Chiral-symmetry breaking in QCD. I. The infrared domain

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The Dyson-Schwinger equation for the quark propagator is handled in the infrared in three ways: (i) by a sharp cutoff; (ii) by an automatic infrared cutoff, which yields a pseudolinear equation; (iii) by considering the full nonlinearity. The last treatment is rendered possible by detailed comparison with the pseudolinear case; and it is found that chiral symmetry can be dynamically broken if the effective coupling is strong enough.

I. INTRODUCTION

We continue the study of dynamical breaking of chiral symmetry in massless QCD in the context of Dyson-Schwinger equations for the quark propagator. Our previous work involved a truncation scheme in which the quark-gluon vertex and gluon propagator remain free,[1,2] and we found that dynamical symmetry breaking occurs when both infrared (gluon mass) and ultraviolet (Pauli-Villars) cutoffs are made.[3] The ultraviolet cutoff is necessary because of divergences in the integration over an internal loop in the Dyson-Schwinger equation. Actually, the behavior of Green's functions in the ultraviolet is subject to constraints imposed by asymptotic freedom in the perturbative regime, and one can develop a natural ultraviolet truncation scheme by using these constraints.[4] However, there is no unequivocal understanding of the infrared behavior of Green's functions in non-Abelian theories.

Some time ago, it was suggested that an artificial infrared cutoff is unnecessary, in that the nonlinearity present in the Dyson-Schwinger equation for the quark propagator provides a natural truncation in the infrared region.[5] Indeed, such an insensitivity to infrared effects would clarify and simplify certain issues in dynamical symmetry breaking. When a constituent gluon mass is introduced in the truncation scheme,[6] one must analyze a coupled system of integral equations for the two scalar functions in the quark propagator, even in the Landau gauge.[7,8] Furthermore, the results may be sensitive to the infrared structure of the gluon propagator; especially in the continuum limit as the ultraviolet cutoff is removed. Nonlinearity in the Dyson-Schwinger equation may well play a vital role in the character of chiral-symmetry breaking. We have made a study of a truncation based upon spectral *Ansätze* of Salam and Delbourgo,[9] in which a linear equation for the quark propagator spectral function is obtained.[10] An ultraviolet cutoff is required in this approach, in which chiral symmetry is spontaneously broken. However, in the continuum limit as the cutoff is removed, chiral symmetry is restored.

In this paper we consider the Dyson-Schwinger equation in the Landau gauge with free gluon propagator and vertex. We show in Secs. II and III that a linearized form of the truncated Dyson-Schwinger equation is sensitive to an infrared truncation, but the sensitivity is removed when linearization is effected by replacing the quark propagator by its value at zero momentum, in accordance with the approach of Miransky and co-workers in Ref. 5. In Sec. IV we show that solutions to the nonlinear truncated equation exhibit chiral-symmetry breaking and are insensitive to the infrared. The results are discussed in Sec. V.

II. SHARP INFRARED AND ULTRAVIOLET CUTOFFS

When the vertex function and the gluon propagator are replaced by bare values, the truncated Dyson-Schwinger equation for the quark propagator S in momentum space is

$$S^{-1}(p)=\not{p}-\frac{ig^2}{(2\pi)^4}\int d^4p'\gamma_\mu S(p')\gamma_\nu D^{\mu\nu}(p'-p) \ . \tag{2.1}$$

In the Landau gauge, for which

$$D^{\mu\nu}(k)=\frac{-g^{\mu\nu}+k^\mu k^\nu/k^2}{k^2+i\epsilon} \ , \tag{2.2}$$

we obtain

$$S^{-1}(p)=\not{p}+\alpha(-p^2) \ , \tag{2.3}$$

where α satisfies the nonlinear integral equation

$$\alpha(x)=\lambda\int_0^\infty \frac{dy}{x_{\max}}\frac{y\alpha(y)}{y+\alpha^2(y)} \ , \tag{2.4}$$

with the notations $\lambda=3g^2/16\pi^2$ and $x_{\max}=\max(x,y)$.

