Diffusion of charmed quarks in the quark-gluon plasma

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We calculate the classical drag and diffusion coefficients for a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when the plasma is hot will be stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to *increase* the yield of J/ψ mesons in relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate the dynamics of a $c\bar{c}$ pair.

I. INTRODUCTION

In the effort to create a quark-gluon plasma (QGP) in nuclear collisions,[1] a persistent problem is the scarcity of signatures that would indicate that a plasma has indeed been formed. A recent suggestion[2] is to expect suppression of J/ψ production, basically because the outstanding feature of the QGP is the absence of quark confinement. Any charmed-quark pair formed by either a hard or a soft process has only a screened Coulomb potential to hold it together, and at a high enough temperature, that potential is supposed to be too short-ranged to permit a bound state.

The fate of a $c\bar{c}$ pair created in the quark plasma depends on a variety of factors. For instance, recombination will occur if there is a significant density of charmed quarks created in the initial stages of the collision; this can perhaps be studied adequately with simple statistical methods.[3] A more interesting question is whether a *single* $c\bar{c}$ pair will somehow stay together long enough to form a J/ψ at hadronization. This question requires study of the dynamics by which a heavy quark propagates in the QGP.

In this paper I present a study of the drag and diffusion forces which act on a charmed quark in the QGP. My conclusion is rather a startling one: even though the charmed quark is quite heavy on the scale of the plasma temperature, drag is very strong and diffusion is very fast. The classical analysis implies that a charmed quark created in the early stages of a nuclear collision, whatever its momentum, will be stopped before it traverses 1 fm and will undergo diffusion thereafter, with a violence which *increases* as the plasma cools. The implications for a $c\bar{c}$ pair are anything but obvious; a pair formed with high relative momentum will come to rest before attaining a great separation, but the subsequent Brownian motion could well overwhelm the potential that binds them together and prevent recombination. The strong drag would act to *increase* the yield of J/ψ particles at the expense of $D\bar{D}$ pairs above threshold; the rapid diffusion would act in the opposite direction.

The problem of the motion of a charmed quark in the QGP will look familiar to anyone who has looked at the classic "test-particle problem" in plasma physics.[4] That problem in turn traces its ancestry to the problem of Brownian motion, which was studied by the classical physicists around the turn of the century. The starting point is the Boltzmann equation for the density $f(\mathbf{x},\mathbf{p},t)$ of charmed quarks in phase space:

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{E}\cdot\frac{\partial}{\partial \mathbf{x}}+\mathbf{F}\cdot\frac{\partial}{\partial \mathbf{p}}\right]f(\mathbf{x},\mathbf{p},t)=\left[\frac{\partial f}{\partial t}\right]_{\text{collisions}} . \tag{1.1}$$

$\mathbf{F}$ represents external forces acting on the charmed quark, such as the interaction with large concentrations of color charge; the right-hand side contains interactions with nearby plasma particles, gluons and light quarks and antiquarks, which are taken to be significant only in the course of a collision of short duration. In principle, one could use (1.1) to solve for the evolution of the charmed-quark distribution under the influence of the $c\bar{c}$ potential by inserting the latter into $\mathbf{F}$. We will, however, solve a simpler problem: We will neglect all interactions with other heavy quarks and with background color fields, and set $\mathbf{F}=0$. (This is the opposite of the collisionless Vlasov approximation considered in conventional plasma physics.[4]) Further assuming that the plasma is uniform, that is, that the distribution functions of light particles appearing in the right-hand side of (1.1) are $\mathbf{x}$ independent, we can average (1.1) over $\mathbf{x}$. Defining

$$f(\mathbf{p},t)=\frac{1}{V}\int d^3\mathbf{x}\, f(\mathbf{x},\mathbf{p},t) , \tag{1.2}$$

which is the normalized probability distribution in momentum space, we have

$$\frac{\partial}{\partial t}f(\mathbf{p},t)=\left[\frac{\partial f}{\partial t}\right]_{\text{collisions}} . \tag{1.3}$$

In the absence of external forces, all the variation of f with time is due to collisions.

The right-hand side of (1.3) is a linear integral operator acting on f. An approximation due to Landau[5] is to allow only soft scattering in the collision integral, which turns the integral into a differential operator, so that (1.3) becomes a Fokker-Planck equation. We review this formalism in Sec. II, showing how the collision terms may be interpreted as due to drag forces and diffusion induced by random collisions. We evaluate the drag and diffusion coefficients in Sec. III and present numerical results in Sec. IV. Appendix A contains background material about Brownian motion and the Langevin and Fokker-Planck equations.[6] Appendixes B and C contain details of the calculation of the collision integral.

II. FORMALISM

As discussed in the Introduction, our starting point is the Boltzmann equation (1.3), derived in the absence of external forces. Defining $w(\mathbf{p},\mathbf{k})$ to be the rate of collisions which change the momentum of the charmed quark from $\mathbf{p}$ to $\mathbf{p}-\mathbf{k}$, we have

$$R(\mathbf{p},t)\equiv\left[\frac{\partial f}{\partial t}\right]_{\text{collisions}} = \int d^3k[w(\mathbf{p}+\mathbf{k},\mathbf{k})f(\mathbf{p}+\mathbf{k})-w(\mathbf{p},\mathbf{k})f(\mathbf{p})] \; . \tag{2.1}$$

