Zero-point energy in flux-tube confinement

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We discuss the zero-point energy associated with a heavy-quark–antiquark system confined to a cylindrical cavity through the dielectric vacuum picture of confinement. The correction to the string tension and the universal Coulomb-type behavior produced by the Casimir forces are treated.

I. INTRODUCTION

A complete analysis of the long-distance behavior of quantum chromodynamics[1] (QCD) has proven to be elusive, and models attempting to represent the important features of QCD continue to be of value. A sufficient number of rather different quarkonium potentials exist[2] to make it clear that the pure phenomenology of quarkonia will never serve as a decisive arbiter for the nonrelativistic potential between quarks, let alone for the behavior of QCD. This is partly because quarkonia exist in the delicate transition region between long and short distances, as well as in a region which is neither unambiguously relativistic nor nonrelativistic. Nevertheless, it is important to continue to explore differences between and consequences of these models in order to sort out the universal from the particular predictions, as well as to search for critical tests of them.

Perhaps the simplest kind of dynamics which such models can explore is the static potential between a very massive (fixed) quark-antiquark pair at large distances. Since all models (and perhaps even QCD itself) lead to potentials which at large distances grow linearly with separation between the sources, what distinguishes different models for the potential are the coefficient of the leading term and the behavior of the nonleading terms. One such model,[3] which formed the basis for earlier calculations[4] on the static potential, treats the nonlinear and non-Abelian aspects of the gluon field and possible effects of light quark-antiquark pairs through the creation of a cavity in the QCD vacuum; the cavity is carved out when $F^{\mu\nu}F_{\mu\nu} \sim \mathbf{E}^2 - \mathbf{B}^2$ (**E** and **B** are the color fields) exceeds a certain critical value. The vacuum has the property that the gluon fields within the cavity, which are now treated as Abelian, cannot penetrate it, because in the region where the vacuum occurs the dielectric constant vanishes ($\epsilon = 0$) and the magnetic permeability is infinite ($\mu = \infty$) (maintaining the relativistic condition $\epsilon\mu = 1$). Inside the cavity, on the other hand, $\epsilon = \mu = 1$. We refer to this model as a "dielectric vacuum model" (DVM). This model has many elements in common with the so-called bag models,[5] although, among other differences, there is no analog to the surface tension in the DVM.

The version of the DVM which we treat is characterized by a discontinuous transition from the cavity to the vacuum. When there is a color-singlet quark-antiquark pair with a fixed separation, the shape of the cavity is determined by a constant critical value on the boundary for the electric field **E**, which is tangential to the surface. These boundary conditions have been treated numerically in earlier work.[6] The resulting cavity has cusplike singularities near the sources, and an analytic solution of this problem in three dimensions is not known.

In earlier work,[4] we treated cavities of fixed shape. This approach has the virtue that it is possible to find analytic results and that it is possible to check whether a particular fixed cavity approaches the "true" cavity, the cavity that would satisfy the boundary conditions. Such a check can be made by seeing how well the analytic solution for the fixed cavity satisfies the boundary conditions. We found an analytic solution to the case where the cavity is an open-ended cylinder of radius R with opposite charges separated by a distance a on the axis. The results salient to the question of bag shape are the following: first, for large a, the bag shape is arbitrarily close to that of a cylinder, and second, the fields fall off exponentially to the outside of the charges with a scale that depends on R, not a. For large a, the bag is a closed cylinder, and how the cylinder is closed at the ends is independent of a.

Our calculation was nothing more than a simple exercise in classical electrostatics. However, the results are both transparent and interesting. The leading behavior of the potential for large separations is linear, as expected, and the nonleading force terms—below the linear and constant terms—fall off exponentially as a function of the parameter $\lambda = a/2R$, where a is the charge separation and R is the cylinder radius. The spin-spin forces also fall off exponentially. The question then arises as to whether there are quantum corrections to this result, as expressed by a nonzero vacuum expectation value of the energy of oscillating fields within the cavity. This is the question that is examined in this paper.

A note of disclaimer and motivation is perhaps appropriate at this point. Because of the lack of "end caps" and cusplike behavior, we certainly do not claim that the prediction of our earlier calculation is a prediction of the DVM with sharp dielectric cutoff, but in any case the sharp cutoff case is itself unlikely to be an exact prediction of QCD. The infinite-mass restriction is also not very realistic (although the success of flavor-independent potential theory suggests that it is better than might naively be expected). As stated above, putting end caps on is a step which is required by the fields calculated in Ref. 4, and their details cannot change the a dependence. Nevertheless, highly detailed application of our calculation to phenomenology would not make much sense. Instead, the fact that our model can give analytic results makes it sufficiently valuable as a theoretical tool to allow it to stand on its own. For example, we can pose questions about the relation[7] of the cylindrical flux tube to the hadronic string, or compare its features with those that are predicted by lattice gauge theories. This is the same spirit in which we calculate the quantum corrections. The picture can also shed light on the significance of certain qualitative features of the phenomenology; like so much quarkonium phenomenology, even the quantitative aspects turn out to be surprisingly relevant.

The crudest measure of the quantum corrections to the energy of a classical system such as the cylinder is contained in the so-called zero-point energy, or the Casimir effect.[8,9] We shall be studying the oscillating solutions which are allowed in the context of the original geometry, but with the cylinder closed at the charges, as the fields we previously calculated require. The vacuum expectation value of the energy associated with these solutions is the zero-point energy.

Oscillating solutions with baglike boundary conditions have also been studied by Laperashvili and Nielsen,[10] although their aims were quite different from ours; much of our formulation of the problem is based on theirs. The concept of oscillating solutions to the bag geometry was also discussed[11] earlier in the context of strings and of the sine-Gordon equation.

In Sec. II we formulate the boundary conditions and find the allowed modes and corresponding electric and magnetic fields. Zero-mass modes must be given special attention. Section III contains a discussion of the zero-point energy, first for the correction to the string tension, and second for a term proportional to $1/a$ with a coefficient which is a pure number. Zero-mass modes lead to this term, which in the context of the string is known as the Lüscher term.[12] Usually it is associated with transverse vibrations of the string. The connection follows from the relation between the zero-mass modes and the vanishing width of the tube. We conclude with some discussion of the relation of our results to others, and of phenomenological implications. It is perhaps worth stating here that the $1/a$ term has a significant coefficient, $\simeq -0.26$, which implies a contribution surprisingly close to the phenomenological value in the region around 0.3 fm, where physical quarkonia are relevant. As was discussed in Ref. 4, the extension of the large distance, cylindrical, geometry down to distances this small works unexpectedly well.