Equation (2.4) always has the trivial solution $\alpha\equiv 0$, but the existence and behavior of a family of nontrivial solutions may well be sensitive to cutoffs. With sharp infrared and ultraviolet cutoffs, the functional derivative of Eq. (2.4) evaluated at $\alpha(x)\equiv 0$, is

$$f(x)=\lambda\int_\mu^\Lambda \frac{dy}{x_{\max}}f(y) \ , \tag{2.5}$$

$\mu\le x\le\Lambda$. This integral equation is equivalent to the Sturm-Liouville system consisting of the differential equation

$$[xf'(x)]'+\lambda f(x)=0 \ , \tag{2.6}$$

with boundary conditions

$$f'(\mu)=0, \quad (xf)'(\Lambda)=0 . \tag{2.7}$$

Let us analyze this system in three regimes of the parameter λ.

(i) For $\lambda<\frac{1}{4}$, (2.6) has the solution

$$f(x)=Ax^{-1/2+\sigma}+Bx^{-1/2-\sigma} , \tag{2.8}$$

$$\sigma=(\tfrac{1}{4}-\lambda)^{1/2} . \tag{2.9}$$

The boundary conditions require that

$$\frac{\frac{1}{2}-\sigma}{\frac{1}{2}+\sigma}=\left[\frac{\Lambda}{\mu}\right]^{\sigma} , \tag{2.10}$$

which cannot be satisfied for $\sigma>0$. Consequently, (2.5) has only the trivial solution for $\lambda<\frac{1}{4}$.

(ii) For $\lambda=\frac{1}{4}$, (2.6) has the solution

$$f(x)=x^{-1/2}(C+D\ln x) , \tag{2.11}$$

and the boundary conditions require that

$$\ln\frac{\Lambda}{\mu}+4=0 . \tag{2.12}$$

Since this relation is not satisfied, there is no nontrivial solution of (2.5) for $\lambda=\frac{1}{4}$.

(iii) The case $\lambda>\frac{1}{4}$ may be treated by making the replacement $\sigma\to i\rho$ in (2.8) and (2.10), where

$$\rho=(\lambda-\tfrac{1}{4})^{1/2} . \tag{2.13}$$

The boundary conditions are met for ρ in the monotonic sequence $\{\rho_1,\rho_2,\ldots\}$, where ρ_n is the unique positive solution of

$$\rho_n\ln\frac{\Lambda}{\mu}+2\arctan 2\rho_n=2n\pi . \tag{2.14}$$

Therefore, nontrivial solutions occur for λ in the set $\{\lambda_n=\frac{1}{4}+\rho_n^2\}$. As the ratio Λ/μ becomes large, these eigenvalues become dense over the whole domain $\lambda\geq\frac{1}{4}$.

III. AUTOMATIC INFRARED CUTOFF

One obtains the integrand of the linear integral equation (2.5) by making the replacement $y+\alpha^2(y)\to y$ in the denominator of the nonlinear equation (2.4). Such a replacement is reasonable at large y when the function $\alpha(y)$ is uniformly bounded, but at small y it is unreliable unless $\alpha(0)=0$. In fact, the divergence of the solutions of (2.6) at small x [Eq. (2.8) with $\sigma\to i\rho$] is an artifact of this uncontrolled approximation. It would be much more reliable at small y to make the replacement $y+\alpha^2(y)\to y+\alpha^2(0)$, and to eliminate the infrared, but not the ultraviolet, cutoff.

