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Quasi-invariance and central extensions

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Motivated by the theory of anomalies, the theory of classical dynamical systems described by quasi-invariant Lagrangians is reexamined in the present paper. A mathematical structure similar to the one describing anomalies in quantum field theory is found in systems for which an invariant Lagrangian description requires central extensions of the symmetry groups of the equations of motion. The case in which the symmetry group does not allow for nontrivial central extensions is also discussed.

I. INTRODUCTION: QUASI-INVARIANCE

In the study of dynamical systems described by a Lagrangian $\mathcal{L}(q,\dot{q})$ there are many cases where both the Lagrangian itself and the equations of motion derived from it are invariant under a point-transformation group G acting on the coordinates with its lift acting on the velocities as well.

It happens ever so often that we have a symmetry group G acting on $(q,\dot{q})$ which leaves the equations of motion unchanged but does not leave the Lagrangian invariant; instead, transformations change it by a total time derivative. These cases deserve special study as they are likely cases of central extensions of the group G. The best known examples of this are mechanical systems which have equations of motion which are invariant under the 10-parameter Galilei group but for which the Lagrangian is only quasi-invariant, changing by a total time derivative. In these cases we know that we have to consider the central extension into the 11-parameter extended Galilei group, the eleventh generator corresponding to the total mass of the system. We shall discuss the simplest of such Galilei systems, the free particle, in Sec. III.

The general situation is as follows. (As for notations, we follow as closely as possible Refs. 1 and 2.) Let the configuration space be an n-dimensional manifold Q, let TQ be the tangent bundle with natural coordinates $(q^j,\dot{q}^j)$, and let $\mathcal{L}$ be a Lagrangian function on it, $\mathcal{L}=\mathcal{L}(q,\dot{q})$. A group G acting on Q has infinitesimal generators, i.e., vector fields $X_1, X_2, \dots, X_\mu$, $\dim G=\mu$, such that they close on the Lie algebra of the group:

$$[X_j,X_k]=C^e_{jk}X_e\ . \tag{1}$$

This action of G on Q has a natural lift[1,2] to TQ with infinitesimal generators $\{X_j^T\}$ which still close on the same Lie algebra $\underline{G}$ of G, i.e.,

$$[X_j^T,X_k^T]=C^e_{jk}X_e^T\ . \tag{2}$$

In local coordinates if we write $X_j=A_j^k\partial/\partial q^k$, then the lifted vector fields X_j^T are given by

$$X_j^T=A_j^k\frac{\partial}{\partial q^k}+\frac{\partial A_j^k}{\partial q^l}\dot{q}^l\frac{\partial}{\partial \dot{q}^k}\ .$$

Now we can state our problem. Let us assume that

$$L_{X_j^T}\mathcal{L}=\frac{d}{dt}f_j\ , \tag{3}$$

where f_j is a function of q only. Since the Lagrangian is changed only by a total time derivative, the equations of motion

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^k}-\frac{\partial\mathcal{L}}{\partial q^k}=0 \tag{4}$$

are unchanged.

One may try to compensate this quasi-invariance of the Lagrangian by changing it with the addition of a total time derivative $\dot{F}$ of a function $F\in\mathcal{F}(Q)$, hoping that

$$\bar{\mathcal{L}}(q,\dot{q})=\mathcal{L}(q,\dot{q})-\frac{d}{dt}F \tag{5}$$

is actually invariant, i.e.,

$$L_{X_k^T}\mathcal{L}=0 \tag{6}$$

for any vector field X_j^T. It follows that we are looking for a function F such $L_{X_k^T}\dot{F}=\dot{f}_k$ or, equivalently,

$$L_{X_k}F=f_k+c_k \ , \tag{7}$$

where c_k is some additive constant. Now we can think of Eq. (7) as a system of partial differential equations for F, where X_k is given and f_k is given up to an additive constant c_k.

These equations may or may not admit of integration. Necessary and sufficient conditions for their integrability have been given[3] when Q is a homogeneous space of G; i.e., G acts on Q transitively. When the integrability conditions are not satisfied, we shall go to central extensions[4] of our group action in a way that is explained in the following sections, and obtain an invariant Lagrangian description on an enlarged space.