II. OSCILLATING FIELDS IN A CYLINDRICAL CAVITY

A. Conditions determining the fields

We start with a cylinder of radius R, choosing its axis to be the z axis and placing the charges $\pm g$ at $z=\pm a/2$, as in Fig. 1. The cylinder is closed with flat end caps at the charges. In this static configuration, the electric field satisfies $\hat{\mathbf{n}}\cdot\mathbf{E}=0$, where $\hat{\mathbf{n}}$ is a unit three-vector perpendicular to the surface, part of a unit four-vector n_μ normal to the surface given by $n_\mu=(n_0,\hat{\mathbf{n}})=(0,1,0,0)$. The boundary is determined by a particular constant electric field strength $\mathbf{E}_0=(0,0,E_0)$. We are working here in cylindrical coordinates (ρ,θ,z); a vector is written as $\mathbf{V}=(V_\rho,V_\theta,V_z)$. The static configuration defines the zeroth order of the calculation; fields containing corrections must reduce to this in the limit where these vanish. The static configuration corresponds to $\mathbf{E}=(0,0,E_0)$, and the spirit of our calculation is that E_0 is a constant throughout, even when the boundary of cavity changes; in other words, it is the *changes* in the fields which reflect the boundary conditions.

The oscillating corrections to the electric field, as well as some new magnetic fields, are determined by the boundary conditions, by a gauge choice, and by Maxwell's equations. Let us list these in order. The co-

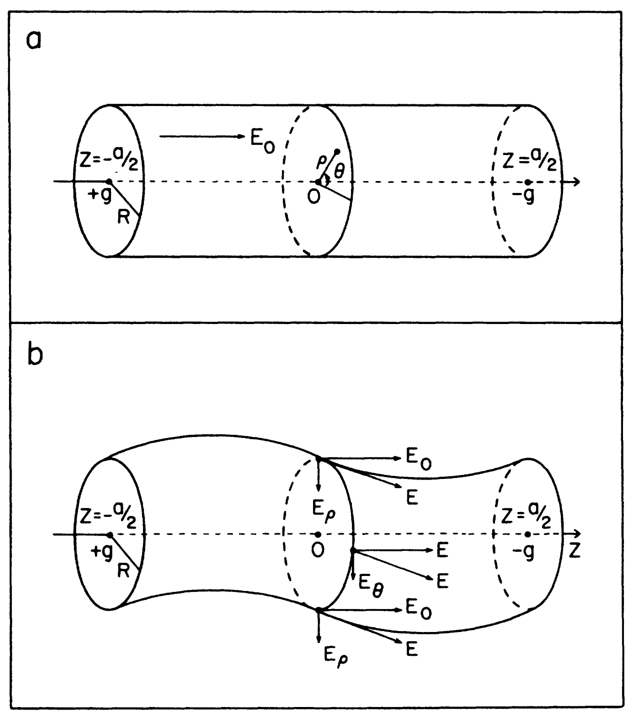

FIG. 1. Schematic representation of a flux tube of length a and radius R. The charges $\pm g$ are placed at $z=\pm a/2$. (a) shows the tube in the static configuration. (b) shows the zero-mass mode (much exaggerated) of Eqs. (2.45) and (2.46) with $\kappa=ka/2\pi=1$.

variant boundary conditions are

$$n_\mu F^{\mu\nu}=0 \text{ on the surface} \tag{2.1}$$

and

$$F_{\mu\nu}F^{\mu\nu}=\mathbf{E}^2-\mathbf{B}^2=E_0^2 \text{ on the surface .} \tag{2.2}$$

Equation (2.1) can be rewritten in terms of the fields as

$$\hat{\mathbf{n}}\cdot\mathbf{E}=0 \ , \tag{2.3}$$

$$n_0\mathbf{E}+\hat{\mathbf{n}}\times\mathbf{B}=0 \ . \tag{2.4}$$

Note that n_μ is not necessarily the zeroth-order form when oscillating fields are present, although it should reduce to this form when first-order terms are neglected. Indeed, the work of Ref. 10, for which the context is the bag model, begins with the assumption that the surface is oscillating, and that the oscillating fields inside the cavity must be consistent with the oscillating surface. While this assumption is not necessary, it should indeed be considered. One more condition for the surface must be chosen, and that has to do with the conditions at the two ends. We showed in Ref. 4 that the fields die exponentially outside the two charges. We can thus incorporate an end cap without materially changing the fields, and we in addition take periodic boundary conditions at the two ends:

$$\mathbf{A}(z=a/2)=\mathbf{A}(z=-a/2) \ . \tag{2.5}$$

We choose a gauge in which the vector potential A_μ satisfies

$$A_0=0 \quad \text{and} \quad \nabla\cdot\mathbf{A}=0 \ . \tag{2.6}$$

In cylindrical coordinates, this gauge-fixing condition, the Lorentz condition, reads

$$\partial_z A_z+\partial_\rho A_\rho+\frac{1}{\rho}A_\rho+\frac{1}{\rho}\partial_\theta A_\theta=0 \ . \tag{2.7}$$

In this gauge, the fields are given in terms of the vector potential by

$$\mathbf{E}=-\partial_0\mathbf{A} \quad \text{and} \quad \mathbf{B}=\nabla\times\mathbf{A} \ . \tag{2.8}$$

Finally, Maxwell's equations reduce to

$$(\nabla^2-\partial_0^2)\mathbf{A}=0 \ , \tag{2.9}$$

or, in cylindrical coordinates,

$$\left[\partial_\rho^2+\frac{1}{\rho}\partial_\rho+\frac{1}{\rho^2}\partial_\theta^2+\partial_z^2\right]\mathbf{A}=\partial_0^2\mathbf{A} \ . \tag{2.10}$$

