The reaction $\gamma\gamma \to \pi^0\pi^0$ and chiral loops

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Two-photon production of a neutral-pion pair is uniquely predicted near threshold by the theory of chiral symmetry. The prediction vanishes at the tree level and is nonzero only at one-loop order, yielding a finite result without any unknown counterterms. In this paper we calculate the cross section for both on-shell and off-shell photons, as both cases can both be studied at e^+e^- storage rings. This reaction is the most accessible process which directly probes the loop structure of chiral-SU(2) symmetry.

I. INTRODUCTION

At the very lowest energies, QCD is a theory of pions and photons. There exists a rigorous method for calculating all of the interactions of these particles in this energy regime. This is the theory of chiral symmetry,[1] made especially useful by the framework of effective chiral Lagrangians.[2] All of the lowest-order interactions are entirely determined in terms of the pion decay constant F_π, the pion mass m_π, and, of course, the electric charge e.

Phenomenologically, the reactions $\pi\pi \to \pi\pi$ and $\pi^0 \to \gamma\gamma$ have had the most impact on the development of the low-energy theory. In this paper we point out the value of the $\gamma\gamma \to \pi^0\pi^0$ process and evaluate its cross section. Basically, $\gamma\gamma \to \pi^0\pi^0$ is unique is that this amplitude tests the chiral-symmetry–effective-Lagrangian approach at loop level. Alternate processes have contributions from tree-level couplings. In addition they contain counterterms which are needed to handle divergences which occur when loop diagrams are considered. By contrast, $\gamma\gamma \to \pi^0\pi^0$ cannot be generated by tree diagrams at either order E^2 or order E^4 in the energy expansion. As a consequence, its one-loop contribution is necessarily finite and can be expressed only in terms of F_π, m_π, and e. It is the only experimentally accessible process of which we are aware which tests this aspect of chiral SU(2), and hence should provide a unique system for studying the loop structure of chiral symmetry.

The experimental study of two-photon reactions is richer than might naively be expected. The prime sources of information on $\gamma\gamma$ processes[3] are the high-energy e^+e^- storage rings, through reactions such as $e^+e^- \to e^+e^-\gamma\gamma \to e^+e^-\pi^0\pi^0$. In addition to studying on-shell two-photon scattering, these reactions can also be measured with one photon off shell by detecting a final e^+ or e^- scattered at a nonzero q^2. These are often called "single-tag" experiments and have been used to study resonance production by off-shell photons, $\gamma\gamma^* \to M$. "Double-tag" experiments, with both photons at nonzero values of q^2, are also, in principle, possible, but are much more difficult. We will also display our results for the off-shell scattering, focusing on the single-tag case, as this is also uniquely predicted by the chiral-symmetry approach.

In Sec. II we provide a brief review of the effective-Lagrangian technique in chiral symmetry and the interactions of pions and photons which results therefrom. Section III derives the matrix-element calculation for $\gamma\gamma \to \pi^0\pi^0$. We present cross-section results in Sec. IV along with a discussion of possible corrections. Section V is a brief conclusion.

II. EFFECTIVE LAGRANGIANS

If the Lagrangian (current) mass of up and down quarks were to vanish, QCD would possess an exact global $SU(2)_L \times SU(2)_R$ chiral symmetry

$$\psi_L \to e^{i\alpha\cdot\tau}\psi_L \equiv U_L\psi_L \ , \quad \psi_R \to e^{i\beta\cdot\tau}\psi_R \equiv U_R\psi_R \tag{1}$$

with ψ being an SU(2) doublet

$$\psi_{L,R} = \tfrac{1}{2}(1\pm\gamma_5)\psi \ , \quad \psi = \begin{bmatrix} u \\ d \end{bmatrix} \ . \tag{2}$$

This Lagrangian is dynamically broken in the real world, with pions being the associated Goldstone bosons. Predictions of this dynamically broken chiral symmetry can be worked out using current-algebra techniques. However, it is simpler to utilize effective Lagrangians which incorporate the desired symmetry behavior.[2,4] These yield equivalent predictions, but one needs to use only simple algebraic calculations. Basically, the equivalence occurs because if the predictions follow from the symmetry structure, two theories which have the same symmetry behavior must lead to the same physics. In the effective-Lagrangian approach to chiral $SU(2)_L \times SU(2)_R$ the pion is incorporated within an SU(2) matrix Σ having transformation properties

$$\Sigma \to U_L^\dagger \Sigma U_R \tag{3}$$

such that $\Sigma^\dagger\Sigma = 1$ is preserved. The Lagrangian invariant under this symmetry, which contains the fewest number of derivations, is then

$$\mathcal{L}_0 = C\,\mathrm{Tr}(\partial_\mu\Sigma\,\partial^\mu\Sigma^\dagger) \ . \tag{4}$$

The fluctuations in Σ are the pion fields

$$\Sigma = \exp\left[i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{F} \right] . \tag{5}$$

The normalization C is fixed to be $F^2/4$ by the requirement that the pion kinetic energy be correctly normalized,

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu \boldsymbol{\pi}\, \partial^\mu \boldsymbol{\pi} + \cdots , \tag{6}$$

and F is identified with $F_\pi = 94$ MeV by consideration of the axial-vector current. This results in the effective Lagrangian

$$\mathcal{L}_0 = \frac{F_\pi^2}{4}\mathrm{Tr}(\partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger) \tag{7}$$

appropriate for massless pions.