This paper is organized as follows.

In Sec. II we give a general construction, and apply it to an example where the Lagrangian is quasi-invariant, showing how the procedure gives rise to a central extension of the Euclidean group E(2).

In Sec. III we consider the corresponding problem on the phase space, where a novel problem arises because the kinematics and the dynamics, both carried by the Lagrangian function on the $(q,\dot{q})$ space, now are separately associated with the Poisson brackets and the Hamiltonian function. Here we show again how central extensions arise.

In Sec. IV we exhibit an example where central extensions are trivial and therefore strictly invariant Lagrangians can be defined. In Sec. V we discuss the case in which global projective group representations play a role in the analysis of quasi-invariant dynamical systems.

Section VI is devoted to the discussion of how one can build up a momentum map in a Lagrangian context, and how the problem of the central extensions is connected to the equivariance of the (Lagrangian) momentum map. The paper ends with a short section dealing with conclusions and further comments. A certain number of mathematical concepts employed in the text are defined and briefly discussed in the Appendix.

II. A LAGRANGIAN WITH QUASI-INVARIANCE

A systematic method of dealing with quasi-invariant Lagrangians is to enlarge the configuration space from Q to $Q\times R$, i.e., from coordinates (q) to $(q;s)$ where s is a new variable.

We now claim that it is possible to define an extended Lagrangian on $T(Q\times R)$, say

$$\tilde{\mathcal{L}}=\mathcal{L}-\dot{s} \tag{8}$$

(with abuse of notation we write $\mathcal{L}$ for what should be $\pi^*\mathcal{L}$, with π: $TQ\times TR\rightarrow TQ$) and a central extension of G, say $\overline{G}$, such that $\tilde{\mathcal{L}}$ is strictly invariant under the $\overline{G}$ action on $TQ\times TR$.

We first exhibit a central extension of the Lie algebra of G (in terms of vector fields on Q) to the Lie algebra of $\overline{G}$ in terms of vector fields on $Q\times R$.

We have the correspondence

$$X_k\rightarrow\tilde{X}_k=X_k+(f_k+c_k)\frac{\partial}{\partial s} \ . \tag{9}$$

By computing Lie-bracket commutators, we find

$$[\tilde{X}_j,\tilde{X}_k]=[X_j,X_k]+(L_{X_j}f_k-L_{X_k}f_j)\frac{\partial}{\partial s} \ .$$

To express the right-hand side in terms of $\tilde{X}$ we need a preliminary result.

From $L_{X_j^T}\mathcal{L}=\dot{f}_j$ we derive

$$L_{[X_j^T,X_k^T]}\mathcal{L}=L_{X_j^T}\dot{f}_k-L_{X_k^T}\dot{f}_j$$

or, by using Eq. (2),

$$C_{jk}^l L_{X_l^T}\mathcal{L}=C_{jk}^l\dot{f}_l=L_{X_j^T}\dot{f}_k-L_{X_k^T}\dot{f}_j$$

[this is similar to the Wess-Zumino consistency condition, and defines a one-cocycle (see the Appendix)] or equivalently, upon integration,

$$C_{jk}^l(f_l+c_l)=(L_{X_j}f_k-L_{x_k}f_j)+a_{jk}$$

the a_{ik}'s being integration constants.

Using this relation we can write

$$[\tilde{X}_j,\tilde{X}_k]=C_{jk}^l\tilde{X}_l-a_{jk}\frac{\partial}{\partial s} \tag{10}$$

showing that the new algebra generated by vector fields $\tilde{X}$ is actually a central extension of the Lie algebra by vector field X.

Remark. The formula preceding Eq. (10) shows clearly that when it is possible to solve for the arbitrary additive constants c_k the algebraic equations

$$C_{jl}^k c_k=a_{jl} \ ,$$

then it is possible, by adding such solutions to f_k, to get rid of a_{jk} in formula (10). This means that our central extension is a trivial one.[4] This technique has been used by Bargmann[5] in discussing the Lie algebra extensions of the given Lie algebra by the one-dimensional Lie algebra of U(1); he was able to exponentiate his results to the whole (simply connected) universal covering group.

