Hadronic transitions of D-wave quarkonium

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The decays $\Upsilon(1D)\to\Upsilon\pi\pi$ are calculated within the framework of the multipole expansion in QCD. Using a description of the conversion of gluons into pions based on soft-pion theorems, we find $\Gamma(\Upsilon(1\,^3D_J)\to\Upsilon\pi\pi)=0.07$ keV; the rates are identical for $J=1,2,3$. The smallness of these rates means that the ^3D states can probably only be observed using their electric-dipole transitions $\Upsilon(1D)\to\Upsilon(1P)\gamma$.

I. INTRODUCTION

The spectrum and decays of the Υ system of $b\bar{b}$ bound states are described well by potential models and QCD decay-rate formulas.[1–3] A further test of these models would be the observation of D-wave states (i.e., states having orbital angular momentum $l=2$). Potential models predict the existence of $\Upsilon(1D)$ states with mass $M\approx10.16$ GeV; the ^3D_J states ($J=1,2,3$) could be produced as a result of electromagnetic transitions from the $\Upsilon(3S)$. The most promising signatures[1] of the decay of the $1D$ would be a photon of energy ~250 MeV from the decay $\Upsilon(1D)\to\Upsilon(1P)\gamma$ (plus two photons of energy ~100 MeV produced in the cascade from the $3S$) or a pion pair plus missing mass from $\Upsilon(1D)\to\Upsilon\pi^+\pi^-$. In addition, the ^3D states can decay into three gluons—these rates have been calculated[4] and found to be small.

The techniques for calculating electromagnetic transitions in quarkonium, based on the usual multipole expansion, are well established. The various potential models are in rough agreement among themselves concerning electric-dipole rates; for the transitions $\Upsilon(1D)\to\Upsilon(1P)\gamma$ a typical model[5] predicts rates of several tens of keV, but these decays may be difficult to observe due to the energy resolution of detectors currently in use.

At present, it might be more promising to look for two pions, plus missing mass equal to the Υ mass, in the decay $\Upsilon(1D)\to\Upsilon\pi\pi$. Unfortunately, there are conflicting theoretical predictions in the literature concerning the magnitude of these decays for ^3D states. Kuang and Yan[6] predict a very large rate of 24 keV, while Billoire *et al.*[7] predict a rate that is smaller by a factor of 10^{-3}. We shall reexamine the calculation of these rates in an attempt to resolve this discrepancy.

At first sight, the calculation of hadronic rates seems intractable, in comparison with that of the electromagnetic transitions, due to the intrusion of long-distance physics. The emission of two gluons, for example, can be calculated using the multipole expansion in QCD (Refs. 8–10), but somehow one has to take into account the conversion of the gluons into pions. It is the use of various more-or-less arbitrary models for this long-distance process that accounts for the disagreement in the literature.

In our view this confusion is unnecessary, since the low-energy theorems of QCD provide what is virtually a "first-principles" description of the conversion of gluons into pions, as shown by Voloshin and Zakharov[11] and refined by Novikov and Shifman.[12] This method has been applied[13] to show that the branching ratio for the transition $\Upsilon(3S)\to\Upsilon(1\,^1P_1)\pi\pi$ must be as small as 10^{-4} or less. In this paper we use the techniques of Refs. 11 and 12 to calculate $\Gamma(\Upsilon(1D)\to\Upsilon\pi\pi)$, and compare the results and techniques with those reported previously. We find that these rates are less than 0.1 keV, implying that these transitions are probably unobservable in present experiments.

II. HADRONIC AMPLITUDES

The relevant term in the Hamiltonian derived from the multipole expansion in QCD is[8]

$$\mathcal{H}\approx-\tfrac{1}{2}g\,\mathbf{r}\cdot\mathbf{E}^a\xi^a\ . \tag{2.1}$$

Here g is the QCD coupling constant, $\mathbf{r}$ is the radius vector, $\mathbf{E}^a$ is the chromoelectric field strength ($1\le a\le8$), and $\xi^a=\frac{1}{2}(\lambda_1^a+\lambda_2^a)$, where λ_1^a,λ_2^a are Gell-Mann matrices. (The subscripts 1 and 2 label the quark and antiquark.) Regarding (2.1) as a perturbation on the Hamiltonian describing the quark-antiquark interaction gives the required amplitude, at second order in perturbation theory. The amplitude factorizes into a quarkonium matrix element times a hadronic amplitude. Writing $g^2=4\pi\alpha_s$, we have

$$A=\langle f\mid\xi^a r_i G r_j\xi^b\mid i\rangle\langle\pi^+\pi^-\mid\pi\alpha_s E_i^aE_j^b\mid0\rangle\ . \tag{2.2}$$

Here i,j are three-vector indices, $\mid i\rangle$ and $\langle f\mid$ denote the initial and final quarkonium states, and G is the Green's function of the unperturbed system:

$$G(M)=\sum_k\frac{\mid k\rangle\langle k\mid}{M_k-M}\ , \tag{2.3}$$

where the summation is over octet as well as singlet quark-antiquark intermediate states, and M and M_k are the masses of the initial and intermediate states, respectively. The operator ξ^a turns a singlet state into an octet state, and vice versa, in view of which only the octet states in the summation in (2.3) contribute to the amplitude (2.2); we call the corresponding octet Green's function G_8. Using the rule[9]

$$\langle \text{singlet} \mid \xi^a \xi^b \mid \text{singlet} \rangle = \tfrac{2}{3}\delta^{ab} , \quad (2.4)$$

we obtain

$$A = \tfrac{2}{3}\langle f \mid r_i G_8 r_j \mid i \rangle \langle \pi^+\pi^- \mid \pi\alpha_s E_i^a E_j^a \mid 0 \rangle . \quad (2.5)$$

Voloshin and Zakharov[11] showed the correct way to obtain the hadronic amplitude in (2.5); the careful treatment by Novikov and Shifman[12] of higher-order corrections gives the amplitude for emitting the $\pi^+\pi^-$ pair in a D wave as well as the corrections to S-wave emission. We will review in some detail the techniques of Refs. 11 and 12, since we will need the hadronic amplitude in a slightly more general form than the one presented in those papers, where only transitions between S-wave quarkonium states were considered, and where expressions for absolute rates were not presented. A self-contained discussion will also facilitate comparison with other methods.