The first term in the integrand represents gain of probability through collisions which knock the charmed quark into the element of momentum space at $\mathbf{p}$, and the second term represents loss out of that element. w is a sum of contributions from gluon scattering and from light-quark and antiquark scattering:

$$w(\mathbf{p},\mathbf{k})=w^g(\mathbf{p},\mathbf{k})+w^q(\mathbf{p},\mathbf{k})+w^{\bar q}(\mathbf{p},\mathbf{k}) \; . \tag{2.2}$$

w^g is given by an integral over the momentum of the incident gluon:

$$w^g(\mathbf{p},\mathbf{k})=\gamma_g\int\frac{d^3\mathbf{q}}{(2\pi)^3}\hat{f}_g(\mathbf{q})v_{\mathbf{q},\mathbf{p}}\sigma^g_{\mathbf{p},\mathbf{q}\to\mathbf{p}-\mathbf{k},\mathbf{q}+\mathbf{k}} \; , \tag{2.3}$$

where $\hat{f}_g$ is the gluon distribution in phase space (assumed to be position and time independent) and $v_{\mathbf{q},\mathbf{p}}\equiv|\mathbf{v_q}-\mathbf{v_p}|$ is the relative velocity. The degeneracy factor for gluons $\gamma_g=2\times 8$ appears because we must sum over spin and color of the incident gluon. σ^g is the differential cross section, as usual summed over the spin and color of the final particles and averaged over those of the incident particles. It is given by[7]

$$\sigma^g_{\mathbf{p},\mathbf{q}\to\mathbf{p}-\mathbf{k},\mathbf{q}+\mathbf{k}}=\frac{1}{(2\pi)^6}\frac{1}{v_{\mathbf{q},\mathbf{p}}}\frac{1}{2E_\mathbf{q}}\frac{1}{2E_\mathbf{p}}\frac{1}{\gamma_g\gamma_c}\sum|\mathcal{M}_{gc}|^2\frac{1}{2E_{\mathbf{q}+\mathbf{k}}}\frac{1}{2E_{\mathbf{p}-\mathbf{k}}}(2\pi)^4\delta(E_\mathbf{p}+E_\mathbf{q}-E_{\mathbf{p}-\mathbf{k}}-E_{\mathbf{q}+\mathbf{k}}) \; . \tag{2.4}$$

(w^q and $w^{\bar q}$, which are obviously equal, are given by similar expressions; since w^g and w^q are to be treated exactly alike, we will discuss only w^g and henceforth drop the superscript.) Combining (2.1)–(2.4), one gets the familiar expression[8]

$$R=\frac{1}{2E_\mathbf{p}}\int\frac{d^3\mathbf{q}}{(2\pi)^3 2E_\mathbf{q}}\int\frac{d^3\mathbf{q}'}{(2\pi)^3 2E_{\mathbf{q}'}}\int\frac{d^3\mathbf{p}'}{(2\pi)^3 2E_{\mathbf{p}'}}\frac{1}{\gamma_c}\sum|\mathcal{M}|^2(2\pi)^4\delta^4(p+q-p'-q')[f(\mathbf{p}')\hat{f}(\mathbf{q}')-f(\mathbf{p})\hat{f}(\mathbf{q})] \; , \tag{2.5}$$

where we have let $\mathbf{p}'=\mathbf{p}-\mathbf{k}$ and $\mathbf{q}'=\mathbf{q}+\mathbf{k}$.

Equation (1.3) is a linear equation in f, but a rather unmanageable one because of the integral operator appearing in R. It may be simplified by working in the Landau approximation,[4,5] which is physically motivated by noting that most of the quark-gluon scattering is soft. In other words, $w(\mathbf{p},\mathbf{k})$ falls off rapidly with $|\mathbf{k}|$. One is thus entitled to expand the integrand in (2.1) in powers of $\mathbf{k}$, viz.,

$$w(\mathbf{p}+\mathbf{k},\mathbf{k})f(\mathbf{p}+\mathbf{k})\approx w(\mathbf{p},\mathbf{k})f(\mathbf{p})+\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}}(wf) +\tfrac{1}{2}k_ik_j\frac{\partial^2}{\partial p_i\partial p_j}(wf) \; , \tag{2.6}$$

giving

$$R\approx\int d^3\mathbf{k}\left[\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}}+\tfrac{1}{2}k_ik_j\frac{\partial^2}{\partial p_i\partial p_j}\right](wf) \; . \tag{2.7}$$

Equation (1.3) now takes the form of a Fokker-Planck equation[6]

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial p_i}\left[A_i(\mathbf{p})f+\frac{\partial}{\partial p_j}[B_{ij}(\mathbf{p})f]\right] , \tag{2.8}$$

where we have defined the kernels

$$A_i=\int d^3\mathbf{k}\, w(\mathbf{p},\mathbf{k})k_i \; , \tag{2.9}$$

$$B_{ij}=\tfrac{1}{2}\int d^3\mathbf{k}\, w(\mathbf{p},\mathbf{k})k_ik_j \; . \tag{2.10}$$

To see the significance of A and B, consider the $\mathbf{p}\to 0$ limit. In that limit, we take $B_{ij}\to D\delta_{ij}$ and $A_i\to\gamma p_i$, and ignore derivatives of A and B. Then the Fokker-Planck equation (2.8) reduces to

$$\frac{\partial f}{\partial t}=\gamma\frac{\partial}{\partial\mathbf{p}}\cdot(\mathbf{p}f)+D\left[\frac{\partial}{\partial\mathbf{p}}\right]^2 f \; . \tag{2.11}$$

