Chiral-symmetry breaking in QCD. II. Running coupling constant

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The quark-propagator Dyson-Schwinger equation, with a running coupling constant to provide an ultraviolet regulator, with no infrared cutoff, is calculated numerically in the Landau gauge. It is shown that, for one or more generations of quarks, the chiral symmetry of the bare QCD Lagrangian is dynamically broken, so that quark masses are engendered.

I. INTRODUCTION

The notion that quark masses are generated by a mechanism of dynamical chiral-symmetry breaking has been widely studied.[1-10] In the preceding paper,[11] we examined the infrared domain in detail and we found that, even without the infrared cutoff that we and others had used,[6-9] there exists a critical value of the QCD coupling, above which chiral symmetry can be dynamically broken. We used a sharp ultraviolet cutoff in Ref. 11, so that numerical results were not of interest; the principle was however of importance.

In this paper, we refine the treatment by introducing a running coupling constant that has the inverse logarithmic ultraviolet behavior characteristic of the asymptotic freedom of QCD. The mass scale is set by the renormalization point; and the effective coupling is expressed in terms of the ultraviolet asymptotic behavior of the quark propagator. It is found numerically that this effective coupling is indeed greater than the critical coupling for the onset of chiral-symmetry breakdown and consequent mass generation.

The basic method consists in the investigation of the truncated Dyson-Schwinger equation for the quark propagator in the Landau gauge. We use a bare vertex and gluon propagator, along with the running coupling, to which we have already alluded. In earlier work,[8,9] we encountered difficulties in other gauges, associated with ultraviolet divergences of loop integrals. These difficulties can plausibly be traced to the replacement of the vertex function by its bare value: such a replacement is inconsistent with the Ward identity, except in the Landau gauge, as we now show.

In general, the inverse of the quark propagator can be written

$$S^{-1}(p)=\alpha(p^2)+\not p\beta(p^2) , \tag{1.1}$$

where p is the momentum. The quark-gluon vertex function satisfies the Ward identity

$$k_\nu\Gamma_\nu(p,p-k)=S^{-1}(p)-S^{-1}(p-k) , \tag{1.2}$$

as a consequence of gauge invariance. This condition is met when $\beta(p^2)\equiv 1$ and

$$\Gamma_\nu(p,p-k)=\gamma_\nu+\frac{k_\nu}{k^2}[\alpha(p^2)-\alpha((p-k)^2)] . \tag{1.3}$$

Moreover, the contribution of the term involving k_ν drops out of the Dyson-Schwinger equation for $S^{-1}(p)$ in the Landau gauge because the gluon propagator $D_{\mu\nu}(k^2)$ is orthogonal to k_ν in the Landau gauge. Hence only the bare γ_ν contributes to the Dyson-Schwinger equation, and in this case it is known that $\beta(p^2)$ is identically equal to unity (this result is true in any number of dimensions). Admittedly, this is not a proof that $\beta\equiv 1$ and $\Gamma_\nu\equiv\gamma_\nu$ in the Landau gauge; but at least there is consistency with the Ward identity, in terms of the ansatz (1.3), which is lacking in other gauges.

In Sec. II we introduce the Dyson-Schwinger equation, with the above approximations. The only parameters we use are the renormalization mass, which is implicit, since it can be scaled out of the equation, the number of quark flavors, and the infrared limit of the effective coupling. In Sec. III we reduce the integral equation to a nonlinear differential equation, with ultraviolet and infrared boundary conditions. Guided by the general findings of our analytical study of the equation with a sharp ultraviolet cutoff,[11] we set up a numerical method to look for nontrivial, non-chiral-symmetric solutions. A fourth-order Runge-Kutta algorithm is used to integrate the differential equation back from large p^2 down to $p^2=0$. To begin the integration at a very large value of p^2, an asymptotic series is used, as we explain in Sec. IV.

Section V is devoted to the presentation and discussion of the numerical results, which are given for two, four, and six quark flavors.

II. DYSON-SCHWINGER EQUATION WITH RUNNING COUPLING

We replace the vertex function and the gluon propagator by their bare values, and introduce a multiplicative running coupling function,

$$\omega(k^2)=\left[\ln\left[\tau+\frac{k^2}{M^2}\right]\right]^{-1} . \tag{2.1}$$

Here k is the Euclidean momentum, M is the renormalization mass scale, and $\tau>1$. The inverse logarithmic decrease in the ultraviolet is prescribed by asymptotic freedom, but the behavior of $\omega(k^2)$ in subasymptotic domains is unknown. By varying the parameter τ, we can assess the relative importance of the asymptotic and subasymptotic domains.