The idea is to assume initially that the pions are massless and use soft-pion theorems[14] which hold in the chiral-symmetry limit. Let the pion four-momenta be $p_1 = (\epsilon_1, \mathbf{p}_1)$ and $p_2 = (\epsilon_2, \mathbf{p}_2)$, and let $q = p_1 + p_2$ be the total four-momentum of the two-pion system. Since $E_i^a = -G_{0i}^a$, where $G_{\mu\nu}^a$ is the field-strength tensor, what we need is the matrix element of the operator $G_{\mu\lambda}^a G_{\nu\sigma}^a$. The form of the matrix element of $G_{\mu\lambda}^a G_{\nu\sigma}^a$ is fixed by the soft-pion theorems and the symmetry of the indices; up to terms quadratic in the momenta we have[13]

$$\begin{aligned}\langle \pi^+\pi^- \mid \alpha_s G_{\mu\lambda}^a G_{\nu\sigma}^a \mid 0 \rangle = {} & A q^2 (g_{\mu\nu} g_{\lambda\sigma} - g_{\mu\sigma} g_{\nu\lambda}) \\ & + B (g_{\mu\nu}\tau_{\lambda\sigma} - g_{\mu\sigma}\tau_{\lambda\nu} + g_{\lambda\sigma}\tau_{\mu\nu} \\ & - g_{\lambda\nu}\tau_{\mu\sigma}) , \end{aligned} \quad (2.6)$$

where we have introduced the symmetric tensor $\tau_{\mu\nu} = p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}$, and A and B are constants. (The other symmetric combination $p_{1\mu}p_{1\nu} + p_{2\mu}p_{2\nu}$ is excluded by the requirement[12] that the amplitude vanish when the four-momentum of either pion goes to zero.) The wave function of the pions in (2.6) is assumed to be symmetric under the interchange of π^+ and π^-.

It turns out that A and B in (2.6) are fixed if we know the matrix element of the operator $G_{\mu\lambda}^a G_{\nu\lambda}^a$ with a pair of indices contracted.[13] This operator is symmetric in the indices μ, ν, and the soft-pion theorems restrict its matrix element to have the form

$$\langle \pi^+\pi^- \mid \alpha_s G_{\mu\lambda}^a G_{\nu\lambda}^a \mid 0 \rangle = C q^2 g_{\mu\nu} + D \tau_{\mu\nu} . \quad (2.7)$$

Contracting a pair of indices in (2.6) and comparing with (2.7) gives the relations

$$A = \tfrac{1}{3}(C - \tfrac{1}{2}D), \quad B = \tfrac{1}{2}D . \quad (2.8)$$

It remains to express C and D in terms of known quantities; this can be done using the matrix element of the gluonic part of the energy-momentum tensor.[12] First we consider the full QCD energy-momentum tensor (with massless quarks):

$$\begin{aligned}\theta_{\mu\nu} = {} & G_{\mu\lambda}^a G_{\nu\lambda}^a + \tfrac{1}{4} g_{\mu\nu} G_{\lambda\sigma}^a G_{\lambda\sigma}^a \\ & + (i/2) \sum_q (\bar{q}\gamma_\mu D_\nu q + \bar{q}\gamma_\nu D_\mu q) . \end{aligned} \quad (2.9)$$

This quantity is symmetric in the indices μ, ν, and so its matrix element has a form similar to the right-hand side of (2.7); the two arbitrary constants in the corresponding expression can be fixed by imposing the conditions

$$q_\mu \langle \pi^+\pi^- \mid \theta_{\mu\nu} \mid 0 \rangle = 0 , \quad (2.10)$$

$$\langle \pi^+ \mid \theta_{\mu\nu} \mid \pi^+ \rangle \mid_{p_1 = p_2 = p} = 2 p_\mu p_\nu . \quad (2.11)$$

The first condition of course corresponds to the fact that $\theta_{\mu\nu}$ is conserved. As for the second condition, its kinematic structure is obvious from Lorentz invariance, while the factor of 2 is due to the fact that the pion field is charged. [The energy-momentum tensor of a massless complex scalar field ϕ is $\partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi - g_{\mu\nu}\partial_\lambda\phi^*\partial_\lambda\phi$; substituting $\phi \sim e^{-ipx}$ for a plane wave and using $p^2 = 0$ gives (2.11).] Applying the conditions (2.10) and (2.11) fixes the matrix element of $\theta_{\mu\nu}$ to be

$$\langle \pi^+\pi^- \mid \theta_{\mu\nu} \mid 0 \rangle = \tfrac{1}{2} q^2 g_{\mu\nu} - \tau_{\mu\nu} ; \quad (2.12)$$

the trace is $\langle \pi^+\pi^- \mid \theta_{\mu\mu} \mid 0 \rangle = q^2$.