B. Determination of the fields from the equations of motion

Our procedure here will be first to find the field solutions corresponding to the Lorentz condition and the Maxwell equations. The boundary conditions will be applied last, in Sec. II C, and will be used as much to help determine the boundary as to restrict the field, always, of course, keeping in mind that the cylinder is the zeroth-order boundary. We expand the fields in an orthonormal set of functions which are natural for the given boundary conditions. Beginning with the time and z dependence, we write

$$\mathbf{A}(\rho,\theta,z,t)=(2\omega a)^{-1/2}\sum_k[\mathbf{C}(\rho,\theta,k)e^{i(kz-\omega t)}+\mathbf{C}^*(\rho,\theta,k)e^{-i(kz-\omega t)}] \ . \tag{2.11}$$

The periodic boundary condition (2.5) requires that

$$e^{ika/2}=e^{-ika/2} \ ,$$

so that

$$k=\frac{2\pi\kappa}{a}, \quad \kappa=0,\pm1,\pm2,\ldots \tag{2.12}$$

and the sum over k will be understood to be a summation over κ. The factors $1/\sqrt{a}$ and $1/\sqrt{2\omega}$ are in Eq. (2.11) for orthonormality. Note that the values of ω will in general depend on k and on the eigenvalues μ associated with the other boundary conditions:

$$\omega_k^2=k^2+\mu^2 \ . \tag{2.13}$$

This expression follows from the insertion of the decomposition (2.11) into the Maxwell equations (2.10). Maxwell's equations take the new form

$$\left[\partial_\rho^2+\frac{1}{\rho}\partial_\rho+\frac{1}{\rho^2}\partial_\theta^2+\mu^2\right]C(k,\rho,\theta)=0 \ . \tag{2.14}$$

It is convenient at this point to define right- and left-handed combinations by

$$C_R=\frac{C_x+iC_y}{2} \quad \text{and} \quad C_L=\frac{C_x-iC_y}{2} \ . \tag{2.15}$$

The cylindrical components are expressed in terms of these by

$$C_\rho=e^{-i\theta}C_R+e^{i\theta}C_L \quad \text{and} \quad C_\theta=-ie^{-i\theta}C_R+ie^{i\theta}C_L \ . \tag{2.16}$$

Having accomplished this transformation, we expand the θ dependence of the components of C in a Fourier series:

$$e^{-i\theta}C_R=\sum_n e^{in\theta}C_{Rn}(\rho,k) \ , \tag{2.17a}$$

$$e^{i\theta}C_L=\sum_n e^{in\theta}C_{Ln}(\rho,k) \ , \tag{2.17b}$$

$$C_z=\sum_n e^{in\theta}C_{zn}(\rho,k) \ . \tag{2.17c}$$

The summation is over all positive and negative integer values of n.

The Lorentz condition and the Maxwell equations with Eqs. (2.17) must now hold for each value of n, and the $e^{in\theta}$ factor cancels in each term. This leaves, for the Lorentz condition,

$$\left[\partial_\rho+\frac{n+1}{\rho}\right]C_{Rn}+\left[\partial_\rho-\frac{n-1}{\rho}\right]C_{Ln}+ikC_{zn}=0 \ . \tag{2.18}$$

The Maxwell equations become

$$\left[\partial_\rho^2+\frac{1}{\rho}\partial_\rho-\frac{(n+1)^2}{\rho^2}+\mu^2\right]C_{Rn}(\rho,k)=0 , \tag{2.19a}$$

$$\left[\partial_\rho^2+\frac{1}{\rho}\partial_\rho-\frac{(n-1)^2}{\rho^2}+\mu^2\right]C_{Ln}(\rho,k)=0 , \tag{2.19b}$$

$$\left[\partial_\rho^2+\frac{1}{\rho}\partial_\rho-\frac{n^2}{\rho^2}+\mu^2\right]C_{zn}(\rho,k)=0 . \tag{2.19c}$$

Given that the fields must be finite, the solution to the Maxwell equations is generally given in terms of Bessel functions:

$$C_{Rn}(\rho,k)=A_{Rnk}J_{n+1}(\mu\rho) , \tag{2.20a}$$

$$C_{Ln}(\rho,k)=A_{Lnk}J_{n-1}(\mu\rho) , \tag{2.20b}$$

$$C_{zn}(\rho,k)=B_{nk}J_n(\mu\rho) . \tag{2.20c}$$

The Lorentz condition, Eq. (2.18), provides us with a relation between the coefficients B and A if we use the Bessel function relations

$$\left[\partial_\rho+\frac{n+1}{\rho}\right]J_{n+1}(\mu\rho)=\mu J_n(\mu\rho) \tag{2.21a}$$

and

$$\left[\partial_\rho-\frac{n-1}{\rho}\right]J_{n-1}(\mu\rho)=-\mu J_n(\mu\rho) . \tag{2.21b}$$

The resulting relation is

$$A_{Rnk}-A_{Lnk}=-\frac{ik}{\mu}B_{nk} . \tag{2.22}$$

In summary, the vector potential is given by

$$A_\rho=(2\omega a)^{-1/2}\sum_{k,n}\{e^{i(kz-\omega t)}e^{in\theta}[A_{Rnk}J_{n+1}(\mu\rho)+A_{Lnk}J_{n-1}(\mu\rho)]+\text{c.c}\} , \tag{2.23}$$

$$A_\theta=(2\omega a)^{-1/2}\sum_{k,n}\{e^{i(kz-\omega t)}e^{in\theta}(-i)[A_{Rnk}J_{n+1}(\mu\rho)-A_{Lnk}J_{n-1}(\mu\rho)]+\text{c.c.}\} , \tag{2.24}$$

$$A_z=(2\omega a)^{-1/2}\sum_{k,n}\left[e^{i(kz-\omega t)}e^{in\theta}\left[\frac{i\mu}{k}\right](A_{Rnk}-A_{Lnk})J_n(\mu\rho)+\text{c.c.}\right] . \tag{2.25}$$

