$M1$ transitions involving the D states of quarkonia

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We derive expressions for the decay rates of the $M1$ transitions (1) $n\,^1D_2 \leftrightarrow n'\,^3S_1$, (2) $n\,^3D_1 \leftrightarrow n'\,^1S_0$, and (3) $n\,^1D_2 \leftrightarrow n'\,^3D_1$ including their leading relativistic corrections. Using the potential proposed by Gupta, Radford, and Repko we also numerically calculate the decay widths for some of these transitions in both the $c\bar{c}$ and the $b\bar{b}$ bound systems. In general the $M1$ decay rates in the $b\bar{b}$ systems are found to be extremely small, of the order of a few electron volts or less. But in the $c\bar{c}$ system the $M1$ decay $1\,^1D_2 \leftrightarrow 1\,^3S_1$ may have a significant branching ratio, its partial decay width being more than 2 keV. The inclusion of coupled-channel mixing leads to only small corrections.

I. INTRODUCTION

In the literature there exist several papers[1–4] regarding the $S \to S$ $M1$ decays of quarkonia and especially of charmonium. The interest was mostly restricted to S states because only those transitions were seen or expected to be seen for charmonium in the e^+e^- colliding-beam experiments. But when experiments involving $\bar{p}p$ collisions are performed at Fermilab in the near future there is a distinct possibility[5] that D states of charmonium will be formed and their decays can be studied. Furthermore the singlet D state of charmonium, namely, the $^1D_2$ is expected to be narrow and so its radiative $M1$ decays to $1\,^3S_1$ and $2\,^3S_1$ and its $E1$ decays to $^1P_1$ should have observable branching ratios. Even though the $^1D_2$ state is predicted to be above the charm threshold in most potential models, it is expected to be very narrow since the decay to $D_0+\bar{D}_0$ is forbidden by conservation of parity and the prediced mass of $^1D_2$ is such that the decay to $D_0^*+\bar{D}_0$ is energetically forbidden. Moreover in the $b\bar{b}$ system all the $2D$ and the $1D$ states are predicted to lie below the B threshold in most potential models. So the calculation of their radiative decay rates may be of some interest.

In this paper we first derive the expressions for the $M1$ decay rates of quarkonia involving the D states. We then use these expressions to numerically calculate the $M1$ decay widths of the $c\bar{c}$ and the $b\bar{b}$ systems in the potential model of Gupta, Radford, and Repko[6–8] (GRR), which has been very successful in predicting the energy spectra of both the $c\bar{c}$ and the $b\bar{b}$ bound systems. The calculated rate in charmonium for $1\,^1D_2 \to 1\,^3S_1+\gamma$ is about 2.13 keV and for $1\,^1D_2 \to 2\,^3S_1+\gamma$ is about 0.041 keV. Even the branching ratio for $1\,^1D_2 \to 1\,^3S_1+\gamma$ should be rather small since the rate of the $E1$ decay $1\,^1D_2 \to 1\,^1P_1+\gamma$ is about 661 keV in the GRR model. The $M1$ decay rates for the $b\bar{b}$ system are found to be extremely small in all cases. This is to be expected since these decays are either forbidden in the nonrelativistic limit or when they are allowed nonrelativistically the photon energy is too small to give a significant decay rate, as the decay rate is proportional to the cube of the photon energy.

The format of the rest of the paper is as follows. In Sec. II we derive the expressions for the $M1$ decay rates of the following transitions in quarkonia in terms of integrals involving the radial wave functions of the states in question: (a) $n\,^1D_2 \to n'\,^3S_1+\gamma$, (b) $n'\,^3S_1 \to n\,^1D_2+\gamma$, (c) $n\,^3D_1 \to n'\,^1S_0+\gamma$, (d) $n\,^3D_1 \to n'\,^1D_2+\gamma$, and (e) $n'\,^1D_2 \to n\,^3D_1+\gamma$. In Sec. III we use the results of Sec. II to estimate the decay widths of these transitions in the $c\bar{c}$ and in the $b\bar{b}$ systems using the potential suggested by Gupta, Radford, and Repko.[6–8] In Sec. IV we provide corrections due to coupled-channel mixing. Finally, in Sec. V we make some concluding remarks.

II. DERIVATION OF THE FORMULAS FOR THE $M1$ DECAY RATES

In order to derive the $M1$ transition rates we use a formula derived earlier[9] by one of us (K.J.S.). According to this formula, the $M1$ transition rate is given by

$$W_{BA}^{M1} = \tfrac{4}{3}k^3 \,|\, {}_I\langle A \,|\, \mathbf{y}_0+\mathbf{y}_1 \,|\, B\rangle_I \,|^2 , \tag{1}$$

where k is the wave number of the emitted photon. The ket vectors $|A\rangle_I$ and $|B\rangle_I$ are eigenstates of the internal Hamiltonian[9] h which is also the full Hamiltonian of the isolated quarkonium in its center-of-mass frame where the total momentum $\mathbf{p}$ is zero. In order to get the measured rates we have to sum over the final spins of $|A\rangle_I$ and average over the initial spins of $|B\rangle_I$. The operator $\mathbf{y}_0$ is the nonrelativistic $M1$ transition operator and $\mathbf{y}_1$ is its relativistic correction of relative order $1/c^2$. The expressions for $\mathbf{y}_0$ and $\mathbf{y}_1$ were given in Ref. 9. Since the $M1$ transitions are possible only between the singlet and the triplet states because of the charge-conjugation parities, only the spin-dependent terms in $\mathbf{y}_0$ and $\mathbf{y}_1$ need be considered. We can show that $\mathbf{W}^{(1)}$ defined in Ref. 9 has no spin-dependent part. The proof makes use of the commutation relation between $\mathbf{W}^{(1)}$ and $h^{(0)}$ derived by

Sebastian and Yun.[10] We will first consider the $D \to S$, $|\Delta L| = 2\, M1$ transitions.