Now it is a simple matter to show that $\tilde{\mathcal{L}}$ is invariant under the action of $\overline{G}$, indeed

$$L_{\tilde{X}_k^T}\tilde{\mathcal{L}}=L_{X_k^T}\mathcal{L}-L_{\tilde{X}_k^T}\dot{s}=0 \ . \tag{11}$$

Thus now with $\overline{G}$ acting on $T(Q\times R)$ we have assured that both the dynamics and the Lagrangian $\tilde{\mathcal{L}}$ are invariant.

As for the dynamics, we recognize that $\tilde{\mathcal{L}}$ is a constrained Lagrangian since

$$\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{s}}=-1 \ , \tag{12}$$

and the behavior of s as a function of time is entirely arbitrary.

We shall not analyze these "constrained Lagrangians" that we are going to get: symmetries in the presence of constrained Lagrangians have been analyzed by Mukunda[6] and, in the geometrical context, by Marmo, Mukunda, and Samuel.[7]

As an example of a quasi-invariant Lagrangian we consider a charged particle in a constant magnetic field: $Q=R^2$ and the symmetry group is E(2). TQ has coordinates $(x,y;\dot{x},\dot{y})$ and the Lagrangian function is

$$\mathcal{L}(x,y;\dot{x},\dot{y})=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)+eB(\dot{x}y-x\dot{y}) . \tag{13}$$

This Lagrangian is strictly invariant under rotations

$$X_j^T=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}+\dot{x}\frac{\partial}{\partial \dot{y}}-\dot{y}\frac{\partial}{\partial \dot{x}}$$

but under translations

$$X_x^T=\frac{\partial}{\partial x},\quad X_y^T=\frac{\partial}{\partial y}$$

the Lagrangian changes by total time derivatives

$$L_{X_x^T}\mathcal{L}=-eB\dot{y},\quad L_{X_y^T}\mathcal{L}=+eB\dot{x} .$$

Let us try then our enlargement from R^2 to $R^2\times R$. We enlarge the configuration space and extend the Lagrangian: then $(x,y;\dot{x},\dot{y})$ is replaced by $(x,y;\dot{x},\dot{y};s,\dot{s})$, $\mathcal{L}$ is replaced by $\tilde{\mathcal{L}}=\mathcal{L}-\dot{s}$, and for our vector fields we have

$$X_j^T\to\tilde{X}_j^T=X_j^T , \tag{14a}$$

$$X_x^T\to\tilde{X}_x^T=\frac{\partial}{\partial x}-eBy\frac{\partial}{\partial s}-eB\dot{y}\frac{\partial}{\partial \dot{s}} , \tag{14b}$$

$$X_y^T\to\tilde{X}_y^T=\frac{\partial}{\partial y}+eBx\frac{\partial}{\partial s}+eB\dot{x}\frac{\partial}{\partial \dot{s}} . \tag{14c}$$

Note that $[\tilde{X}_x,\tilde{X}_y]=2eB\partial/\partial s\neq 0$. Obviously on $(x,y;\dot{x},\dot{y})$ they act as before; and moreover

$$L_{\tilde{X}^T}\tilde{\mathcal{L}}=0 .$$

It is an interesting question whether we could have achieved this exact invariance of the Lagrangian without enlarging the configuration space. We shall show later that this is related to whether there are nontrivial central extensions of $\underline{G}$.

For the Lagrangian at hand it is a simple matter to show that there is no function $F=F(x,y)$ whose time derivative could be added to $\mathcal{L}$ to make it into a strictly invariant Lagrangian. Indeed $L_{X_x}F=-eBy$ and $L_{X_y}F=eBx$ would require that $L_{X_y}L_{X_x}F=-eB$ be equal to $L_{X_x}L_{X_y}F=eB$.