The electric and magnetic fields can be expressed in terms of these coefficients. Let us write these fields in the form

$$\mathbf{E}=(E_\rho,E_\theta,E_0+E_{z1}),\quad \mathbf{B}=(B_\rho,B_\theta,B_z) , \tag{2.26}$$

where all the fields except the background field E_0 are first order small and we work only to this order. Then we have

$$E_\rho=\partial_0A_\rho=(2\omega a)^{-1/2}\sum_{k,n}\{e^{i(kz-\omega t)}e^{in\theta}(i\omega)[A_{Rnk}J_{n+1}(\mu\rho)+A_{Lnk}J_{n-1}(\mu\rho)]+\text{c.c.}\} , \tag{2.27}$$

$$E_\theta=\partial A_\theta=(2\omega a)^{-1/2}\sum_{k,n}\{e^{i(kz-\omega t)}e^{in\theta}(-i\omega)[A_{Rnk}J_{n+1}(\mu\rho)-A_{Lnk}J_{n-1}(\mu\rho)]+\text{c.c.}\} , \tag{2.28}$$

$$E_{z1}=\partial_0A_z=(2\omega a)^{-1/2}\sum_{k,n}\left[e^{i(kz-\omega t)}e^{in\theta}\left[-\frac{\mu\omega}{k}\right](A_{Rnk}-A_{Lnk})J_n(\mu\rho)+\text{c.c.}\right] , \tag{2.29}$$

$$\begin{aligned}B_\rho&=\frac{1}{\rho}\partial_\theta A_z-\partial_zA_\theta\\&=(2\omega a)^{-1/2}\sum_{k,n}\left\{e^{i(kz-\omega t)}e^{in\theta}\left[\left[\frac{-\mu n}{k\rho}\right](A_{Rnk}-A_{Lnk})J_n(\mu\rho)-k[A_{Rnk}J_{n+1}(\mu\rho)-A_{Lnk}J_{n-1}(\mu\rho)]\right]+\text{c.c.}\right\} ,\end{aligned} \tag{2.30}$$

$$\begin{aligned}B_\theta&=\partial_zA_\rho-\partial_\rho A_z\\&=(2\omega a)^{-1/2}\sum_{k,n}\left\{e^{i(kz-\omega t)}e^{in\theta}\left[ik[A_{Rnk}J_{n+1}(\mu\rho)+A_{Lnk}J_{n-1}(\mu\rho)]+\left[\frac{-i\mu^2}{k}\right](A_{Rnk}-A_{Lnk})J_n(\mu\rho)\right]+\text{c.c.}\right\} ,\end{aligned} \tag{2.31}$$

$$B_z=\left[\partial_\rho-\frac{1}{\rho}\right]A_\theta-\frac{1}{\rho}\partial_\theta A_\rho=(2\omega a)^{-1/2}\sum_{k,n}[e^{i(kz-\omega t)}e^{in\theta}(-i\mu)(A_{Rnk}+A_{Lnk})J_n(\mu\rho)+\text{c.c.}] . \tag{2.32}$$

C. Application of the boundary conditions

The boundary conditions lead us to choose certain eigenvalues μ. In this section we apply these conditions. The zero-mass modes $\mu=0$ will require special care.

In applying the boundary conditions, we keep in mind that n_μ is not necessarily the normal to the constant cylinder. To do this, we adopt the following notation: the zeroth-order solution to the surface is a cylinder with time-independent radius R. Corrections to R are given in terms of the first-order quantity $\sigma(\theta,z,t)$. The four-vector n_μ takes the general form, in terms of σ,

$$n_\mu=\left[\frac{\partial\sigma}{\partial t},1,-\frac{1}{R}\frac{\partial\sigma}{\partial\theta},-\frac{\partial\sigma}{\partial z}\right] . \qquad (2.33)$$

The boundary condition (2.2) concerns only the longitudinal component E_{z1}, since the contributions of the other fields to this equation are automatically second order. Since $(E_0+E_{z1})^2\simeq E_0^2+2E_0E_{z1}$, and since by Eq. (2.2) this quantity must equal E_0^2 to leading order, we have, to this order,

$$E_{z1}(\rho,\theta,z,t)\,|_{\rho=R}=0 . \qquad (2.34)$$

The first boundary condition, Eq. (2.1), gives us, to first order, conditions on the field components E_ρ, B_θ, and B_z, namely,

$$E_\rho(\rho,\theta,z,t)=E_0\frac{\partial\sigma}{\partial z} , \qquad (2.35)$$

$$B_\theta(\rho,\theta,z,t)=-E_0\frac{\partial\sigma}{\partial t} , \qquad (2.36)$$

$$B_z(\rho,\theta,z,t)=0 . \qquad (2.37)$$

Equation (2.1) implies no conditions to this order on B_ρ, because a zeroth-order radial B field is allowed by this boundary condition; the same remark holds for an electric field in the θ direction. In other words, B_ρ and E_θ can have undetermined first-order values. Note also that the dependence of σ on θ does not enter into these first-order conditions, which might have a consequence for the possible surfaces allowed in these modes.

By going back to the fields, Eqs. (2.27)–(2.32), we can see the two configurations which will satisfy the boundary conditions. We refer to these configurations as modes I and II. We must in addition consider zero-mass modes separately.

Mode I:

$$J_n(\mu R)=0 \text{ and } A_{Rnk}-A_{Lnk}=0 . \qquad (2.38)$$

The masses μ are the zeros of the Bessel functions, $\mu=\gamma_{n\lambda}/R$. This solution leads to trivial (zero) fields for $\mu=0$. With this solution, we have for the oscillating fields, up to normalization factors,

$$\mathbf{E}=(2\omega a)^{-1/2}\times\sum_{k,n,\lambda}[e^{i(kz-\omega t)}e^{in\theta}2\omega A_{nk}(0,-J_n'(\mu_{n\lambda}\rho),0)+\text{c.c.}] , \qquad (2.39)$$

$$\mathbf{B}=(2\omega a)^{-1/2}\times\sum_{k,n,\lambda}[e^{i(kz-\omega t)}e^{in\theta}2kA_{nk}(J_n'(\mu_{n\lambda}\rho),0,0)+\text{c.c.}] . \qquad (2.40)$$

We have here set $A_{nk}\equiv A_{Rnk}=A_{Lnk}$. So long as $\mu\neq0$, both E_ρ and B_θ vanish *on the original stationary surface* $\rho=R$. In other words, this mode occurs within the original cylinder, and the surface does not oscillate at all. This mode was not considered in Ref. 10. Note that a σ dependent on θ alone is formally allowed, although this does not represent a very interesting variation, because we might well have started with such a "scalloped" stationary surface. Only time dependence is of real interest here.