A. $n\, ^1D_2 \to n'\, ^3S_1 + \gamma$

Since we are considering $|\Delta L| = 2$ spin-flip transitions the operators in $\mathbf{y}_0$ and $\mathbf{y}_1$ contributing to the matrix element should be tensor operators of rank two in the coordinate space and proportional to $(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$. Even though in $\mathbf{y}_0$ there is no such term we have to include $\mathbf{y}_0$ if the observed $n'\,^3S_1$ state is a linear combination of $^3S_1$ and $^3D_1$ states. The constant term proportional to $(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$ in $\mathbf{y}_0$ can connect between $^3D_1$ and $^1D_2$ states. The $|\Delta L| = 0$ terms in $\mathbf{y}_1$ need not be considered if we assume that the mixing coefficient of $^3D_1$ is of order v^2/c^2 (or at least small). It should be noted that we are calculating the transition amplitudes correct to relative order $1/c^2$ and $\mathbf{y}_1$ is of relative order $1/c^2$ compared to $\mathbf{y}_0$. So the contributing terms are given by

$$\mathbf{y}_{0c} = \frac{e_q}{2mc}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \,, \tag{2}$$

$$\mathbf{y}_{1c} = -\frac{ie_q k}{16m^2c^2}[\mathbf{q}\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)]\boldsymbol{\pi} + \frac{e_q k^2}{80mc}\mathbf{q}[\mathbf{q}\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)] + \frac{e_q}{16m^2c^3}[q\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)]\nabla(V_p + V_c)$$
$$+ \frac{e_q a}{4mc}\left[\tfrac{1}{20}k^2\mathbf{q}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\mathbf{q} - \frac{1}{m^2c^2}\boldsymbol{\pi}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\pi} - \frac{ik}{2mc}[q\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)]\boldsymbol{\pi}\right] + \frac{e_q}{8m^3c^3}[(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\pi}]\boldsymbol{\pi} \,. \tag{3}$$

In Eqs. (2) and (3) e_q is the electric charge of the quark, m is its mass. The symbol a in Eq. (3) is the dimensionless parameter representing any anomalous magnetic moment the quark may have. V_p and V_c are the perturbative QCD potential and the confining potential, respectively. Also,

$$\mathbf{q} = (\mathbf{r}_1 - \mathbf{r}_2)\,|_{p=0} \tag{4}$$

and $\boldsymbol{\pi}$ is the momentum canonically conjugate to $\mathbf{q}$. The last term in the expression for $\mathbf{y}_{1c}$ in Eq. (3) owes its origin to the recoil momentum of quarkonium in its final state and to the use of the relativistic relation[9,11] between the constituent and the center-of-mass (c.m.) and internal variables. We will find that this term makes a significant contribution to the rates.

In order to calculate the matrix element of $\mathbf{y}_{1c}$ and $\mathbf{y}_{0c}$ we first note that the initial and the final relativistically corrected quarkonium states can be written as

$$|\,n\,^1D_{2m}\rangle \to R_{nD}(q)Y_{2m}(\theta,\phi)\chi_{00} = \psi_{n\,^1D_{2m}} \tag{5}$$

and

$$|\,n'\,^3S_{1m'}\rangle \to R_{n'S}Y_{00}\chi_{1m'} + a_D R_D \mathcal{Y}_{211}^{m'}$$
$$= \psi_{n'\,^3S_{1m'}} + a_D\psi_{^3D_{1m'}} \,, \tag{6}$$

where $a_D R_D \mathcal{Y}_{211}^{m'}$ represents the D-state admixture which is probably small. In Eq. (6), $\mathcal{Y}_{211}^{m'}$ is the angular-spin part of the wave function which is written as $\mathcal{Y}_{LSJ}^{m_J}$. If we assume that the D-state admixture in Eq. (6) is due to the relativistic terms of order v^2/c^2 in the Hamiltonian, in first-order perturbation theory it will be given by

$$a_D^* R_D = \sum_l \frac{\langle n'\,^3S_1\,|\,V_T S_{12}\,|\,l^3D_1\rangle R_{lD}^{(0)}}{\mathrm{E}_{n'S}^{(0)} - E_{lD}^{(0)}} \,, \tag{7}$$

where $V_T S_{12}$ is the so-called tensor term in the Hamiltonian and the superscript (0) refers to nonrelativistic quantities. In the infinite sum over l in Eq. (7), in practice we need to keep only one or two terms. We should also note that the $^1D_2$ state cannot mix with any other orbital angular momentum state since the total angular momentum is to be conserved. Since the coefficient a_D is of order v^2/c^2 and since we are calculating the matrix element only to order v^2/c^2 the matrix element of $\mathbf{y}_{1c}$ need be considered only between $R_{n'S}Y_{00}\chi_{1m'}$ and $R_{nD}Y_{2m}\chi_{00}$ and that of $\mathbf{y}_{0c}$ between $a_D R_D \mathcal{Y}_{211}^{m'}$ and $R_{nD}Y_{2m}\chi_{00}$. Since one of the states involved is an S state, using the Hermitian property of the momentum operator $\boldsymbol{\pi}$ and the fact that

$$\boldsymbol{\pi}\psi_{n'\,^3S_{1m'}} = -\frac{i}{q}\frac{dR_{1S}}{dq}Y_{00}\chi_{1m'}\mathbf{q} \tag{8}$$

we can easily show that

$$(\psi_{n'\,^3S_{1m'}}, \mathbf{q}\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\boldsymbol{\pi}\psi_{n\,^1D_{2m}})$$
$$= (\psi'_{n'\,^3S_{1m'}}, \mathbf{q}\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\mathbf{q}\psi_{n\,^1D_{2m}}) \,, \tag{9}$$

where

$$\psi'_{n'\,^3S_1} = -\frac{i}{q}\frac{dR_{n'S}}{dq}Y_{00}\chi_{1m} = -iR'_{n'S}Y_{00}\chi_{1m'} \,. \tag{10}$$

