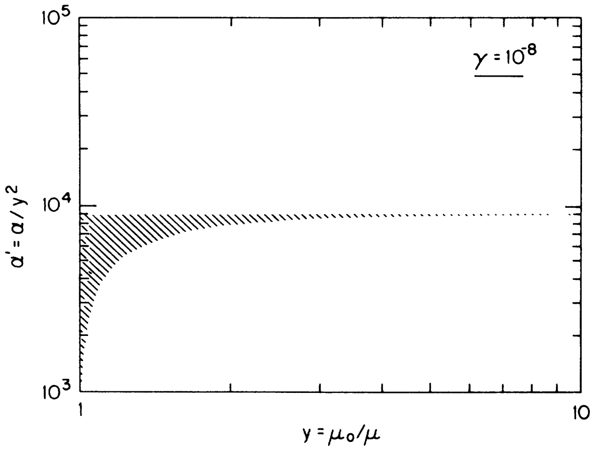

FIG. 8. The allowed α' parameter space as a function of y for a global string with $\gamma = 10^{-8}$.

2. *The gauged case*

The full Hamiltonian per length for the gauged string is

$$\langle \tilde{H} \rangle = \frac{\pi \bar{v}^2}{4} + \pi \bar{v}^2 G(s) + \frac{4\pi \bar{v}^2 \ln(\frac{9}{8})}{abs^2} + \frac{89}{288}\pi \bar{v}^2 a - \frac{3\pi\mu^2}{\lambda_\sigma x^2}[\alpha x^2 - \beta x^2 F(x) - 1]^2 . \quad (2.62)$$

We redefine variables as in the global case, again use $y = \mu_0/\mu$, and further define $s' = sy$. We replace a with $a'y^2$, where a' is the value of a in the *concrete-vortex* approximation. Dropping constant factors and additive constants we now have the variational Hamiltonian in y, x', and s':

$$\langle \tilde{H} \rangle = G(s'/y) + \frac{4\ln(\frac{9}{8})}{a'bs'^2} + \frac{89}{288}a'y^2 - \frac{\gamma}{x'^2}[\alpha' x'^2 - \beta' x'^2 F(x'/y) - 1]^2 . \quad (2.63)$$

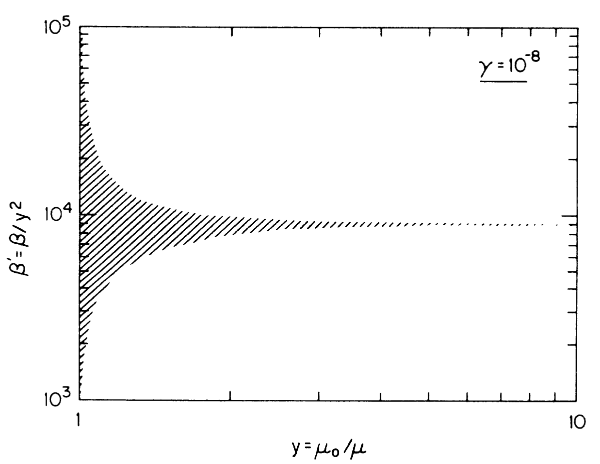

FIG. 9. The allowed β' parameter space as a function of y for a global string with $\gamma = 10^{-8}$.

We have defined γ as before, and in this case $\gamma = \lambda_\Phi / 2\lambda_\sigma a'$. The constraint on α', which follows from constraint (1), now takes the form

$$\alpha' < \left[\frac{a'}{2\gamma} \right]^{1/2} , \quad (2.64)$$

and, as in the global case, we require $\beta' > \alpha'$ [constraint (2)] and $W = \alpha' x'^2 - \beta' x'^2 F(x'/y) - 1 > 0$ [constraint (3)]. The method of solution in this case was to numerically search for a global minimum in the energy (and, of course, checking to see that the charged condensate actually lowers the energy per length of the original flux tube[16]). Results very similar to the global case were obtained. Again, the main result is that if we are to have a superconducting solution and $y \gg 1$, we require

$$\frac{m_\Phi^4}{\lambda_\Phi} \approx \frac{m_\sigma^4}{\lambda_\sigma} \text{ and } \alpha' \approx \beta' . \quad (2.65)$$

To summarize the issue of back reaction, we can say that for all but a small portion of the parameter space of solutions back reaction of the σ condensate upon the vortex itself is not important. Only for $\alpha \approx \beta$ is back reaction potentially significant.

E. Coleman-Weinberg effective potentials

It is interesting to consider the possibility of vortices and associated superconductivity, in the case of Coleman-Weinberg[17] symmetry breaking. Here the fields Φ and σ have zero renormalized mass, but the radiative corrections due to the interactions with their gauge bosons produce an unstable effective potential at the one-loop level. We may consider the effective potential to be[17]

$$V(\Phi,\sigma) = \frac{\lambda_\Phi}{3!}|\Phi|^4 + \frac{3q^4}{16\pi^2}|\Phi|^4[\ln(2|\Phi|^2/v^2) - \tfrac{25}{6}] + \frac{\lambda_\sigma}{3!}|\sigma|^4 + \frac{3e^4}{16\pi^2}|\sigma|^4[\ln(2|\sigma|^2/v'^2) - \tfrac{25}{6}] + f|\Phi|^2|\sigma|^2 . \quad (2.66)$$

The condition that $\Phi = ve^{i\theta}/\sqrt{2}$ minimize the Φ part of the potential implies that

$$0 = \frac{\lambda_\Phi}{6} - \frac{11q^4}{16\pi^2} , \quad (2.67)$$

at which point the vacuum energy density is

$$E_{\rm vac} = -\frac{3q^4}{128\pi^2}v^4 \quad (2.68)$$

[for simplicity we have not added a constant term to $V(\Phi,\sigma)$ to make the vacuum energy vanish at the desired minimum].

In general we wish for the σ field to minimize the potential at some other mass scale, $v'/\sqrt{2}$. Consequently, we have the relationship

$$0 = \frac{\lambda_\sigma}{6} - \frac{11e^4}{16\pi^2} . \quad (2.69)$$

At the σ minimum we have

$$E_{\rm vac} = -\frac{3e^4}{128\pi} v'^4 \qquad (2.70)$$

and the stability of the theory at the $\sigma=0$, $\Phi\neq 0$ minimum requires [constraint (1')]

$$qv > ev' \ , \qquad (2.71)$$

which is the analog of constraint (1). Besides $f>0$, no further constraints are obtained. Unlike the case with ordinary scalar potentials, we find that the second nontrivial constraint arises from requiring that the extremum $\sigma=v_1/\sqrt{2}$, $\Phi=v_2 e^{i\theta}/\sqrt{2}$, $(v_1,v_2\neq 0)$ not be the true minimum. Upon extremizing the potential, v_1, and v_2 can be solved from

$$8A(v_1/v)^2\ln(v_1/v)+C(v_2/v')^2=0 \ , \qquad (2.72)$$

$$8(v_2/v')^2\ln(v_2/v')+C(v_1/v)^2=0 \ , \qquad (2.73)$$

where $A=(qv/ev')^4>1$, $C=\beta/X>0$, and we have introduced the parameter X:

$$X=\frac{3e^4 v'^2}{64\pi^2\mu^2} \ , \qquad (2.74)$$

which is the analog of α (β is defined as before, i.e., $\beta=fv^2/2\mu^2$). Because of the positivity of A and C, it is clear that $v_1<v$ and $v_2<v'$. To find the parameter space (A,C) that represents the global minimum, we scan through the space $(v_1/v, v_2/v')$, solve for A and C, and then check that the energy is lower than that given by Eq. (2.68). This parameter space, which is *not* acceptable, is shown in Fig. 10. We see from Fig. 10 that a necessary condition (though not sufficient if $A\approx 1$) for stability of the theory at the $\sigma=0$, $\Phi\neq 0$ minimum is

$$\beta \gtrsim 1.2X \ , \qquad (2.75)$$

which is the analog for Coleman-Weinberg breaking of constraint (2).