After a Wick rotation, the Dyson-Schwinger equation for the quark propagator S can be written in four-dimensional polar coordinates as follows:

$$S^{-1}(p)=\not p+\frac{\mu}{(2\pi)^4}\int_0^\infty p'^3dp'\gamma_\mu S(p')\gamma_\nu \times\int d^3\Omega'\omega((p'-p)^2) \times D_{\mu\nu}(p'-p)\ , \qquad (2.2)$$

with the scale parameter μ to be set presently by a renormalization-group argument. After the further standard approximation

$$\omega((p-p)^2)\simeq\omega(\max(p^2,p'^2))\ , \qquad (2.3)$$

the angular integral in (2.2) may be evaluated. In the Landau gauge, one finds that

$$S^{-1}(p)=\not p+\alpha(p^2)\ , \qquad (2.4)$$

where the mass function α satisfies

$$\alpha(x)=\frac{3\mu}{16\pi^2}\int_0^\infty dy\frac{\omega(x_{\max})}{x_{\max}}\frac{y\alpha(y)}{y+\alpha^2(y)}\ , \qquad (2.5)$$

with

$$x_{\max}=\max(x,y)\ . \qquad (2.6)$$

Renormalization-group analysis yields the UV asymptotic behavior

$$\alpha(x)\underset{x\to\infty}{\sim}\frac{1}{x}\left[\ln\left[\frac{x}{M^2}\right]\right]^{-1+\lambda}\ , \qquad (2.7)$$

where

$$\lambda=\frac{12}{33-2f}\ , \qquad (2.8)$$

with f the number of quark flavors that are involved in the calculation of the gluon β function.[12] It is easy to check that (2.5) is consistent with the regular asymptote (2.7), on condition that

$$\frac{3\mu}{16\pi^2}=\lambda\ . \qquad (2.9)$$

Equations (2.8) and (2.9) are used to eliminate the parameter μ.

The mass M is arbitrary, as may be seen by introducing the scaling $x/M^2\to x$, $y/M^2\to y$, $\alpha(x)/M\to\alpha(x)$: (2.5) remains unchanged, except that M is replaced in (2.1) by unity. In subsequent sections we take $M=1$, so that the renormalization mass M sets the scale of momentum.

III. DIFFERENTIAL EQUATION AND RUNGE-KUTTA METHOD

It follows from (2.1), (2.5), and (2.9) that

$$\frac{d}{dx}[\alpha(x)]=\lambda\frac{d}{dx}\left[\frac{\omega(x)}{x}\right]\int_0^x dy\frac{y\alpha(y)}{y+\alpha^2(y)}\ , \qquad (3.1)$$

$$\frac{d}{dx}\left[\frac{x}{\omega(x)}\alpha(x)\right]=\lambda\frac{d}{dx}\left[\frac{x}{\omega(x)}\right]\int_x^\infty dy\,\omega(y)\frac{\alpha(y)}{y+\alpha^2(y)}\ , \qquad (3.2)$$

$$\frac{d}{dx}\left[\frac{\frac{d}{dx}\left[\frac{x\alpha(x)}{\omega(x)}\right]}{\frac{d}{dx}\left[\frac{x}{\omega(x)}\right]}\right]+\lambda\omega(x)\frac{\alpha(x)}{x+\alpha^2(x)}=0\ . \qquad (3.3)$$

In general, $\alpha(0)$ is not determined by the integral equation (2.5), but, as can be seen from (3.1),

$$\alpha'(0)=-\frac{\lambda}{2\alpha(0)\ln\tau}\ . \qquad (3.4)$$

As $x\to\infty$,

$$\alpha(x)\sim\mu f_R(x)+\nu f_I(x)\ , \qquad (3.5)$$

where

$$f_R(x)=\frac{1}{x}(\ln x)^{-1+\lambda}\ , \qquad (3.6)$$

$$f_I(x)=(\ln x)^{-\lambda}\ . \qquad (3.7)$$

On physical grounds, namely, the conservation of the axial-vector QCD currents in the absence of a bare mass,[3] the irregular solution f_I must be suppressed, and this can be done by choosing $\alpha(0)$ in such a way that $\nu=0$ in the asymptotic formula (3.5). Indeed, this requirement constitutes a kind of nonlinear eigenvalue constraint on $\alpha(0)$.