Also,

$$(\psi_{n'\,^3S_{1m'}}, [(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\pi}]\boldsymbol{\pi}\psi_{n\,^1D_{2m}})$$
$$= (\psi''_{n'\,^3S_{1m'}}, [(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\mathbf{q}]\mathbf{q}\psi_{n\,^1D_{2m}}) \,, \tag{11}$$

where

$$\psi''_{n'\,^3S_{1m'}} = -\frac{1}{q}\frac{d}{dq}\left[\frac{1}{q}\frac{dR_{n'S}}{dq}\right]Y_{00}\chi_{1m'} = R''_{n'S}Y_{00}\chi_{1m'} \,. \tag{12}$$

The observed decay rate is given by

$$W^{M1}_{{}^1D_2\to{}^3S_1}=\tfrac{4}{3}k^3\tfrac{1}{3}\sum_{m,m'}|\langle n'\,{}^3S_{1m'}|\mathbf{y}|n\,{}^1D_{2m}\rangle|^2 . \quad (13)$$

At this point we note that

$$|\langle A|\mathbf{y}|B\rangle|^2=\tfrac{1}{2}\langle A|y_+|B\rangle\langle A|y_+B\rangle^* +\tfrac{1}{2}\langle A|y_-|B\rangle\langle A|y_-|B\rangle^* +\tfrac{1}{2}\langle A|y_z|B\rangle\langle A|y_z|B\rangle^* , \quad (14a)$$

where

$$y_\pm=y_x\pm iy_y . \quad (14b)$$

Using Eqs. (3) and (9)–(14) it becomes clear that the problem reduces to the calculation of

$$\langle n'\,{}^3S_{1m'}|q_r\mathbf{q}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)|n\,{}^1D_{2m}\rangle ,$$

where q_r could be $q_\pm$ or q_z. It is also useful to make the observation

$$\mathbf{q}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)=\tfrac{1}{2}(\sigma_1-\sigma_2)_+q_-+\tfrac{1}{2}(\sigma_1-\sigma_2)_-q_+ +(\sigma_1-\sigma_2)_zq_z .$$

We can then do the spin and angular parts in these matrix elements easily by making use of relations such as

$$\chi^\dagger_{1m'}\tfrac{1}{2}(\sigma_1-\sigma_2)_\pm\chi_{00}=\mp\sqrt{2}\,\delta_{m',\pm1} , \qquad \int Y^*_{00}q_-q_-Y_{2m}d\Omega=(\tfrac{8}{15})^{1/2}q^2\delta_{m2} , \quad (15)$$

and so on. We find that these matrix elements are given by the expressions

$$\langle n'\,{}^3S_{1m'}|q_\pm\mathbf{q}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)|n\,{}^1D_{2m}\rangle=\left[\pm\frac{2}{3}\frac{1}{\sqrt{10}}\delta_{m',\pm1}\delta_{m0}\pm\frac{2}{\sqrt{15}}\delta_{m',\mp1}\delta_{m,\mp2}\pm(\tfrac{2}{15})^{1/2}\delta_{m'0}\delta_{m,\mp1}\right]\times\text{radial part} , \quad (16)$$

$$\langle n'\,{}^3S_{1m'}|q_z\mathbf{q}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)|n\,{}^1D_{2m}\rangle=\left[\frac{2}{3}\frac{1}{\sqrt{5}}\delta_{m'0}\delta_{m0}+\frac{1}{\sqrt{15}}\delta_{m',-1}\delta_{m,-1}+\frac{1}{\sqrt{15}}\delta_{m'1}\delta_{m1}\right]\times\text{radial part} . \quad (17)$$

Usings Eqs. (2), (3), (9)–(12), (14), (16), and (17) we can express Eq. (13) entirely in terms of radial integrals. Our final result for the decay rates are given by

$$W^{M1}_{{}^1D_2\to{}^3S_1}=\frac{1}{90}\alpha\left[\frac{e_q}{e}\right]^2\times\left[\frac{\omega}{mc^2}\right]^2\omega|J_0+J_1+J_2+J_3+J_4|^2 . \quad (18)$$

In Eq. (18) J_0 originates from the matrix element of $\mathbf{y}_0$ between the wave functions $R_{nD}Y_{2m}\chi_{00}$ and $a_DR^{m'}_{D211}$ and it is given by

$$J_0=\sqrt{72}\,a^*_D\int R_DR_{nD}q^2dq . \quad (19a)$$

The other dimensionless integrals $J_i(i=1,2,3,4)$ are given by

$$J_1=\frac{k^2}{10}(1+a)\int_0^\infty R_{n'S}R_{nD}q^4dq , \quad (19b)$$

$$J_2=\frac{1}{2mc}(1+2a)\int_0^\infty R'_{n'S}R_{nD}q^4dq , \quad (19c)$$

$$J_3=\frac{1}{m^2c^2}(1-2a)\int_0^\infty R''_{n'S}R_{nD}q^4dq , \quad (19d)$$

$$J_4=\frac{1}{2mc^2}\int_0^\infty R_{n'S}\frac{1}{q}\frac{\partial}{\partial q}(V_p+V_c)R_{nD}q^4dq , \quad (19e)$$

where

$$R'_{n'S}=\frac{1}{q}\frac{dR_{n'S}}{dq},\quad R''_{n'S}=-\frac{1}{q}\frac{d}{dq}\left[\frac{1}{q}\frac{dR_{n',S}}{dq}\right] .$$

The integral J_3 when $a=0$ gives the recoil contribution[3,9] to the $M1$ amplitude. Later we will see that it makes a significant contribution to the $1\,{}^1D_2\to n\,{}^3S_1$ ($n=1,2$) $M1$ decay rates in charmonium.