Now we use the variational *Ansatz* of the preceding

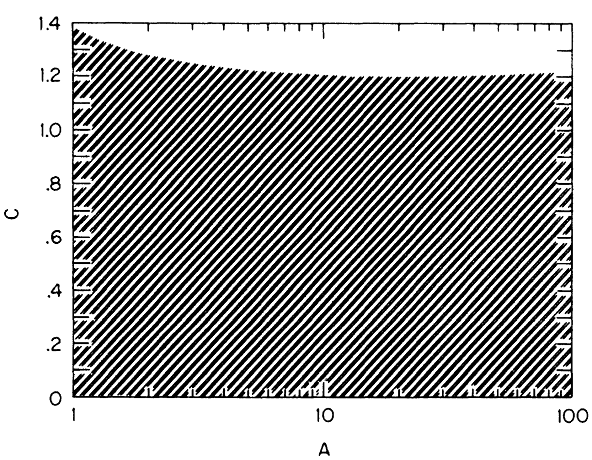

FIG. 10. The region of parameter space $[A=(qv/ev')^4$, $C=\beta/X]$ that corresponds to a true vacuum with $\langle\sigma\rangle$, $\langle\Phi\rangle\neq 0$. This region, which corresponds to electromagnetism being broken far from the vortex, is strictly disallowed.

analysis. We again recover Eq. (2.9) and find presently, for μ,

$$\mu^2=\frac{q^4v^2}{\pi^2}\left[\tfrac{3}{16}-\tfrac{5}{36}+\tfrac{11}{256}-O(10^{-3})\right]\approx 0.092\frac{q^4v^2}{\pi^2} \qquad (2.76)$$

in the global case. The series does not terminate because of the expansion of the logarithm; we keep here terms to $O(e^{-4\mu r})$.

Similarly, for the σ field we find the energy functional per unit length:

$$\tilde{H}(\sigma_0,\kappa)=\frac{\pi}{4}[1+\beta x^2F(x)]\sigma_0^2 - \frac{9e^4x^2\sigma_0^4}{1024\mu^2\pi}+\frac{3e^4x^2\sigma_0^4}{512\mu^2\pi}\ln(\sigma_0^2/v'^2) \ . \qquad (2.77)$$

We see here another manifestation of the modified constraint (2); if $\beta\to 0$ this energy is unbounded below for $\kappa\to 0$; the lower limit to f prevents this catastrophe.

We rescale the above Hamiltonian per unit length by letting $\sigma_0^2=v'^2\zeta_0^2/X$. The energy per length then takes the form

$$\tilde{H}=\pi v'^2\left\{2\zeta_0^2[1+\beta x^2F(x)]+\zeta_0^4x^2\left[\ln\left[\frac{\zeta_0^2}{X}\right]-\frac{3}{2}\right]\right\}\Big/ 8X. \qquad (2.78)$$

Upon extremization of the energy per length with respect to ζ_0 and x we obtain

$$\beta[2F(x)+xF'(x)]+\zeta_0^2[\ln(\zeta_0^2/X)-\tfrac{1}{2}]=0 \ , \qquad (2.79)$$

$$1+\beta x^2F(x)+\zeta_0^2x^2[\ln(\zeta_0^2/X)-1]=0 \ . \qquad (2.80)$$

In this case the upper boundary to the parameter space of solutions does not correspond to $\zeta_0^2=0$, and the boundary is more difficult to locate. Given values of X and β, ζ_0^2 typically has two extremal solutions. However, the solution of interest can be obtained by requiring that the condensate is bound to the vortex: $\tilde{H}<0$. The extremal equations can be solved for β and X in terms of x and ζ_0. It is then possible to scan over the variational parameter space (x,ζ_0) and search for solutions satisfying the modified constraint (2). Results from such a scan are shown in Fig. 11. We see that the parameter space (X,β) is very similar to the parameter space (α,β) obtained with ordinary quartic potentials.

III. DYNAMICS OF BOSONIC SUPERCONDUCTIVITY

To recapitulate what we have done to this point, using our variational *Ansätze* we have mapped out the regions of parameter space that allow bosonic superconductivity, for both ordinary and Coleman-Weinberg scalar potentials (see Figs. 4 and 11). Furthermore, we have explored the regions of parameter space for the scalar potential where the σ condensate significantly modifies the vortex solution itself, which occurs for $\alpha\approx\beta$ and $m_\Phi^4/\lambda_\Phi\approx m_\sigma^4/\lambda_\sigma$. Throughout these analyses the super-

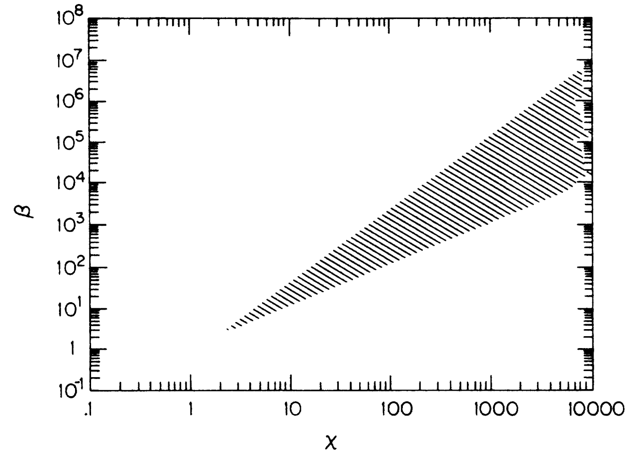

FIG. 11. The χ,β parameter space of solutions for superconducting vortices for Coleman-Weinberg potentials.

conducting current was taken to be zero. Presently we obtain the expression for the energy associated with superconducting currents, i.e., the kinetic energy of the charge carriers and the energy in the magnetic field. Throughout we will use our truncated *Ansatz*. First we directly solve Maxwell's equation without resort to a Green's-function expression (which Witten[1] does); however, the usual UV singularities still occur and must be dealt with in a self-consistent manner.

Consider phase fluctuations about the σ condensate obtained above,

$$\sigma(r,\theta,z,t)\rightarrow\frac{\sigma_0(r)}{\sqrt{2}}\exp[i\phi(z,t)] \ , \tag{3.1}$$

and we obtain the effective action for $\phi(z,t)$ in the case of an infinite straight string along the z axis:

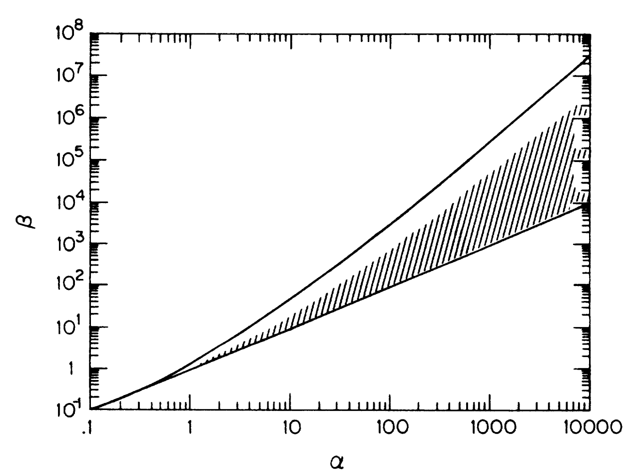

FIG. 12. The α,β parameter space of solutions consistent with $K\lambda_\sigma > 100$. In this region the magnetic field energy dominates that of the KE of the charge carriers. Solid lines indicate the entire parameter space of solutions.