The pion mass is generated by shifting the up- and down-quark masses away from zero. This removes the separate $SU(2)_L \times SU(2)_R$ invariances, but a diagonal SU(2) invariance survives, i.e., that with $U_L = U_R$. To the extent that the quark masses are small they can be included linearly in the Lagrangian. The lowest-order term of the proper form is

$$\mathcal{L}_m = \frac{F_\pi^2}{2}\mathrm{Tr}(m_\pi^2 \Sigma) . \tag{8}$$

Expansion of $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m$ to order π^4 easily gives the conventional Weinberg scattering lengths for $\pi\pi$ scattering.

The addition of photons requires the inclusion of local U(1) gauge invariance

$$\Sigma \to e^{-i\Lambda(x)Q}\Sigma e^{+i\Lambda(x)Q} \tag{9}$$

and is accomplished by turning usual derivatives ∂_μ into covariant derivatives

$$D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu[Q,\Sigma] . \tag{10}$$

However, in addition the triangle anomaly of QCD must be taken into account. This is contained in the Wess-Zumino effective Lagrangian[5]

$$\mathcal{L}_A = \frac{-N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}[eA_\mu \mathrm{Tr}(QL_\nu L_\alpha L_\beta + QR_\nu R_\alpha R_\beta) + ie^2\partial_\mu A_\nu A_\alpha \mathrm{Tr}(2Q^2L_\beta + 2Q^2R_\beta + QUQU^{-1}R_\beta + QU^{-1}QUL_\beta)] \tag{11}$$

with

$$L_\mu = \Sigma^{-1}\partial_\mu \Sigma , \quad R_\mu = (\partial_\mu \Sigma)\Sigma^{-1} . \tag{12}$$

Note that the purely hadronic portion of the Wess-Zumino action vanishes in SU(2) and that the terms involving photons can be written in terms of a local Lagrangian. Expansion of this Lagrangian yields a $\pi \to \gamma\gamma$ amplitude,

$$A(\pi^0 \to \gamma\gamma) = \frac{\alpha N_c}{6\pi F_\pi}\epsilon^{\mu\nu\alpha\beta}\epsilon_\mu k_\nu \epsilon'_\alpha k'_\beta , \tag{13}$$

in close agreement with experiment.

Overall the lowest-order effective Lagrangian for chiral $SU(2)_L \times SU(2)_R$ possesses the structure

$$\mathcal{L} = \frac{F_\pi^2}{4}\mathrm{Tr}(D_\mu \Sigma\, D^\mu \Sigma^\dagger) + \frac{F_\pi^2}{2}\mathrm{Tr}(m_\pi^2 \Sigma) + \mathcal{L}_A \tag{14}$$

and has considerable experimental as well as theoretical support. It is this which is our starting point.

There can in addition exist chirally invariant Lagrangians which contain higher powers of derivatives and/or masses. The complete set at next order (called E^4 below) has been written out and discussed by Gasser and Leutwyler.[6] The result is

$$\begin{aligned}\mathcal{L}_4 = {} & \gamma_1[\mathrm{Tr}(D_\mu \Sigma\, D^\mu \Sigma^\dagger)]^2 + \gamma_2 \mathrm{Tr}(D_\mu \Sigma\, D_\nu \Sigma^\dagger)\mathrm{Tr}(D^\mu \Sigma\, D^\nu \Sigma^\dagger) + \gamma_3[\mathrm{Tr}(m_\pi^2 \Sigma)]^2 + m_\pi^2 \gamma_4 \mathrm{Tr}(D_\mu \Sigma\, D^\mu \Sigma^\dagger m_\pi^2 \Sigma) \\ & + \gamma_5 m_\pi^4[\mathrm{Tr}(\tau_3 \Sigma)]^2 + \gamma_6 \mathrm{Tr}(F_{\mu\nu} D^\mu \Sigma\, D^\nu \Sigma^\dagger - D^\mu \Sigma\, F_{\mu\nu} D^\nu \Sigma^\dagger) + \gamma_7 \mathrm{Tr}(\Sigma^\dagger F_{\mu\nu} \Sigma F^{\mu\nu}) ,\end{aligned} \tag{15}$$

where

$$(D_\mu D_\nu - D_\nu D_\mu)\Sigma \equiv F_{\mu\nu}\Sigma \tag{16}$$

and the γ_i are dimensionless coefficients typically of order 10^{-3}. The γ_5 term arises because of isospin breaking from $m_u \neq m_d$. We will need this general structure in our work, but the specific values of the coefficients will not be important.