Next we consider the gluonic part of the energy-momentum tensor:

$$\theta_{\mu\nu}^G = -G_{\mu\lambda}^a G_{\nu\lambda}^a + \tfrac{1}{4} g_{\mu\nu} G_{\lambda\sigma}^a G_{\lambda\sigma}^a . \quad (2.13)$$

We again assume the matrix element of $\theta_{\mu\nu}^G$ to have a form similar to the right-hand side of (2.7), but of course we cannot use the conservation condition since we are no longer dealing with the full energy-momentum tensor; also, the condition (2.11) will be modified. In this case, the two arbitrary constants are fixed using the following conditions:[12]

$$\langle \pi^+\pi^- \mid \theta_{\mu\mu}^G \mid 0 \rangle = \frac{\beta^G}{\beta} \langle \pi^+\pi^- \mid \theta_{\mu\mu} \mid 0 \rangle , \quad (2.14)$$

$$\langle \pi^+ \mid \theta_{\mu\nu}^G \mid \pi^+ \rangle \mid_{p_1 = p_2 = p} = 2\rho^G p_\mu p_\nu , \quad (2.15)$$

where ρ^G is the glue fraction of the pion, β is the QCD β function,

$$\beta(\alpha_s) = -\frac{(11 - \frac{2}{3} n_f)\alpha_s^2}{2\pi} + \cdots , \quad (2.16)$$

where henceforth we set the number of quark flavors to be $n_f = 3$, and the gluonic part of the β function β^G is given by the right-hand side of (2.16) with $n_f = 0$. The quantities α_s and ρ^G are to be taken at a mass scale of the order of the inverse size of the quarkonium system.

The condition (2.14) is a consequence of the trace anomaly,

$$\theta_{\mu\mu} = \frac{\beta}{4\alpha_s} G_{\lambda\sigma}^a G_{\lambda\sigma}^a , \quad \theta_{\mu\mu}^G = \frac{\beta^G}{4\alpha_s} G_{\lambda\sigma}^a G_{\lambda\sigma}^a , \quad (2.17)$$

while the condition (2.15) is consistent with the definition of ρ^G as the moment of a structure function as used in deep-inelastic scattering.[15] These conditions fix the corresponding matrix element to be

$$\langle \pi^+\pi^- \mid \theta^G_{\mu\nu} \mid 0 \rangle = \frac{1}{4}\left[\frac{\beta^G}{\beta}+\rho^G\right] q^2 g_{\mu\nu} - \rho^G \tau_{\mu\nu} \,. \quad (2.18)$$

Solving (2.13) for $G^a_{\mu\lambda}G^a_{\nu\lambda}$ we can evaluate its matrix element using (2.17), (2.18), and (2.12); the result is that the constants in (2.7) are fixed to be

$$C = \frac{\alpha_s^2}{\beta} - \frac{\alpha_s}{4}\left[\frac{\beta^G}{\beta}+\rho^G\right], \quad D = \alpha_s \rho^G \,. \quad (2.19)$$

The constants A and B in (2.6) are then found using (2.8). Using $\beta/\beta^G \approx \frac{11}{9} + O(\alpha_s)$, the result for the hadronic amplitude appearing in Eq. (2.5) is

$$\begin{aligned}\langle \pi^+\pi^- \mid \pi\alpha_s E^a_i E^a_j \mid 0 \rangle &= \langle \pi^+\pi^- \mid \pi\alpha_s G^a_{0i} G^a_{0j} \mid 0 \rangle \\ &= \frac{1}{3}\left[\frac{2\pi^2}{9} + O(\alpha_s)\right] q^2 \delta_{ij} \\ &\quad + \frac{\pi\alpha_s\rho^G}{2}\tau_{ij} \,. \end{aligned} \quad (2.20)$$

Defining the relative four-momentum of the pions to be $p = p_1 - p_2$, we can write $\tau_{\mu\nu} = \frac{1}{2}(q_\mu q_\nu - p_\mu p_\nu)$. To decompose (2.20) into terms corresponding to emission of the $\pi^+\pi^-$ pair in an S and in a D wave, we note that the D-wave amplitude must be proportional to $p_i p_j - \frac{1}{3}(q_i q_j + q^2\delta_{ij})$, since in the rest frame of the pion pair this goes over to $p_i p_j - \frac{1}{3}\mathbf{p}^2\delta_{ij}$. [The corresponding covariant expression is $p_\mu p_\nu - \frac{1}{3}(q_\mu q_\nu - q^2 g_{\mu\nu})$.] We mention that this definition of the D-wave part of the amplitude does not distinguish between the "internal" orbital angular momentum of the pion pair and that due to the motion of the pions relative to the final quarkonium state. The final results for the hadronic amplitudes are

$$\langle (\pi^+\pi^-)_S \mid \pi\alpha_s E^a_i E^a_j \mid 0 \rangle = \frac{2\pi^2}{27} q^2 \delta_{ij} \,, \quad (2.21)$$

$$\begin{aligned}\langle (\pi^+\pi^-)_D \mid \pi\alpha_s E^a_i E^a_j \mid 0 \rangle \\ = \frac{\pi\alpha_s\rho^G}{4}\left[p_i p_j - \tfrac{1}{3}(q_i q_j + q^2\delta_{ij})\left[1 - \frac{4m_\pi^2}{q^2}\right]\right] .\end{aligned} \quad (2.22)$$

We only give the leading term in each case; the question of the correction to (2.21) is addressed in Ref. 12. In (2.22) we include the kinematic correction $\sim m_\pi^2$ giving the second term in the D-wave amplitude the correct threshold behavior; for arguments concerning the smallness of the remaining $O(m_\pi^2)$ corrections, we again refer to Ref. 12. This concludes our review of the methods of Refs. 11 and 12. It only remains to consider the quarkonium matrix element in (2.5).