$$I=\tfrac{1}{2}\int 2\pi r\, dr\, dz\, dt\, \sigma_0(r)^2[(\partial_t\phi-eA_0)^2-(\partial_z\phi-eA_z)^2]$$
$$-\tfrac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}+O(1/\kappa^2) \ . \tag{3.2}$$

We thus obtain the equation of motion for the ϕ field and the vector potential:

$$\partial_0^2\phi-\partial_z^2\phi=e(\partial_0 A_0-\partial_z A_z) \ , \tag{3.3}$$

$$\partial_\mu F^{\mu\nu}=eK(\partial^z\phi-eA^z)\delta^2(x_\perp)\delta^{z\nu} \ , \tag{3.4}$$

where

$$K=\int 2\pi r\, dr\, \sigma_0(r)^2 \quad (=\pi\sigma_0{}^2/2\kappa^2) \ , \tag{3.5}$$

the latter result holding with our simple *Ansatz*. The interesting solution for our purposes corresponds to a conducting wire of length L with ϕ topological winding number N; that is,

$$\phi=\frac{2\pi N}{L}z \ , \quad A^0=A^x=A^y=0 \ , \tag{3.6}$$

and we obtain the remaining component of the vector potential,

$$A_z=-\text{const}\times\ln(\sqrt{x^2+y^2}/L) \ , \tag{3.7}$$

which describes a circumferential B field:

$$B_x=-\text{const}\times\frac{y}{x^2+y^2} \ ,$$
$$B_y=\text{const}\times\frac{x}{x^2+y^2} \ , \tag{3.8}$$

and the constant is determined by self-consistency with the current by way of Stokes's theorem. The current is

$$j_z=eK[\partial_z\phi-eA_z(0)]\delta^2(x_\perp) \tag{3.9}$$

and we must interpret $A_z(0)$. (Witten similarly encounters this subtlety in his Green's-function solution.) We simply define

$$A_z(0)=\text{const}\times\ln(\kappa L) \tag{3.10}$$

(the lower cutoff is the size of our "wire," $\sim\kappa^{-1}$). Then using Stokes's theorem we obtain

$$\text{const}=\frac{eKN}{L}\left[1+\frac{e^2K}{2\pi}\ln(\kappa L)\right]^{-1} \ . \tag{3.11}$$

We make the definition

$$\omega=\frac{1}{e^2K}\left[1+\frac{e^2K}{2\pi}\ln(\kappa L)\right] \ , \tag{3.12}$$

which is the "inductance per unit length" and the result for the B field becomes

$$B_x=-\frac{N}{eL\omega}\frac{y}{x^2+y^2} \ ,$$
$$B_y=\frac{N}{eL\omega}\frac{x}{x^2+y^2} \ . \tag{3.13}$$

Now, we may compute the resulting energy from the

Hamiltonian:

$$H=\tfrac{1}{2}\int 2\pi r\,dr\,dz\,\sigma_0(r)^2(\partial_z\phi-eA_z)^2+\int d^3x\frac{B^2}{2}\ . \tag{3.14}$$

The $\partial_z\phi-eA_z$ contribution is just $2\pi^2N^2/Le^4\omega^2K$ and the $B^2/2$ contribution is logarithmically divergent in the transverse dimensions which we truncate at L, which yields $\pi N^2\ln(\kappa L)/Le^2\omega^2$. These terms combine to give the effective Hamiltonian per unit length associated with the current:

$$\langle\tilde{H}\rangle=\frac{2\pi^2N^2}{L^2K\omega^2e^4}\left[1+\frac{e^2K}{2\pi}\ln(\kappa L)\right]=\frac{2\pi^2N^2}{L^2e^2\omega}=\tfrac{1}{2}\omega I^2\ , \tag{3.15}$$

where the current $I=\int dA\,j_z=2\pi N/eL\omega$. We use electromagnetic units here, and elsewhere, that correspond to $e^2/4\pi=\alpha_{\rm EM}$, where $\alpha_{\rm EM}\simeq\frac{1}{137}$ is the electromagnetic coupling constant. For reference, e GeV$=2.43\times10^5$ A.

The ratio of the energy in the magnetic field to the kinetic energy (KE) of the charge carriers is $e^2K\ln(\kappa L)/2\pi$, and with our simple *Ansatz* $K=\pi\sigma_0^2/2\kappa^2$. Upon using our previous extremal solution for σ_0^2 and taking $\lambda_\sigma\lesssim1$ (perturbativity) we find that for most of the solution parameter space $K\gg1$, as shown in Fig. 12. For the loops of interest L is a macroscopic (or even cosmological) length, implying that $L\kappa\gg1$, and so the field energy is the dominant contribution when $K\gg1$. The energy associated with the superconducting current, in this case, is no different from that of an ordinary wire with current I. The only region of our parameter space (α,β) that the kinetic energy of the charge carriers can have a significant contribution is near the upper boundary of solutions, where the σ condensate is starting to become energetically unfavorable and $K\rightarrow0$.

Superconducting currents can be induced in a cosmic string if primeval magnetic flux is present in the early Universe. Whether or not there were primordial magnetic fields in the early Universe is still an open and very important question which has been considered elsewhere.[18] We mention, however, that superconducting strings inherently have a nonzero winding number in the σ field (and hence current) since the phase of the σ field can have a correlation length ξ no larger than the scale of the horizon ($H^{-1}\sim t$) at the time of the spontaneous symmetry breaking ($t=t_{\rm SSB}$) that gave rise to cosmic strings: $\xi\lesssim t_{\rm SSB}$. This results in a winding number in a length of string L of at least $\sim(L/\xi)^{1/2}$, which leads to a minimum current that must be present (and which one might refer to as "the Kibble current").

IV. CRITICAL CURRENTS

We now examine the breakdown of bosonic superconductivity. Collecting terms in the Hamiltonian associated with the σ condensate gives

$$\langle\tilde{H}_\sigma\rangle=-\frac{\pi}{4}[\alpha x^2-\beta x^2F(x)-1]\sigma_0^2+\frac{\pi\lambda_\sigma x^2\sigma_0^4}{192\mu^2}+\frac{2\pi^2N^2K}{L^2[1+(e^2K/2\pi)\ln(\kappa L)]} \tag{4.1}$$

using our simple *Ansatz* for σ and using the ordinary scalar potential. (Since this is a Hamiltonian per length, the last term is of the order of $1/L^2$.) The current cannot be arbitrarily large because there will be a (critical) current beyond which it will be energetically favorable for the σ field to become zero everywhere. Since $\langle\tilde{H}_\sigma\rangle$ vanishes for $\sigma=0$, it becomes energetically favorable for the current to dissipate when $\langle\tilde{H}_\sigma\rangle$ becomes nonnegative.[19] Although N is topological in nature, it can unwind in processes where the σ field can go through zero (where the phase ϕ is not well defined), which on energetic grounds should occur when the above Hamiltonian approaches zero.