III. HAMILTONIAN REFORMULATION

The problem of quasi-invariance can be reformulated in terms of Hamiltonian variables, i.e., on T^*Q rather than on TQ. We consider directly our previous example. On TR^2 we start with

$$\mathcal{L}=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)+eB(\dot{x}y-x\dot{y})$$

and find the Legendre transformation from TR^2 to T^*R^2 to provide us with

$$p_x=\frac{\partial\mathcal{L}}{\partial\dot{x}}=m\dot{x}+eBy , \tag{15a}$$

$$P_y=\frac{\partial\mathcal{L}}{\partial\dot{y}}=m\dot{y}-eBx , \tag{15b}$$

and the Hamiltonian function

$$H=p_x\dot{x}+p_y\dot{y}-\mathcal{L}=\frac{1}{2m}[(p_x-eBy)^2+(p_y+eBx)^2] . \tag{15c}$$

To find out about invariance of this Hamiltonian we consider now the canonical lift of X_x and X_y. We recall[1,2] that lifts of vector fields on Q are defined to be vector fields on T^*Q which act on the q's in the same way as they act on Q, while the action on p is derived by imposing that

$$L_{X^*}\theta_0=0 .$$

We have called X^* the lift to T^*Q of X and θ_0 the canonical one-form $p_j dq^j$. In our case

$$X_x^*=\frac{\partial}{\partial x},\quad X_y^*=\frac{\partial}{\partial y} ,$$

and

$$X_j^*=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}+p_x\frac{\partial}{\partial p_y}-p_y\frac{\partial}{\partial p_x} .$$

It is clear that the Hamiltonian of our example has rotational invariance but not obvious translational invariance.

If we consider a different lift of vector fields from Q to T^*Q, for instance, those that we would get by "transporting" X^T from TQ to T^*Q by using the Legendre map,[8] we would get vector fields $\bar{X}_x$ and $\bar{X}_y$ which, in our specific example, are

$$\bar{X}_x=\frac{\partial}{\partial x}-eB\frac{\partial}{\partial p_y},\quad \bar{X}_y=\frac{\partial}{\partial y}+eB\frac{\partial}{\partial p_x} .$$

It is a simple matter to verify that

$$L_{\bar{X}_x}H=0,\quad L_{\bar{X}_y}H=0 ,$$

but now

$$L_{\bar{X}_x}\theta_0=-eBdy,\quad L_{\bar{X}_y}\theta_0=+eBdx .$$

On T^*Q we are presented with a situation which is different from the one on TQ [this is the manifestation of the well-known fact that, in terms of $(q,\dot{q})$, the Lagrangian possesses both dynamical and kinematical information; in terms of (q,p) the dynamical information is carried by the symplectic structure or its "potential" $p_j dq^j$]. When dealing with dynamical symmetries on the configuration space, we may either lift vector fields from Q to T^*Q by requiring them to preserve the one-form $p_j dq^j$ (but then they need not preserve the Hamiltonian) or by requiring them to preserve the Hamiltonian (but then they need not preserve the one-form $p_j dq^j$).

This last situation is reminiscent of "anomalies" in quantum field theory where the action functional loses an invariance since the measure ceases to be invariant.[9]

Again we can try to cure the pathology by adding first an additional variable to Q, i.e., we go from Q to $Q\times R$ and from $T^*(Q)$ to $T^*(Q)\times(R)$, and replacing the group G (or its Lie algebra $\underline{G}$) with $\bar{G}$ (or $\bar{\underline{G}}$).

Here we do not want to change the Hamiltonian as we did for the Lagrangian on TQ; thus we have to consider the lift of vector fields from Q to T^*Q so that we can preserve the Hamiltonian function and try to add "something" to the one-form so that the new one is actually invariant. Going back to our example we see that the vector fields

$$\tilde{X}_x=\frac{\partial}{\partial x}-eB\frac{\partial}{\partial p_y}-eB(y+c_x)\frac{\partial}{\partial s}\ ,$$

$$\tilde{X}_y=\frac{\partial}{\partial y}+eB\frac{\partial}{\partial p_x}+eB(x+c_y)\frac{\partial}{\partial s}\ ,$$

preserve

$$\tilde{\theta}=p_x dx+p_y dy-ds$$

and the Hamiltonian H.