III. QUARKONIUM TRANSITION RATES

In the cases we are considering, where the quarkonium states are S or D waves, only the P-wave part of the octet Green's function contributes to the amplitude (2.5). We shall use Cartesian wave functions, constructing the "double-dipole" matrix elements in (2.5) starting from the dipole matrix elements:

$$\langle (^3P_0) \mid r_l \mid (^3S_1)_i \rangle = \tfrac{1}{3}\langle P \mid r \mid S \rangle \delta_{il} \,, \quad (3.1)$$

$$\langle (^3P_1)_j \mid r_l \mid (^3S_1)_i \rangle = -\frac{1}{\sqrt{6}}\langle P \mid r \mid S \rangle \epsilon_{ijl} \,, \quad (3.2)$$

$$\langle (^3P_2)_{jk} \mid r_l \mid (^3S_1)_i \rangle = \frac{\langle P \mid r \mid S \rangle}{2\sqrt{3}}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{il}\delta_{jk}) \,, \quad (3.3)$$

$$\langle (^3P_0) \mid r_l \mid (^3D_1)_i \rangle = \frac{2}{3\sqrt{2}}\langle P \mid r \mid D \rangle \delta_{il} \,, \quad (3.4)$$

$$\langle (^3P_1)_j \mid r_l \mid (^3D_1)_i \rangle = \frac{1}{2\sqrt{3}}\langle P \mid r \mid D \rangle \epsilon_{ijl} \,, \quad (3.5)$$

$$\langle (^3P_2)_{jk} \mid r_l \mid (^3D_1)_i \rangle = \frac{1}{10\sqrt{6}}\langle P \mid r \mid D \rangle (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{il}\delta_{jk}) \,, \quad (3.6)$$

$$\langle (^3P_1)_k \mid r_l \mid (^3D_2)_{ij} \rangle = -\frac{3}{2\sqrt{30}}\langle P \mid r \mid D \rangle (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) \,, \quad (3.7)$$

$$\langle (^3P_2)_{km} \mid r_l \mid (^3D_2)_{ij} \rangle = \frac{\langle P \mid r \mid D \rangle}{4\sqrt{15}}(\epsilon_{ilk}\delta_{jm} + \epsilon_{ilm}\delta_{jk} + \epsilon_{jlk}\delta_{im} + \epsilon_{jlm}\delta_{ik}) \,, \quad (3.8)$$

$$\langle (^3P_2)_{mn} \mid r_l \mid (^3D_3)_{ijk} \rangle = \frac{1}{3\sqrt{10}}\langle P \mid r \mid D \rangle [\delta_{im}\delta_{jn}\delta_{kl} + \delta_{im}\delta_{jl}\delta_{kn} - \tfrac{2}{5}(\delta_{il}\delta_{jk}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{in}\delta_{jk}\delta_{lm})] + \text{c.p.} \quad (3.9)$$

[The indicated permutations in (3.9) are over the indices i,j,k.] We neglect the fine structure of the octet P-wave states, so that the masses M_k of the intermediate states [cf. Eq. (2.3)] are identical for the ^3P_J octet states having $J=0,1,2$. This assumption has the important result that the Green's function transforms as a scalar. Defining

$$I_{i,f} = \langle f \mid r G_{8,P} r \mid i \rangle \,, \quad (3.10)$$

we find

$$\langle ({}^3S_1)'_j \mid r_k G_{8,P} r_l \mid ({}^3S_1)_i \rangle = \frac{I_{S,S'}}{3}\delta_{ij}\delta_{kl} \;, \tag{3.11}$$

$$\langle ({}^3S_1)_j \mid r_k G_{8,P} r_l \mid ({}^3D_1)_i \rangle = \frac{\sqrt{2}I_{D,S}}{10}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\tfrac{2}{3}\delta_{ij}\delta_{kl}) \;, \tag{3.12}$$

$$\langle ({}^3S_1)_m \mid r_k G_{8,P} r_l \mid ({}^3D_2)_{ij} \rangle = \frac{\sqrt{5}I_{D,S}}{60}[\epsilon_{ilm}\delta_{jk}+\epsilon_{ilk}\delta_{jm}+\epsilon_{jlm}\delta_{ik}+\epsilon_{jlk}\delta_{im}+3(\epsilon_{jkm}\delta_{il}+\epsilon_{ikm}\delta_{jl}-\tfrac{2}{3}\epsilon_{lkm}\delta_{ij})] \;, \tag{3.13}$$

$$\langle ({}^3S_1)_n \mid r_l G_{8,P} r_m \mid ({}^3D_3)_{ijk} \rangle = \frac{\sqrt{30}I_{D,S}}{90}[\delta_{in}\delta_{jl}\delta_{km}+\delta_{il}\delta_{jn}\delta_{km}-\tfrac{2}{5}(\delta_{jk}\delta_{il}\delta_{mn}+\delta_{jk}\delta_{im}\delta_{ln}+\delta_{jk}\delta_{in}\delta_{lm})]+\text{c.p.} \tag{3.14}$$

Contracting the indices on the radius vectors in the expressions above, we find that the matrix element vanishes in the case of a ${}^3D_J$ initial state, $J=1,2,3$, so that these states can decay only by emitting the pions in a D wave. For ${}^3D_2$ and ${}^3D_3$ this is obvious from conservation of angular momentum. For the ${}^3D_1$ state it follows because the spin variable appears neither in the interaction (2.1) nor in the Hamiltonian of the unperturbed system (due to the assumption mentioned above that the fine structure of the intermediate states is neglected). The spin dependence factorizes and therefore the orbital angular momentum is conserved; this in turn implies the vanishing of the amplitude for S-wave emission.