Notice, however, that we should display the full σ_0 dependence in this expression by restoring the expression for K obtained previously:

$$\langle\tilde{H}_\sigma\rangle=-\frac{\pi}{4}[\alpha x^2-\beta x^2F(x)-1]\sigma_0^2+\frac{\pi\lambda_\sigma x^2\sigma_0^4}{192\mu^2}+\frac{\pi^3N^2\sigma_0^2}{\kappa^2L^2[1+e^2\sigma_0^2\ln(\kappa L)/4\kappa^2]}\ . \tag{4.2}$$

If we could neglect the logarithm term in the last term in Eq. (4.2), the $\sigma_0\neq0$ superconducting minimum would smoothly go to zero as N/L is increased to $(N/L)_{\rm max}$ (where $\sigma_0=0$), and the current would achieve its maximum at $(N/L)=(N/L)_{\rm max}/\sqrt{3}$ and go to zero at $(N/L)_{\rm max}$. The quench transition would be second order. However, the logarithm is typically large and *cannot* be neglected. In fact, one can see by plotting $\tilde{H}_\sigma$ as a function of σ_0 (for fixed x) that, for a range of currents, the local minimum at $\sigma_0\neq0$ persists even when $\tilde{H}_\sigma(\sigma_0)>0$. Thus, it is possible for the superconducting state to be metastable even when $\tilde{H}_\sigma(\sigma_0)>0$. This also suggests that the transition should be *first-order,* and proceed through the nucleation of bubbles (inside which $\sigma_0=0$). The full analysis requires considering variations in both x and σ_0, and computing tunneling rates; this analysis is currently in progress.[20] We will content ourselves to defining absolute stability when $\langle\tilde{H}_\sigma\rangle<0$ (however, see Ref. 19). In analogy with finite-temperature SSB phase transitions, we define the critical current to be that where $\tilde{H}_\sigma=0$ (holding N fixed).

This critical current can easily be estimated by using the unperturbed extremal solution for σ_0^2 from Eq. (2.49) and equating the right-hand side of Eq. (4.2) to zero. The result for the critical current found in this way is

$$I_{\rm crit}=\mu\left[\frac{6\pi}{\lambda_\sigma\omega}\right]^{1/2}[\alpha x-\beta F(x)x-x^{-1}]\ . \tag{4.3}$$

In the limit $K\gg1$ the critical current takes the form

$$I_{\rm crit}=\pi\mu\left[\frac{12}{\lambda_\sigma\ln(\kappa L)}\right]^{1/2}[\alpha x-\beta F(x)x-x^{-1}]\ . \tag{4.4}$$

This limit is convenient to study since the λ_σ and logarithmic dependence is now multiplicative. For $K\lambda_\sigma > 100$ and $\alpha < 10^4$ we find that the coefficient involving x, α, and β is roughly proportional to $\alpha^{1.8}/\beta^{3/4}$. Using this approximation, the critical current for the aforementioned range of parameters is

$$I_{\rm crit} \simeq 5.5\mu[\lambda_\sigma \ln(\kappa L)]^{-1/2}\alpha^{1.8}/\beta^{3/4} \; , \qquad (4.5)$$

which is accurate to about $\approx 30\%$. Note that, $I_{\rm crit} \lesssim 2\bar{v}[a/\ln(\kappa L)]^{1/2}$, the value $=O(\bar{v})$ being achieved for $\alpha \simeq \beta \simeq (a/2\gamma)^{1/2}$. The critical current as defined is shown in Fig. 13.

We note that one could equally well choose to define the critical current to be that for which the lifetime of the metastable, superconducting state becomes shorter than some relevant cosmological time scale. However, as we shall now describe, that definition can differ with ours by at most a factor of a few because the local $\sigma_0 \neq 0$ minimum only persists for currents at most a factor of a few greater than our critical current.

To see this we study the one-dimensional potential $\tilde{H}_\sigma(\sigma_0)$, always choosing x to minimize the energy per length and holding N fixed. The current is varied by varying the length of the loop L. The energy per length of the condensate, Eq. (4.1), is shown as a function of σ_0 and the current in Fig. 14 for a choice of parameters, $\alpha = 10^4$, $\beta = 10^5$, $N = 10^4$, and $\lambda_\sigma = 1$. For small currents ($I \ll I_{\rm crit}$), the current has little effect on the condensate, i.e., the value of σ_0, which minimizes $\tilde{H}_\sigma$, is almost independent of the current. As the current increases $\tilde{H}_\sigma$ becomes minimized for $\sigma_0 = 0$, while a local superconducting minimum still exists where the σ field becomes trapped for a supercritical current. For currents not much greater than the critical current (in all cases less than a factor of a few) the superconducting minimum disappears, making the superconducting state classically unstable. A search through many different parameters was made, and it appears that the properties exhibited by this particular choice of parameters are quite general.

Qualitatively, the features seen in Fig. 14 are quite simple to understand. Picture the two-dimensional potential $\tilde{H}_\sigma(x,\sigma_0)$. The surface $\tilde{H}_\sigma(x,\sigma_0)$ has a dimple near $x \sim 1$ and $\sigma_0 = \sigma_0^{\rm sc} \neq 0$ (the superconducting minimum). However, for values of σ_0 very different from $\sigma_0^{\rm sc}$, $\tilde{H}_\sigma$ (x, fixed σ_0) is minimized for $x \sim 0$, and for $x = 0$, $\tilde{H}_\sigma \propto \sigma_0^2$. Thus the potential plotted in Fig. 14, $\tilde{H}_\sigma(\sigma_0)$ [$\equiv \tilde{H}_\sigma(x,\sigma_0)$, minimized as a function of the condensate size x], varies as σ_0^2, except near the dimple, $\sigma_0 \sim \sigma_0^{\rm sc}$. As the current increases the dimple flattens, and eventually disappears (in Fig. 14, for $I = 1.0046 I_{\rm crit}$); for currents greater than this $\tilde{H}_\sigma \propto \sigma_0^2$, and the metastability of the superconducting state is lost.

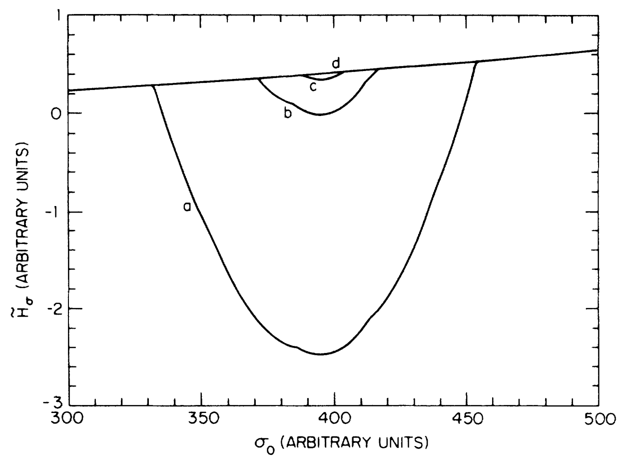

FIG. 14. The energy per length of the condensate $\tilde{H}_\sigma$ (arbitrary units) as a function of σ_0 and the current, for the parameters $\alpha = 10^4$, $\beta = 10^5$, $N = 10^4$, and $\lambda_\sigma = 1$. For curve a, $I = 0.97 I_{\rm crit}$ and the superconducting state is stable; for curve b, $I = I_{\rm crit}$ ($\tilde{H}_\sigma = 0$); for curve c, $I = 1.004 I_{\rm crit}$ and the superconducting state is metastable; for curve d, $I \gtrsim 1.0046 I_{\rm crit}$, the superconducting local minimum disappears and the metastability of the superconducting state is lost. The behavior illustrated here for a particular set of parameters appears to be generic.