Mode II:

$$J_n(\mu R)=0 \text{ and } A_{Rnk}+A_{Lnk}=0 , \qquad (2.41)$$

$$\mathbf{E}=(2\omega a)^{-1/2}\times\sum_{k,n,\lambda}[e^{i(kz-\omega t)}e^{in\theta}2\omega A_{nk}(-iJ_n'(\mu_{k\lambda}\rho),0,0)+\text{c.c.}] , \qquad (2.42)$$

$$\mathbf{B}=(2\omega a)^{-1/2}\times\sum_{k,n,\lambda}\left[e^{i(kz-\omega t)}e^{in\theta}\left[\frac{2\omega^2}{k}\right]A_{nk}(0,-iJ_n'(\mu_{n\lambda}\rho),0)+\text{c.c.}\right] . \qquad (2.43)$$

Note that this time, $A_{nk}\equiv A_{Rnk}=-A_{Lnk}$. Again, there are no nontrivial zero-mass modes.

Given these fields, we can use Eqs. (2.35) and (2.36), together with the statement that σ dependent on θ alone is without interest, to specify the boundary. We find

$$\sigma(\theta,z,t)=\frac{1}{E_0}(2\omega a)^{-1/2}\times\sum_{k,n,\lambda}\left[e^{i(kz-\omega t)}e^{in\theta}\left[-\frac{2\omega}{k}\right]A_{nk}J_n'(\mu_{k\lambda}R)+\text{c.c.}\right] . \qquad (2.44)$$

This configuration, apart from the zero-mass modes, is the one considered in Ref. 10, although in that reference the surface plays a privileged role and there is a surface tension term in the energy.

Zero-mass modes. The possibility of zero-mass modes is best handled by looking directly at the fields in the limit $\mu\to0$. There is, we shall find, only one such mode, and its field configuration is distinct from either of the two modes considered above. Note that all J_n have a zero at zero except for J_0. We find from Eqs. (2.27)–(2.32) that for $n=0$, the fields all vanish in this limit; to show this we use the symmetry $J_{-1}=-J_1$. This is also true for $n\geq2$. For $n=1$, however, $J_0(0)=1$ appears in the fields, and there is a nontrivial configuration of the field which survives: namely,

$$\mathbf{E}=(2\omega a)^{-1/2}\sum_k [e^{ik(z-t)}e^{i\theta}\omega A_{L1k}(i,-1,0)+\text{c.c.}] \,, \tag{2.45}$$

$$\mathbf{B}=(2\omega a)^{-1/2}\sum_k [e^{ik(z-t)}e^{i\theta}k A_{L1k}(1,i,0)+\text{c.c.}] \,. \tag{2.46}$$

This configuration of fields is contained in neither mode I nor II. It will be significant for the $1/a$ term in the zero-point energy that there is one and only one zero-mass mode, even in the absence of surface tension. From Eqs. (2.35) and (2.36), we find a surface oscillation

$$\sigma(\theta,z,t)=\frac{1}{E_0}(2\omega a)^{-1/2}\sum_k (e^{ik(z-t)}e^{i\theta}A_{L1k}+\text{c.c.}) \,. \tag{2.47}$$

This surface corresponds to a snakelike oscillation of the original cylinder, in which the circular cross section is undisturbed, but there are transverse undulations along the tube. If seen from a distance, it would resemble the transverse oscillations of the string.

There is a slight subtlety in the discussion of the zero-mass modes having to do with the normalization of the fields. If the normalization of the fields is performed with some factors of μ, then μ could cancel in leading terms on the expansion in $\mu\rho$ which is treated here. In this case, we would be left with power behavior in ρ, growing to a maximum up to the boundary, where the boundary conditions would insist on a drop to zero. The only power that would in fact be allowed by the boundary conditions yet still give nonzero fields is in fact constant behavior in ρ; this is indeed the behavior picked out by the solutions of Eqs. (2.45) and (2.46).

[It is perhaps worth mentioning here that the second boundary condition that led us to set $E_{z1}=0$ rules out still another possible solution with a stationary surface. In this third solution, $J_n'(\mu R)=0$ and $A_{Rnk}+A_{Lnk}=0$ on the original surface, and there is a vanishing E_ρ, while both E_θ and E_z have a first-order contribution, and $B_z=0$, while the other two components of $\mathbf{B}$ are nonzero.]

D. Quantization and normalization of the fields

Quantization is performed by treating the A's as creation and annihilation operators. This means that they become noncommuting variables, that the complex-conjugate operation becomes the Hermitian conjugate operation, and that the *size* of the various commutators is fixed. It is this last requirement that normalizes the oscillating fields into, hopefully, small quantities. An equivalent approach to normalization is to insist that the energy of the oscillating fields in the cavity, proportional to the integral of E^2+B^2, is the sum over the ω_i. In this paper, we are in fact interested only in this latter quantity, so, having already found the spectrum, we do not need the fields correctly normalized. Nevertheless, it may be helpful to see how this works, so we illustrate the normalization procedure with the fields of mode I.

The energy W of the oscillating fields in the cavity is

$$W=\frac{1}{8\pi}\int dz\int d\theta\int \rho\, d\rho(\mathbf{E}^2+\mathbf{B}^2) \,. \tag{2.48}$$

Both $\mathbf{E}$ and $\mathbf{B}$ are sums over k, n, and λ. It is rather direct to see, because of the orthogonality of the functions e^{ikz} and $e^{in\theta}$, that cross terms in k and n disappear. The integrals over z and θ are simple. It is less straightforward that this is the case in the sum over λ, the index for the zeros of the Bessel functions. Nevertheless, this can be shown, and we find

$$W=\frac{1}{8\pi}2\pi\sum_{k,n,\lambda}\int_0^R \rho\, d\rho\{A_{kn\lambda},A_{kn\lambda}^\dagger\}\left\{[\omega+(k^2/\omega)][J_{n-1}^2(\mu_{n\lambda}\rho)+J_{n+1}^2(\mu_{n\lambda}\rho)]+\frac{2\mu_{n\lambda}^2}{\omega}J_n^2(\mu_{n\lambda}\rho)\right\}$$
$$=\tfrac{1}{2}\sum_{k,n,\lambda}\omega\{A_{kn\lambda},A_{kn\lambda}^\dagger\}R^2J_{n+1}^2(\mu_{n\lambda}R) \,. \tag{2.49}$$

