Beta-function computation without the use of normal coordinates

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The complete one-loop regularization of a two-dimensional generalized σ model is performed by means of the Schwinger proper-time method. The result for the β functions is in agreement with the previous computations. The explicit expression of the wave-function regularization, required to make the theory finite off shell, is also obtained.

I. INTRODUCTION

The recent interest in two-dimensional generalized σ models is motivated mainly by their connection with the string theory.[1,2] In the above framework, one of the classical problems is to compute the β function of the σ model.[3]

This problem is not trivial. There are first some technical difficulties because the Lagrangian in general contains infinitely many interaction vertices with two space-time derivatives. Second, some troubles may arise on what is meant here by renormalizability of the models.[4] The point is that in general the infinitely many coupling constants of the model are not related to each other by some symmetry. Therefore, in carrying out the renormalization there are *a priori* infinitely many constants to be fixed. In standard terms, this means that the theory looks like a nonrenormalizable theory. However, as shown by Friedan,[3] with some extensions of the standard criteria of renormalizability it is still possible to study the properties of the models. Needless to say, the fact that the generalized σ models represent an extension of a standard renormalizable field theory makes the whole subject more interesting.

In Ref. 3 the form of the β function has been argued from the knowledge of the two-point function, whereas all the subsequent computations appearing in the literature make use of the normal-coordinate expansion.[5] Therefore, a complete computation of the the divergences of the model (on shell *and* off shell, in the terminology of Ref. 5) is still lacking. The purpose of the present paper is to fill this gap.

In particular, I compute all the one-loop ultraviolet divergences (that is, the quadratic and logarithmic divergences) for a σ model with torsion without making use of the normal-coordinate expansion. I use the Schwinger proper-time regularization;[6] this regularization is particularly convenient at the one-loop level and can also be extended to higher loops, of course.

The result for the β functions is essentially in agreement with the previous computations. More precisely, the regularization of the logarithmic divergences of the model induces a change in the metric of the target manifold and in the antisymmetric tensor. These modifications of the metric and of the antisymmetric tensor coincide with the already computed expressions of the β functions modulo an appropriate reparametrization of the target space which can be interpreted as a wave-function regularization. The precise expression of the wave-function regularization, which is required in order to make the theory finite off shell, was missing in all the previous computations.

II. THE MODEL

The action of the generalized σ model with torsion in flat two-dimensional Euclidean space-time is

$$S=\frac{1}{2}\int d^2x\,[G_{ij}(\phi)\partial_\mu\phi^i\partial_\mu\phi^j-i\epsilon^{\mu\nu}K_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j]\ . \tag{2.1}$$

The interactions of the field $\phi^i(x)$ $(i=1,2,\ldots,D)$ with itself are described by the two functions $G_{ij}(\phi)$ and $K_{ij}(\phi)$; $G_{ij}=G_{ji}$ just represents the "metric" of the target manifold, whereas the Wess-Zumino term in (2.1) is constructed in terms of the antisymmetric function $K_{ij}(\phi)$. In practice, the presence of a nontrivial Wess-Zumino term in (2.1) introduces a torsion on the target space

$$T_{ijk}=\frac{1}{2}(\partial_i K_{jk}+\partial_k K_{ij}+\partial_j K_{ki})\ . \tag{2.2}$$

The equations of motion following from (2.1) are

$$\mathcal{D}_\mu\partial_\mu\phi^i\equiv\partial_\mu\partial_\mu\phi^i+\Gamma^i_{jk}\partial_\mu\phi^j\partial_\mu\phi^k+iT^i_{jk}\epsilon^{\mu\nu}\partial_\mu\phi^j\partial_\nu\phi^k=0\ , \tag{2.3}$$

where

$$\Gamma^i_{jk}=\frac{1}{2}G^{il}(\partial_j G_{lk}+\partial_k G_{lj}-\partial_l G_{jk}) \tag{2.4}$$

is the usual Christoffel connection.