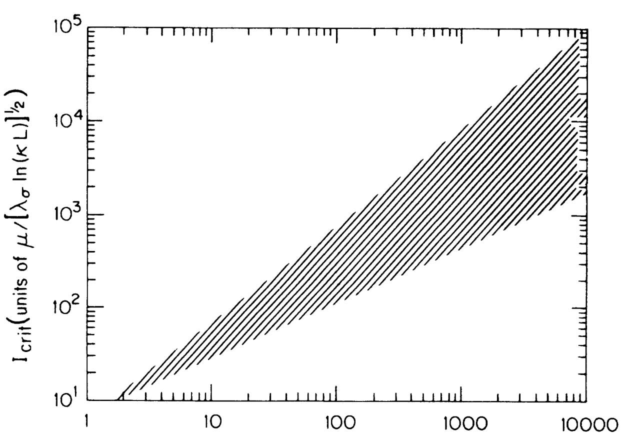

FIG. 13. The critical currents of bosonic superconducting strings are shown as a function of α, for all possible choices of β consistent with $K\lambda_\sigma > 100$.

The precise details of how a loop in a cosmological setting quenches involves not only the microphysics of the quench transition for a loop of fixed current (that is, fixed length), but also the rate at which the loop is shrinking (and the current is increasing), whether or not cusps are present on the loop which can dissipate phase twist in σ, etc. From our work it is already clear that supercritical loops can only be metastable for currents of at most a few times $I_{\rm crit}$. And the metastability of these supercritical loops depends upon the tunneling rate (which has been addressed in Ref. 20).

V. STATIC LOOPS

We will now explore the possibility that a loop of string can be stabilized against its string tension by electromagnetic stresses and achieve a static, or floating state.[4,5] That this could occur is easy to see. Neglecting numerical factors, the energy of a loop is ~(string tension $\sim \bar{v}^2$)$\times L + LI^2/2$, the first term representing the energy due to the string tension and the second due to the electromagnetic field. As a loop oscillates it radiates both electromagnetic and gravitational radiation, and in the process must shrink in size. Conservation of the winding number N means that the supercurrent $I \propto N/L$ must increase, and so the magnetic field energy varies as $1/L$, whereas the potential energy of the string varies as L. Assuming that N remains constant (and the string superconducting) the loop will reach a state of minimum energy for $L = L_{\text{static}} \sim N/\bar{v}$, where it can no longer decrease its energy by shrinking, and the string tension is balanced by electromagnetic stresses.

To consider loops of string in our present framework which is strictly only applicable to infinitely long strings, we require that the scale of the fields be much smaller than the curvature of the loop: $\mu L \gg 1$, $\kappa L \gg 1$, $hL \gg 1$. We also implicitly assume that the loop is "relatively smooth," so that $\partial\phi/\partial z \sim$ const. Our analysis may be invalid if the loop has cusps where $\partial\phi/\partial z$ is large, as superconductivity may quench at these points long before $I \simeq I_{\text{crit}}$, dissipating phase twist and perhaps preventing the current from ever getting close to I_{crit}.

The ideal approach to search for static loops would be to extremize the full Hamiltonian with respect to σ, κ, μ, and L, and search for solutions consistent with our constraints. However, we will restrict ourselves to the case where the loop is stabilized by currents that are smaller than the critical current, and so here we will not address the question of metastable, supercritical static loops. By so restricting ourselves, we are able to use our previous results, i.e., our variational parameters do not change in the presence of the current. This amounts to requiring that the energy in the current not "back react" upon the rest of the Hamiltonian. In what follows we consider the case of the usual quartic potential, Eq. (2.41), although most of the results also apply to Coleman-Weinberg potentials.

We write the energy of a superconducting loop, in units of $\pi\bar{v}^2$, as

$$E(L) = [A_\Phi + w\ln(\mu L)]L - B_\sigma L + \frac{C_\sigma}{L[1+(e^2K/2\pi)\ln(\kappa L)]}, \tag{5.1}$$

where B_σ includes all the σ-dependent terms except for those associated with the charge carriers and magnetic field, which are given by C_σ, and A_Φ contains the remaining terms depending only upon Φ, which are independent of the loop length L. In the global case $w=1$, and in the gauge case $w=0$. The quantities A_Φ, B_σ, and C_σ are determined from Eqs. (2.21) or (2.12) and (4.1). In general, the energy per length of a superconducting string (with subcritical current) must be less than that of an ordinary cosmic string since the σ contribution to Eq. (5.1) is necessarily negative.

Note that the coefficients A_Φ, B_σ, and C_σ are all positive. The length of loop that minimizes the above energy, and represents the static state, is

$$L_{\text{static}} = \left[\frac{C_\sigma \delta}{[A_\Phi + w + w\ln(\mu L_{\text{static}}) - B_\sigma][1+(e^2K/2\pi)\ln(\kappa L_{\text{static}})]}\right]^{1/2}, \tag{5.2}$$

where

$$\delta = 1 + \frac{e^2K/2\pi}{1+(e^2K/2\pi)\ln(\kappa L_{\text{static}})}. \tag{5.3}$$

Requiring L_{static} to be real gives us the condition that $A_\Phi + w[1+\ln(\mu L)] > B_\sigma$.

To obtain a stable, static configuration with subcritical current, the sum of the last two terms in Eq. (5.1) must be negative. We define the critical loop size, L_{critical}, to be the loop size for which the loop current is equal to the critical current:

$$L_{\text{critical}} = \left[\frac{C_\sigma}{B_\sigma[1+(e^2K/2\pi)\ln(\kappa L_{\text{critical}})]}\right]^{1/2}. \tag{5.4}$$

(In calculating L_{critical} we have ignored the back reaction of the current upon the vortex and the condensate; including the back reaction may modify this result slightly.) Since we are seeking subcritical floating solutions, we must have $L_{\text{static}} > L_{\text{critical}}$ (to be perfectly safe we should probably require $L_{\text{static}} \gtrsim 3L_{\text{critical}}$, so that the back reaction of the current upon the vortex is less than $\sim 10\%$ and can be neglected). Using $L_{\text{static}} > L_{\text{critical}}$ as the criterion, we see that static loops are possible only if

$$0 < A_\Phi + w[1+\ln(\mu L_{\text{static}})] - B_\sigma < \delta B_\sigma \frac{1+(e^2K/2\pi)\ln(\kappa L_{\text{critical}})}{1+(e^2K/2\pi)\ln(\kappa L_{\text{static}})}. \tag{5.5}$$

If we require $\kappa L/\pi > 10$ so that our variational approach is not invalidated by curvature effects, we must have $1 \le \delta < 1.3$. It is then apparent that the energy in the vortex is very close to that in the condensate (true for global or gauged loops), which is not surprising since a static string implies an equal balance of the energy between the σ and Φ fields. The energy per length of a static loop is

$$\frac{E(L)}{L} = [A_\Phi + w\ln(\mu L) - B_\sigma]\left[1+\frac{1}{\delta}\right] + \frac{w}{\delta}. \tag{5.6}$$

For a given set of parameters that specify the potential A_Φ and B_σ are fixed. However, C_σ depends upon the winding number. In the canonical scenario of cosmological loop production, loops of size L_0 are continuously formed by breaking off from infinite strands of string when the age of the Universe is about $t_0 \sim L_0$ (Refs. 21 and 22). This results in a loop having a winding number N of at least $\sim(t_0/\xi)^{1/2}$ (the "Kibble current"). Taking $N \propto (t_0/\xi)^{1/2}$ leads to a spectrum of static length sizes: $L_{\text{static}} \propto t_0^{1/2}$.