The correctly normalized creation operator is then

$$a_{kn\lambda}=A_{kn\lambda}R\,|J_{n+1}(\mu_{n\lambda}R)| \,. \tag{2.50}$$

If we divide A in our equations for the fields by the additional factor on the right-hand side of Eq. (2.50), we will have a properly normalized set of fields, in terms of which Eq. (2.49) gives for an expectation value

$$\text{zero-point energy}\equiv\langle 0|W|0\rangle=\tfrac{1}{2}\sum_{k,n,\lambda}\omega_{kn\lambda} \,. \tag{2.51}$$

III. ZERO-POINT ENERGY

We want to compute the sum of Eq. (2.51). It will be helpful in doing so to compare the result to the background, classical, energy associated with the cylindrical cavity,[4] or with the energy in the standard quarkonium phenomenology.[2] The standard phenomenological potentials have the form

$$V(a)=\kappa a-\frac{\alpha}{a}+V_0 \,. \tag{3.1}$$

κ is known as the string tension and is associated with the slope of Regge trajectories. α is the coefficient of an attractive Coulomb-type term. Without going into details, κ and α take the phenomenological values

$$0.3<\alpha<0.5 \quad\text{and}\quad 0.15<\kappa<0.25\ \text{GeV}^2 \,. \tag{3.2}$$

The potential for the charges in the cylinder is, for large separation,

$$V_{\rm cyl}(a)=\frac{2g^2}{R^2}a+D+O(e^{-a/2R}) . \quad (3.3)$$

Here g is the magnitude of the charge of the heavy source, and D is a numerical constant times $4g^2/R$. That there is no power correction to the linear and constant term in a is one of the remarkable features of this cavity.

These potentials make it clear that it is the linear and the Coulomb terms in a which are the interesting quantities to extract from the zero-point energy. We can get a feeling for the nature of these terms by reminding ourselves that the eigenenergies $\omega_{kn\lambda}$ are given by

$$\omega_{kn\lambda}=[(2\pi\kappa/a)^2+(\gamma_{n\lambda}/R)^2]^{1/2} . \quad (3.4)$$

where

$$J_n(\gamma_{n\lambda})=0, \quad \lambda=1,2,3 . \quad (3.5)$$

The sum over the ω must, for dimensional reasons, be of the form $f(R/a)/a$. Our job will be to extract the various terms in the limit $a \gg R$, which involves the evaluation of singular sums; we shall in particular worry only about the linear and the Coulomb terms. The extraction of these terms is in fact a problem with a long lineage, and we shall have occasion to refer to some of it as we proceed.

The Euler-MacLaurin series[13] is a useful tool for the evaluation of the sums, and it also provides us with the desired expansion in the small parameter R/a. The sum over the ω is, as it stands, divergent. We shall eventually have to continue, or subtract, this sum in such a way that it is finite, so we replace the exponent $\frac{1}{2}$ in Eq. (3.4) by $-s/2$, and, *after* evaluation of expressions with positive s has taken place, we replace s by -1. Because $a \gg R$, it is appropriate to turn the sum over κ into an integral with a remainder. We have

$$\sum_\kappa \left[\frac{4\pi^2\kappa^2}{a^2}+\Delta\right]^{-s/2}=\frac{a}{2\pi}\int_0^\infty dk\,F(k)+\tfrac{1}{2}[F(\infty)+F(0)]+\frac{\pi}{2a}B_2[F'(\infty)-F'(0)]$$
$$+\frac{\pi^2}{2a^2}B_2\sum_{l=0}^{\infty}F''\left[\left[\frac{l\pi}{a}\right]+\left[\frac{\theta\pi}{a}\right]\right]+\cdots . \quad (3.6)$$

Here B_2 is a Bernoulli number, θ is a number between 0 and 1, and

$$F(k)=(k^2+\Delta)^{-s/2} . \quad (3.7)$$

The quantity Δ is proportional to $1/R^2$, but for now it is unnecessary to be more explicit.

The terms of Eq. (3.6), which still need to be summed over the n,λ index to give the zero-point energy, show their a dependence clearly. The first term is linear in a; the coefficient must be proportional, ultimately, to R^{-2}, and is a correction to the string tension. The second term is independent of a, must be proportional to R^{-1}, and represents the constant term in the potential. The third term is the a^{-1} term, and the fourth term represents higher-order corrections to the energy.

A. Contribution to the string tension

The contribution $\delta\kappa$ of the zero-point energy to the string tension κ is, from Eq. (3.6),

$$\delta\kappa=\frac{1}{2\pi}\sum_{n,\lambda}\int_0^\infty dk\,F(k) . \quad (3.8)$$

We continue this integral in the number of longitudinal dimensions p, i.e., $\int dk \to \int k^{p-1}dk$. Later we can let $p\to 1$. In addition, we use the identity

$$F(k)=\frac{1}{\Gamma(s/2)}\int_0^\infty du\,u^{(s-2)/2}e^{-u(k^2+\Delta)} . \quad (3.9)$$

This gives us

$$\int_0^\infty k^{p-1}dk\,F(k)=\frac{1}{\Gamma(s/2)}\int_0^\infty du\,u^{(s-2)/2}\int_0^\infty dk\,k^{p-1}e^{-u(k^2+\Delta)}$$
$$=\frac{1}{2\Gamma(s/2)}\int_0^\infty du\,u^{(s-2)/2}e^{-u\Delta}\int_0^\infty dv\,v^{(p-2)/2}e^{-uv}$$
$$=\frac{\Gamma(p/2)}{2\Gamma(s/2)}\int_0^\infty du\,u^{(s-2)/2}u^{-p/2}e^{-u\Delta}=\frac{1}{2}\frac{\Gamma(p/2)\Gamma((s-p)/2)}{\Gamma(s/2)}\Delta^{(p-s)/2} . \quad (3.10)$$

Thus

$$\delta\kappa=\frac{1}{2\pi}\frac{1}{2}\frac{\Gamma(p/2)\Gamma((s-p)/2)}{\Gamma(s/2)}\sum_{n,\lambda}(\gamma_{n\lambda}/R)^{p-s}\,, \quad (3.11)$$

where we have used Eq. (3.4) to specify Δ.

