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Invariant Berry connections

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Berry connections are shown to be invariant under the action of symmetry groups on parameter space. This observation allows one to use the theory of invariant Yang-Mills potentials to evaluate these connections without computing the instantaneous energy eigenstates. Hamiltonians belonging to the algebra of a Lie group are examined in this light. They provide a wide class of systems admitting nontrivial adiabatic holonomy. The special case of the generalized harmonic oscillator is analyzed and the SO(2,1) invariance of the associated Berry connection is exhibited.

As shown by Berry[1] and then Wilczek and Zee,[2] gauge potentials arise in the description of quantum-mechanical systems whose Hamiltonians $H(\alpha)$ depend on a set of parameters $\{\alpha_i\}$ which vary slowly.[3] Let $\{|n,a,\alpha(t)\rangle\}$ be a choice of bases for the instantaneous eigenspaces of $H(\alpha(t))$,

$$H(\alpha)\,|\,n,a;\alpha\rangle = E_n(\alpha)\,|\,n,a;\alpha\rangle\ . \qquad (1)$$

The index a accounts for possible degeneracies. As shown in Refs. 1 and 2, the adiabatic evolution of $|\,n,a,\alpha(0)\rangle$ is given by

$$|\,t,a\rangle = \exp\left[-i\int_0^t dt' E_n(\alpha(t'))\right] \times U_{ab}(t)\,|\,n,b;\alpha(t)\rangle\ , \qquad (2)$$

where $U(t)$ is the path-ordered integral $P\exp[\int_0^t A(t')]$ and A is given by

$$A_{ab} = \langle n,a;\alpha\,|\,d\,|\,n,b;\alpha\rangle\ . \qquad (3)$$

The matrix one-form A shall be referred to as the Berry (local) connection. Under a change of bases $|\,n\rangle' = h\,|\,n\rangle$, it transforms like a Yang-Mills potential $A' = h^{-1}Ah + h^{-1}dh$. When the system is slowly cycled around a closed circuit C in parameter space, A can lead to a nontrivial holonomy, i.e., to a nontrivial Wilson loop

$$P\exp\left[\int_{C=\partial S}A\right] = P\exp\left[\int_S F\right] \qquad (4)$$

with $F = dA + A\wedge A$ the curvature two-form.

Various authors have provided examples of systems where such a phenomenon occurs. In some of these examples, the Hamiltonian is invariant under transformations of the dynamical variables if these are accompanied by appropriate transformations of the parameters. When this is so, the Berry connection proves to be invariant up to gauge transformations under the group action thus induced on parameter space. Let us mention three of these cases. The first one analyzed by Berry (see Ref. 1) deals with spins $\mathbf{s}$ in slowly varying magnetic fields. When $\mathbf{s}$ is rotated, the Hamiltonian $H(\mathbf{B}) = \kappa\mathbf{B}\cdot\mathbf{s}$ is obviously invariant if the magnetic field is transformed by the same rotation.[4] The U(1) connection A corresponding to this problem is that of a magnetic monopole located at the origin of B space and is well known to be invariant under rotations. The second case has been obtained by Moody, Shapere, and Wilczek.[5] They showed that the Born-Oppenheimer Hamiltonian for nuclear motion in a diatom contains a U(2) gauge field which was subsequently proven to be invariant under rotations of nuclear coordinates by Jackiw.[6] The last example that we would like to quote arises in chiral gauge theories when the fermions are treated as fast variables and the gauge fields as slow variables. Again, the U(1) Berry connection defined on the space of static Yang-Mills potentials is seen to be invariant under the action of the gauge group.[7] The purpose of this note is to demonstrate that this is a general pattern: Whenever a symmetry group acts on parameter space, the Berry connections will be invariant up to gauge transformations under this action.