Next we consider the related $M1$ decays $n'\,{}^3S_1\to n\,{}^1D_2+\gamma$ and $n\,{}^3D_1\to n'\,{}^1S_0+\gamma$ of quarkonia.

B. $n'\,{}^3S_1\to n\,{}^1D_2+\gamma$

The decay rate for this transition can be obtained from the results of Sec. II A by the following simple observations:

$$\langle n\,{}^1D_{2m}|\mathbf{y}_c|n'\,{}^3S_{1m'}\rangle^*=\langle n'\,{}^3S_{1m'}|y_c^\dagger|n\,{}^1D_{2m}\rangle , \quad (20)$$

$$|\langle n\,{}^1D_{2m}|\mathbf{y}_c|n'\,{}^3S_{1m'}\rangle^*|^2 =|\langle n\,{}^1D_{2m}|\mathbf{y}_c|n'\,{}^3S_{1m'}\rangle|^2 , \quad (21)$$

and that $\mathbf{y}_c$ is Hermitian except for one term corresponding to the integral J_2. In fact we find

$$W^{M1}_{{}^3S_1\to{}^1D_2}=\tfrac{5}{3}\times\tfrac{1}{90}\alpha\left[\frac{e_q}{e}\right]^2\left[\frac{\omega}{mc^2}\right]^2\times\omega|J_0+J_1-J_2+J_3+J_4|^2 . \quad (22)$$

The extra factor $\frac{5}{3}$ in Eq. (22) comes from the fact that the final state ${}^1D_2$ now has five spin states instead of three for ${}^3D_1$ in the previous case.

C. $n\,{}^3D_1\to n'\,{}^1S_0+\gamma$

Here the state $n'\,{}^1S_0$ cannot mix with any other orbital angular momentum, but the state $n\,{}^3D_1$ can. In fact we expect

$$|n\,{}^3D_1\rangle \to R_{nD}\mathcal{Y}_{211}+C_S R_S \mathcal{Y}_{011}\ , \tag{23}$$

where the coefficient c_S is expected to be small. In the usual potential models it is of order v^2/c^2, and comes from the so-called tensor term in the Hamiltonian. In first-order perturbation theory $C_S R_S$ will be given by, just as in Eq. (7),

$$C_S R_S=\sum_l \frac{{}_0\langle l\,{}^3S_1 | V_T S_{12} | n\,{}^3D_1\rangle_0 R_{ls}^{(0)}}{E_{nD}^{(0)}-E_{ls}^{(0)}}\ . \tag{24}$$

In practice in the infinite sum over l only one or two terms need be considered. By doing a calculation similar to Sec. II A we find

$$W^{M1}_{{}^3D_1\to{}^1S_0}=\frac{1}{54}\left[\frac{e_q}{e}\right]^2\alpha\left[\frac{\omega}{mc^2}\right]^2 \times\omega\,|J_0'+J_1+J_2+J_3+J_4|^2\ , \tag{25}$$

where J_1, J_2, J_3, and J_4 are given by Eqs. (19) and J_0' is defined as

$$J_0'=\sqrt{72}\,C_S\int_0^\infty R_S R_{n'S}q^2dq\ . \tag{26}$$

In Eqs. (22) and (25) J_3 gives the recoil contribution.

Next we turn to the $D\to D$ $M1$ decays of quarkonia. We will consider the decays $n\,{}^3D_1\to n'\,{}^1D_2+\gamma$ and $n'\,{}^1D_2\to n\,{}^3D_1+\gamma$.

D. $n\,{}^3D_1\to n'\,{}^1D_2+\gamma$

Since this is a $|\Delta L|=0$, $D\to D$ spin-flip $M1$ transition, both the scalar and the tensor terms in coordinate space and proportional to $(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)$ in the expression for $\mathbf{y}_0$ and $\mathbf{y}_1$ given in Ref. 9 will contribute. In addition to the terms given in Eqs. (2) and (3) for $\mathbf{y}_0$ and $\mathbf{y}_1$ the following additional terms will contribute to $\mathbf{y}_1$ which we will write as $\mathbf{y}_{1A}$:

$$\mathbf{y}_{1A}=\frac{e_q}{4mc}\left[\frac{2k}{mc}(1+a)-3\frac{\pi^2}{m^2c^2}\right](\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)+\frac{ie_qk}{8m^2c^2}(1+2a)(\mathbf{q}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)-\frac{e_qk^2q^2}{20mc}(1+a)(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)$$
$$-\frac{e_q}{8m^2c^3}q\frac{\partial(V_p+V_c)}{\partial q}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)-\frac{e_q}{2m^2c^2}V_s(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\ . \tag{27}$$

The contributing $\mathbf{y}_0$ and $\mathbf{y}_1$ will now be given as

$$\mathbf{y}_0=\mathbf{y}_{0c} \tag{28}$$

and

$$\mathbf{y}_1=\mathbf{y}_{1c}+\mathbf{y}_{1A}\ , \tag{29}$$

where $\mathbf{y}_{0c}$ and $\mathbf{y}_{1c}$ were given earlier by Eqs. (2) and (3). The observed decay rates will be given by

$$W^{M1}_{{}^3D_1\to{}^1D_2}=\tfrac{4}{3}k^3\tfrac{1}{3}\sum_{m,m'}|\langle n'\,{}^1D_{2m'}|\mathbf{y}_0+\mathbf{y}_1|n\,{}^3D_{1m}\rangle|^2$$
$$=\tfrac{4}{3}k^3\sum_{m'}|\langle n'\,{}^1D_{2m'}|\mathbf{y}_0+\mathbf{y}_1|n\,{}^3D_{11}|^2\ . \tag{30}$$