Accordingly, we shall consider the linear integral equation

$$\beta(x)=\lambda\int_0^{\Lambda}\frac{dy}{x_{\max}}\frac{\beta(y)}{y+m^2} \tag{3.1}$$

for $0\leq x\leq\Lambda$, where we make the identification

$$\beta(0)=m \tag{3.2}$$

in comparison with a solution of the nonlinear equation (2.4). Equation (3.1) is equivalent to the Sturm-Liouville system[11] consisting of the differential equation

$$\frac{d}{dx}\left[x^2\frac{d\beta}{dx}\right]+\frac{\lambda x}{x+m^2}\beta(x)=0 , \tag{3.3}$$

along with the infrared condition (3.2) and the ultraviolet condition

$$B(\Lambda)=0 , \tag{3.4}$$

where

$$B(x)=\frac{d}{dx}[x\beta(x)] . \tag{3.5}$$

Note that

$$B(0)=m . \tag{3.6}$$

The solution to (3.3) subject to the infrared condition (3.2) may be expressed in terms of hypergeometric functions[12] as

$$\beta(x,m)=mF\left[\tfrac{1}{2}+\sigma,\tfrac{1}{2}-\sigma;2;-\frac{x}{m^2}\right] , \tag{3.7}$$

with σ given in (2.9). We shall make use of several simple relations involving hypergeometric functions, which are given in Appendix A. Applying Eq. (A6) of Appendix A, we obtain

$$\beta'(x,m)=-\frac{\lambda}{2m}F\left[\tfrac{3}{2}+\sigma,\tfrac{3}{2}-\sigma;3;-\frac{x}{m^2}\right] . \tag{3.8}$$

Equation (A8) with $c=2$ implies

$$B(x,m)=mF\left[\tfrac{1}{2}+\sigma,\tfrac{1}{2}-\sigma;1;-\frac{x}{m^2}\right] , \tag{3.9}$$

and finally we apply (A6) to (3.9) to get

$$B'(x,m)=-\frac{\lambda}{m}F\left[\tfrac{3}{2}+\sigma,\tfrac{3}{2}-\sigma;2;-\frac{x}{m^2}\right] . \tag{3.10}$$

Consider first formulas (3.7)–(3.10) for the case $0<\lambda<\frac{1}{4}$ so that the parameter σ is real. The parameters $(a,b,c,)$ in the hypergeometric functions in (3.7)–(3.10) are real, and conditions (A1) are met, so that these hypergeometric functions are all positive, viz., (A3). Consequently, we have established that, for $0<x<\infty$ and all m,

$$\beta(x)>0, \quad \beta'(x)<0, \quad B(x)>0, \quad B'(x)<0 . \tag{3.11}$$

The asymptotic behavior of $\beta(x)$ or $B(x)$ at large x can be obtained from Eq. (A9). The leading asymptotes are of the form given in (2.8) for $\lambda<\frac{1}{4}$, and for $\lambda=\frac{1}{4}$ it is of the form given in (2.11). For $\lambda\leq\frac{1}{4}$ the functions $\beta(x)$ and $B(x)$ are positive at all x and decrease monotonically to zero at large x. In particular, because of the third relation in (3.11), condition (3.4) cannot be met for any cutoff parameter Λ when $\lambda\leq\frac{1}{4}$.

For $\lambda>\frac{1}{4}$ one must make the replacement $\sigma\to i\rho$, with ρ defined by (2.13), in the formulas (3.7)–(3.10). The function $\beta(x)$ becomes infinitely oscillatory at large x —a precise asymptotic formula can be obtained from (A9). The corresponding asymptotic formula for $B(x)$, valid for $x \gg m^2$, is

$$B(x,m)=2m\,\mathrm{Re}\left[\frac{\Gamma(2i\rho)}{\Gamma^2(\frac{1}{2}+i\rho)}\left[\frac{x}{m^2}\right]^{-1/2+i\rho}\right] . \tag{3.12}$$

There is an infinite set of zeros of $B(x)$ at locations $x_n=m^2 r_n$, $n=1,2,\ldots,$ where the values of the monotonic sequence $\{r_n\}$ can be estimated from (3.12). In particular, the first zero occurs at $x=m^2 r_1$, where

$$\ln r_1 \sim \frac{1}{\rho}\left[\frac{\pi}{2}-\arg\frac{\Gamma(2i\rho)}{\Gamma^2(\frac{1}{2}+i\rho)}\right] . \tag{3.13}$$

The parameter m in Eqs. (3.1) and (3.2) is arbitrary, and it may be chosen so that, say, the nth zero of $B(x)$ occurs at $x=\Lambda$; i.e., take $m=(\Lambda/r_n)^{1/2}$, so that

$$B\left[\Lambda,\left[\frac{\Lambda}{r_n}\right]^{1/2}\right]=0 . \tag{3.14}$$

Therefore, the system (3.1) and (3.2) has an infinite number of solutions for every value of $\lambda>\frac{1}{4}$.