Next, we insert the expressions (3.11)–(3.14) into Eq. (2.5) to obtain the required amplitudes. Let ϵ_i be a polarization vector for the final ${}^3S_1$ state, satisfying

$$\sum_{\text{polar}} \epsilon_i \epsilon_j = \delta_{ij} \;. \tag{3.15}$$

Then the amplitudes are

$$A_i({}^3S_1 \to {}^3S'_1(\pi\pi)_S) = \frac{4\pi^2}{81} I_{S,S'}\epsilon_i q^2 \;, \tag{3.16}$$

$$A_i({}^3D_1 \to {}^3S_1(\pi\pi)_D) = \frac{\pi\alpha_s\rho^G\sqrt{2}}{30} I_{D,S}\left[p_i \mathbf{p}\cdot\boldsymbol{\epsilon} - \frac{\epsilon_i}{3}\mathbf{p}^2 - \tfrac{1}{3}\left[q_i \mathbf{q}\cdot\boldsymbol{\epsilon} - \frac{\epsilon_i}{3}\mathbf{q}^2\right]\left[1-\frac{4m_\pi^2}{q^2}\right]\right] \;, \tag{3.17}$$

$$A_{ij}({}^3D_2 \to {}^3S_1(\pi\pi)_D) = \frac{\sqrt{5}\pi\alpha_s\rho^G}{90} I_{D,S}\left[p_i(\mathbf{p}\times\boldsymbol{\epsilon})_j + p_j(\mathbf{p}\times\boldsymbol{\epsilon})_i - \tfrac{1}{3}[q_i(\mathbf{q}\times\boldsymbol{\epsilon})_j + q_j(\mathbf{q}\times\boldsymbol{\epsilon})_i]\left[1-\frac{4m_\pi^2}{q^2}\right]\right] \;, \tag{3.18}$$

$$\begin{aligned} A_{ijk}({}^3D_3 \to {}^3S_1(\pi\pi)_D) = \frac{\sqrt{30}\pi\alpha_s\rho^G}{270} I_{D,S} \Bigg[& \epsilon_i p_j p_k - \tfrac{1}{5}\epsilon_i\delta_{jk}\mathbf{p}^2 - \tfrac{2}{5}p_i\delta_{jk}(\mathbf{p}\cdot\boldsymbol{\epsilon}) \\ & - \tfrac{1}{3}[\epsilon_i q_j q_k - \tfrac{1}{5}\epsilon_i\delta_{jk}\mathbf{q}^2 - \tfrac{2}{5}q_i\delta_{jk}(\mathbf{q}\cdot\boldsymbol{\epsilon})]\left[1-\frac{4m_\pi^2}{q^2}\right]\Bigg] + \text{c.p.} \end{aligned} \tag{3.19}$$

It remains to square the amplitudes, sum over polarizations using (3.15), and integrate over phase space. Already from (3.16)–(3.19) we see that the ${}^3D$ rates are suppressed relative to the ${}^3S_1$ rates, since the corresponding amplitudes receive no contribution from the trace of the QCD energy-momentum tensor and so are proportional to α_s.

First, let us consider the case where the initial and final quarkonium states are both S waves. Squaring (3.16) and using (3.15), and dividing by 3 to average over initial polarizations, we obtain

$$\langle \mid A \mid^2 \rangle = \frac{2^4\pi^4}{3^8} \mid I_{S,S'} \mid^2 q^4 \;. \tag{3.20}$$

A simple analytic form can be found for the phase-space integral in the limit in which the quarkonium is very heavy (so that there is no recoil) and in which the pion mass is neglected. In that limit we have $q^2=\Delta^2$, where Δ is the difference in masses of the initial and final quarkonium states, so that the amplitude squared is a constant. Then the rate is given by

$$\Gamma = \langle \mid A \mid^2 \rangle \int \frac{1}{(2\pi)^5}\delta(\epsilon_1+\epsilon_2-\Delta)\delta^{(3)}(\mathbf{P}+\mathbf{p}_1+\mathbf{p}_2) \times \frac{1}{2\epsilon_1 2\epsilon_2} d^3P\,d^3p_1\,d^3p_2 \;, \tag{3.21}$$

where $\mathbf{P}$ is the momentum (which is equal to zero in this approximation) of the final quarkonium state. Note that the denominator in (3.21) contains a factor 2ϵ only for each pion; the quarkonium states are normalized as in nonrelativistic quantum mechanics. Using the integral over d^3P to eliminate the momentum δ function, and writing

$$d^3p_1 d^3p_2 \to 16\pi^2\epsilon_1^2\epsilon_2^2 d\epsilon_1 d\epsilon_2 \;,$$

we get

$$\Gamma=\langle\,|A|^2\rangle\frac{1}{8\pi^3}\int\delta(\epsilon_1+\epsilon_2-\Delta)\epsilon_1\epsilon_2 d\epsilon_1 d\epsilon_2$$
$$=\langle\,|A|^2\rangle\frac{1}{8\pi^3}\int_0^{\Delta}\epsilon_1(\Delta-\epsilon_1)d\epsilon_1$$
$$=\langle\,|A|^2\rangle\frac{\Delta^3}{48\pi^3}\ . \qquad (3.22)$$

Inserting (3.20) into (3.22) gives

$$\Gamma({}^3S_1\to{}^3S_1'\pi^+\pi^-)\approx\frac{\pi\,|I_{S,S'}|^2\Delta^7}{3^9}\ , \qquad (3.23)$$

where Δ is the mass difference. This turns out to be a poor approximation but serves to illustrate the strong dependence on the mass splitting Δ. A considerable amount of suppression with respect to this formula results when the physical masses of the particles are taken into account in evaluating the phase space. A numerical calculation, using the physical pion mass, and allowing for the recoil of the final-quarkonium state, gives suppression factors of 0.08, 0.02, and 0.09, respectively, for the transitions $2S\to1S$, $3S\to2S$, and $3S\to1S$ in the Υ system. We have taken the masses of the $1S$, $2S$, and $3S$ to be 9.46, 10.02, and 10.355 GeV, respectively.