We now check to see if the *concrete-vortex* approximation is appropriate for our study of static loops (and find that it is not). The *concrete-vortex* approximation is valid when

$$B_\sigma \ll A_\Phi + w \ln(\mu L) \ . \quad (5.7)$$

Because we are considering static loops, $A_\Phi + w \ln(\mu L) \approx B_\sigma$, and inequality (5.7) cannot be satisfied, indicating that there is always significant back reaction of the superconducting condensate onto the vortex. This immediately locates the only possible region of parameter space where static loops *might* exist:

$$\frac{m_\Phi^4}{\lambda_\Phi} \approx \frac{m_\sigma^4}{\lambda_\sigma} (\Longrightarrow \alpha \approx \sqrt{a/2}\,\gamma^{-1/2}) \ \text{ and } \alpha \approx \beta \ . \quad (5.8)$$

Since there is back reaction we must now consider the global and gauged case separately.

A. The global case

For a global loop, $w=1$, and, for an ordinary scalar potential,

$$A_\Phi = \tfrac{1}{4} - \ln(y) + \frac{y^2}{2} \ . \quad (5.9)$$

Upon using our simple *Ansatz* for σ,

$$B_\sigma = \frac{\gamma}{x'^2}[\alpha' x'^2 - \beta' x'^2 F(x'/y) - 1]^2 \ . \quad (5.10)$$

From Eq. (5.5) we develop a criterion for stable static global loops is

$$A_\Phi + 1 + \ln(\mu_0 L) - B_\sigma < 1.3 B_\sigma \ . \quad (5.11)$$

Since the size of the loop L enters into the criterion, for a given potential there is always a maximum length $L_{\max}$ beyond which stable, static global loops do not exist:

$$\mu_0 L_{\max} = \exp(2.3 B_\sigma - A_\Phi - 1) \ . \quad (5.12)$$

For lengths bigger than $\sim L_{\max}$ the critical current is reached before the loop shrinks to its static length. Recall that there is a minimum length we can consider without having to worry about loop curvature effects:

$$\mu_0 L_{\min} = 10\pi y \ . \quad (5.13)$$

This leads to yet another constraint: $L_{\max} > L_{\min}$. We note that small values of the ratio $L_{\max}/L_{\min}$ imply that static global loops were only produced during a very short cosmological time, early in the history of the Universe.

As mentioned before, the variational equations can be solved for the parameters α' and β' in terms of x and y. For the choice $\gamma = 10^{-4}$ we show the α', β' parameter space for $L_{\max}/L_{\min} \geq 1$, 1000, and 10^{10} in Figs. 15–17. We see that loops much larger than the minimum length occupy an increasingly smaller portion of parameter space.

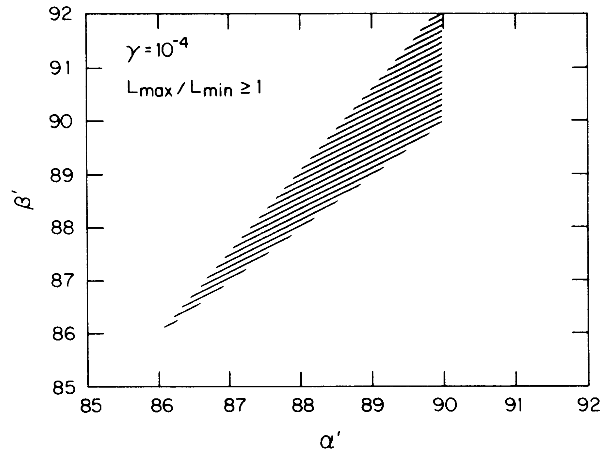

FIG. 15. The allowed α', β' parameter space for a floating global string with $\gamma = 10^{-4}$ and $L_{\max}/L_{\min} \geq 1$.

B. The gauged case

In the gauged case we do not encounter the peculiarity encountered with static global loops; i.e., the energy per length of the vortex is independent of the size of the loop. Static loops are more natural in this sense, as the existence of floating solutions only depends upon the parameters of the potential, and not also upon L_{static} as it does in the global case. However, here, too, floating solutions only occupy a very small portion of parameter space. From Eq. (5.5) it follows that our criterion for static gauge loops is

$$\frac{A_\Phi}{B_\sigma} < 2.3 \ . \quad (5.14)$$

The full Hamiltonian, given by Eq. (2.62), was numerically minimized for $\gamma = 10^{-4}$ for three different values of b: 0.01, 1, and 100. Superconducting solutions satisfying the above constraint are shown in Figs. 18–20,

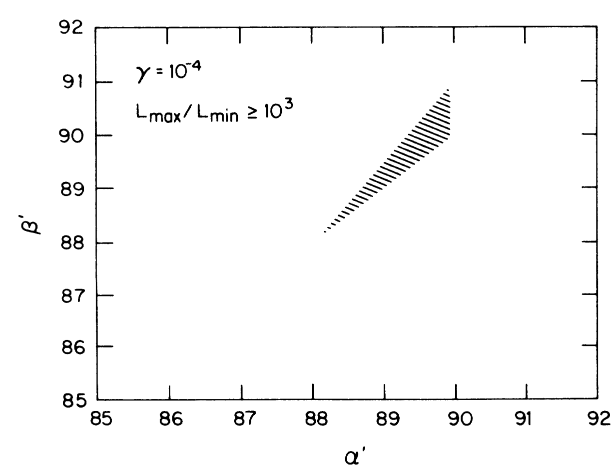

FIG. 16. The allowed α', β' parameter space for a floating global string with $\gamma = 10^{-4}$ and $L_{\max}/L_{\min} \geq 1000$.

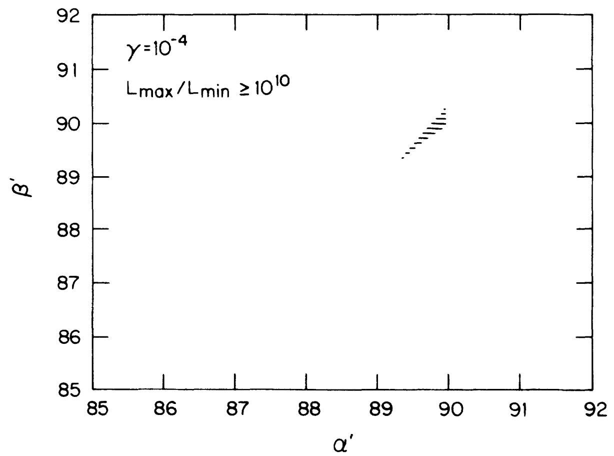

FIG. 17. The allowed α',β' parameter space for a floating global string with $\gamma=10^{-4}$ and $L_{\max}/L_{\min}\geq 10^{10}$.

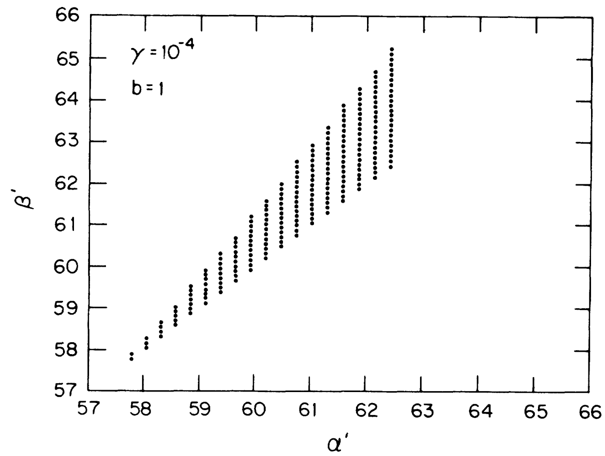

FIG. 19. The allowed α',β' parameter space for a floating gauged string with $\gamma=10^{-4}$ and $b=1$.

where we plot α' vs β' for each value of b. In each case the allowed region of parameter space for floating loops is very small. To gain a proper perspective of just how small this parameter space is, we note that for the aforementioned cases the ratio of the area of our static solutions to that of our superconducting solutions is $\sim$1:1000. [Note, the area of α-β space where static loops exist is of order unity (independent of γ), whereas the area of solution space is $\sim(a/2\gamma)^{3/2}/4$. For $\gamma \ll 1$ static solutions occupy a small part of solution space; for $\gamma \to 1$, they occupy a large fraction.]