Loop diagrams can be handled within this framework. Indeed, they *must* be included if unitarity is to be respected. The renormalization program is somewhat different in nonlinear theories such as this, due to the "nonrenormalizable" character of the theory. One-loop divergences arising from $\mathcal{L}_0$ will generate modifications not only to $\mathcal{L}_0$ itself, but also to the γ_i coefficients which characterize higher-order interactions. One needs then to absorb the divergences into a new value of γ_i to determine the renormalized γ_i from experiment. The nonrenormalizable nature is apparent in that at each subsequent loop order more unknown constants appear. Nevertheless, a perfectly sensible low-energy theory emerges, as all of those constants are negligible [suppressed by $(E/\Lambda)^n$ for $\Lambda\sim 1$ GeV] at low-enough energies. At the very lowest energies only F_π, m_π, and e are required, while at somewhat higher energies, γ_i, $i=1,7$, becomes significant. The energy scale where this expansion breaks down is empirically of order 1 GeV.

The theory can be extended to chiral $SU(3)_L\otimes SU(3)_R$ (including kaons and η's) in a straightforward fashion. The Σ matrix becomes a 3×3 element of SU(3):

$$\Sigma=\exp\left[i\frac{\lambda^A\phi^A}{F_\pi}\right], \tag{17}$$

where λ^A ($A=1,\ldots,8$) are the Gell-Mann SU(3) matrices and ϕ^A are the fields of the pseudoscalar octet. The lowest-order Lagrangian then has a form similar to Eq. (14):

$$\mathcal{L}=\frac{F_\pi^2}{4}\mathrm{Tr}(D_\mu\Sigma D^\mu\Sigma^\dagger)-\frac{F_\pi^2}{2}\mathrm{Tr}\left[m\left[\frac{\Sigma+\Sigma^\dagger}{2}\right]\right], \tag{18}$$

where now (assuming approximate isospin symmetry)

$$m=\begin{bmatrix} m_\pi^2 & 0 & 0 \\ 0 & m_\pi^2 & 0 \\ 0 & 0 & 2m_K^2-m_\pi^2 \end{bmatrix}. \tag{19}$$

Chiral SU(3) is not expected to be as good a symmetry as chiral SU(2), because the kaon and η masses are not small compared to the chiral scale. Nevertheless, a good deal of work has been performed with this effective Lagrangian and some of the chiral SU(3) loop predictions have been worked out. One result is particularly interesting in relation to our present calculation. The weak decay $K_S\to\gamma\gamma$ shares with $\gamma\gamma\to\pi^0\pi^0$ the property that it is first generated by one-loop diagrams which must be finite as there are no E^4 counterterms possible which could absorb any divergences. The finite result is a direct test of the loop structure of chiral SU(3), and it has recently been confirmed, within experimental errors, by the experimental measurement of $K_S\to\gamma\gamma$ at LEAR.

The one-loop renormalization for $\pi\pi\to\pi\pi$ has been worked out in detail by Gasser and Leutwyler.[4] The one-loop effects in $\pi^0\to\gamma\gamma$, which turn out to be finite, have been given by us in Ref. 7. In the next section, we turn our attention to the corresponding one-loop calculation of $\gamma\gamma\to\pi^0\pi^0$.

III. THE $\gamma\gamma\to\pi^0\pi^0$ AMPLITUDE

The process $\gamma\gamma\to\pi^0\pi^0$ involves only neutral particles. It follows from this fact that none of the terms in either $\mathcal{L}_0$ or $\mathcal{L}_4$ can produce a tree-level amplitude for $\gamma\gamma\to\pi^0\pi^0$. This is obvious for the terms involving the covariant derivatives, as the commutator of Q with the neutral components vanishes. It is less obvious for the γ_7 term in $\mathcal{L}_4$, but is easily seen to be true by direct calculation. Thus the tree-level amplitudes vanish at order E^2 and E^4. This has been previously discovered by Terent'ev.[8] In fact, the first nonvanishing contribution occurs at order E^6, and arises from terms such as

$$\mathrm{Tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)F_\lambda^\mu F^{\lambda\nu}\,,\quad \mathrm{Tr}(D_\mu\Sigma D^\mu\Sigma^\dagger)F_{\lambda\sigma}F^{\lambda\sigma}\,. \tag{20}$$

At low-enough energies these forms will be unimportant and can be distinguished from loop corrections by their distinctive energy dependence.

Despite the lack of tree-level couplings, $\gamma\gamma\to\pi^0\pi^0$ can (and must) be generated by loops. The most obvious diagram is that required by unitarity: $\gamma\gamma\to\pi^+\pi^-\to\pi^0\pi^0$. At low energies one-loop corrections will be the dominant contribution to the energy expansion, as they are formally of order E^4. The lack of tree-level couplings at order E^4 yields then a very powerful result: The one-loop correction must in fact be finite. This is required because there exist no possible counterterms with which to absorb any divergences. Thus the cross section is entirely expressible in terms of F_π, m_π, and e. This makes $\gamma\gamma\to\pi^0\pi^0$ a direct probe of loop effects in chiral theories.