The one-dimensional version of (2.11) is known as Rayleigh's equation,

$$\frac{\partial f}{\partial t}=\gamma\frac{\partial}{\partial p}(pf)+D\frac{\partial^2 f}{\partial p^2} \; , \tag{2.12}$$

which describes the evolution of the momentum distribution of a particle undergoing Brownian motion. For the initial condition $f(p,t=0)=\delta(p-p_0)$, the solution of (2.12) is

$$f(p,t)=\left[\frac{\gamma}{2\pi D}(1-e^{-2\gamma t})\right]^{-1/2}\times\exp\left[-\frac{\gamma}{2D}\frac{(p-p_0e^{-\gamma t})^2}{1-e^{-2\gamma t}}\right], \tag{2.13}$$

which shows a drag force acting on the mean momentum of the particle,

$$\langle p\rangle=p_0e^{-\gamma t}, \tag{2.14}$$

along with a diffusion process in momentum space,

$$\langle p^2\rangle-\langle p\rangle^2=\frac{D}{\gamma}(1-e^{-2\gamma t}), \tag{2.15}$$

which has as its limit the Maxwell distribution if we demand that $\gamma/D=\beta/M$.

Under certain reasonable assumptions about the diffusion process,[6] the diffusion in momentum space can be shown to lead to diffusion in position, with

$$\langle x^2\rangle-\langle x\rangle^2\sim\frac{2D}{m^2\gamma^2}t\equiv D_xt \tag{2.16}$$

for $t\gg\gamma^{-1}$. An explicit example of a diffusion process satisfying these assumptions is described by the Langevin equation, and is discussed in Appendix A.

Thus A_i yields the drag coefficient while B_{ij} gives the diffusion constant. In this example there is a simple relation between the two, which comes from the requirement of thermodynamic equilibrium in the $t\to\infty$ limit. In general, when the momentum dependence of A and B is not neglected, the relation is somewhat more complex. It may be derived by demanding that $f(\mathbf{p})=\exp(-\beta E_{\mathbf{p}})$ satisfy (2.8) in the steady state, $\partial f/\partial t=0$, and serves as a check on the consistency of the results below.

III. EVALUATION OF THE DIFFUSION AND DRAG COEFFICIENTS

We proceed to evaluate A_i and B_{ij}, which are given explicitly by the expressions

$$A_i=\frac{1}{2E_{\mathbf{p}}}\int\frac{d^3\mathbf{q}}{(2\pi)^32E_{\mathbf{q}}}\int\frac{d^3\mathbf{q}'}{(2\pi)^32E_{\mathbf{q}'}}\int\frac{d^3\mathbf{p}'}{(2\pi)^32E_{\mathbf{p}'}}\frac{1}{\gamma_c}\sum|\mathcal{M}|^2(2\pi)^4\delta^4(p+q-p'-q')\times\hat{f}(\mathbf{q})[(p-p')_i]\equiv\langle\langle(p-p')_i\rangle\rangle, \tag{3.1a}$$

$$B_{ij}=\tfrac{1}{2}\langle\langle(p'-p)_i(p'-p)_j\rangle\rangle. \tag{3.1b}$$

Since A_i and B_{ij} depend only on the vector $\mathbf{p}$, they may be decomposed according to

$$A_i=p_iA(p^2), \tag{3.2a}$$

$$B_{ij}=\left[\delta_{ij}-\frac{p_ip_j}{p^2}\right]B_0(p^2)+\frac{p_ip_j}{p^2}B_1(p^2), \tag{3.2b}$$

with

$$A=p_iA_i/p^2=\langle\langle 1\rangle\rangle-\langle\langle\mathbf{p}\cdot\mathbf{p}'\rangle\rangle/p^2, \tag{3.3a}$$

$$B_0=\frac{1}{2}\left[\delta_{ij}-\frac{p_ip_j}{p^2}\right]B_{ij}=\tfrac{1}{4}[\langle\langle p'^2\rangle\rangle-\langle\langle(\mathbf{p}'\cdot\mathbf{p})^2\rangle\rangle/p^2], \tag{3.3b}$$

$$B_1=\frac{p_ip_j}{p^2}B_{ij}=\tfrac{1}{2}[\langle\langle(\mathbf{p}'\cdot\mathbf{p})^2\rangle\rangle/p^2-2\langle\langle\mathbf{p}'\cdot\mathbf{p}\rangle\rangle+p^2\langle\langle 1\rangle\rangle]. \tag{3.3c}$$

Note that A is positive definite. According to (A11), this assures a dissipative drag force.

The momentum-conserving δ function enables us to evaluate part of the integral in (3.1) by solving the kinematics in the center-of-momentum frame (see Appendix B). The result is an integral over the momentum of the incident gluon and over the c.m. scattering angles:

$$\langle\langle F(\mathbf{p}')\rangle\rangle=\frac{1}{(2\pi)^5}\frac{1}{2E_{\mathbf{p}}}\times\int\frac{d^3\mathbf{q}}{2E_{\mathbf{q}}}\int d\Omega_{\mathrm{c.m.}}\frac{s-m^2}{8s}\frac{1}{\gamma_c}\times\sum|\mathcal{M}|^2\hat{f}(\mathbf{q})F(\mathbf{p}'), \tag{3.4}$$

where $\mathbf{p}'$ is a function of $\mathbf{p}$, $\mathbf{q}$, and the c.m. scattering angles, and as usual $s=(E_{\mathbf{p}}+E_{\mathbf{q}})^2-(\mathbf{p}+\mathbf{q})^2$.