We can replace *either* p or s by their naive values, but not both. For example, while we can safely set $s=-1$ here, we cannot naively replace p by 1; the sum is divergent, as would be the Γ function of argument $(s-p)/2$. The sum on the zeros of the Bessel functions is a special case of the more general Minakshisundaram-Pleijel ζ function,[14] or spectral ζ function,

$$Z_\Delta(\beta)=\sum_n |\gamma_n|^{-\beta}\,, \quad (3.12)$$

where γ_n label the eigenvalues of the Laplacian operator acting on a curved surface. This formula is closely related to the Selberg trace formula,[15] which either in its general form or in the guise of Eq. (3.12) appears in a large variety of contexts, including the "quantum billiards" problem,[16] problems involving chaos,[17] and in sum-rule methods for the calculation of bound-state energies.[18] The continuation of this function, and particularly the sum in Eq. (3.11), to values of the argument which allow it to be defined, would in principle follow from a symmetry relation, or reflection formula, obeyed by the function.[19] Although we can conjecture the form of this relation, we cannot prove it except in the case of Euclidean geometry. This unproven relation was already discussed by Weyl.[18]

To illustrate this relation in the case of Euclidean geometry, we draw on the work of Ref. 9, in which the Casimir pressures in various Euclidean cavities are discussed, using an appropriate symmetry relation. Use of this relation with continuation in the variable s is known as "ζ-function regularization," and is discussed in detail elsewhere.[20] Continuation in p rather than s is referred to as dimensional regularization, and leads to equivalent results. We consider a long tubelike box, whose cross section is a square in t transverse dimensions with sides of length L and whose length is a. The longitudinal dimension will be p (at the end we can set $p=1$) and the square root which gives the energy, as in Eq. (3.4), will change to a power $-s/2$ (at the end we can set $s=-1$). Then the linear term in a in the zero-point energy has string tension

$$\kappa=\frac{1}{2\pi}\sum_{n_1,\ldots,n_t}\int_0^\infty dk\,k^{p-1}[k^2+\pi^2(n_1^2+\cdots+n_t^2)/L^2]^{-s/2}=\frac{\pi}{4}\frac{\Gamma(p/2)\Gamma((s-p)/2)}{\Gamma(s/2)}\times\sum_{n_1,\ldots,n_t}[(n_1^2+\cdots+n_t^2)/L^2]^{(p-s)/2}\,. \quad (3.13)$$

The function appearing on the right-hand side of Eq. (3.13) is the Epstein ζ function[21]

$$Z_t(1/L_1,\ldots,1/L_t;m)=\sum_{n_1,\ldots,n_t}[(n_1/L_1)^2+\cdots+(n_t/L_t)^2]^{-m/2}\,. \quad (3.14)$$

The string tension for the Euclidean geometry is then

$$\kappa=\frac{\pi}{4}\frac{\Gamma(p/2)\Gamma((s-p)/2)}{\Gamma(s/2)}Z_t(1/L,\ldots,1/L;(s-p))\,. \quad (3.15)$$

The reflection formula obeyed by this function, allowing it to be continued to the appropriate values of p and/or s, is

$$\Gamma\left[\frac{m}{2}\right]\pi^{-m/2}Z_t(L_1,\ldots,L_t;m)=\frac{1}{L_1\times\cdots\times L_t}\Gamma\left[\frac{t-m}{2}\right]\pi^{(m-t)/2}Z_t(1/L_1,\ldots,1/L_t;t-m)\,. \quad (3.16)$$

For example, suppose we set $s=-1$ in Eq. (3.15). Then

$$\kappa=\frac{\pi}{4}\frac{\Gamma(p/2)\Gamma((-1-p)/2)}{\Gamma(-1/2)}Z_t(1/L,\ldots,1/L;(-1-p))\,. \quad (3.17)$$

We cannot set $p=1$ in this expression, since both the sum and a Γ function are ill defined. However, using the reflection symmetry (3.16) we have

$$\kappa=\frac{\pi}{4}\pi^{-p-1-t/2}\frac{\Gamma(p/2)\Gamma((t+p+1)/2)}{\Gamma(-1/2)}L^t\times Z_t(L,\ldots,L;t+p+1)\,. \quad (3.18)$$

Everything here is nicely behaved. We can set $p=1$, and, for example, $t=2$, to find

$$\kappa=-\left[\frac{1}{8\pi^2L^2}\right]Z_2(1,1,4)=-\left[\frac{1}{8\pi^2L^2}\right]\times 4\zeta(2)\times 0.915\,. \quad (3.19)$$

0.915... is Catalan's constant, and $\zeta(2)=\pi^2/6$.

The reflection symmetry obeyed by the ζ function for the cylindrical geometry,

$$Z_{\rm cyl}(1/R,\,1/R;\,s-p)\equiv\sum_{n,\lambda}(\gamma_{n\lambda}/R)^{p-s}\,, \quad (3.20)$$

would , we conjecture, obey the same reflection symmetry as for the Euclidean case. For $t=2$, this would read

$$\Gamma(m/2)\pi^{-m/2}Z_{\rm cyl}(1/R,1/R;m) = R^2\Gamma((2-m)/2)\pi^{(m-2)/2}Z_{\rm cyl}(R,R;2-m) . \quad (3.21)$$

The correction to the string tension then involves only finite quantities and is calculable, at least numerically. Unfortunately, the hope that the reflection formula is obeyed "is still unsubstantiated."[22]

We shall discuss the numerical size of the correction to the string tension in the next section.