Since f_I is asymptotically much larger than f_R, a stepwise integration of the differential equation (3.3) from $x=0$ to $x=\infty$, with the requirement that the UV asymptotics be regular (i.e., $\nu=0$), is inherently unstable. Accordingly, we rather begin at asymptotically large x with

$$\alpha(x)\sim\mu f_R(x)\ , \qquad (3.8)$$

and integrate back to the origin. For general values of μ, condition (3.4) will not be met, so that μ must be changed iteratively until (3.4) is satisfied to a preassigned accuracy.

It is convenient to define the functions

$$\mu_1(x)=x[\omega(x)]^{-1+\lambda}\alpha(x)\ , \qquad (3.9)$$

$$\mu_2(x)=\frac{\frac{d}{dx}\left[\frac{x}{\omega(x)}\alpha(x)\right]}{\frac{d}{dx}\left[\frac{x}{\omega(x)}\right]}\ . \qquad (3.10)$$

Equation (3.3) is then equivalent to the coupled system

$$\mu_1'(x)=\mu_2(x)[\omega(x)]^{\lambda}\frac{d}{dx}\left[\frac{x}{\omega(x)}\right]-\frac{\lambda\omega(x)\mu_1(x)}{x+\tau}\ , \qquad (3.11)$$

$$\mu_2'(x)=-\lambda\omega(x)\frac{\alpha(x)}{x+\alpha^2(x)}\ , \tag{3.12}$$

with

$$\alpha(x)=\frac{\mu_1(x)}{x}[\omega(x)]^{1-\lambda}\ . \tag{3.13}$$

The "eigenvalue condition" (3.4) is now replaced by

$$\mu_1(0)=0\ . \tag{3.14}$$

The numerical method proceeds from a starting value of μ in (3.8). This involves setting

$$\mu_1(\infty)=\mu\ , \tag{3.15}$$

$$\mu_2(\infty)=0\ , \tag{3.16}$$

and integrating the system (3.11) and (3.12) by a fourth-order Runge-Kutta routine from $x=\infty$ to $x=0$. Condition (3.14) will not be satisfied in general, so μ is changed and the whole procedure is repeated again and again, in such a way as to decrease $|\mu_1(0)|$ systematically. When (3.14) is satisfied, $\mu_2(0)=\alpha(0)$, and (3.4) is then also true.

IV. ASYMPTOTIC SERIES

The Runge-Kutta method is not well suited to the very first step, from $x=\infty$ to some large, finite value of x. For this initial step, we use an asymptotic series expansion for

$$\beta(z)=x\ln x\,\alpha(x)\sim z^{\lambda}\sum_{n=0}^{\infty}\beta_n z^{-n}\ , \tag{4.1}$$

where $z=\ln x$. For large x, the parameter τ in (2.1), and the function $\alpha^2(x)$ in the denominator in (3.3) may be dropped, and one obtains the following linear differential equation for β:

$$\left[z^2(1+z)\left[\frac{d}{dz}\right]^2-z^2(2+z)\frac{d}{dz}+\lambda(1+z)^2\right]\beta(z)=0\ . \tag{4.2}$$

The starting value (3.15) implies $\beta_0=\mu$, and then (4.2) yields $\beta_1=\lambda(1-\lambda)\mu$, and, for $n\geq 0$,

$$\beta_{n+2}=-\frac{1}{n+2}\{[(n-\lambda)(n-\lambda+1)+\lambda]\beta_n+[(n-\lambda+1)(n-\lambda+4)+2\lambda]\beta_{n+1}\}\ . \tag{4.3}$$

Although the series (4.1) is divergent, it is strongly asymptotic,[13] and it can be used to determine $\beta(z)$, and thence $\mu_1(x)$ and $\mu_2(x)$, for sufficiently large x, to adequate precision. As a check on the accuracy, the second derivative of $\beta(z)$ can also be estimated from (4.1); and, thus armed with α, α', and α'', we calculate the left-hand side of (3.3), which should be zero. The larger x is, the better the estimate given by the series; on the other hand, one is constrained by the necessity of avoiding unacceptably large rounding errors in the first few Runge-Kutta integrations. In practice, we evaluate the series for *two* large values, say $x=x_a$ and $x=x_b$, $x_a>x_b$; and in addition we run the Runge-Kutta routine from x_a to x_b, comparing the result with that obtained from the series, evaluated at x_b. The integration of the differential equation, and the imposition of the eigenvalue condition (3.14), are under good numerical control.