It is impossible to calculate reliably the Green's function $G_{8,P}$ since we have no information on the octet sector of quark-antiquark states. We shall simply approximate the Green's function by a constant G_0 so that

$$I_{i,f}\approx G_0\langle f\,|r^2|\,i\rangle\ . \qquad (3.24)$$

[Neglecting the nonlocality of the Green's function in this way is in fact justified in the limiting case of extremely large quark mass.[9] While this assumption is certainly not literally valid for the Υ system, the simple model (3.24) should suffice for the present purpose of estimating the order of magnitude of the transition rates under discussion.]

A numerical calculation in the potential model of Ref. 5 gives

$$|\langle 1S\,|r^2|\,2S\rangle|=1.25\ \text{GeV}^{-2}\ , \qquad (3.25)$$
$$|\langle 2S\,|r^2|\,3S\rangle|=4.18\ \text{GeV}^{-2}\ , \qquad (3.26)$$
$$|\langle 1S\,|r^2|\,3S\rangle|=0.30\ \text{GeV}^{-2}\ . \qquad (3.27)$$

The constant G_0 can be fixed using the observed rate[16] $\Gamma(\Upsilon(2S)\to\Upsilon\pi\pi)=6.62$ keV, obtaining $G_0=4.38\ \text{GeV}^{-1}$. Through Eq. (2.3), this value implies a rather low value for the octet state masses, probably indicating the crudeness of the assumption (3.24). Nonetheless, our calculation gives an excellent result for the $3S\to2S$ transition; namely, 0.51 keV, to be compared with the experimental number 0.49 ± 0.14 keV.

The model appears to do very badly, however, for the $3S\to1S$ transition, giving a rate of 11 keV as compared with the experimental result of 0.81 ± 0.14 keV. But in fact the potential-model wave functions are fixed by the spectrum only to an accuracy $\sim10\%$, in view of which the small value (3.27) as compared with (3.25) or (3.26) could be considered as equal to zero, within the accuracy mentioned. Thus the presence of zeros in the $3S$ wave function is a likely explanation for the small value of the rate for $\Upsilon(3S)\to\Upsilon\pi\pi$. We mention that, of course, the $1D$ wave function has no zero, so the simple model (3.24) should work quite well for estimating the hadronic states of the ${}^3D$ states.

Squaring the amplitudes (3.17)–(3.19), and dividing by $2J+1$ to average over initial polarizations, we find that the squared amplitudes are identical for $J=1,2,3$ and equal to

$$\langle\,|A|^2\rangle=\frac{\pi^2(\alpha_s\rho^G)^2\,|I_{D,S}|^2}{3^4\times5^2}\times\left[\mathbf{p}^4+\tfrac{1}{9}\mathbf{q}^4\left[1-\frac{4m_\pi^2}{q^2}\right]^2+[\tfrac{1}{3}\mathbf{p}^2\mathbf{q}^2-(\mathbf{p}\cdot\mathbf{q})^2]\left[1-\frac{4m_\pi^2}{q^2}\right]\right]\ . \qquad (3.28)$$

The equality of the rates for the three ${}^3D$ states was proved in Ref. 17. It is somewhat more simply explained just by looking at the amplitudes, as follows.

In terms of spherical tensors, we are dealing with the matrix element of two irreducible rank-2 tensors coupled together to give a scalar. That is, we have a tensor $f_2^{Q\bar{Q}}$ corresponding to the Cartesian tensor $r_iG_{8,P}r_j$, and a tensor $f_2^{\pi\pi}$ corresponding to $E_i^aE_j^a$. Using a standard formula from the theory of addition of angular momenta,[18] we have

$$\langle{}^3S_1(\pi\pi)_D\,|(f_2^{Q\bar{Q}}f_2^{\pi\pi})_{00}|\,{}^3D_J\rangle\sim\begin{Bmatrix}1 & J & 2\\ 0 & 2 & J\end{Bmatrix}\langle{}^3S_1\,|f_2^{Q\bar{Q}}|\,{}^3D_J\rangle\langle(\pi\pi)_D\,|f_2^{\pi\pi}|\,0\rangle$$
$$\sim\frac{1}{(2J+1)^{1/2}}\langle{}^3S_1\,|f_2^{Q\bar{Q}}|\,{}^3D_J\rangle\langle(\pi\pi)_D\,|f_2^{\pi\pi}|\,0\rangle\ , \qquad (3.29)$$

where we have used the following formula for the $6j$ symbol:

$$\begin{Bmatrix}a & b & c\\ 0 & c & b\end{Bmatrix}=\frac{(-1)^{(a+b+c)}}{[(2b+1)(2c+1)]^{1/2}}\ . \qquad (3.30)$$

The J dependence of the quarkonium matrix element given by[18]

$$\langle{}^3S_1\,|f_2^{Q\bar{Q}}|\,{}^3D_J\rangle\sim\sqrt{2J+1}\begin{Bmatrix}J & 2 & 1\\ 0 & 1 & 2\end{Bmatrix}\langle S\,|f_2^{Q\bar{Q}}|\,D\rangle$$
$$\sim\sqrt{2J+1}\langle S\,|f_2^{Q\bar{Q}}|\,D\rangle\ . \qquad (3.31)$$

Combining the expressions (3.29) and (3.31), we see that the factors of $\sqrt{2J+1}$ cancel, and so the squared amplitudes are independent of J. The physical reason why the rates are independent of J is that the Hamiltonian has been assumed to be completely independent of the spin, so that the rates cannot depend on how the spin is coupled to the orbital angular momentum of the initial state.