From the classical action (2.1) it is also easy to see that an infinitesimal coordinates transformation of the target space

$$\Delta\phi^i(x)=V^i(\phi(x)) \tag{2.5}$$

induces the following transformations on the metric $G_{ij}(\phi)$ and on the antisymmetric tensor $K_{ij}(\phi)$:

$$\Delta G_{ij}(\phi)=\nabla_i V_j(\phi)+\nabla_j V_i(\phi)\ , \tag{2.6}$$

$$\Delta K_{ij}(\phi)=2V_k(\phi)T^k_{ij}(\phi)\ . \tag{2.7}$$

Our purpose now is to compute all the one-loop ultraviolet divergences of the model (2.1). The one-loop effective action $\Gamma[\phi]$,

$$\tfrac{1}{2}\mathrm{Tr}\ln\left[\frac{\delta^2 S}{\delta\phi^i\delta\phi^j}\right]\ , \tag{2.8}$$

is regularized in the manner of Schwinger:[6]

$$\Gamma[\phi]=-\frac{1}{2}\int_\epsilon^\infty \frac{d\tau}{\tau}\mathrm{Tr}(e^{-\tau H})\ . \tag{2.9}$$

The trace Tr in Eq. (2.9) means the sum on the indices of the target space as well as the integral over the space-time coordinates. The H operator in Eq. (2.9) is of the form

$$H_{ij}(p,q)=p_\mu A_{ij}(q)p_\mu+i[p_\mu B^\mu_{ij}(q)+B^\mu_{ij}(q)p_\mu]+C_{ij}(q)\ , \tag{2.10}$$

where

$$p_\mu\equiv -i\frac{\partial}{\partial q^\mu} \tag{2.11}$$

and from the expression (2.1) one finds

$$A_{ij}(q)=A_{ji}(q)=G_{ij}(\phi(q))\ , \tag{2.12}$$

$$B^\mu_{ij}=-B^\mu_{ji}=\tfrac{1}{2}(\partial_i G_{jk}-\partial_j G_{ik})\partial^\mu\phi^k-iT_{kij}\epsilon^{\mu\nu}\partial_\nu\phi^k\ , \tag{2.13}$$

$$C_{ij}=C_{ji}=\tfrac{1}{2}\partial_i\partial_j G_{kl}\partial_\mu\phi^k\partial_\mu\phi^l-\tfrac{1}{2}\partial_\mu[(\partial_i G_{jk}+\partial_j G_{ik})\partial_\mu\phi^k]$$
$$-\frac{i}{2}(\partial_i T_{klj}+\partial_j T_{kli})\epsilon^{\mu\nu}\partial_\mu\phi^k\partial_\nu\phi^l\ . \tag{2.14}$$

The cutoff dependence of $\Gamma[\phi]$ can be obtained by considering the logarithmic derivative of Γ with respect of ϵ:

$$\epsilon\frac{d\Gamma}{d\epsilon}=\tfrac{1}{2}\mathrm{Tr}(e^{-\epsilon H})\ . \tag{2.15}$$

In the $\epsilon\to 0$ limit, only the first two terms of the asymptotic expansion of $\mathrm{Tr}[\exp(-\epsilon H)]$ survive:

$$\frac{1}{\epsilon}b_{-1}+b_0\ . \tag{2.16}$$

Clearly, b_{-1} is related to the quadratic divergences of the model and b_0 to the logarithmic divergences. Note that in this formalism it is not required that ϕ^i satisfies the equations of motion. In fact, the effective action $\Gamma[\phi]$ is, by definition, a functional of the classical and arbitrary functions $\phi^i(x)$. So, the divergences that we shall find are all the one-loop divergences of the theory.

Let us compute now b_{-1} and b_0. The computation is done with the general form of H given in Eq. (2.10); only at the end we substitute the particular values of A_{ij}, B^μ_{ij}, and C_{ij} displayed in Eqs. (2.12)–(2.14). In the following we also omit the target indices i and j, and write simply A, B^μ, C to indicate the corresponding matrices. One can write

$$\mathrm{Tr}(e^{-\epsilon H})=\int d^2x\,\mathrm{tr}\langle x\,|\,e^{-\epsilon H}\,|\,x\rangle\ , \tag{2.17}$$

where the trace tr refers to the target indices and the states $|\,x\rangle$ satisfy

$$q^\mu\,|\,x\rangle=x^\mu\,|\,x\rangle\ . \tag{2.18}$$

The amplitude $\mathrm{tr}\langle x\,|\exp[-\epsilon H(p,q)]\,|\,x\rangle$ is computed by using the method introduced in Ref. 7. The details of the computation can be found also in Ref. 8, where the amplitude $\langle x,i\,|\exp(-\epsilon H)\,|\,x,j\rangle$ is computed in presence of an arbitrary gravitational background. One first notices that

$$\mathrm{tr}\langle x\,|\,e^{-\epsilon H(p,q)}\,|\,x\rangle=\mathrm{tr}\langle 0\,|\,e^{-\epsilon H(p,q+x)}\,|\,0\rangle\ , \tag{2.19}$$

where

$$q^\mu\,|\,0\rangle=0\ . \tag{2.20}$$

Then, by means of a Taylor expansion of $H(p,q+x)$ in powers of q^μ one gets

$$H(p,q+x)=H_0(p;x)+H_{\mathrm{I}}(p,q;x)\ , \tag{2.21}$$

where

$$H_0=A(x)p^2\ , \tag{2.22}$$

and

$$H_{\mathrm{I}}=\partial_\nu A(x)p_\mu q^\nu p_\mu+\tfrac{1}{2}\partial_\nu\partial_\sigma A(x)p_\mu q^\nu q^\sigma p_\mu$$
$$+2iB^\mu(x)p_\mu+i\partial_\nu B^\mu(x)(p_\mu q^\nu+q^\nu p_\mu)$$
$$+C(x)+\cdots\ . \tag{2.23}$$