Briefly, the reason for this state of affairs is the following: Let g^{-1} be the transformation of the dynamical variables which renders the Hamiltonian invariant when the external parameter α is transformed into α^g. (When there can be no confusion, we shall denote by g, indifferently, the abstract group element and its representation on state vectors.) The pullback of the Berry connection under $\alpha\to\alpha^g$ is obtained by substituting $|\,n,\alpha\rangle^g \equiv g\,|\,n,\alpha\rangle$ for $|\,n,\alpha\rangle$ in Eq. (3). This will correspond to a change of basis with respect to the initial choice $|\,n,\alpha^g\rangle$ and hence lead to a gauge transformation of the original connection. From here on, I shall consider the case of a Hamiltonian that belongs to the algebra of a Lie group G with the action of G on parameter space now given by the adjoint representation. This will allow for a more concrete discussion and no generality will be lost in the conclusions. Moreover, we shall thus describe a fairly wide class of systems with nontrivial adiabatic holonomy.

Let $\{x_i\}$ be (some anti-Hermitian representation of) a basis for the Lie algebra $\mathfrak{g}$ of G and $\{c_l, l=1,\ldots,\text{rank}\,\mathfrak{g}\}$ a basis for one of its Cartan subalgebras. We shall consider Hamiltonians of the form

$$H(\alpha) = \sigma(\alpha)H_D\sigma^{-1}(\alpha)\ , \qquad (5)$$

with $H_D = \varepsilon_l c_l$ and ε_l imaginary constants. In the above formula, σ represents local sections of $G\to G/\mathcal{C}(H_D)$

with $\mathscr{C}(H_D)$ the centralizer of H_D. The parameter space M is thereby identified with $G/\mathscr{C}(H_D)\times(i\mathbb{R})^{\mathrm{rank}\,\mathfrak{g}}$. We shall denote by $|n,a\rangle_D$, $a=1,\ldots,n$ the eigenstates of H_D. They are independent of the parameters and form a representation space for $\lambda:\mathscr{C}(H_D)\to U(n)$. Given Eq. (5), a natural choice for the states $|n,a,\alpha(t)\rangle$ in the adiabatic approximation is

$$|n,a,\alpha(t)\rangle=\sigma(\alpha(t))\,|n,a\rangle_D \;. \tag{6}$$

From Eq. (3), it is clear that the Berry connection does not depend on the ε parameters. For our purposes, we may therefore omit these parameters and simply take $M=G/\mathscr{C}(H_D)$.

The adjoint representation defines a natural G action on the parameters. Indeed, we may define $\alpha\to\alpha^g$ through

$$H(\alpha^g(t))=\alpha_i^g(t)x_i=g(t)H(\alpha(t))g^{-1}(t) \;. \tag{7}$$

This action can further be identified with left multiplication on the cosets $G/\mathscr{C}(H_D)$. Note also that the eigenvalues $E_n(\alpha)$ are invariant under these transformations, $E_n(\alpha^g)=E_n(\alpha)$, since they are not affected by conjugations.

We would now like to know how the Berry connection on M, defined in Eq. (3), transforms under this action of G on the parameters. As already mentioned, the pullback of A under $\alpha\to\alpha^g$ is given by

$$(g^*A)_{ab}={}^g\langle n,a,\alpha\,|\,d\,|\,n,b,\alpha\rangle^g \;, \tag{8}$$

where

$$|n,a,\alpha\rangle^g\equiv g\,|n,a,\alpha\rangle=g\sigma(\alpha)\,|n,a\rangle_D \;. \tag{9}$$

Let $h:G\times M\to\mathscr{C}(H_D)$ denote the transition function which relates σ and its image under G; that is,

$$g\sigma(\alpha)=\sigma(\alpha^g)h(g,\alpha) \;. \tag{10}$$

For consistency under group composition, h must satisfy the two-cocycle condition

$$h(g_1,\alpha^{g_2})h(g_2,\alpha)=h(g_1g_2,\alpha),\quad g_1,g_2\in G \;. \tag{11}$$