The one-loop diagrams for this process are shown in Fig. 1. Figures 1(a) and 1(b) are related to $\gamma\gamma\to\pi^+\pi^-$, while Figs. 1(c) and 1(d) also are required by the chiral-symmetric couplings. The five vertices which enter these diagrams are given in Fig. 2. We define the transition amplitude as

$$\mathcal{A}(\gamma\gamma\to\pi^0\pi^0)=\epsilon_\mu(k_1)\epsilon_\nu(k_2)M^{\mu\nu}\,, \tag{21}$$

where k_1 and k_2 are the momenta of the two photons.

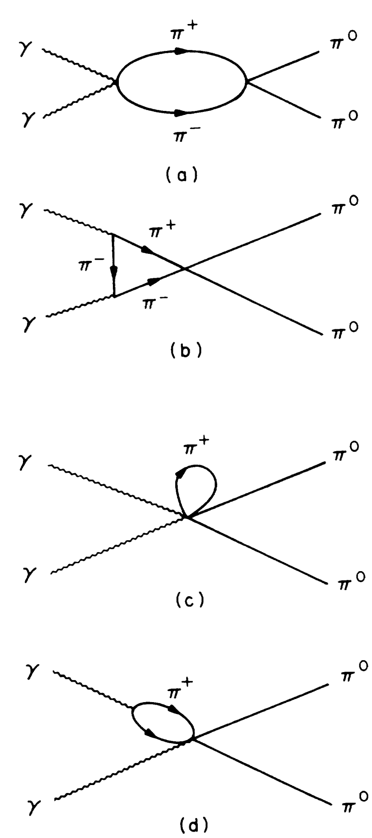

FIG. 1. The Feynman diagrams for $\gamma\gamma\to\pi^0\pi^0$ in the chiral-Lagrangian approach.

The general S-wave Lorentz structure of $M^{\mu\nu}$ is restricted by gauge invariance to be

$$M^{\mu\nu}=A(g^{\mu\nu}k_1\cdot k_2-k_2^\mu k_1^\nu)+B(g^{\mu\nu}k_1^2k_2^2-k_1^\mu k_1^\nu k_2^2-k_2^\mu k_2^\nu k_1^2+k_1\cdot k_2 k_1^\mu k_2^\nu)\ . \tag{22}$$

For on-shell photons only the first term contributes. In e^+e^- interactions the off-shell behavior of $\gamma\gamma$ scattering may be studied. Single-tag experiments, with one photon at $k^2\neq 0$, also involve only the amplitude A, although its value changes with k^2. The B amplitude only is relevant if both photons are off shell, unless there are $1/k^2$ terms in B, which does not occur in our calculation.

The four diagrams for $\gamma\gamma\to\pi^0\pi^0$ yield, respectively,

$$\begin{aligned}
M_a^{\mu\nu}&=\frac{2e^2g^{\mu\nu}}{3F_\pi^2}\int\frac{d^4l}{(2\pi)^4}\frac{2s-m_\pi^2-2l\cdot(l-p_1-p_2)}{(l^2-m_\pi^2)[(l-p_1-p_2)^2-m_\pi^2]}\ ,\\
M_b^{\mu\nu}&=\frac{-e^2}{3F_\pi^2}\int\frac{d^4l}{(2\pi)^4}\frac{(2l+k_1)^\mu(2l-k_2)^\nu[2s-m_\pi^2-2(l+k_1)\cdot(l-k_2)]}{(l^2-m_\pi^2)[(l+k_1)^2-m_\pi^2][(l-k_2)^2-m_\pi^2]}+(k_1,\mu)\leftrightarrow(k_2,\nu)\ ,\\
M_c^{\mu\nu}&=\frac{4e^2}{3F_\pi^2}g^{\mu\nu}\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2-m_\pi^2}\ ,\\
M_d^{\mu\nu}&=-\frac{2e^2}{3F_\pi^2}\int\frac{d^4l}{(2\pi)^4}\left[\frac{(2l-k_1)^\mu(2l-k_1)^\nu}{(l^2-m_\pi^2)[(l-k_1)^2-m_\pi^2]}+\frac{(2l-k_2)^\mu(2l-k_2)^\nu}{(l^2-m_\pi^2)[(l-k_2)^2-m_\pi^2]}\right]\ .
\end{aligned}\tag{23}$$

The terms in M_a and M_c have similar structures, as do those in M_b and M_d. They can be combined overall to determine the complete matrix element

$$M_{\mu\nu}=\frac{2e^2}{F_\pi^2}(s-m_\pi^2)\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l+k_1)^2-m_\pi^2}\frac{1}{(l-k_2)^2-m_\pi^2}\frac{1}{l^2-m_\pi^2}[g_{\mu\nu}(l^2-m_\pi^2)-(2l+k_1)^\mu(2l-k_2)^\nu]\ , \tag{24}$$

which is easily seen to be finite since the potential logarithmic divergence, i.e., the term proportional to $(g_{\mu\nu}l^2-4l^\mu l^\nu)$, cancels. The amplitude $M_{\mu\nu}$ can also be seen to be gauge invariant, via