In the heavy-quarkonium limit we have $\mathbf{q}=0$ and the squared amplitudes are equal to

$$\langle |A|^2\rangle \approx \frac{\pi^2(\alpha_s\rho^G)^2}{3^4\times 5^2}|I_{D,S}|^2\mathrm{p}^4 . \tag{3.32}$$

In this approximation we have $\mathrm{p}^2=\Delta^2$; also setting $m_\pi=0$ and doing the phase space as before, we obtain the approximate formula

$$\Gamma(^3D\to {}^3S_1\pi^+\pi^-)\approx\frac{(\alpha_s\rho^G)^2|I_{D,S}|^2\Delta^7}{2^4\times 3^5\times 5^2\pi} . \tag{3.33}$$

A numerical calculation of the phase-space integral using the exact amplitude (3.28) gives a suppression factor of 0.13 with respect to Eq. (3.33). The mass of the $1D$ was taken to be 10.16 GeV. In the model of Ref. 5 we find

$$|\langle 1D|r^2|1S\rangle|=1.65\ \mathrm{GeV}^{-2} \tag{3.34}$$

and using this value we obtain

$$\Gamma(\Upsilon(1^3D)\to\Upsilon\pi\pi)=1.83(\alpha_s\rho^G)^2\ \mathrm{keV} . \tag{3.35}$$

The only point remaining is to fix the value of the parameter $\alpha_s\rho^G$. It is expected that the glue fraction of the pion should be about the same as that of the nucleon,[12] i.e., $\rho^G\approx\frac{1}{2}$. As for α_s, we shall take the value at the scale ($\sim$0.5 GeV) of the inverse radius of the J/ψ used in Ref. 12, $\alpha_s(J/\psi)\approx 0.7$ and let the coupling run to the scale ($\sim$1 GeV) of the inverse radius of the Υ, giving $\alpha_s(\Upsilon)\approx 0.4$ Using this value in Eq. (3.35), we finally obtain our prediction for the width for decay of the $\Upsilon(1D)$ into the Υ plus charged pions:

$$\Gamma(\Upsilon(1^3D)\to\Upsilon\pi\pi)=0.07\ \mathrm{keV} . \tag{3.36}$$

We repeat that the rates are identical for the states having $J=1,2,3$.

IV. DISCUSSION

We compare our result for $\Gamma(\Upsilon(1D)\to\Upsilon\pi\pi)$ with those of Kuang and Yan[6] and Billoire, *et al.*[7] in Table I. (The value from Ref. 7 was obtained using the ratio of the formulas for ^3D and $^3S_1$ decay and the experimental value for the $2S\to 1S$ transition.) The reason for the large value obtained by Kuang and Yan is very clear. Although they adopted the soft-pion approach, they did not fix the constants in the corresponding hadronic amplitude [as in Eq. (2.6)] from first principles. Rather they fixed one constant using experimental input and the second by appealing to the case of two-gluon emission (which should be valid only in a much higher-energy range). In fact, it is this second constant which governs the magnitude of the amplitude for emitting the pion pair in a D wave and hence controls the size of the hadronic rates of ^3D states.

TABLE I. Rate in keV for $\Upsilon(1\,^3D)\to\Upsilon\pi^+\pi^-$.

This paper	Kuang and Yan	Billoire *et al.*
0.07	24	0.03

It is easy to obtain the two-gluon rates in the formalism of the present paper. The hadronic amplitude is simply

$$\langle 2g|\pi\alpha_sE_i^aE_j^a|0\rangle=\pi\alpha_s\omega_1\omega_2(e_{1i}^ae_{2j}^a+e_{2i}^ae_{1j}^a) , \tag{4.1}$$

where ω_1,ω_2 are the energies and e_{1i}^a,e_{2i}^a the Coulomb-gauge polarization vectors of the gluons. Decomposing (4.1) into S- and D-wave parts, we have

$$\langle(2g)_{0^+}|\pi\alpha_sE_i^aE_j^a|0\rangle=\frac{2\pi\alpha_s}{3}\omega_1\omega_2\mathbf{e}_1^a\cdot\mathbf{e}_2^a\delta_{ij} , \tag{4.2}$$

$$\langle(2g)_{2^+}|\pi\alpha_sE_i^aE_j^a|0\rangle=\pi\alpha_s\omega_1\omega_2(e_{1i}^ae_{2j}^a+e_{2i}^ae_{1j}^a-\tfrac{2}{3}\mathbf{e}_1^a\cdot\mathbf{e}_2^a\delta_{ij}) . \tag{4.3}$$

The quarkonium decay amplitudes are computed as before; the ^3D states can decay only by emitting the gluons in a 2^+ state. After squaring the amplitudes, the sum over polarizations is done using

$$\sum_{\text{polar}} e_i^ae_j^a=\delta_{ij}-n_in_j , \tag{4.4}$$

where $\hat{\mathbf{n}}$ is a unit vector in the direction of the gluon momentum. One has to divide by a factor of 8 for the number of colors and by an additional factor of 2 to account for the identity of the gluons, and then the phase space can be done as in the two-pion case. Restricting ourselves to the heavy-quarkonium limit, we find

$$\Gamma(^3S_1\to {}^3S_1'(2g)_{0^+})\approx\frac{\alpha_s^2|I_{S,S'}|^2\Delta^7}{2^7\times 3^3\pi} , \tag{4.5}$$

$$\Gamma(^3D\to {}^3S_1(2g)_{2^+})\approx\frac{\alpha_s^2|I_{D,S}|^2\Delta^7}{2^3\times 3^3\times 5^2\pi} . \tag{4.6}$$