IV. NONLINEAR EQUATION

We make an ultraviolet (but not infrared) cutoff in (2.4) to obtain

$$\alpha(x)=\lambda\int_0^\Lambda \frac{dy}{x_{\max}}\frac{y\alpha(y)}{y+\alpha^2(y)} , \tag{4.1}$$

to be considered for $0\le x\le\Lambda$. For any real (integrable) function α, the magnitude of the right side of (4.1) is bounded by

$$\frac{\lambda}{2}\int_0^\Lambda \frac{dy\sqrt{y}}{x_{\max}}\le\lambda\sqrt{\Lambda} , \tag{4.2}$$

so one need consider only bounded, continuous functions α. Any solution of (4.1) will also satisfy the differential equation

$$\frac{d}{dx}\left[x^2\frac{d\alpha}{dx}\right]+\lambda x\frac{\alpha(x)}{x+\alpha^2(x)}=0 , \tag{4.3}$$

along the boundary condition

$$A(\Lambda)=0 , \tag{4.4}$$

with

$$A(x)=\frac{d}{dx}[x\alpha(x)] . \tag{4.5}$$

We shall seek solutions of (4.1) by integrating Eq. (4.3), starting from the initial value

$$\alpha(0)=m , \tag{4.6}$$

with $m>0$ by convention.

The solutions of (4.3) with arbitrary initial value τ may be obtained by a scale transformation:

$$\bar{\alpha}(z)\equiv\frac{\tau}{m}\alpha\left[\left[\frac{m}{\tau}\right]^2 z\right] . \tag{4.7}$$

The solution $\alpha(x)$ of (4.3) and (4.6) satisfies the integral equation

$$\alpha(x)=m-\lambda\int_0^x dy\left[1-\frac{y}{x}\right]\frac{\alpha(y)}{y+\alpha^2(y)} , \tag{4.8}$$

from which one obtains

$$\alpha'(x)=-\frac{\lambda}{x^2}\int_0^x dy\frac{y\alpha(y)}{y+\alpha^2(y)} \tag{4.9}$$

and

$$A(x)=m-\lambda\int_0^x dy\frac{\alpha(x)}{y+\alpha^2(y)} . \tag{4.10}$$

Let us define the domain

$$\mathcal{D}(\alpha)=\{x\mid x\ge 0;\alpha(y)\ge 0 \text{ for } y\in[0,x]\} . \tag{4.11}$$

For y in $\mathcal{D}$, the integrals in (4.8)–(4.10) are non-negative, so that $\alpha(x)$ and $A(x)$ are monotonically decreasing in $\mathcal{D}$ and subject to the constraint

$$m\ge\alpha(x)\ge A(x) . \tag{4.12}$$

The smallest positive zero of $A(x)$ must occur at x to the left of all positive zeros of $\alpha(x)$.

Since the Sturm-Liouville system (3.2) and (3.3) is a relatively reliable approximation to the nonlinear system (4.3)–(4.6) in both the infrared and the ultraviolet, the behavior of the function $\beta(x)$ is expected to be similar to that of the function $\alpha(x)$. It is shown in Appendix B that, for x in $\mathcal{D}(\alpha)$ the domain of positivity of α,