$$\begin{aligned}
k_{1\mu}M^{\mu\nu}&=\frac{2e^2}{F_\pi^2}[s-m_\pi^2]\int\frac{d^4l}{(2\pi)^4}\left[\frac{(2l+k_1-k_2)^\nu}{[(l+k_1)^2-m_\pi^2][(l-k_2)^2-m_\pi^2]}-\frac{(2l-k_2)^\nu}{[(l-k_2)^2-m_\pi^2](l^2-m_\pi^2)}\right]\\
&=\frac{2e^2}{F_\pi^2}[s-m_\pi^2]\int\frac{d^4l}{(2\pi)^4}l_\nu\left[\frac{1}{\{[l+\frac12(k_1+k_2)]^2-m_\pi^2\}\{[l-\frac12(k_1+k_2)]^2-m_\pi^2\}}\right.\\
&\qquad\left.-\frac{1}{[(l-k_2/2)^2-m_\pi^2][(l+k_2/2)^2-m_\pi^2]}\right]=0\ .
\end{aligned}\tag{25}$$

Here the integrands are both odd functions of l_ν and hence integrate to zero. (In going to the second line, a shift of integration variable was used. We have also checked gauge invariance by a more tedious method without using the shift of the variable.)

At this stage the integral may be parametrized and integrated using standard Feynman-diagram techniques. We find that the A amplitude, which fully describes on-shell or single off-shell photon amplitudes, is given by

$$M_{\mu\nu}^A=\frac{2e^2}{F_\pi^2}(s-m_\pi^2)\left[\frac{-i}{16\pi^2}\right]\frac{(g_{\mu\nu}k_1\cdot k_2-k_2^\mu k_1^\nu)}{k_1\cdot k_2}(1+2I) \tag{26}$$

with

$$I=\int_0^1dz_1\int_0^1dz_2\,\theta(1-z_1-z_2)\frac{m_\pi^2-z_1(1-z_1)k_1^2-z_2(1-z_2)k_2^2}{2z_1z_2k_1\cdot k_2+z_1(1-z_1)k_1^2+z_2(1-z_2)k_2^2-m_\pi^2+i\epsilon}\ . \tag{27}$$

Let us agree to keep photon number one as the possibly off-shell photon, and set $k_2^2=0$. In this case the integral I may be reduced to a simpler form by performing the z_2 integration

$$I(s,k_1^2)=\frac{m_\pi^2}{s-k_1^2}[F(s)-F(k_1^2)]-\frac{k_1^2}{s-k_1^2}[G(s)-G(k_1^2)], \tag{28}$$

where

$$F(a)=\int_0^1 \frac{dz}{z}\ln\left[\frac{m_\pi^2-a(1-z)z-i\epsilon}{m_\pi^2}\right],$$
$$G(a)=\int_0^1 dz\, z\ln\left[\frac{m_\pi^2-a(1-z)z-i\epsilon}{m_\pi^2}\right]. \tag{29}$$

for $a>4m^2$, the integrals may be calculated by first factorizing

$$m_\pi^2-a(1-z)z=a(z-z_+)(z-z_-),$$
$$z_\pm=\frac{1\pm(1-m_\pi^2/a)^{1/2}}{2}, \tag{30}$$

with the result

$$F(a)=\tfrac{1}{2}(\ln z_+/z_- -i\pi)^2,$$
$$G(a)=-1+\tfrac{1}{2}(1-4m_\pi^2/a)^{1/2}(\ln z_+/z_- -i\pi). \tag{31}$$

For $0<a<4m^2$, we find

$$F(a)=-\frac{1}{2}\left[\pi-2\arctan\left[\frac{4m_\pi^2}{a}-1\right]^{1/2}\right]^2,$$
$$G(a)=-1+\frac{1}{2}\left[\frac{4m_\pi^2}{a}-1\right]\times\left[\pi-2\arctan\left[\frac{4m_\pi^2}{a}-1\right]^{1/2}\right], \tag{32a}$$

and, for $a<0$,

$$F(a)=2\left[\operatorname{arcsinh}\left[\frac{a}{4m_\pi^2}\right]^{1/2}\right]^2,$$
$$G(a)=-1+\left[\frac{4m_\pi^2}{a}+1\right]^{1/2}\operatorname{arcsinh}\left[\frac{a}{4m_\pi^2}\right]^{1/2}. \tag{32b}$$

For clarity, we display the full on-shell amplitude

$$M_{\mu\nu}=\frac{-i}{16\pi^2}\left[\frac{2e^2}{F_\pi^2}\right]\frac{s-m_\pi^2}{s}(g_{\mu\nu}s-2k_{2\mu}k_{1\nu})\times\left\{1+\frac{m_\pi^2}{s}\left[\ln\left[\frac{1+(1-4m_\pi^2/s)^{1/2}}{1-(1-4m_\pi^2/s)^{1/2}}\right]-i\pi\right]^2\right\}. \tag{33}$$

Note that the amplitude has a factorized structure, in that the amplitude for $\pi^+\pi^-\to\pi^0\pi^0$ scattering

$$A(\pi^+\pi^-\to\pi^0\pi^0)=\frac{1}{F_\pi^2}(s-m_\pi^2) \tag{34}$$

appears outside of the loop integration, so that the chiral prediction is that $\gamma\gamma\to\pi^0\pi^0$ is strictly proportional to $\pi^+\pi^-\to\pi^0\pi^0$, at least to lowest order.