The scattering matrix element[9–11] comes from the diagrams in Fig. 1. We introduce a mass in the gluon propagator of Figs. 1(a) and 1(d) in order to regulate an infrared divergence in (3.4); physically, this mass should be on the order of the Debye screening mass,[4] which cuts off scattering at large impact parameters. The summed and averaged matrix element squared takes the form

$$\frac{1}{\gamma_c}\sum|\mathcal{M}|^2=M(\hat{p},t)\ , \tag{3.5}$$

where $\hat{p}$ is the c.m. momentum of the incident gluon (or quark) and $t=2\hat{p}^{\,2}(\cos\theta_{\mathrm{c.m.}}-1)$.

The integral in (3.4) is simplified by noting that we only need $\langle\langle F(\mathbf{p}')\rangle\rangle$ when F is a scalar function, parametrically dependent on $\mathbf{p}$. This means that if we choose the polar axis in the $\mathbf{q}$ integral to be along $\mathbf{p}$, the integrand does not depend on the azimuth of $\mathbf{q}$. Furthermore, we note that $\sum|\mathcal{M}|^2$ depends only on the polar angle $\theta_{\mathrm{c.m.}}$ and not on the azimuth $\phi_{\mathrm{c.m.}}$. Thus we find

$$\langle\langle F(\mathbf{p}')\rangle\rangle=\frac{1}{1024\pi^4}\frac{1}{E_p}\int_0^\infty q\,dq\,d(\cos\chi)\frac{s-m^2}{s}\hat{f}(q)\int_{-1}^{1}d(\cos\theta_{\mathrm{c.m.}})\frac{1}{\gamma_c}\sum|\mathcal{M}|^2\int_0^{2\pi}d\phi_{\mathrm{c.m.}}F(\mathbf{p}')\ . \tag{3.6}$$

The integral over $\phi_{\mathrm{c.m.}}$ is elementary for the cases needed (see Appendix B). The remaining three-dimensional integral may be evaluated numerically. We recall that implicit in (3.6) is the need to add up the contributions from cg, cq, and $c\bar{q}$ scattering. We take the u and d to be massless and ignore s quarks.

IV. RESULTS

We show in Fig. 2 the drag coefficient $A(p)$ for $T=\mu=200$ MeV. (We have set $\alpha_s=0.6$ and $m=1.5$ GeV.) Two features to note are that (1) the quark contribution is of the same order as, though smaller than, the gluon contribution, and (2) the variation in A as the momentum is varied between 0 and 1.5 GeV is about 10%. In B_0 and B_1, shown in Fig. 3, the quark and gluon contributions are likewise comparable; the variation in B_0 is 50% of its value at $p=0$, and B_1-B_0, which is the coefficient of p_ip_j/p^2 in B_{ij}, grows to 30% of B_0. These features persist at higher temperatures and suggest that detailed studies of the dynamics of charmed quarks may legitimately take A and B_0, but perhaps not B_1, to be constant in the phenomenologically relevant momentum range.

We show in Fig. 4 the temperature variation of A and B_0 evaluated at $p=0$. To indicate the sensitivity of the result to μ, we have plotted curves for $\mu=T\pm100$ MeV around the central curves which assume $\mu=T$. Defining $\gamma=A$ and $D=B_0$, the approximate consistency condition $\gamma/D=\beta/m$ is satisfied.

To see the physical importance of the values shown in Fig. 4, consider the central region of a high-energy nucleus-nucleus collision. We suppose[12] that a plasma is formed at an initial temperature $T=500$ MeV, and that a charmed quark is created in this plasma with momentum[13] $p=800$ MeV. Let us henceforth neglect both B_1-B_0 and the momentum variation of A and B_0. We

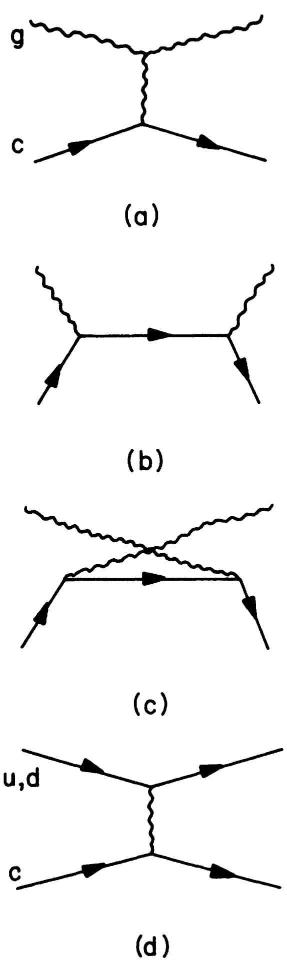

FIG. 1. Feynman diagrams for scattering of (a)–(c) gluons by charmed quarks and of (d) light quarks by charmed quarks.

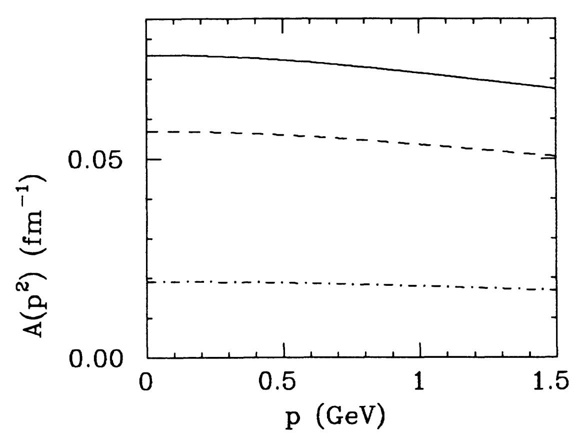

FIG. 2. Drag coefficient $A(p^2)$ at temperature $T=200$ MeV, assuming QCD coupling $\alpha_s=0.6$ and Debye screening mass $\mu=200$ MeV. The dashed-dotted curve is the contribution of quark and antiquark scattering, the dashed curve that of gluon scattering, and the solid line the sum of the two.