$$\alpha(x)\le\beta(x)\le m \tag{4.13}$$

and

$$A(x)\le B(x)\le m , \tag{4.14}$$

the functions A and B being defined in (4.5) and (3.5), respectively. It follows from (4.13) that if $\beta(x)$ develops its first zero at $x=x_1$, then $\alpha(x)$ must have a zero at a point $x_0\le x_1$. Similarly, it follows from (4.14) that the first zero of $A(x)$ must precede any zeros of $B(x)$. Since we have established in Sec. III that $B(x)$ has finite zeros for every value of λ greater than $\frac{1}{4}$ [viz. Eq. (3.12)], $A(x)$ must have at least one zero. Let the first zero of $A(x)$ occur at $x=x_0$. By making the scale transformation (4.7) with $\tau=m(x_0/\Lambda)^{1/2}$, we obtain a function $\bar{\alpha}(x)$ which satisfies the boundary condition (4.4), and therefore is a solution of the nonlinear integral equation (4.1). In summary, Eq. (4.1) has a nontrivial positive solution $\bar{\alpha}(x)$ for all $\lambda>\frac{1}{4}$.

To analyze the case $\lambda\le\frac{1}{4}$, we first establish that $\alpha(x)$ cannot become small for sufficiently small x. For $y\in\mathcal{D}(\alpha)$, $\alpha(y)$ is positive, and the bound

$$\frac{\alpha(y)}{y+\alpha^2(y)}\le\frac{1}{2\sqrt{y}} \tag{4.15}$$

may be used in the integrand of (4.8) to show that

$$\alpha(x) \geq m - \frac{2\lambda}{3}\sqrt{x} \ . \tag{4.16}$$

Under the restrictions $\lambda \leq \frac{1}{4}$ and $a < 36m^2$, we have

$$\alpha(x) \geq M \equiv m - \frac{\sqrt{a}}{6} > 0 \tag{4.17}$$

for $x \in [0,a]$.

Let us define $\gamma(x,M)$ by means of the differential equation

$$\frac{d}{dx}\left[x^2 \frac{d\gamma}{dx}\right] = -\lambda\gamma(x)s(x) \ , \tag{4.18}$$

where

$$s(x) = 1 - \frac{M^2}{x+m^2}\theta(a-x) \ , \tag{4.19}$$

along with the boundary condition

$$\gamma(0,M) = m \ . \tag{4.20}$$

Also, we define the auxiliary function

$$G(x,M) = \frac{d}{dx}[x\gamma(x,M)] \ . \tag{4.21}$$

In Appendix B the functions γ and G are determined, and it is shown that, for $\lambda \leq \frac{1}{4}$ and $0 \leq x < \infty$,

$$0 < \gamma(x,M) \leq \alpha(x) \tag{4.22}$$

and

$$0 < G(x,M) \leq A(x) \ . \tag{4.23}$$

The domain of positivity of $\alpha(x)$, $\mathcal{D}(\alpha)$, consists of all $x \geq 0$, and $A(x)$ cannot have any finite zeros, for $\lambda \leq \frac{1}{4}$. The bounds (4.13) and (4.14) guarantee that $\alpha(x)$ and $A(x)$ approach zero as $x \to \infty$ for $\lambda \leq \frac{1}{4}$, since $\beta(x)$ and $B(x)$ vanish in that limit. Therefore, boundary condition (4.4) cannot be met for $\lambda \leq \frac{1}{4}$, and nontrivial solutions of (4.1) do not exist.

In summary, we have shown that the nonlinear integral equation (4.1) has only the trivial solution for $\lambda \leq \frac{1}{4}$, whereas for all $\lambda > \frac{1}{4}$ there is a nontrivial, positive solution of that equation.