If one extends the chiral Lagrangian to include kaons, as outlined above, there will arise additional loop contributions. These also, of course, must yield a finite answer, and we find the result

$$M_{\mu\nu}^A=\frac{e^2}{2F_\pi^2}\left[\frac{-i}{16\pi^2}\right]\frac{g_{\mu\nu}k_1\cdot k_2-k_2^\mu k_1^\nu}{k_1\cdot k_2}(1+2I_K), \tag{35}$$

where I_K is the same integral as given in Eq. 27, except with m_π^2 replaced with m_K^2. Since one is working far below $K\bar{K}$ threshold, one can Taylor expand the Feynman integral to find

$$1+2I_K=-\frac{s}{12m_K^2}+\cdots. \tag{36}$$

We observe that this result is in accord with the general principles of chiral Lagrangians. Thus the arguments which we gave above imply that pion loops must give the full result at order E^4 in the $SU(2)_L\times SU(2)_R$ limit. Hence kaon loops should enter only at one-higher power of s. The fact that $1+2I_K$ vanishes at $s=0$ makes manifest this result. Numerically the kaon correction is small and we will drop it in what follows.

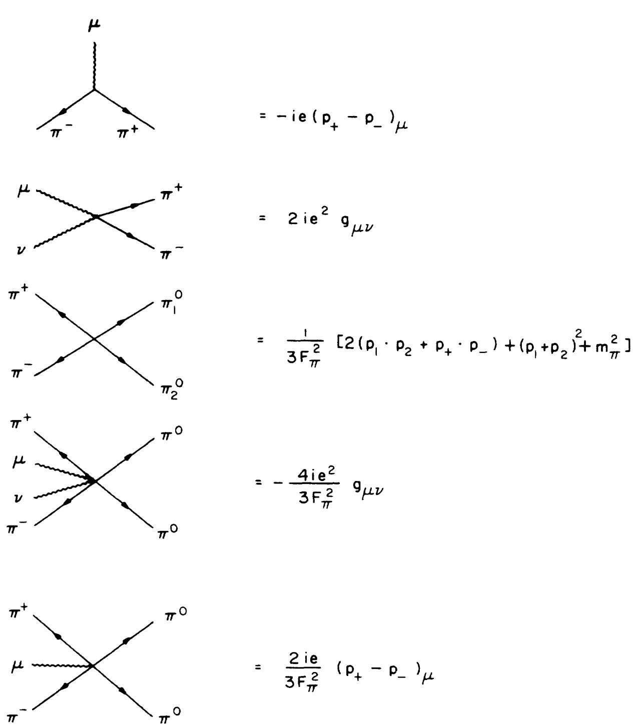

FIG. 2. The vertices needed to calculate $\gamma\gamma\to\pi^0\pi^0$.

IV. CROSS SECTIONS

The threshold value of the on-shell scattering cross section can be easily determined to be

$$\sigma(\gamma\gamma\to\pi^0\pi^0)=\frac{\alpha^2}{256\pi^3F_\pi^4}\frac{(s-m_\pi^2)^2}{s}\times\left[1-\frac{4m_\pi^2}{s}\right]^{1/2}\left[1+\frac{m_\pi^2}{s}f(s)\right] \tag{37}$$

with

$$f(s)=2[\ln^2(z_+/z_-)-\pi^2]+\frac{m_\pi^2}{s}[\ln^2(z_+/z_-)+\pi^2]^2 \ . \tag{38}$$

As used previously,

$$z_\pm \equiv \frac{1\pm(1-4m_\pi^2/s)^{1/2}}{2} \ . \tag{39}$$

It is important to address the question of what the expected limits of validity of this formula are expected to be. This can be answered most naturally by examination of the related process of $\pi\pi$ scattering, because, as noted previously, the $\gamma\gamma\to\pi^0\pi^0$ amplitude is directly proportional to the $\pi^+\pi^-\to\pi^0\pi^0$ amplitude. Indeed an alternate and useful way of presenting the chiral-symmetry prediction is

$$\sigma(\gamma\gamma\to\pi^0\pi^0)=\frac{\alpha^2}{8\pi^2}\left[1-\frac{4m_\pi^2}{s}\right]^{1/2} \times\left[1+\frac{m_\pi^2}{s}f(s)\right]\sigma(\pi^+\pi^-\to\pi^0\pi^0) \ , \tag{40}$$

where

$$\sigma(\pi^+\pi^-\to\pi^0\pi^0)=\frac{1}{32\pi sF_\pi^4}(s-m_\pi^2)^2 \tag{41}$$

is the lowest-order result for $\pi\pi$ scattering.