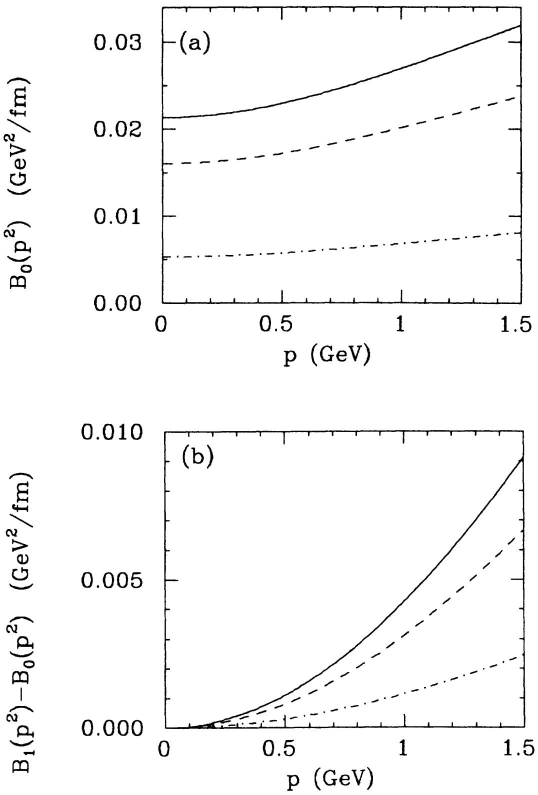

FIG. 3. Momentum-space diffusion coefficients $B_0(p^2)$ and $B_1(p^2)-B_0(p^2)$ at $T=200$ MeV, plotted as in Fig. 2.

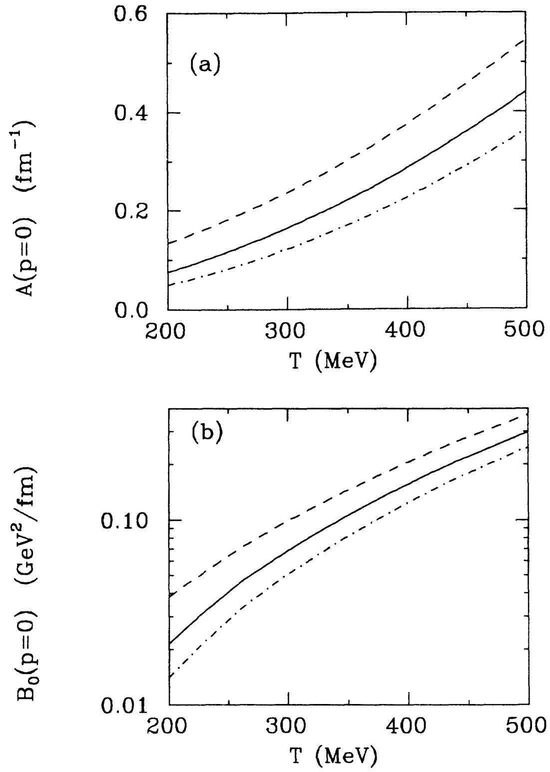

FIG. 4. Coefficients A and B_0 evaluated at $p=0$, as functions of temperature. The solid curve is for $\mu=T$, the dashed curve for $\mu=T-100$ MeV, and the dashed-dotted curve for $\mu=T+100$ MeV.

recall the discussion of Rayleigh's equation (2.12). The drag coefficient $\gamma\equiv A$ gives a decay time for the initial momentum of just 2 fm; for the initial momentum of 800 MeV we obtain from (A12) a penetration length of 1 fm, a very rapid damping indeed. Once the charmed quark is stopped it diffuses. According to (2.15), $\langle p^2\rangle$ reaches its thermal value in $t\sim 1$ fm. Diffusion in position takes place with the diffusion constant $D_x=2dD/m^2\gamma^2\sim 4.2$ fm, where we have generalized (2.16) to d dimensions. Thus we have $\langle x^2\rangle - |\langle \mathbf{x}\rangle|^2 \sim (4.2 \text{ fm})t$.

Since the damping constant γ is so large, the charmed quark will remain in kinetic equilibrium as the plasma cools. Thus its mean momentum drops. However, γ is dropping as well, as seen in Fig. 4(a). This means that the quark does not change direction as often, and hence that spatial diffusion is more rapid. The explicit expression for D_x shows that at $T=200$ MeV, we have $D_x\sim 9$ fm: Diffusion actually accelerates as the plasma cools.

These numerical estimates are of course merely illustrative; their dependence on plasma temperature, Debye length, and initial charmed-quark momentum may be read off the figures and equations given. We note that both γ and D are proportional to α_s^2 via (C1) and (C2); thus the time scales for damping and thermalization are proportional to α_s^{-2}, and so is the spatial diffusion constant D_x.

A prediction for the rate of J/ψ and charm production in central nucleus-nucleus collisions will require adoption of models for the initial creation of charm, the evolution of the QGP, and the $c\bar{c}$ recombination process. We will content ourselves with two observations. The first is that it makes no difference exactly how a given charmed-quark pair is created. If resonance creation takes place, and the pair is created with a wave function near that of the J/ψ, then the initial relative momentum will be small. If it is created with large invariant mass, as in $D\bar{D}$ creation above threshold,[13] then plasma drag will stop the pair before it has separated much. Subsequent motion of the pair will be dominated by Brownian motion, although effects of the Debye-screened Coulomb interaction remain to be included.