Taking the ratio of these expressions, and assuming for simplicity that $|I_{S,S'}|\approx|I_{D,S}|$, we find

$$\frac{\Gamma_{1D}^{2g}}{\Gamma_{2S}^{2g}}\approx\frac{16}{25}\left[\frac{M_{1D}-M_\Upsilon}{M_{2S}-M_\Upsilon}\right]^7 . \tag{4.7}$$

Using (4.7) and taking Γ_{2S}^{2g} equal to the experimental two-pion rate gives $\Gamma_{1D}^{2g}\approx 20$ keV, which is comparable to the result of Kuang and Yan. Thus, although Kuang and Yan set out to work within the soft-pion framework, their calculation of the ^3D rate is actually a soft-pion–two-gluon hybrid, and this results in their large value for the rate.

As for the result of Billoire *et al.*, they are working within the framework of two-gluon decays, but they impose the *ad hoc* constraint that the gluons be admitted in a 0^+ state. (This is allowed in their calculation even for initial ^3D states due to their definition of the D-wave part

TABLE II. Total widths and branching ratios for decays of the $\Upsilon(1D)$.

	Γ_{tot} (keV)	$\Upsilon(1\,^3P_0)\gamma$	$\Upsilon(1\,^3P_1)\gamma$	$\Upsilon(1\,^3P_2)\gamma$	$\Upsilon\pi\pi$	3g
$^3D_1$	58	0.62	0.32	0.02	1×10^{-3}	0.04
$^3D_2$	47		0.79	0.20	1×10^{-3}	0.01
$^3D_3$	42			0.97	2×10^{-3}	0.03

of the two-gluon wave function, which refers only to the "internal" orbital angular momentum of the gluon pair.) As would be expected, the projection onto the 0^+ two-gluon state reduces the width by a large factor, and this is why Billoire *et al.* obtain such a small rate for the ^3D decay. If they had not made this projection, they would have obtained a rate comparable to that of Kuang and Yan. It is interesting that their model gives a value for the ratio of the ^3D and $^3S_1$ rates that almost coincides with that of the present work, which is based on the soft-pion approach.

Finally, Table II gives the result of a calculation of the total widths of the three ^3D_J states, $J=1,2,3$, and the branching ratios for various decays. The electromagnetic rates were calculated using standard formulas for electric-dipole transitions[19] and the value $|\langle 1P\,|\,r\,|\,1D\rangle|=2.75\ \mathrm{GeV}^{-1}$ from the model of Ref. 5. The $1P$ masses have been measured to be $M(1\,^3P_0)=9860$, $M(1\,^3P_1)=9892$, and $M(1\,^3P_2)=9913$ MeV. For the $1D$ masses we used the theoretical values given in Ref. 1: $M(1\,^3D_1)=10149$, $M(1\,^3D_2)=10156$, and $M(1\,^3D_3)=10161$ MeV. The values for the three-gluon annihilation widths were taken from Ref. 4.

In conclusion, the result of our calculation using the soft-pion approach is that the ^3D states have very small rates for the transition to $\Upsilon\pi\pi$. As a result, the $\Upsilon(1D)$ states can probably be seen only by using their electric-dipole transitions. A calculation[20] of the probability for producing the $\Upsilon(1D)$ in electromagnetic cascades from the $\Upsilon(3S)$ will clarify the prospects for observing these states.

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[1] W. Kwong, J. L. Rosner, and C. Quigg, Annu. Rev. Nucl. Part. Sci. **37**, 325 (1987).

[2] W. Buchmüller and S. Cooper, M.I.T. Report No. MIT-LNS-159, 1987 (unpublished).

[3] J. H. Kühn, in *Proceedings of the International Symposium on Production and Decay of Heavy Hadrons,* Heidelberg, 1986, edited by K. R. Schubert and R. Waldi (DESY, Hamburg, 1986).

[4] G. Bélanger and P. Moxhay, Phys. Lett. B **199**, 575 (1987).

[5] P. Moxhay and J. L. Rosner, Phys. Rev. D **28**, 1132 (1983); P. Moxhay, Ph.D. thesis, University of Minnesota, 1983.

[6] Y.-P. Kuang and T.-M. Yan, Phys. Rev. D **24**, 2874 (1981).

[7] A. Billoire *et al.*, Nucl. Phys. **B155**, 493 (1979).

[8] K. Gottfried, Phys. Rev. Lett. **40**, 598 (1978).

[9] M. B. Voloshin, Nucl. Phys. **B154**, 365 (1979).

[10] M. E. Peskin, Nucl. Phys. **B156**, 365 (1979).

[11] M. Voloshin and V. Zakharov, Phys. Rev. Lett. **45**, 688 (1980).

[12] V. A. Novikov and M. A. Shifman, Z. Phys. C **8**, 43 (1981).

[13] M. B. Voloshin, Yad. Fiz. **43**, 1571 (1986) [Sov. J. Nucl. Phys. **43**, 1011 (1986)].

[14] S. L. Adler, Phys. Rev. **137B**, 1022 (1965)].

[15] See, e.g., A. J. Buras, Rev. Mod. Phys. **52**, 199 (1980).

[16] All experimental rates and masses are taken from Ref. 1.

[17] T.-M. Yan, Phys. Rev. D **22**, 1652 (1980).

[18] See, e.g., L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1977), Sec. 109.

[19] See, e.g., V. B. Berestetskii *et al.*, *Quantum Electrodynamics* (Pergamon, New York, 1982), Sec. 49.

[20] W. Kwong and J. L. Rosner, Enrico Fermi Institute Report No. EFI 88-02, 1988 (unpublished).