At this point, a perturbative expansion in H_{I} of the amplitude (2.19) is performed. The results, order by order in H_{I}, are the following.

Zero order:

$$(0)=\mathrm{tr}\langle 0\,|\,e^{-\epsilon H_0}\,|\,0\rangle=\frac{1}{(2\pi)^2}\int d^2k\,\mathrm{tr}(e^{-\epsilon Ak^2})$$
$$=\frac{1}{4\pi\epsilon}\mathrm{tr}(A^{-1})\ . \tag{2.24}$$

First order:

$$(\mathrm{I})=-\epsilon\int_0^1 d\alpha\,\mathrm{tr}\langle 0\,|\,e^{-\epsilon(1-\alpha)H_0}H_{\mathrm{I}}e^{-\epsilon\alpha H_0}\,|\,0\rangle$$
$$=-\frac{1}{4\pi}[\mathrm{tr}(A^{-1}C)+\tfrac{1}{3}\mathrm{tr}(A^{-1}\partial_\mu\partial_\mu A)]\ . \tag{2.25}$$

Second order:

$$(\mathrm{II})=\epsilon^2\int_0^1 \alpha\, d\alpha \int_0^1 d\beta\, \mathrm{tr}\langle 0|e^{-\epsilon(1-\alpha)H_0}H_1 e^{-\epsilon\alpha(1-\beta)H_0}H_1 e^{-\epsilon\alpha\beta H_0}|0\rangle$$
$$=\frac{1}{4\pi}[\tfrac{1}{12}\mathrm{tr}(A^{-1}\partial_\mu A A^{-1}\partial_\mu A)-\mathrm{tr}(A^{-1}B_\mu A^{-1}B_\mu)] \ . \tag{2.26}$$

There are no other contributions in the $\epsilon\to 0$ limit. Note also that in Eqs. (2.25) and (2.26) terms which vanish in the $\epsilon\to 0$ limit have been omitted.

From Eqs. (2.24)–(2.26) one obtains finally

$$\tfrac{1}{2}\mathrm{Tr}(e^{-\epsilon H})|_{\epsilon\to 0}=\frac{1}{8\pi\epsilon}\int d^2x\, \mathrm{tr}(A^{-1})-\frac{1}{8\pi}\int d^2x\, \mathrm{tr}(A^{-1}C+A^{-1}B_\mu A^{-1}B_\mu+\tfrac{1}{4}A^{-1}\partial_\mu A A^{-1}\partial_\mu A) \ . \tag{2.27}$$

III. THE β FUNCTIONS

By inserting in the expression (2.27) the particular values (2.12)–(2.14) of the matrix elements of A, B^μ, and C, one finds

$$\epsilon\frac{d\Gamma}{d\epsilon}=\frac{1}{8\pi\epsilon}\int d^2x\, G^{ii}(\phi(x))+\frac{1}{8\pi}\int d^2x[R_{ij}-T_{ikl}T_j^{kl}+\nabla_i(\nabla_j \ln\sqrt{G}-G^{kl}\partial_k G_{lj})]\partial_\mu\phi^i\partial_\mu\phi^j$$
$$-\frac{i}{8\pi}\int d^2x[-\nabla^k T_{kij}+T_{ij}^k(\nabla_k\ln\sqrt{G}-G^{ml}\partial_m G_{lk})]\epsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j \ , \tag{3.1}$$

where R_{ij} is the Ricci tensor constructed with the metric G_{ij} and ∇_i is the covariant derivative defined in terms of the Christoffel connection (2.4).

Equation (3.1) shows that Γ contains first a quadratic divergence

$$-\frac{1}{8\pi\epsilon}\int d^2x\, G^{ii}(\phi(x)) \ , \tag{3.2}$$

which corresponds to the well-known divergence of the integration measure in the Feynman path integral.[9] The divergence (3.2) is simply eliminated by the counterterm

$$S_c=\frac{1}{8\pi\epsilon}\int d^2x\, G^{ii}(\phi(x)) \ . \tag{3.3}$$

More precisely, in deriving the Feynman path integral from the canonical operator formalism one finds that, with the action (2.1), the measure of the integral over the paths $\phi^i(x)$ contains the counterterm (3.3) which cancels the quadratic divergence (3.2).