The limits of validity of the lowest-order PCAC (partial conservation of axial-vector current) prediction are very obvious in the study of $\pi\pi$ scattering.[9] In the partial-wave decomposition of the amplitude, only the S wave is relevant at these energies, and we have

$$\sigma(\pi^+\pi^-\to\pi^0\pi^0)=\frac{8\pi}{9q^2}\,|T|^2 \ , \tag{42}$$

where

$$|T|^2=\sin^2(\delta_0-\delta_2) = |e^{i\delta_0}\sin\delta_0-e^{i\delta_2}\sin\delta_2|^2 \tag{43}$$

with δ_0 (δ_2) being the $I=0$ ($I=2$) phase shift and

$$4q^2=s-4m_\pi^2 \ . \tag{44}$$

The lowest-order PCAC prediction corresponds to

$$T=(e^{i\delta_0}\sin\delta_0-e^{i\delta_2}\sin\delta_2) =\frac{3}{32\pi F_\pi^2}(s-m_\pi^2)\left[1-\frac{4m_\pi^2}{s}\right]^{1/2} \ . \tag{45}$$

It is clear that this lowest-order prediction, being real and not yet unitarized will disagree with the unitarity of the S matrix at some level. In fact, the simplest consequence of unitarity

$$|T|^2<1 \tag{46}$$

is violated at $\sqrt{s}=600$ MeV, so that clearly no results can be trusted above this energy. However, there should exist deviations from the lowest-order results even at smaller energies when the imaginary art of T becomes important. As a crude estimate we could take this to be when $\sin\delta\approx\frac{1}{2}$, which occurs around $\sqrt{s}=450$ MeV. At such energies one expects unitarity corrections to begin to become relevant. These are supplied in chiral theories by one-loop calculations, and have been performed by Gasser and Leutwyler. While unitarity is restored, a unique prediction does not result because of the appearance of counterterms from the E^4 terms in the Lagrangian. If we turn to experiment for guidance, we find that the various measurements are not in agreement. Figure 3 displays the relevant matrix element for those $\pi^-p\to\pi^0\pi^0\eta$ and $\pi^-p\to\pi^0\pi^0\Delta^0$ experiments[10] which extrapolated the scattering amplitude to the pion pole (a crucial requirement). (Experiments which do not extrapolate are in no better agreement.) The PCAC prediction is shown, continued above $\sqrt{s}=450$ MeV as a dotted line. Assuming that the lowest-order result is valid below $\sqrt{s}\approx450$ MeV, a smooth continuation up to $\sqrt{s}\approx800$ MeV similar to the hand-drawn solid curve is pretty much forced by the requirement of saturating the unitarity limit around $\sqrt{s}\approx700$ MeV. We do not at this time understand the origin of the experimental disagreement.

The above analysis suggests that one can expect modifications of $\sigma(\gamma\gamma\to\pi^0\pi^0)$ to become significant in the $\sqrt{s}=450-600$ MeV region. A heuristic way to deal with this problem is to use the alternate form of our result, Eq. (40), and simply to impose the unitarity restrictions on $\sigma(\pi^+\pi^-\to\pi^0\pi^0)$ by hand. We suspect that this is sensible physics, but we emphasize that it is not yet a

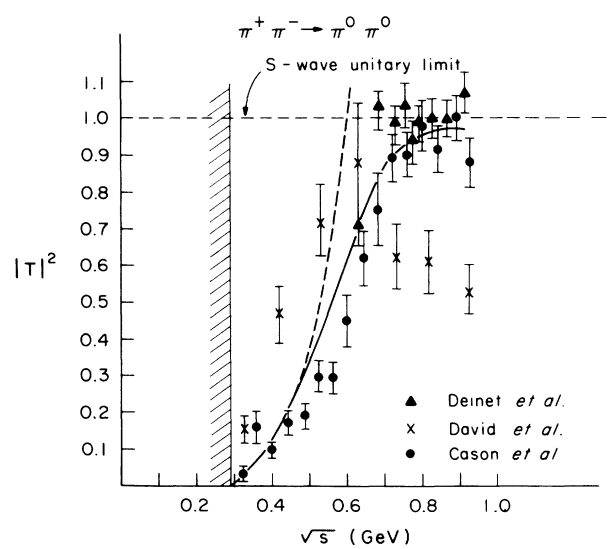

FIG. 3. The $\pi^+\pi^-\to\pi^0\pi^0$ scattering amplitude. The data points are from Ref. 10 [●, Cason *et al.*; ×, David *et al.*; Δ, Deinet *et al.*]. Below $\sqrt{s}=450$ MeV the theoretical curve is that of lowest-order chiral symmetry. This is continued above $\sqrt{s}=450$ MeV as the dashed line, in a region where unitarity corrections should be important. The solid line above $\sqrt{s}=450$ MeV represents a proposed continuation of the amplitude to take into account unitarity.