From (8), we then have

$$g^*A={}_D\langle n\,|\,h^{-1}(g,\alpha)\sigma^{-1}(\alpha^g)d[\sigma(\alpha^g)h(g,\alpha)]\,|\,n\rangle_D \;, \tag{12}$$

leading to

$$g^*A=Ad\lambda^{-1}[h(g)]A+\lambda^{-1}[h(g)]d\lambda[h(g)] \;, \tag{13}$$

where $\lambda:\mathscr{C}(H_D)\to U(n)$ is the representation of $\mathscr{C}(H_D)$ on the considered eigenspace of H_D with degree of degeneracy n. From (13), we conclude that the transformation of A under the symmetry-group action on parameter space reproduces the original A up to a compensating gauge transformation which is provided by the transition function h. The theory of gauge potentials that are invariant in this sense is by now well understood[8] and can therefore be used to compute Berry connections when symmetries are present. It will allow bypass of the evaluation of the instantaneous energy eigenstates. Also, when the external parameters are quantized and treated in the Born-Oppenheimer approximation, the work of Jackiw and Manton[9] will provide the conservation laws.

In the case we are considering, we need $\mathfrak{u}(n)$-valued one-forms A on $G/\mathscr{C}(H_D)$ that are invariant under left-G-multiplications on the cosets. The general theory[8] immediately gives the answer. Decompose $\mathfrak{g}$ according to

$$\mathfrak{g}=\mathfrak{c}(H_D)\oplus\mathfrak{m} \;. \tag{14}$$

Let $\omega^\sigma=\sigma^{-1}d\sigma$ stand for the pullback under $\sigma(\alpha)$ of the left-invariant Maurer-Cartan forms on G. Relative to the reductive decomposition (14) of $\mathfrak{g}$ we shall write

$$\omega^\sigma=\omega_{\mathfrak{c}}^\sigma+\omega_{\mathfrak{m}}^\sigma \;. \tag{15}$$

The invariant connections are then of the form

$$A=\lambda_*\omega_{\mathfrak{c}}^\sigma+\Phi\cdot\omega_{\mathfrak{m}}^\sigma \;, \tag{16}$$

where $\lambda_*:\mathfrak{c}\to\mathfrak{u}(n)$ is the differential of λ and $\Phi:\mathfrak{m}\to\mathfrak{u}(n)$ a linear map satisfying the constraints

$$\Phi[x_c,x]=[\lambda_*(x_c),\Phi(x)],\quad \forall x_c\in\mathfrak{c} \;. \tag{17}$$

In the specific case of invariant Berry connections, we further have $\Phi(x)=\Psi(\pi x\pi)$ with π the projection on the λ-representation space and Ψ an application into $\mathfrak{u}(n)$ of the corresponding restriction of the representation of $\mathfrak{g}$ on the eigenstates of H_D. The local forms A on parameter space define a connection one-form on the $U(n)$-principal bundle $E^\lambda\to G/\mathscr{C}(H_D)$ obtained by factoring the trivial $U(n)$ bundle $G\times U(n)$ by the equivalence relation

$$(g,u)\sim(gk,\lambda^{-1}(k)u),\quad g\in G,\quad k\in\mathscr{C}(H_D) \;. \tag{18}$$

When the homomorphism $\lambda:\mathscr{C}(H_D)\to U(n)$ extends smoothly to a homomorphism $\Lambda:G\to U(n)$ the bundle E^λ is trivial.[8] Under this proviso, there is a case for which the Berry connection is irrelevant. Indeed for $\Phi:\mathfrak{m}\to\mathfrak{u}(n)$ the restriction to $\mathfrak{m}$ of the differential Λ_* of Λ, the potential A is given by

$$A=\Lambda_*\omega^\sigma \;, \tag{19}$$

and has zero curvature owing to the Maurer-Cartan structure equations.[10]

Let us now consider an example where this analysis applies: namely, the case of the generalized harmonic oscillator originally discussed by Hannay[11] and Berry.[12] The Hamiltonian is

$$H=\tfrac{1}{2}[X(t)q^2+Y(t)(qp+pq)+Z(t)p^2] \;. \tag{20}$$

We shall limit ourselves to the bound-state situation by demanding that the parameters satisfy

$$XZ-Y^2>0 \;. \tag{21}$$

It is well known[13] that the operators q^2, $qp+pq$, and p^2 realize the so(2,1) [or sp(1)] algebra. Indeed, let

$$U_1=-\frac{i}{4}(p^2-q^2) \;,\qquad U_2=\frac{i}{4}(qp+pq) \;,\qquad U_3=\frac{i}{4}(p^2+q^2) \;. \tag{22}$$