When we use Eqs. (14) in Eq. (30) it becomes a calculation of

$$\langle n'\,{}^2D_{2m'}|y_\pm|n\,{}^3D_{11}\rangle$$

and

$$\langle n'\,{}^1D_{2m'}|y_z|n\,{}^3D_{11}\rangle\ .$$

Using our expressions for $\mathbf{y}$ in Eqs. (28), (29), (27), (2), and (3) and going through a rather long but straightforward calculation similar to the one in Sec. II A (but more difficult because both the initial and the final states are D states) we finally obtain

$$W^{M1}_{{}^3D_1\to{}^1D_2}=\frac{4}{3}\left[\frac{e_q}{e}\right]^2\alpha\left[\frac{\omega}{mc^2}\right]^2\omega\,|L_1+L_2+L_3+L_4+L_5+L_6+L_6'|^2\ , \tag{31}$$

where the L's are dimensionless integrals given by the expressions

$$\begin{aligned}
L_1&=\left[1+\frac{k}{8mc}(5+6a)\right]\int_0^\infty R_{n'D}R_{nD}q^2dq\ ,\\
L_2&=-\frac{k^2}{30}(1+a)\int_0^\infty R_{n'D}R_{nD}q^4dq\ ,\\
L_3&=\frac{k}{24mc}(1+2a)\int_0^\infty R_{n'D}\frac{\partial R_{nD}}{\partial q}q^3dq\ ,\\
L_4&=-\frac{1}{24mc^2}\int_0^\infty R_{n'D}\frac{\partial(V_p+V_c)}{\partial q}R_{nD}q^3dq\ ,\\
L_5&=-\frac{1}{2mc^2}\int_0^\infty R_{n'D}V_sR_{nD}q^2dq\ ,\\
L_6&=-\frac{1}{2m^2c^2}\left[1+\frac{2}{3}a\right]\left[\int_0^\infty\frac{\partial R_{n'D}}{\partial q}\frac{\partial R_{nD}}{\partial q}q^2dq+6\int_0^\infty R_{n'D}R_{nD}dq\right]\ ,\\
L_6'&=-\frac{1}{12m^2c^2}\left[\int_0^\infty\frac{\partial R_{n'D}}{\partial q}\frac{\partial R_{nD}}{\partial q}q^2dq+6\int_0^\infty R_{n'D}R_{nD}dq\right]\ .
\end{aligned} \tag{32}$$

The integral L_6' has been written separately, just to show that it is the contribution due to the recoil of the quarkonium, the so-called "recoil term."[3,9] The integrals L_2 to L_6' have coefficients which are of order v^2/c^2 and so in doing their evaluation we can use the nonrelativistic radi-

TABLE I. $D \to S$ $M1$ transition rates of charmonium.

Decay	Photon energy (GeV)	Dimensionless integrals J_0	J_1	J_2	J_3	J_4	$\sum J_i$	Predicted decay rate (keV)
$1\,^1D_2 \to 1\,^3S_1$	0.643	−0.2294	0.2738	−0.4056	−0.4266	0.1646	−0.6232	2.13
$1\,^1D_2 \to 2\,^3S_1$	0.129	−0.3159	−0.0294	0.0358	−0.3938	−0.2600	−0.9634	0.041
$1\,^3D_1 \to 1\,^1S_0$	0.697	0.2294	0.3219	−0.4399	−0.4266	0.1646	−0.1506	0.265
$1\,^3D_1 \to 2\,^1S_0$	0.171	0.3160	−0.0515	0.0473	0.3938	−0.2600	−0.3420	0.020

al wave functions. In the evaluation of L_1 we have to be more careful. It comes from the matrix element of (constant) $(\sigma_1-\sigma_2)$ between $n\,^3D_1$ and $n'\,^1D_2$ states. Since part of the constant is of order 1 we must calculate this radial integral to order v^2/c^2. Writing the Hamiltonian of the isolated quarkonium (without any interaction with the radiation field)

$$H=H_{SI}+U_{SS}(\mathbf{q})\mathbf{S}_1\cdot\mathbf{S}_2+U_{LS}(\mathbf{q})\mathbf{L}\cdot\mathbf{S}+U_T(\mathbf{q})S_{12}\ , \tag{33}$$

where S_{12} is the tensor operator defined by

$$S_{12}=4\left[3\frac{(\mathbf{S}_1\cdot\mathbf{q})(\mathbf{S}_2\cdot\mathbf{q})}{\mathbf{q}^2}-\mathbf{S}_1\cdot\mathbf{S}_2\right]\ . \tag{34}$$

We next note that the states $n\,^3D_{1m}$ and $n'\,^1D_{2m'}$ can be written as

$$|n\,^3D_{1m}\rangle \to R^t_{nD}\mathcal{Y}^m_{211}+a_S R_S \mathcal{Y}^m_{011}\ , \tag{35}$$

$$|n'\,^1D_{2m'}\rangle \to R^s_{n'D}\mathcal{Y}^{m'}_{202}\ . \tag{36}$$

It should be noted that when we take into account relativistic correction terms in the Hamiltonian we have to distinguish between the singlet $(R^s_{n'D})$ and the triplet (R^t_{nD}) radial wave functions since the differential equations obeyed by them are slightly different because of those relativistic terms in the Hamiltonian. Writing the energy-eigenvalue equations for the states in Eqs. (35) and (36) and using the Hamiltonian of Eq. (33) we get the integral in L_1 as

$$(R^s_{n'D},R^t_{nD})=\int_0^\infty R_{n'D}R_{nD}q^2dq=\frac{1}{(E^t_n-E^s_{n'})}\int_0^\infty R^s_{n'D}(U_{SS}-2U_T-3U_{LS})R^t_{nD}q^2dq+\sqrt{8}\,a_S\frac{\int_0^\infty R^s_{n'D}U_T R_S q^2dq}{E^t_n-E^s_{n'}}\ . \tag{37}$$

When $n\neq n'$ the second term on the right-hand side is of order v^4/c^4 and can be dropped. Then the radial integral is entirely given by the first term where we can use nonrelativistic radial wave functions since U_{SS}, U_T, and U_{LS} are of order v^2/c^2. When $n=n'$, the radial integral will be 1 to order v^2/c^2.