To summarize floating loops, we find that there is a *small* region of the parameter space of solutions where stable, static solutions *may* exist, as specified by

$$f \simeq \sqrt{\lambda_\Phi \lambda_\sigma}/3\ , \quad m_\sigma \simeq m_\Phi(\lambda_\sigma/\lambda_\Phi)^{1/4}\ . \tag{5.15}$$

Because of the potential importance of back reaction of both the charged condensate and the current upon the vortex, a more detailed analysis is needed before one can confidently make a definitive statement about the existence of floating loops, even in this tiny region of parameter space.

VI. MODEL BUILDING

In this section we describe a procedure for choosing the five parameters of the potential

$$V(v,u) = -m_\Phi^2 v^2/2 + \frac{\lambda_\Phi v^4}{4!} - m_\sigma^2 u^2/2 + \frac{\lambda_\sigma u^4}{4!} + f v^2 u^2/4 + \frac{3m_\Phi^4}{2\lambda_\Phi}\ , \tag{6.1}$$

such that superconducting vortex solutions exist (see Tables I and II). However, we first discuss the natural values for the parameters f, λ_Φ, and λ_σ. The coupling f can be arbitrarily small since it is multiplicatively renormalized [note that this requires no mixing between the U(1) and U(1)′ gauge bosons and is special to our model; it may not be a general feature of this mechanism in other settings], but it cannot be larger than $\sim$1, the point perturbativity is lost. It is clear that λ_σ and λ_Φ cannot be arbitrarily small since the exchange of σ and Φ loops

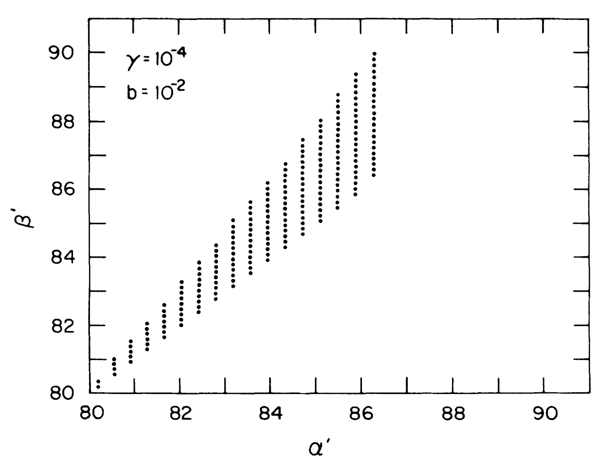

FIG. 18. The allowed α',β' parameter space for a floating gauged string with $\gamma=10^{-4}$ and $b=0.01$.

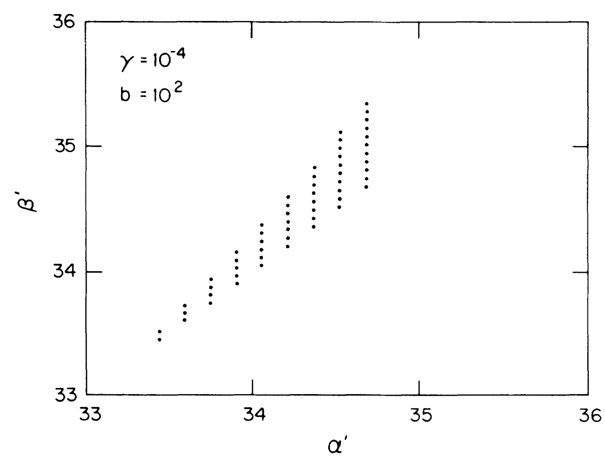

FIG. 20. The allowed α',β' parameter space for a floating gauged string with $\gamma=10^{-4}$ and $b=100$.

TABLE I. Summary of parameters for the scalar potential: $V(v,u)=-m_\Phi^2 v^2/2+\lambda_\Phi v^4/4!+3m_\Phi^4/2\lambda_\Phi-m_\sigma^2 u^2/2+\lambda_\sigma u^4/4!+fu^2v^2/4$. Primed quantities are related to their unprimed counterparts by $a'=a/y^2$, $s'=sy$, $\alpha'=\alpha/y^2$, $x'=yx$, and $\beta'=\beta/y^2$ (see Sec. II D).

Parameter	Definition	Comments
$\bar{v}$	$\bar{v}^2=6m_\Phi^2/\lambda_\Phi$	VEV of the real part of Φ
a	m_Φ^2/μ^2	Size of vortex $\approx\sqrt{a}/m_\Phi$ for a global string $a\simeq 1.6$; for a gauged string $a\sim$order unity
b	$q^2\bar{v}^2/m_\Phi^2=6q^2/\lambda_\Phi$	2 times the square of the ratio of the vector to scalar masses
s	μ/h	Ratio of the size of the magnetic flux tube to that of the vortex
α	$m_\sigma^2/\mu^2=am_\sigma^2/m_\Phi^2$	Convenient normalization for m_σ^2 by vortex scale μ^2
β	$f\bar{v}^2/2\mu^2=3af/\lambda_\Phi$	Convenient normalization for interaction by vortex scale μ^2
x	μ/κ	Ratio of the size of σ condensate to that of vortex
y	μ_0/μ	True size of vortex to that obtained in the *concrete-vortex* approximation
γ	$3\mu_0^2/\bar{v}^2\lambda_\sigma=(\lambda_\Phi/\lambda_\sigma)/2a$	In terms of γ constraint (1) is $\alpha<(a/2\gamma)^{1/2}$
χ	$3e^4v'^2/64\pi^2\mu^2$	Analog of α in the Coleman-Weinberg case

require $\sim f^2$ counterterms. If we try to define the renormalized theories with λ_σ and λ_Φ smaller than $\sim f^2$ we will have the values of physical quantities such as σ determined by effective potentials of the Coleman-Weinberg type rather than by the tree-level potentials, and we will effectively recover the same constraints. Moreover, for small f the gauge-loop corrections require that λ_σ exceed $\sim e^4$ and λ_Φ exceed $\sim q^4$. For global loops $q=0$, and for gauged loops $q^2=bm_\Phi^2/\bar{v}^2$ (which requires $\lambda_\Phi\lesssim b^{-2}$). We then see, in the gauged case, that the natural range for b is $10^{-2}\lesssim b\lesssim 10^2$ (where we also assume that q is of the general order of e). For $b\gtrsim 10^2$ the appropriate effective potential is that of the Coleman-Weinberg type, and for $b\lesssim 10^{-2}$ a nonlinear σ model approximation becomes appropriate.

We presently describe a method, though not unique, that allows one to construct a potential which permits bosonic superconductivity. This can easily be done with the aid of several of our graphs for the case that there is negligible back reaction—which is a good approximation unless $\alpha\approx\beta$, and in any case is always a good starting point.

One can first pick the quartic couplings and f (consistent with perturbativity and the Coleman-Weinberg limit), and b if the string is gauged. This determines β:

TABLE II. Summary of constraints on a bosonic superconducting cosmic string with unstable quartic scalar potential.