V. DISCUSSION

Since the loop integral in the truncated Dyson-Schwinger equation (2.1) for the quark propagator is potentially divergent in the ultraviolet, the nonlinear integral equation (2.4) for the quark mass function $\alpha(x)$ is sensitive to large x. The linearized version (2.1) also exhibits a sensitivity to the small-x region. We have established that, in fact, such infrared sensitivity is not a feature of the nonlinear problem, in that the solutions of the nonlinear equation (4.1) are insensitive to small x. Equation (4.1), which has a sharp ultraviolet cutoff, exhibits dynamical symmetry breaking: only the trivial solutions exist for couplings $\lambda \leq \lambda_c$; and nontrivial solutions of (4.1) occur for every value $\lambda > \lambda_c$ — in the continuum limit $\Lambda \to \infty$, these solutions become a one-parameter family corresponding to solutions of the nonlinear differential equation (4.3) with arbitrary initial value $\alpha(0)$.

By making a less abrupt ultraviolet cutoff we would expect to obtain more restricted and more realistic solutions for the dynamical quark mass function in the appropriate continuum limit. From a physical point of view, one should make the ultraviolet truncation procedure consistent with the constraints imposed by asymptotic freedom. We shall explore this latter problem in a later paper.

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APPENDIX A

Selected properties of the hypergeometric function:[12] The hypergeometric function $F(a,b;c;z)$, under the restrictions

$$0 < \mathrm{Re}\,a < \mathrm{Re}\,c \quad \text{and} \quad -\infty < z \leq 0 \ , \tag{A1}$$

has the absolutely convergent integral representation

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt\, t^{a-1}(1-t)^{c-a-1} \times (1-tz)^{-b} \ . \tag{A2}$$

When the additional restriction is made that (a,b,c) are all real, the integrand in (A2) is non-negative and so

$$F(a,b;c;z) > 0 \ . \tag{A3}$$

The function F satisfies the differential equation

$$\left[z(1-z)\frac{d^2}{dz^2} + [c-(a+b+1)z]\frac{d}{dz} - ab\right] \times F(a,b;c;z) = 0 \ , \tag{A4}$$

is symmetric under interchange of arguments a and b, and obeys the relation

$$F(a,b,c;0) = 1 \ . \tag{A5}$$

The standard series expansion of F in powers of z converges absolutely for $|z| < 1$.

The hypergeometric function satisfies the recursive formulas:

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z) \ , \tag{A6}$$

$$\frac{d}{dz}[z^a F(a,b;c;z)] = az^{a-1}F(a+1,b;c;z) \ , \tag{A7}$$

and

$$\frac{d}{dz}[z^{c-1}F(a,b;c;z)]=(c-1)z^{c-2}F(a,b;c-1;z) . \tag{A8}$$

The behavior of the hypergeometric function at large (negative) z may be determined from the continuation formula:

$$F(a,b;c;z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} \times F(a,1+a-c;1+a-b;z^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{b} \times F(1+b-c,b;1+b-a;z^{-1}) . \tag{A9}$$

APPENDIX B

1. Establishing (4.13) and (4.14)

From the differential equations (4.3) for $\alpha(x)$ and (3.3) for $\beta(x)$, we obtain

$$\frac{d}{dx}\{x^2[\beta(x)\alpha'(x)-\alpha(x)\beta'(x)]\} = r(x) \equiv -\lambda x \alpha(x)\beta(x)\left[\frac{1}{x+\alpha^2(x)}-\frac{1}{x+m^2}\right] . \tag{B1}$$

For $x \in \mathcal{D}(\alpha)\cap\mathcal{D}(\beta)$, on which $\alpha(x)$ and $\beta(x)$ lie between 0 and m, we have

$$r(x)\le 0 . \tag{B2}$$

Integrating (B1) from 0 to x, we obtain

$$\beta(x)\alpha'(x)\le\alpha(x)\beta'(x) , \tag{B3}$$

the contribution from the lower limit being zero. Doing another integration, we get

$$\alpha(x)\le\beta(x) . \tag{B4}$$

As a consequence, $\mathcal{D}(\alpha)\subset\mathcal{D}(\beta)$, and relations (B2)–(B4) hold on $\mathcal{D}(\alpha)$. Furthermore, it follows from (B3) and (B4) that

$$A(x)=x\alpha'(x)+\alpha(x)\le x\beta'(x)+\beta(x)=B(x) . \tag{B5}$$

Relations (4.13) and (4.14) are thus established.