Next we turn to the related $M1$ decay $n'\,^1D_2 \to n\,^3D_1+\gamma$.

E. $n'\,^1D_2 \to n\,^3D_1+\gamma$

The decay rate for this transition can be obtained from the result of Sec. II D by making the simple observations

$$\langle n\,^3D_{1m}|\mathbf{y}|n'\,^1D_{2m'}\rangle^*=\langle n'\,^1D_{2m'}|\mathbf{y}^\dagger|n\,^3D_{1m}\rangle$$

and that

$$|\langle n\,^3D_{1m}|\mathbf{y}|n'\,^1D_{2m'}\rangle^*|^2 = |\langle n\,^3D_{1m}|\mathbf{y}|n'\,^1D_{2m'}\rangle|^2\ .$$

We obtain

$$W^{M1}_{^1D_2\to{}^3D_1} =\frac{4}{5}\left[\frac{e_q}{e}\right]^2\alpha\left[\frac{\omega}{mc^2}\right]^2\omega\,|L_1'+L_2+L_3'+L_4+L_5+L_6+L_6'|^2\ , \tag{38}$$

where

TABLE II. $D\to D$ $M1$ transition rates of charmonium.

$$W_{^1D_2\to{}^3D_1}=\tfrac{4}{5}\alpha\left[\frac{e_q}{e}\right]^2\frac{\omega^3}{mc^2}|L_1'+L_2+L_3'+L_4+L_5+L_6+L_6'|^2$$

Decay	Photon energy (GeV)	Dimensionless integrals L_1'	L_2	L_3'	L_4	L_5	L_6+L_6'	$\sum L_i$	Predicted rate (keV)
$1\,^1D_2 \to 1\,^3D_1$	0.0189	1.002	−0.000	−0.005	−0.026	−0.428	−0.222	−0.322	0.001

TABLE III. $D \leftrightarrow S$ $M1$ transition rates in the $b\bar{b}$ system.

Decay	Photon energy (GeV)	Dimensionless integrals J_0	J_1	J_2	J_3	J_4	$\sum J_i$	Predicted rate (keV)
$1\,^1D_2 \to 1\,^3S_1$	0.667	−0.0611	0.0836	−0.1051	−0.3413	0.0365	−0.3873	0.0175
$1\,^1D_2 \to 2\,^3S_1$	0.126	−0.511	−0.0097	0.0102	−0.2719	−0.0622	−0.3847	0.000 12
$2\,^1D_2 \to 1\,^3S_1$	0.941	−0.1078	0.0402	−0.0627	−0.2236	0.0249	−0.3290	0.0355
$2\,^1D_2 \to 2\,^3S_1$	0.415	−0.0874	0.0823	−0.1127	−0.4234	0.0224	−0.5187	0.0076
$2\,^1D_2 \to 3\,^3S_1$	0.092	−0.1654	−0.0187	0.0259	−0.3341	−0.1198	−0.6122	0.000 11
$3\,^3S_1 \to 1\,^1D_2$	0.203	0.0025	0.0067	0.0379	−0.0639	0.0029	−0.0902	0.000 04
$1\,^3D_1 \to 1\,^1S_0$	0.702	0.0527	0.0927	−0.1107	−0.3413	0.0365	−0.2700	0.0166
$1\,^3D_1 \to 2\,^1S_0$	0.158	0.0441	−0.0151	0.0127	−0.2719	−0.0622	−0.2924	0.0002
$2\,^3D_1 \to 1\,^1S_0$	0.975	0.0447	0.0439	−0.0650	−0.2236	0.0249	−0.1758	0.0188
$2\,^3D_1 \to 2\,^1S_0$	0.446	0.0359	0.0949	−0.1210	−0.4234	0.0224	−0.3911	0.0089

$$L_1' = \left[1 + \frac{1}{8}\frac{k}{mc}(1-2a)\right] \int_0^\infty R_{n'D} R_{nD} q^2 dq \;, \quad (39)$$

$$L_3' = \frac{5}{24}\frac{k}{mc}(1+a) \int_0^\infty R_{n'D} \frac{\partial R_{nD}}{\partial q} q^3 dq \;, \quad (40)$$

and all the other L_i's are given by Eq. (32). The extra factor of $\frac{3}{5}$ comes because the final state $n\,^3D_1$ has only three spin states where as the previous final state $n'\,^1D_2$ had five spin states.

III. NUMERICAL EVALUATIONS OF THE $M1$ DECAY RATES IN THE $c\bar{c}$ AND THE $b\bar{b}$ SYSTEMS

We have used the potential proposed by Gupta, Radford, and Repko[6–8] to calculate the radial wave functions and thus the $M1$ decay rates involving the D states of both charmonium and bottomonium. We have used the same parameters as they have since it gives the energy spectra including their fine structures for both $c\bar{c}$ and $b\bar{b}$ systems in excellent agreement with experiment. We also used the variational method proposed by them to solve the semirelativistic Schrödinger equation and chose the same form for wave functions:

$$\psi_{nl}^m(\mathbf{r}) = \sum_{k=0}^{K} a_{L,nl} \left[\frac{r}{R}\right]^L e^{-r/0} Y_l^m(\Omega_r), \quad L = k + l \;,$$

where the value for K is from 7 to 11 and the value of R is usually in the range 0.4–1.0 GeV^{-1}. The results of our calculations are given in Tables I–IV. In Tables V and VI we also give the predicted energy spectra of the D states in the GRR model. For the $c\bar{c}$ system we have only calculated the rates for the $^1D_2 \to 2\,^3S_1$ and the $^1D_2 \to 1\,^3S_1$ $M1$ transitions since these may be the only D-state $M1$ transitions with observable branching ratios. We also like to point out that we have put $a=0$ in our formulas to make the numerical estimates.