V. DISCUSSION OF THE NUMERICAL RESULTS

We have made an ultraviolet cutoff in the Dyson-Schwinger equation (2.2) for the quark propagator in order to remove ultraviolet instabilities. This cutoff, made in the context of QCD, consists in replacing the bare coupling $g^2/4\pi$ by the running coupling constant

$$\alpha_s(q^2)=\frac{12\pi}{33-2f}\frac{1}{\ln(\tau+q^2)}\ , \tag{5.1}$$

the renormalization mass scale being set to 1. We included the parameter $\tau>1$ to set the strength of coupling in the nonperturbative infrared regime, while maintaining consistency with the ultraviolet asymptote. For all numbers of quark flavors $f\geq 0$ and for $\ln\tau>1$, Eq. (3.1) is found to have a nontrivial solution, so that chiral symmetry is broken. By numerical means, we obtain a quark mass function $\alpha(q^2)$ which approaches a finite limit $\alpha(0)$ in the infrared, and which has the ultraviolet asymptote

$$\alpha_s(q^2)\to\frac{\mu}{q^2}(\ln q^2)^{1-12/(33-2f)}\ , \tag{5.2}$$

as discussed above. The function $\alpha_s(q^2)$ is found to approach its ultraviolet asymptote (5.2) for values of q^2 of order unity. In Table I the values of the parameters $\alpha(0)$ and μ, as determined from numerical solution of (2.5) with λ given by (2.8), are presented for various values of τ.

TABLE I. Values of μ and $\alpha(0)$ for various quark flavors f and various choices of the parameter τ.

f	λ	μ	$\alpha(0)$
		(a) $\ln\tau=1$	
0	0.364	0.0020	0.0420
2	0.414	0.0107	0.0957
4	0.480	0.0417	0.1857
6	0.571	0.1405	0.3326
		(b) $\ln\tau=0.5$	
0	0.364	0.0467	0.3046
2	0.414	0.0849	0.3966
4	0.480	0.1604	0.5219
6	0.571	0.3226	0.7079
		(c) $\ln\tau=0.25$	
0	0.364	0.0865	0.5620
2	0.414	0.1365	0.6675
4	0.480	0.2247	0.8070
6	0.571	0.4048	1.0013
		(d) $\ln\tau=0.1$	
0	0.364	0.1097	0.8556
2	0.414	0.1630	0.9733
4	0.480	0.2580	1.1269
6	0.571	0.4454	1.3370

TABLE II. Critical values for chiral-symmetry breaking with various quark flavors.

f	$\ln\tau$ (maximum)	$\alpha_s(0)$ (minimum)
0	1.25	0.91
2	1.43	0.91
4	1.67	0.91
6	2.01	0.89

When the parameter τ in relation (5.1) for α_s decreases, the infrared coupling increases; whereas an increase in the number of quark flavors leads to an increase in the overall strength of coupling. Our numerical results indicate that, below a certain critical level of coupling, solutions of the differential equation (3.5) cannot satisfy both infrared condition (3.4) and ultraviolet condition (3.8). In such a circumstance, the integral equation (2.5) has only the trivial solution, and chiral symmetry is preserved. In Table II our findings on the critical coupling level are presented for various numbers of quark flavors, as expressed in terms of the (maximum) parameter τ, as well as the (minimum) coupling strength at the infrared point $\alpha_s(0)$, that are required for breaking of chiral symmetry.

On the basis of our model (5.1) for the running coupling constant, we conclude that chiral-symmetry breaking can occur if $\alpha_s(0)$ is at least of order unity (almost independently of the number of quark flavors). Thus the phenomenon of dynamical mass generation is governed by the low- and intermediate-energy domains, and not by the asymptotic UV tail. If the low-energy strong interaction were an order of magnitude less $[\alpha_s(0)\sim 0.1]$, chiral-symmetry breaking would have been impossible.

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