It is not difficult to check that the following commutation relations are satisfied:

$$[U_1,U_2]=-U_3\ ,$$
$$[U_2,U_3]=U_1\ , \qquad (23)$$
$$[U_3,U_1]=U_2\ .$$

Note that U_3 is the compact rotation generator. In the U basis, the Hamiltonian reads

$$H(z)=z_i(t)U_i \qquad (24)$$

with $z_1=-i(X-Z)$, $z_2=-2iY$, $z_3=-i(X+Z)$. The quantity

$$z_1^2+z_2^2-z_3^2=4(XZ-Y^2)\equiv-\Delta^2>0 \qquad (25)$$

is invariant under the adjoint action of SO(2,1) and excluding the (positive-) energy scale, the set of parameters is in correspondence with the points on one sheet of the two-sheeted hyperboloid SO(2,1)/SO(2). A convenient parametrization is given by

$$z_1=\Delta\cos\theta\sinh\beta,\quad z_2=\Delta\sin\theta\sinh\beta,\quad z_3=\Delta\cosh\beta,\quad 0\leqq\theta<2\pi,\quad -\infty\leqq\beta\leqq\infty\ . \qquad (26)$$

The (singular) section

$$\sigma(\theta,\beta)=e^{-\theta U_3}e^{-\beta U_1} \qquad (27)$$

of SO(2,1)→SO(2,1)/SO(2) maps the reference point $(0,0,\Delta)$ into (z_1,z_2,z_3). It can be used to write the Hamiltonian as

$$H(\Delta,\theta,\beta)=z_iU_i=\sigma(\theta,\beta)\Delta U_3\sigma^{-1}(\theta,\beta)\ , \qquad (28)$$

thus showing that it belongs to the class of operators that we have examined in detail. The eigenvalues of U_3 are essentially those of a harmonic oscillator with unit frequency (Ref. 14): $U_3=(i/4)(p^2+q^2)=(i/2)(n+\frac{1}{2})$ and from (28), the energy spectrum is therefore given by

$$E_n=\sqrt{XZ-Y^2}(n+\tfrac{1}{2}),\quad n\in\mathbb{N}\ . \qquad (29)$$

(Positivity of the energy implies that $\Delta=-2i\sqrt{XZ-Y^2}$.)

The centralizer of $H_D=\Delta U_3$ in SO(2,1) is the two-dimensional rotation group generated by U_3. It is represented on the eigenstates $|n\rangle_D$ of H_D by phase multiplication:

$$\lambda(e^{\phi U_3})|n\rangle_D=\exp\left[\frac{i}{2}\phi(n+\tfrac{1}{2})\right]|n\rangle_D\ . \qquad (30)$$

Since SO(2,1) is simple, the only map $\Phi:\{U_1,U_2\}\to u(1)$ that satisfy the constraints (17) is $\Phi=0$. According to formula (16), the Berry connection will thus be proportional to the pullback under σ of the canonical connection on the U(1) fibration of SO(2,1)/SO(2):

$$A=\frac{i}{2}(n+\tfrac{1}{2})\sigma^{-1}d\sigma|_{U_3}\ . \qquad (31)$$

The explicit expression for the U_3 component of the form $\sigma^{-1}d\sigma$ is easily obtained (for instance by using a matrix representation for the generators U) and one finds[15]

$$A=-\frac{i}{2}(n+\tfrac{1}{2})\cosh\beta\, d\theta\ . \qquad (32)$$

For the curvature two-form, one gets

$$F=dA=\frac{i}{2}(n+\tfrac{1}{2})\sinh\beta\, d\theta\wedge d\beta\ . \qquad (33)$$

It is a straightforward exercise to check that the expression for F given by Berry[16] in Ref. 12,

$$F=\frac{i}{4}(n+\tfrac{1}{2})(XZ-Y^2)^{-3/2}$$
$$\times(X\,dY\wedge dZ+Y\,dZ\wedge dX+Z\,dX\wedge dY)\ , \qquad (34)$$

coincides with (33) when X, Y, Z are expressed in terms of the variables Δ, θ, and β.

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