IV. COUPLED-CHANNEL MIXING EFFECT ON THE $M1$ DECAY RATES

In addition to relativistic corrections, coupled-channel mixing could also have an effect on the $M1$ decay rates. A bound state such as $c\bar{c}$ or $b\bar{b}$ can make a transition to a pair of heavy-light mesons ($Q\bar{q}, \bar{Q}q$), which then make a transition either to the same or to a different $Q\bar{Q}$ (for example, $c\bar{c}$) state. Consequently different $Q\bar{Q}$ states can get mixed. The values of the mixing coefficients mainly depend on the light-quark pair ($q\bar{q}$) creation mechanism. We have used the flux-tube-breaking model for the pair creation mechanism,[12,13] and find that coupled-channel mixing has little effect on the D-state $M1$ decay rates.

Let us examine the decays in Table I. For $1\,^1D_2 \to 1\,^3S_1$, since the coupled-channel effect can only mix states of $c\bar{c}$ with the same J^{PC}, where J is the total angular momentum of the $c\bar{c}$ state, P is the parity, and C is the charge conjugation of the state, we only need to consider the admixture of $1\,^3S_1$ with $2\,^3S_1$ and $1\,^3D_1$. The mixing coefficients (using the method in Ref. 13) are listed in Table VII. Using the coefficients in Table VII we find

$$|\psi\rangle = 0.998\,|\,^3S_1\rangle + 0.057\,|\,2\,^3S_1\rangle + 0.002\,|\,1\,^3D_1\rangle \;. \quad (41)$$

The admixture in (41) will modify the decay rate from

TABLE IV. $D \to D$ $M1$ transition rates in the $b\bar{b}$ system.

Decay	Photon energy (GeV)	Dimensionless integrals L_1	L_2	L_3	L_4	L_5	$L_6 + L_6'$	$\sum L_i$	Predicted decay rate (keV)
$1\,^3D_1 \to 1\,^1D_2$	0.012	1.002	−0.000	−0.000	−0.006	−0.062	−0.046	0.887	0.000 07
$2\,^1D_2 \to 1\,^3D_1$	0.294	−0.024	−0.015	−0.034	−0.002	−0.016	−0.030	−0.119	0.0102
$2\,^1D_2 \to 2\,^3D_1$	0.006	1.000	−0.000	−0.001	−0.020	−0.222	−0.063	0.695	0.000 03
$2\,^3D_1 \to 1\,^1D_2$	0.299	−0.024	−0.016	−0.007	−0.002	−0.016	−0.030	−0.094	0.011

TABLE V. Predicted energy levels of the $c\bar{c}$ system in the GRR model. Parameters used: $m_c=1.32$ GeV; $\mu=1.94$ GeV; $\alpha_s=0.36$; $k=0.15$ GeV^2; $c=0.392$ GeV.

State	Predicted energy (MeV)	Experiment (Ref. 17) (MeV)
$1\,^3S_1$ (ψ)	3097	3096.93±0.09
$1\,^1S_0$ (η_c)	2982	2980.6 ±1.5
$1\,^3P_0$ (χ_{c0})	3413	3414.9 ±1.1
$1\,^3P_1$ (χ_{c1})	3510	3510.67±0.51
$1\,^3P_2$ (χ_{c2})	3560	3556.31±0.42
$1\,^1P_1$	3529	
$2\,^3S_1$	3684	3686.00+0.10
$2\,^1S_0$	3590	3594 ±0.5
$1\,^3D_1$	3801	
$1\,^3D_2$	3822	
$1\,^3D_3$	3830	
$1\,^1D_2$	3822	

2.13 keV to 2.14 keV. For the decay $1\,^1D_2 \rightarrow 2\,^3S_1$, using the same method, we find the rate changes from 0.041 to 0.039 keV. For all other $M1$ decay rates we found no significant change.

V. CONCLUDING REMARKS

From Table I we see that the $M1$ decay $1\,^1D_2 \rightarrow 1\,^3S_1$ for the $c\bar{c}$ system has a significant decay rate of about 2.13 keV. This should be compared with the $E1$ decay rate of 661 keV for $1\,^1D_2 \rightarrow 1\,^1P_1$ of charmonium, which we calculated using the GRR model and formulas previously given.[14] So this decay may have a measurable branching ratio. Also the $M1$ decay $1\,^3D_1 \rightarrow 1\,^1S_0$ in the $c\bar{c}$ system has a rate of about 0.265 keV. But since this is a broad resonance its branching ratio is probably too small for observation. A glance at Tables III and IV shows that all the $M1$ decays involving the D states of the $b\bar{b}$ system have widths of the order of a few eV or less. It suggests that the $M1$ decays in the $b\bar{b}$ systems may be difficult to detect. This may be the reason why the e^+e^- experiments to date have not been successful in locating the singlet S states in the $b\bar{b}$ system. One may have to wait for the direct formation of the singlet states in $\bar{p}p$ collisions and the $E1$ decays between them to establish their existence and to measure their energies. In Tables V and VI we give the predicted and the experimental energy spectra of the $c\bar{c}$ and of the $b\bar{b}$ systems in the GRR model including the relevant D-state energies.