	Constraint	Comments
(1)	$m_\Phi^4/\lambda_\Phi>m_\sigma^4/\lambda_\sigma$	Stability of vacuum against breaking of electromagnetism
(2)	$\beta>\alpha$	Stability of vacuum against breaking of electromagnetism
(3)	$\sigma_0^2>0$	Existence of true condensate; for $\alpha\geq 10$ requires $\alpha\leq\beta\leq 0.5\alpha^{1.93}$
(4)	$m_\sigma\simeq(\lambda_\sigma/\lambda_\Phi)^{1/4}m_\Phi$ $f\simeq\sqrt{\lambda_\Phi\lambda_\sigma}/3$	Existence of floating solutions

$$\beta = 3fa/\lambda_\Phi \ , \tag{6.2}$$

where $a(b)$ can be determined from Fig. 2. In the global case, $a \approx 1.6$, while in the gauge case, a is typically of the order of unity. Constraint (1) and Fig. 4 restrict the possible choices of α:

$$\max[\tfrac{1}{7},\alpha(\beta)] \le \alpha \le \min[\beta, a\sqrt{\lambda_\sigma/\lambda_\Phi}] \ , \tag{6.3}$$

where $\alpha(\beta)$ is the upper boundary in Fig. 4, which for $\beta \le 50$ is given by $\alpha(\beta) \simeq 0.81\beta^{0.74}$, and for $\beta \ge 50$ is given by $\alpha(\beta) \simeq 1.43\beta^{0.52}$. For solutions to exist to all, λ_σ, λ_Φ, f, and b must be selected accordingly, i.e., $a\sqrt{\lambda_\sigma/\lambda_\Phi} \ge \frac{1}{7}$. Finally, any values of m_Φ and m_σ may be picked consistent with

$$m_\sigma^2/m_\Phi^2 = \alpha/a \ . \tag{6.4}$$

If for the parameters chosen, $\alpha \simeq \beta$ and $\alpha \simeq a\sqrt{\lambda_\sigma/\lambda_\Phi}$, back reaction is likely to be important, and one may wish to take it into account (see Sec. II D).

As an aside we mention how one can arrange the potential to have static loops. If the parameters λ_σ, λ_Φ, and m_Φ are selected, the other parameters are essentially determined:

$$m_\sigma \simeq m_\Phi \left[\frac{\lambda_\sigma}{\lambda_\Phi}\right]^{1/4} \quad \text{and} \quad f \simeq \tfrac{1}{3}\sqrt{\lambda_\sigma\lambda_\Phi} \ . \tag{6.5}$$

The Coleman-Weinberg limit is automatically satisfied in this case.

VII. CONCLUDING REMARKS

Let us summarize our work. By using simple variational *Ansätze* we have studied a number of important properties of cosmic strings. First, we have computed (to an accuracy of better than 2%) the energy per length of ordinary gauge cosmic strings as a function of $b = 6q^2/\lambda_\Phi = 2m_V^2/m_H^2$; our results are displayed in Figs. 1–3. It is very apparent that the energy per length is insensitive to b. Since the critical temperature for the phase transition which produces cosmic strings $T_c \sim \bar{v}(1+b)^{-1/2} \simeq (1+b)^{-1/2} \times (\text{energy/length})^{1/2}$, this implies that for cosmic strings of fixed string tension, one can by appropriate tuning of λ_Φ (i.e., $b \gg 1$) make T_c much smaller than its natural value $\sim(\text{energy/length})^{1/2}$. This fact may be of some importance if one is interested in producing cosmic string in inflationary universe models,[23] where the temperatures reached after inflation are typically much, much smaller than $\bar{v} \sim 10^{15}-10^{16}$ GeV, the scale associated with the energy per length required for "cosmologically interesting" strings.

Second, we have mapped out the scalar potential parameter space for bosonic superconductivity. The parameter space of solutions is shown in Figs. 4 and 11, and the constraints are summarized in Table II. From our analysis it is quite apparent that bosonic superconductivity does not require a fine-tuning of the parameters in the scalar potential, and in fact may be quite a generic phenomenon.

Our study of the dynamics of bosonic superconductivity included a quantitative discussion of the critical current, which we define to be the current such that the energy associated with the σ condensate becomes nonnegative (at which point it becomes energetically favorable to the system to disperse the σ condensate). Our analysis indicates that, in general, the quench transition is likely to be first order; however, the question of metastability of supercritical currents must still be addressed in detail.[20]

With the exception of a small region of the solution-parameter space, the "back reaction" of the σ condensate upon the vortex itself is small (see Fig. 4), and the σ condensate can be treated as existing on a "concrete flux tube." For $\alpha \approx \beta$ and $m_\sigma^4/\lambda_\sigma \approx m_\Phi^4/\lambda_\Phi$ [so that $f \sim (\lambda_\sigma\lambda_\Phi)^{1/2}/3$] the back reaction can be significant. In this regime, gauge or global strings *may* be able to achieve a static (or floating) configuration for subcritical currents; however, since the back reaction is significant, the analysis is difficult, and it is probably still premature to say with confidence that such states are possible. We can say with confidence that the parameter space which allows floating configurations is very tiny (unless γ is of order unity) see Figs. 15–20. (If loops with supercritical currents are metastable, the parameter space could be slightly larger.) Whether or not floating loops ever formed in the Universe is a separate issue. If the string loops which form cosmologically have cusps and if dissipation of phase twist occurs at the cusps, then the near critical currents required to achieve the floating state may never be achieved.

There are still a number of important issues to be addressed. Precisely how does a superconducting string quench when the critical current is exceeded, and does the quench lead to detectable effects [e.g., ultra-high-energy (UHE) cosmic rays,[2] or the photofission of the light elements formed during primordial nucleosynthesis[24]]? While it has been shown that fermionic loops cannot achieve a floating state by the support of the kinetic energy of the charge carriers alone, no thorough analysis similar to ours has yet been performed which also includes the electromagnetic stresses. Our results for the bosonic case would suggest that the possibility is unlikely.

Finally, there is no particular obstacle to extending variational analyses of this type to a large number of related cosmic string issues. For example, of considerable importance is a microphysical understanding of cusps.[25] Cusps arise as singularities in the world-sheet description of cosmic strings, but are clearly nonsingular configurations of the Φ and σ fields. It seems interesting to us to develop a similar analysis of the σ field in the presence of, say, a *concrete cusp* in the Φ field to answer the question of what, if any, are the changes in the critical current and the energetics of the σ condensate at the cusp. For example, we wish to know if superconductivity is destroyed by a cusp of given extrinsic curvature for a given value of the local current.

Note added in proof. Very recently, P. Amsterdamski and P. Laguna-Castillo, Haws, Hindmarsh, and Turok, and A. Babul, T. Piran, and D. Spergel have completed numerical analyses of some regions of the parameter space of bosonic superconducting cosmic-string solutions found here. In these regions their results are in agreement with ours. In addition Hans *et al.* have emphasized the importance of solution space where $\gamma \gtrsim 1$.

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[16] Recall that in the concrete-vortex approximation, constraint (3) ($\sigma_0^2 > 0$) guaranteed that the energy per length of the vortex with a condensate is less than that of the vortex alone, as the energy per length of the condensate is negative. When there is back reaction, it is not clear that the condition $\sigma_0^2 > 0$ alone guarantees a bound condensate because the condensate changes the vortex configuration. However, we found that whenever constraint (3) is satisfied the energy per length of the vortex with a condensate is less than that of the vortex alone.

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