The other observation applies to a geometric argument[14] based on surface effects in the nuclear collision. This argument states that J/ψ's with large p_T, especially those created in nucleon-nucleon collisions near the nuclear surfaces, will escape the plasma without dissociation. According to our discussion, *any* J/ψ whose flight intersects the plasma region will be stopped there, to share in the fate of $c\bar{c}$ pairs created in the plasma in the first place. Thus if there is suppression of low-p_T J/ψ's, there should be suppression at high p_T as well. As mentioned in the Introduction, however, the large plasma drag could lead to *enhancement* of J/ψ production by preventing separation of charm pairs created in the $D\bar{D}$ continuum. This would obviously only apply to pairs created within the plasma volume.

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APPENDIX A: BROWNIAN MOTION AND THE LANGEVIN EQUATION

One may model the interaction of a heavy particle with the plasma by a random process in which collisions are included via a Gaussian white-noise term in the force acting on the particle. The (nonrelativistic) equations of motion are then

$$\frac{dx_i}{dt}=\frac{p_i}{m}\;, \tag{A1}$$

$$\frac{dp_i}{dt}=F_i(\mathbf{p})+G_i(\mathbf{x})+\eta_i\;. \tag{A2}$$

Here $\mathbf{F}$ is a (nonlinear) drag force and $\mathbf{G}$ is an external force; η_i is a noise variable, specified by its correlation function

$$\langle \eta_i(t)\eta_j(t')\rangle=2N_{ij}\delta(t-t')\;. \tag{A3}$$

The tensor N_{ij} can in general be a function of $\mathbf{p}$ and $\mathbf{x}$; its connection to the physics of the problem will appear presently.

As a continuous-time random process, the general Langevin process shown above is fraught with ambiguities, particularly in the specification of the product $N_{ij}\delta(t-t')$. All these ambiguities are removed when we make time a discrete variable and rewrite (A1)–(A3) with a step size ϵ:

$$x_i(t+\epsilon)=x_i(t)+\epsilon\frac{p_i(t)}{m}\;, \tag{A4}$$

$$p_i(t+\epsilon)=p_i(t)+\epsilon[F_i(\mathbf{p}(t))+G_i(\mathbf{x}(t))]+\sqrt{\epsilon}\eta_i(t)\;, \tag{A5}$$

$$\langle \eta_i(t)\eta_j(t')\rangle=2N_{ij}(\mathbf{x}(t),\mathbf{p}(t))\delta_{tt'}\;. \tag{A6}$$

The specification of the time arguments in (A4)–(A6) is unambiguous, and corresponds to the so-called Stratonovich definition of the Langevin process.

To make contact with the discussion of Sec. II we derive the Fokker-Planck equation for the probability distribution $f(\mathbf{x},\mathbf{p},t)$. The evolution of f is determined by the formula

$$f(\mathbf{x},\mathbf{p},t+\epsilon)=\int d\mathbf{x}'d\mathbf{p}'f(\mathbf{x}',\mathbf{p}',t)\delta\left[\mathbf{x}-\mathbf{x}'-\epsilon\frac{\mathbf{p}'}{m}\right]\langle\delta(\mathbf{p}-\mathbf{p}'-\epsilon[\mathbf{F}(\mathbf{p}')+\mathbf{G}(\mathbf{x}')]-\sqrt{\epsilon}\eta(t))\rangle\;. \tag{A7}$$

Expanding to first order in ϵ and making use of (A6), along with $\langle\eta_i\rangle=0$, we have

$$f(\mathbf{x},\mathbf{p},t+\epsilon)=\int d\mathbf{x}'d\mathbf{p}'f(\mathbf{x}',\mathbf{p}',t)\left[1-\epsilon\frac{\mathbf{p}'}{m}\cdot\frac{\partial}{\partial\mathbf{x}}\right]\delta(\mathbf{x}-\mathbf{x}')\left[1-\epsilon(\mathbf{F}+\mathbf{G})\cdot\frac{\partial}{\partial\mathbf{p}}+\epsilon N_{ij}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\right]\delta(\mathbf{p}-\mathbf{p}')\;. \tag{A8}$$

Integrating, we find

$$f(\mathbf{x},\mathbf{p},t+\epsilon)=f(\mathbf{x},\mathbf{p},t)-\epsilon\left[\frac{\mathbf{p}}{m}\cdot\frac{\partial f}{\partial\mathbf{x}}+\frac{\partial}{\partial\mathbf{p}}\cdot[(\mathbf{F}+\mathbf{G})f]-\frac{\partial^2}{\partial p_i\partial p_j}[N_{ij}f]\right]\;. \tag{A9}$$

Now we take the $\epsilon\to 0$ limit and rearrange terms to obtain

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\frac{\partial}{\partial\mathbf{x}}+\mathbf{G}(\mathbf{x})\cdot\frac{\partial}{\partial\mathbf{p}}\right]f=-\frac{\partial}{\partial\mathbf{p}}\cdot[\mathbf{F}(\mathbf{p})f]+\frac{\partial^2}{\partial p_i\partial p_j}(N_{ij}f)\;. \tag{A10}$$

This coincides with the Boltzmann equation (1.1) with the collision term in the Fokker-Planck form (2.8) if we identify

$$F_i=-A_i,\quad N_{ij}=B_{ij}\;. \tag{A11}$$

Setting $\mathbf{G}(\mathbf{x})=0$ and integrating (A10) over $\mathbf{x}$, we obtain Eq. (2.8) for the momentum distribution function $f(\mathbf{p},t)$.