TABLE VI. Predicted energy levels of the $b\bar{b}$ system in the GRR model. Parameters used: $m_b=4.78$ GeV; $\mu=3.65$ GeV; $k=0.18$ GeV^2; $\alpha_s=0.28$; $c=0.079$ GeV.

State	Predicted energy (MeV)	Experiment (Ref. 17) (MeV)
$1\,^3S_1$	9 460	9460.03±0.19
$1\,^1S_0$	9 415	
$1\,^3P_0$	9 866	9859.8 ±1.3
$1\,^3P_1$	9 890	9891.89±0.68
$1\,^3P_2$	9 906	9913.29±0.63
$1\,^1P_1$	9 897	
$2\,^3S_1$	10 011	10 023.37±0.34
$2\,^1S_0$	9 984	
$1\,^1D_2$	10 150	
$1\,^3D_1$	10 143	
$1\,^3D_2$	10 149	
$1\,^3D_3$	10 153	
$2\,^3P_0$	10 230	10 232.7 ±0.5
$2\,^3P_1$	10 252	10 255.3 ±1.7
$2\,^3P_2$	10 265	10 271.1 ±1.7
$2\,^1P_1$	10 257	
$3\,^3S_1$	10 355	10 355.5 ±0.5
$3\,^1S_0$	10 333	
$2\,^3D_1$	10 440	
$2\,^3D_2$	10 446	
$2\,^3D_3$	10 451	
$2\,^1D_2$	10 447	

We also find that the so-called "recoil term"[3,9] in the $M1$ decay amplitude is quite significant in all decays and especially the $D \leftrightarrow S$ $M1$ decays. For example, had we neglected the recoil contribution, namely, J_3, to the $1\,^1D_2 \rightarrow 1\,^3S_1$ $M1$ decay amplitude in charmonium we would have obtained the corresponding transition rate about ten times smaller and the same procedure in $1\,^1D_2 \rightarrow 2\,^3S_1$ in charmonium would have resulted in a

TABLE VII. Mixing coefficients of $c\bar{c}$ states due to coupled-channel effect.

State		$1\,^3S_1$	$2\,^3S_1$	$1\,^3D_1$
ψ	Re	0.998	0.057	0.002
	Im	0	0	0
ψ'	Re	0.005	0.982	0.017
	Im	0	0	0
$\psi(3770)$	Re	0.002	−0.032	0.928
	Im	−0.004	0.047	−0.358
		$1\,^1S_0$	$2\,^1S_0$	
η_c	Re	0.999	0.049	
	Im	0	0	
η_c'	Re	0.005	0.982	
	Im	0	0	

transition rate about three times smaller. The effect of the recoil term J_3 is even more pronounced in the $D \to S$ $M1$ decays of the $b\bar{b}$ systems. We find that the coupled-channel mixing has little effect on the D-state $M1$ decay rates.

Next, a word about the V_s term, the so-called "pair-creation" term in the $M1$ decay amplitude. This term is present in the $M1$ decay amplitude if we start from the so-called Fermi-Breit covariant equation which includes the scalar and the vector potentials, make the minimal coupling to the electromagnetic field in that equation, and then take the Barker-Glover reduction to order v^2/c^2. It is also present if we include the quark-antiquark pair creation vertex and a photon emission vertex in a scalar exchange graph in a field-theoretic formalism. None of these arguments conclusively prove there is such a term in the $M1$ decay amplitude for the following reasons. First, the covariant two-particle equation *in the presence of an external field* has no proper bound-state solutions because of the problem of the so-called "continuum dissolution."[15] So it is incorrect to start from such an equation. Second, no single scalar-meson exchange can give rise to the observed confining potential (say, linear) in quarkonium. So it is not clear how this scalar pair-creation graph comes about. It is quite probable many higher-order QCD graphs involving virtual gluons and quarks conspire to produce a $q\bar{q}$ scattering amplitude which simulates a scalar exchange graph. In that case we should attach an external photon line to every charged-particle line including the internal quark lines and not just the external quark lines to get the two-body equation in the presence of the external electromagnetic (EM) field. While there are only four external quark lines there are an arbitrarily large number of internal quark lines. Under those circumstances it is not clear whether the "V_s term" will survive in the $M1$ transition operator. On the other hand, if you start from an approximately relativistic Hamiltonian[9] correct to order v^2/c^2 and couple it to an external EM field by means of two principles,[9] namely, (1) minimal gauge invariance and (2) when the internal interaction goes to zero, the external EM interaction is a sum of simple one-particle Dirac Hamiltonians, one does not get the "$V_s \sigma \cdot \mathbf{B}$" term in the interaction Hamiltonian. The upshot of all this is that there is some question whether or not the V_s term in the $M1$ transition operator is present. Now the interesting thing is that in the $D \to S$ $M1$ transitions, the V_s term, even if it exists, will not contribute to the decay amplitude, since V_s is a scalar in coordinate space and cannot contribute to a $\Delta L \neq 0$ transition. So if we see better agreement for $D \to S$ transition than for $S \to S$ transition rate (with V_s term present) it will be an indication that the V_s term may be absent in the $M1$ transition operator. We should also mention that the $S \to S$ $M1$ decay rates in charmonium come out much better[16] when we discard this term in the amplitude and include coupled-channel mixing effects.

Another interesting point to note from Tables I–IV is that in the $D \to S$ $M1$ transitions the single most important integral is J_3, which comes purely from the recoil of the composite system. The use of the relativistic center-of-mass variables[11] was important in deriving[3,9] this part of the $M1$ decay amplitude. This particular term did not appear in early treatments of $M1$ decays. The size of this term relative to other terms in the $D \to S$ $M1$ decay amplitudes suggests they may also be of crucial importance in the $D \leftrightarrow S$ $M1$ transitions of positronium.

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