B. Contribution to the Coulomb term

The coefficient α of the $1/a$ term is, according to Eq. (3.6), given by

$$\alpha=\frac{1}{2}\sum_{n,\lambda}\frac{\pi}{2}B_2[F'(\infty)-F'(0)] . \quad (3.22)$$

The derivative is given by $F'(k)=-sk(k^2+\Delta)^{-(s+2)/2}$, and this vanishes for positive s when $k\to\infty$. It is also true that $F'(0)=0$ for nonzero Δ, as dimensional considerations would imply. Thus only $\Delta=0$, i.e., only the zero-mass modes, contribute. We have, however, seen that only $n=1$ contributes to zero-mass modes, and only for a single λ. To see the contribution of this mode, we can go back to the original expression for the zero-point energy and evaluate the contribution of the zero-mass modes. We have

$$\text{zero-point energy}\,|_{\text{zero-mass modes}}=\frac{1}{2}\sum_{\kappa}\left[\frac{2\pi\kappa}{a}\right] = \frac{\pi}{a}\zeta(-1) . \quad (3.23)$$

This is perhaps the simplest example of the ζ function regularization technique;[20] the Riemann ζ function $\zeta(-1)$ is given by $-B_2/2=-\frac{1}{12}$, so that

$$\alpha=-\frac{\pi}{12} . \quad (3.24)$$

Equation (3.24) is a result well known in the context of the hadronic string, and is known in that context as the Lüscher term.[12] We have seen that it is very simply the consequence of the tubelike geometry and of the fact that zero-mass modes are associated with only one kind of transverse oscillation. Although we have not proven that such a term is a consequence of all tubelike cavities, it seems to us very likely to be the case.

IV. COMMENTS

We begin with a few comments on phenomenology. What is the size of the correction to the string tension which we have deduced? Rather than making a numerical calculation involving the zeros of Bessel functions, we prefer to rely on reasoning which makes physical sense and which is, moreover, backed up by previous calculations of Casimir energies. This reasoning is that the zero-point energy, when given as an energy per unit volume, does not depend strongly on the shape of the volume. This idea is already contained in Ref. 18 if one admits that the major contribution to the zero-point energy comes from high modes; Weyl discussed the shape independence of the high modes, a result which seems quite sensible. To verify that this is the case, we can compare Lukosz's calculation[9] of the energy per unit length (the string tension) in a long Euclidean geometry with the numerical approximation of DeRaad and Milton,[23] using a very different renormalization technique, for a cylindrical geometry. Lukosz found for the tube of square cross section $\kappa=-0.038/\text{area}$ [this follows directly from Eq. (3.19)], while DeRaad and Milton found for the cylindrical tube $\kappa=-0.044/\text{area}$. We can also compare the calculation for a cube of sides L (Lukosz, Ref. 9) with that for a sphere of radius R (Boyer, Ref. 8). For the cube, the energy is $0.092/L$, while for a sphere of the same volume, so that $R=(\frac{3}{4}\pi)^{1/3}L$, the energy is $0.074/L$.

It would therefore seem reasonable, in estimating the size of the correction to the string tension, to just use Eq. (3.19) for a square cross section of the same area of our cylinder. This gives us

$$\left[\frac{\delta\kappa}{\kappa}\right]\sim 2\% .$$

The fact that this correction is small gives us *a posteriori* confidence that our treatment of the zero-point fluctuations within the zeroth-order cavity was justified.

The effect of all this is to change the long-distance ($a<0.5$ fm) potential of the flux-tube picture, Eq. (3.3), to

$$V(a)=\frac{2g^2-0.014}{R^2}a+C-\frac{\pi}{12}\frac{1}{a}+O(R/a^2) . \quad (4.1)$$

While the contribution of the zero-point energy to the string tension is small, the $1/a$ contribution, with its coefficient of $-\pi/12\simeq-0.26$, is very significant, especially since the original, primitive, Coulomb term, $-g^2/a$, has been canceled exponentially by the induced potential of the cavity.[4]

The potential (4.1) is quite consistent with the hadronic string potential,[24,25]

$$V_{\rm string}(a)=\kappa(a^2-R_c^2)^{1/2}+C , \quad (4.2)$$

where

$$R_c^2=\frac{\pi}{12}\left[\frac{R^2}{g^2}\right]=\frac{\pi}{6}\frac{1}{\kappa}\simeq(0.3\ \text{fm})^2 .$$

In Ref. 25 it was shown that the potential (4.2) is also consistent with phenomenology and implies that as a decreases, the "effective" Coulomb potential will rise[25,26] from the universal $-0.26/a$ to twice that value at the deconfinement radius $R_c\simeq0.3$ fm. For lower values of the separation, the perturbative regime takes over, with the effective coupling strength $\alpha(a)$ decreasing again (as a decreases) from $\alpha(a)\simeq0.5$ to vanish logarithmically with a through asymptotic freedom. The phenomenological α, Eq. (3.2), reflects all the stages of this process, and, given that the rms radii of physical quarkonia are in the range 0.2 fm ($b\bar{b}$) to 0.4 fm (charmonium), it is no surprise that α takes the range of values that it does.

Let us note that the flux-tube potential is expanded in power of R/a, where R is the radius of the flux tube. The string potential, however, is expanded in powers of R_c/a, where R_c is the "deconfinement" radius. In the limit where our tube becomes truly stringlike, that is, $R \to 0$, $(2g^2-0.014)/R^2 \to \kappa$, and R_c remains finite. In this limit, the zero-point oscillations cannot contribute beyond the $-0.26/a$ term, since by dimensional arguments, these terms must have numerical coefficients which are positive powers of R.

The higher terms in the large-a expansion of the square-root string potential are not associated with any new modes of oscillation of the string, but rather are a consequence[24] of the imposition of relativistic consistency of the transverse oscillations. In the finite-R flux tube, which is not relativistically invariant, the higher power of R/a in the potential reflects the infinitely many modes of internal oscillations. These modes are frozen out in the string limit. The two models are not equivalent even in the $R \to 0$ limit. In any case, the existence and size of terms in the potential proportional to a^{-n}, for n equal to two or greater, are of no conceivable phenomenological significance.

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[1] See, e.g., J. Kogut and L. Susskind, Phys. Rev. D **9**, 3501 (1974); G. 't Hooft, in *Recent Progress in Lagrangian Field Theory and Applications*, proceedings of the Marseille Colloquium, 1974, edited by C. P. Korthal-Altes *et al.* (Centre de Physique Theorique, Marseille, 1975); H. Pagels and E. Tomboulis, Nucl. Phys. **B143**, 485 (1978); R. Friedberg and T.-D. Lee, Phys. Rev. D **18**, 2623 (1978); C. Callan *et al.*, *ibid.* **19**, 1826 (1979); Y. Nambu, Phys. Lett. **102B**, 149 (1981).

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