The Fokker-Planck equation (A10) describes the time evolution of $f(\mathbf{x},\mathbf{p},t)$ and thus allows the calculation of the expectation value of any function of $\mathbf{x}$ and $\mathbf{p}$ at a given time. The solution of Rayleigh's equation, (2.12)–(2.15), is an example. f does not, however, contain any information about correlation functions such as $\Gamma_p(t)\equiv\langle p_i(t)p_j(t')\rangle$ for $t\neq t'$. Thus, from Rayleigh's equation it is possible to calculate

$$\langle x(t)\rangle=x_0+\int_0^t\frac{\langle p(t')\rangle}{m}dt' =x_0+\frac{p_0}{\gamma m}(1-e^{-\gamma t})\;, \tag{A12}$$

but a calculation of

$$\langle(x-x_0)^2\rangle=\frac{1}{m^2}\int_0^t dt'\int_0^t dt''\langle p(t')p(t'')\rangle \tag{A13}$$

is impossible without knowledge of $\Gamma_p(t)$. Direct solution of the Langevin equation (A1)–(A3) for the case at hand gives

$$\langle p(t)p(t')\rangle = p_0^2 e^{-\gamma(t+t')} + \frac{D}{\gamma} e^{-\gamma(t+t')}(e^{2\gamma t_<} - 1) , \tag{A14}$$

where $t_<$ is defined as the lesser of t and t', whence it is easy to verify that

$$\langle (x - x_0)^2 \rangle \sim \frac{2D}{m^2\gamma^2} t \tag{A15}$$

when $t \gg \gamma^{-1}$.

APPENDIX B: KINEMATICS OF THE QUARK-GLUON COLLISION

We begin with the covariant form of the collision integral (3.1),

$$\langle\langle F(\mathbf{p}')\rangle\rangle = \frac{1}{(2\pi)^5 2E_\mathbf{p}} \int d^4q\,\delta(q^2)\theta(q_0)d^4q'\delta(q'^2)\theta(q_0')d^4p'\delta(p'^2 - m^2)\theta(p_0') \times \delta^4(p+q-p'-q')\frac{1}{\gamma_c}\sum |\mathcal{M}|^2 \hat{f}(q_0)F(\mathbf{p}') , \tag{B1}$$

where we have taken both gluons and light quarks to be massless. The integral over q' is trivial, giving

$$\langle\langle F(\mathbf{p}')\rangle\rangle = \frac{1}{(2\pi)^5 2E_\mathbf{p}} \int d^4q\,\delta(q^2)\theta(q_0)d^4p'\delta(p'^2 - m^2)\theta(p_0')\delta((p+q-p')^2) \times \theta((p+q-p')_0)\frac{1}{\gamma_c}\sum |\mathcal{M}|^2 \hat{f}(q_0)F(\mathbf{p}') . \tag{B2}$$

At this point we go to the center-of-momentum frame. Writing $\hat{\mathbf{p}}$, etc., for c.m. momenta, and also defining $\hat{p} = |\hat{\mathbf{p}}|$, etc., we apply the δ functions in (B2) to reduce the integral over p' to one over c.m. scattering angles. First, we write

$$d^4p'\delta(p'^2 - m^2)\theta(p_0') = \frac{d^3\hat{p}'}{2\hat{E}'} = \frac{\hat{p}'^2 d\hat{p}'d\Omega_{\text{c.m.}}}{2\hat{E}'} , \tag{B3}$$

where $\hat{E}' \equiv (\hat{p}'^2 + m^2)^{1/2}$. To eliminate the integral over $\hat{p}'$, we write

$$\delta((p+q-p')^2) = \delta(2[m^2 + \hat{p}_0\hat{q}_0 + \hat{p}^2 - (\hat{p}_0 + \hat{q}_0)\hat{E}']) , \tag{B4}$$

which is satisfied by the energy-conservation condition $\hat{p}' = \hat{p}$. Equation (B4) thus becomes

$$\left[2(\hat{p}_0 + \hat{q}_0)\frac{\hat{p}'}{\hat{E}'}\right]^{-1} \delta(\hat{p}' - \hat{p}) , \tag{B5}$$

which we combine with the result of (B3) to obtain

$$\int d^4p'\delta(p'^2 - m^2)\theta(p_0')\delta((p+q-p')^2)\theta((p+q-p')_0) = \int \frac{\hat{p}}{4(\hat{p}_0 + \hat{q}_0)} d\Omega_{\text{c.m.}} . \tag{B6}$$

Further we note

$$\hat{p}_0 + \hat{q}_0 = \sqrt{s} , \quad \hat{p} = \frac{s - m^2}{2\sqrt{s}} . \tag{B7}$$

This gives us (3.4).

It remains to evaluate $\mathbf{p}'$, the laboratory recoil momentum of the charmed quark, in terms of the integration variables in (3.4). More precisely, we need the scalar quantities p'^2 and $\mathbf{p}\cdot\mathbf{p}'$ which appear in (3.3). This requires some details of the kinematics of the collision. The Lorentz transformation from the laboratory frame to the c.m. frame is

$$\hat{\mathbf{p}} = \gamma_{\text{c.m.}}(\mathbf{p} - \mathbf{v}_{\text{c.m.}}\hat{E}) , \qquad \hat{E} = \gamma_{\text{c.m.}}(E - \mathbf{v}_{\text{c.m.}}\cdot\hat{\mathbf{p}}) , \tag{B8}$$

where the velocity of the c.m. is

$$\mathbf{v}_{\text{c.m.}} = \frac{\mathbf{p}+\mathbf{q}}{E_\mathbf{p} + E_\mathbf{q}} , \quad \gamma_{\text{c.m.}} = \frac{E_\mathbf{p} + E_\mathbf{q}}{\sqrt{s}} . \tag{B9}$$