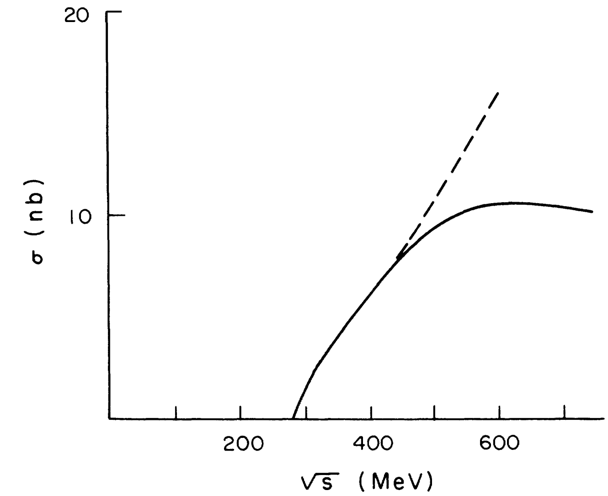

FIG. 4. The $\gamma\gamma \to \pi^0\pi^0$ cross section. The chiral-symmetry prediction is below $\sqrt{s} = 450$ MeV. The dashed line and solid line above $\sqrt{s} = 450$ MeV correspond to the similar curves in Fig. 3.

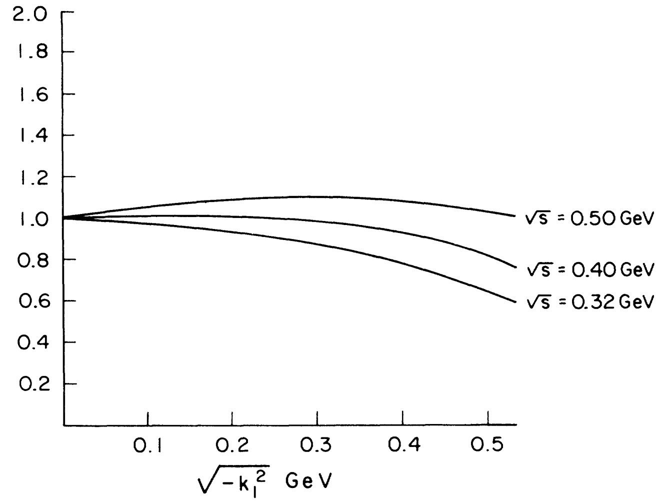

FIG. 5. The off-shell behavior of the $\gamma\gamma \to \pi^0\pi^0$ cross section as a function of the off-shell four-momentum k_1^2.

real prediction of PCAC. To justify this, one would require a two-loop calculation of $\gamma\gamma \to \pi^0\pi^0$. For now let us limit our firm prediction to energies below $\sqrt{s} = 450$ MeV, although we will display one proposed extrapolation (based on the solid curve in Fig. 3) in addition. The result is given in Fig. 4.

This cross section has also been predicted by Morgan and Pennington[11] using the dispersion relation. Their result is larger in the threshold region. We expect that this difference is primarily due to a different parametrization of the $\pi\pi$ scattering amplitude. Whether or not just higher-order chiral corrections would bring these into closer agreement would be an interesting question to study.

The single-tag off-shell behavior is given simply by the modulation of the cross section for transverse photons, σ_{TT} in standard notation.[3] The functional dependence is of the form

$$\frac{\sigma(s,k_1^2)}{\sigma(s,0)} = \frac{|1+2I(s,k_1^2)|}{|1+2I(s,0)|^2} \frac{1}{[1-k_1^2/2s]} ,$$

where $I(s,k_1^2)$ is given in Eq. (28). The second factor is the modification of incident flux. This ratio is plotted in Fig. 5. Note that cross sections have been compared at a common value of s with $s = 2k_1 \cdot k_2 + k_1^2$.

V. CONCLUSIONS

We have calculated the threshold behavior of $\gamma \to \pi^0\pi^0$ for on-shell photons and for single-tag experiments with one off-shell photon. The process requires one-loop diagrams at the lowest nontrivial order. These are necessarily finite in chiral perturbation theory. The result is a prediction solely in terms of known quantities e, F_π, and m_π. The result is small near threshold, and careful knowledge of systematic effects will be required in order to extract the result from experiment. However, results on this process should soon be produced from the Crystal Ball experiment.[12] It will be important to try to compare the PCAC prediction with the data as it becomes available.

On the theoretical side, there remains the possibility of estimating the effect of higher-order contributions to $\pi\pi$ scattering. These do modify the threshold behavior in $\pi\pi \to \pi\pi$ significantly, and they may be of importance in $\gamma\gamma \to \pi^0\pi^0$ also. However, this is not an easy task to carry out completely, and more work will need to be done in order to evaluate its feasibility.

Note added. After the first draft of this paper was written, we learned of a recent paper by J. Bijnens and F. Cornet which has some overlap with the present work. They also considered charged-pion production, while we treat off-shell effects and unitarity modifications of $\pi^+\pi^- \to \pi^0\pi^0$ in addition. Our results for $\gamma\gamma \to \pi^0\pi^0$ appear to agree with theirs. We thank M. Pennington for bringing this paper to our attention.

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