2. Solving Sturm-Liouville system (4.18) and (4.19)

For $x\le a$ the function $\gamma(x)$, which satisfies differential equation (3.3) with $m=M$, is

$$\gamma_r(x)=mF\left[\tfrac{1}{2}+\sigma,\tfrac{1}{2}-\sigma;2,-\frac{x}{M^2}\right] . \tag{B6}$$

For $x>a$, $\gamma(x)$ satisfies differential equation (2.6) and may be written

$$\gamma_l(x)=\begin{cases} A\left[\dfrac{a}{x}\right]^{1/2-\sigma}+B\left[\dfrac{a}{x}\right]^{1/2-\sigma} , & \lambda\ne\tfrac{1}{4} , \\ \left[\dfrac{a}{x}\right]^{1/2}\left[C+D\ln\dfrac{x}{a}\right], & \lambda=\tfrac{1}{4} . \end{cases} \tag{B7}$$

Matching at $x=a$, for $\lambda\ne\frac{1}{4}$, we obtain

$$A=\frac{a}{2\sigma}\frac{d}{dx}\left[\left[\frac{x}{a}\right]^{1/2+\sigma}\gamma_r(x)\right]\Bigg|_a = m\frac{1+2\sigma}{4\sigma}F\left[\tfrac{3}{2}+\sigma,\tfrac{1}{2}-\sigma;2;-\frac{a}{M^2}\right] \tag{B8}$$

and

$$B=-\frac{a}{2\sigma}\frac{d}{dx}\left[\left[\frac{x}{a}\right]^{1/2-\sigma}\gamma_r(x)\right]\Bigg|_a = -m\frac{1-2\sigma}{4\sigma}F\left[\tfrac{1}{2}+\sigma,\tfrac{3}{2}-\sigma;2;-\frac{a}{M^2}\right] ; \tag{B9}$$

whereas for $\lambda=\frac{1}{4}$ we obtain

$$C=mF\left[\tfrac{1}{2},\tfrac{1}{2},2,-\frac{a}{M^2}\right] \tag{B10}$$

and

$$2D=\frac{d}{dx}[x\gamma_r(x)]\,|_a=mF\left[\tfrac{1}{2},\tfrac{1}{2};1;-\frac{a}{M^2}\right] . \tag{B11}$$

We have used identities (A7) and (A8) in establishing these results.

3. Establishing (4.22) and (4.23) for $0<\lambda\le\frac{1}{4}$

The hypergeometric functions in (B6)–(B11) are positive because of (A3), so that $\gamma_l(x)$ is positive, the coefficients A, C, and D are positive, and B is negative. Therefore, the function $\gamma(x)$ is positive for all $x\ge 0$. Let us use Eq. (4.3) for $\alpha(x)$ and (4.18) for $\gamma(x)$ to obtain

$$\frac{d}{dx}\{x^2[\gamma(x)\alpha'(x)-\alpha(x)\gamma'(x)]\}=r(x) , \tag{B12}$$

where

$$r(x)=\lambda\alpha(x)\gamma(x)\left[\frac{\alpha^2(x)}{x+\alpha^2(x)}-\frac{M^2}{x+M^2}\theta(a-x)\right] . \tag{B13}$$

It follows from (4.17) that, for $x\in\mathcal{D}(\alpha)$,

$$r(x)\ge 0 . \tag{B14}$$

Upon integration of the right side of (B12), we obtain inequalities in the opposite sense to (B3)–(B5); namely,

$$\gamma(x)\alpha'(x)\geq\alpha(x)\gamma'(x)\ , \tag{B15}$$

$$\alpha(x)\geq\gamma(x)\ , \tag{B16}$$

and

$$A(x)=x\alpha'(x)+\alpha(x)\geq x\gamma'(x)+\gamma(x)=G(x)\ . \tag{B17}$$

Thus, $\alpha(x)$ is positive for all x, and relations (4.22) and (4.23) are proved.

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