Concerning the logarithmic divergence, a comparison of Eq. (3.1) with the classical action (2.1) shows that

$$4\pi\beta_{ij}^G=R_{ij}-T_{ikl}T_j^{kl}+\nabla_i W_j+\nabla_j W_i \ , \tag{3.4}$$

$$4\pi\beta_{ij}^K=-\nabla^k T_{kij}+2W^k T_{kij} \ , \tag{3.5}$$

where

$$W_i=\tfrac{1}{2}(\nabla_i\ln\sqrt{G}-G^{kl}\partial_k G_{li}) \ . \tag{3.6}$$

Equations (3.4) and (3.5) coincide with the expressions of the β functions found by using the normal coordinates expansion[1–3,5] apart from the additive terms proportional to W_i, which represent the effects of a reparametrization on the metric and on the antisymmetric tensor, see Eqs. (2.6) and (2.7). A reparametrization has no effects on the geometry of the target space; however, a wave-function regularization with the parameters W_i shown in Eq. (3.6) is necessary to make the Green's functions of the theory one-loop finite. The presence of a noncovariant wave-function regularization is easily understood; the point is that the divergences of a field theory are not covariant under general field redefinitions.

Finally, suppose that we are not interested on $\Gamma[\phi]$ as a functional of $\phi^i(x)$ but only on the value of $\Gamma[\phi]$ when $\phi^i(x)$ satisfies the equations of motion. In this case the wave function regularization can be ignored. In fact, Eq. (3.1) can be written as

$$\epsilon\frac{d\Gamma}{d\epsilon}=\frac{1}{8\pi\epsilon}\int d^2x\, G^{ii}(\phi(x))$$
$$+\frac{1}{8\pi}\int d^2x[(R_{ij}-T_{ikl}T_j^{kl})\partial_\mu\phi^i\partial_\mu\phi^j$$
$$+i\nabla^k T_{kij}\epsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j]$$
$$-\frac{1}{8\pi}\int d^2x(\nabla_i\ln\sqrt{G}-G^{kl}\partial_k G_{li})\mathcal{D}_\mu\partial_\mu\phi^i \ . \tag{3.7}$$

By using the equations of motion (2.3), it is clear that the divergence proportional to W_i disappears.

IV. CONCLUSIONS

In the present paper I have computed all the one-loop divergences of a generalized σ model in two dimensions. The technical difficulty related to the fact that the model contains infinitely many interaction vertices has been bypassed by using the most powerful one-loop regularization: the Schwinger proper-time method. The resulting expressions for the β functions are in agreement with the previous computations. In addition, the explicit form of the required wave-function regularization has been obtained.

In conclusion, the results of the present paper confirm the validity of the previous computations and complete also the program of the one-loop regularization of a generalized σ model.

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[1]E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B261**, 1 (1985); Phys. Lett. **158B**, 316 (1985); Pis'ma Zh. Eksp. Teor. Fiz. **41**, 169 (1985) [JETP Lett. **41**, 206 (1985)].

[2]C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, Nucl. Phys. **B262**, 593 (1985).

[3]D. Friedan, Phys. Rev. Lett. **45**, 1057 (1980); Ann. Phys. (N.Y.) **163**, 318 (1985).

[4]I wish to thank C. Becchi and R. Stora for a useful discussion on this subject.

[5]I cannot mention here all the papers in which the normal coordinates expansion has been used to compute the β functions. It is too large a number of publications to be mentioned without forgetting some of them. I refer only to the papers in which (to my knowledge) the normal coordinates method has been introduced in field theory and to the papers in which the same method has been applied to compute the β functions for the σ models in two dimensions for the first time: G. Ecker and J. Honerkamp, Nucl. Phys. **B35**, 481 (1971); J. Honerkamp, *ibid.* **B36**, 130 (1972); L. Alvarez-Gaumè and D. Z. Freedman, Phys. Rev. D **22**, 846 (1980); L. Alvarez-Gaumè, D. Z. Freedman, and S. Mukhi, Ann. Phys. (N.Y.) **134**, 85 (1981).

[6]J. Schwinger, Phys. Rev. **82**, 664 (1951).

[7]E. Guadagnini, Pisa Report No. IFUP-TH 30/87 (unpublished).

[8]E. Guadagnini (in preparation).

[9]T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962).