In order to define scattering angles, we choose axes in the c.m. frame by singling out $\hat{\mathbf{p}}$ and the $\hat{\mathbf{p}}$-$\mathbf{v}_{\text{c.m.}}$ plane,

$$\hat{\mathbf{x}} = \hat{\mathbf{p}}/\hat{p} , \qquad \hat{\mathbf{y}} = N^{-1}[\mathbf{v}_{\text{c.m.}} - (\hat{\mathbf{p}}\cdot\mathbf{v}_{\text{c.m.}})\hat{\mathbf{p}}/\hat{p}^2] , \qquad \hat{\mathbf{z}} = \hat{\mathbf{x}}\times\hat{\mathbf{y}} , \tag{B10}$$

where $N^2 = v_{\text{c.m.}}^2 - (\hat{\mathbf{p}}\cdot\mathbf{v}_{\text{c.m.}})^2/\hat{p}^2$. Energy conservation dictates $\hat{p}'^2 = \hat{p}^2$, so

$$\hat{\mathbf{p}}'=\hat{p}(\cos\theta_{\text{c.m.}}\hat{\mathbf{x}}+\sin\theta_{\text{c.m.}}\sin\phi_{\text{c.m.}}\hat{\mathbf{y}}+\sin\theta_{\text{c.m.}}\cos\phi_{\text{c.m.}}\hat{\mathbf{z}})\ . \tag{B11}$$

Now we can transform $\hat{E}'$ back to the laboratory frame through the inverse of (B8):

$$\begin{aligned}E'&=\gamma_{\text{c.m.}}(\hat{E}'+\mathbf{v}_{\text{c.m.}}\cdot\hat{\mathbf{p}}')\\&=\gamma_{\text{c.m.}}\left[\hat{E}+\hat{p}\left[\cos\theta_{\text{c.m.}}\frac{\mathbf{v}_{\text{c.m.}}\cdot\hat{\mathbf{p}}}{\hat{p}}\right.\right.\\&\qquad\left.\left.+N\sin\theta_{\text{c.m.}}\sin\phi_{\text{c.m.}}\right]\right]\ .\end{aligned} \tag{B12}$$

All the dependence on $\theta_{\text{c.m.}}$ and $\phi_{\text{c.m.}}$ is explicit in (B12). From E' we get

$$p'^2=E'^2-m^2\ , \tag{B13}$$

which is one of the quantities needed.

To get $\mathbf{p}\cdot\mathbf{p}'$, we use

$$\begin{aligned}t&\equiv(p'-p)^2=2m^2-2EE'+2\mathbf{p}\cdot\mathbf{p}'\\&=2\hat{p}^{\,2}(\cos\theta_{\text{c.m.}}-1)\ ,\end{aligned} \tag{B14}$$

whence

$$\mathbf{p}\cdot\mathbf{p}'=EE'-\hat{E}^{\,2}+\hat{p}^{\,2}\cos\theta_{\text{c.m.}}\ . \tag{B15}$$

APPENDIX C: SCATTERING MATRIX ELEMENTS

The scattering matrix elements corresponding to the Feynman diagrams in Fig. 1 are given explicitly in Ref. 10. (Because of a typographical error, the interference terms given in Ref. 11 lack a minus sign.) Labeling them according to Fig. 1, we have

$$\begin{aligned}\sum|\mathcal{M}_a|^2&=3072\pi^2\alpha_s^2\frac{(m^2-s)(m^2-u)}{(t-\mu^2)^2}\ ,\\\sum|\mathcal{M}_b|^2&=\tfrac{2048}{3}\pi^2\alpha_s^2\frac{(m^2-s)(m^2-u)-2m^2(m^2+s)}{(m^2-s)^2}\ ,\\\sum|\mathcal{M}_c|^2&=\tfrac{2048}{3}\pi^2\alpha_s^2\frac{(m^2-u)(m^2-s)-2m^2(m^2+u)}{(m^2-u)^2}\ ,\\\sum\mathcal{M}_a\mathcal{M}_b^*&=\sum\mathcal{M}_b\mathcal{M}_a^*\\&=768\pi^2\alpha_s^2\frac{(m^2-s)(m^2-u)+m^2(u-s)}{(t-\mu^2)(m^2-s)}\ ,\\\sum\mathcal{M}_a\mathcal{M}_c^*&=\sum\mathcal{M}_c\mathcal{M}_a^*\\&=768\pi^2\alpha_s^2\frac{(m^2-u)(m^2-s)+m^2(s-u)}{(t-\mu^2)(m^2-u)}\ ,\\\sum\mathcal{M}_b\mathcal{M}_c^*&=\sum\mathcal{M}_c\mathcal{M}_b^*\\&=\tfrac{256}{3}\pi^2\alpha_s^2\frac{m^2(t-4m^2)}{(m^2-u)(m^2-s)}\end{aligned} \tag{C1}$$

for gluon scattering, and

$$\sum|\mathcal{M}_d|^2=256N_f\pi^2\alpha_s^2\frac{(m^2-s)^2+(m^2-u)^2+2m^2t}{(t-\mu^2)^2} \tag{C2}$$

for quark or antiquark scattering. We have introduced a mass μ into the internal gluon propagator in the t-channel-exchange diagrams, Figs. 1(a) and 1(d), to include the effects of Debye screening.

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