This invariance has been achieved on $T^*Q\times R$, and $\tilde{\theta}$ gives rise to an exact contact structure[1,2] rather than to a symplectic structure. Had we used $\tilde{\mathcal{L}}=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+eB(\dot{x}y-x\dot{y})-\dot{s}$ we would have found

$$\theta_{\tilde{\mathcal{L}}}=\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{q}}dq=p_x dx+p_y dy-ds$$

if

$$p_x=\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}},\quad p_y=\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{y}},\quad p_s=\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{s}}=-1$$

had been used. Thus $\tilde{\theta}$ gives rise to a contact structure rather than a symplectic structure because of the way the Lagrangian constraint due to $\dot{s}$ appears in this Hamiltonian treatment.

If we want to have a Poisson brackets, we have to add two variables instead of only one; i.e., we enlarge our original space to $T^*Q\times T^*R$ rather than to $T^*Q\times R$. Let us try to do so and see what happens with our invariance requirements.

On $T^*R^2\times T^*R$ we have to use the one-form

$$\bar{\theta}=p_x dx+p_y dy+p_x ds\ .$$

We can consider now the canonical lift of X_x and X_y; i.e., we impose $L_{X_x^*}\bar{\theta}=0$, $L_{X_y^*}\bar{\theta}=0$. We find

$$X_x^*=\frac{\partial}{\partial x}-eBy\frac{\partial}{\partial s}+eBp_s\frac{\partial}{\partial p_y}\ ,$$

$$X_y^*=\frac{\partial}{\partial y}+eBx\frac{\partial}{\partial s}-eBp_s\frac{\partial}{\partial p_x}\ ,$$

but as far as the Hamiltonian is concerned we get

$$L_{X_x^*}H=\frac{eB}{m}(1+p_s)(p_y+eBx)\neq 0\ ,$$

$$L_{X_y^*}H=\frac{eB}{m}(1+p_s)(p_x-eBy)\neq 0\ .$$

We see that "on shell" $p_s=-1$ we recover the result that both the Hamiltonian and the one-form are invariant.

On this same carrier space, i.e., $T^*R^2\times T^*R$, we can consider a different lift of X_x and X_y, one which preserves H. If we do so, we find

$$\bar{X}_x=\frac{\partial}{\partial x}-eBy\frac{\partial}{\partial s}-eB\frac{\partial}{\partial p_y},\quad \bar{X}_y=\frac{\partial}{\partial y}+eBx\frac{\partial}{\partial s}+eB\frac{\partial}{\partial p_x}\ ,$$

and obviously

$$L_{\bar{X}_x}H=L_{\bar{X}_y}H=0\ ,$$

but

$$L_{\bar{X}_x}\theta=-eB(1+p_s)dy,\quad L_{\bar{X}_y}\theta=+eB(1+p_s)dx\ .$$

This lift is not canonical on $T^*R^2\times T^*R$, but it coincides with the previous one and gives invariance of θ on shell, where $p_s=-1$.

Before going to the following section, where we deal with actions which admit no nontrivial extensions, we discuss here the physically interesting case of Galilei invariance.[8] We take the simple free particle of mass M.

We consider, on $Q=R^4$ and TR^4 with coordinates $(t,q_j;\dot{t},\dot{q}_j)$, the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}M\dot{q}_j\dot{q}_j\ . \tag{16}$$

The action of $\underline{G}$, with G the Galilei group, is given by the ten vector fields on TR^4:

$$X_t=\frac{\partial}{\partial t},\quad X_j=\frac{\partial}{\partial q_j}\ , \tag{17a}$$

$$X_{jk}=q_j\frac{\partial}{\partial q_k}-q_k\frac{\partial}{\partial q_j}+\dot{q}_j\frac{\partial}{\partial\dot{q}_k}-\dot{q}_k\frac{\partial}{\partial\dot{q}_j}\ , \tag{17b}$$

$$K_j=t\frac{\partial}{\partial q_j}+\frac{\partial}{\partial\dot{q}_j}\ , \tag{17c}$$

having set $\dot{t}=1$.

Commutation relations are well known. In particular,

$$[X_j,K_k]=0\ . \tag{18}$$

We recognize that

$$L_{K_j}\mathcal{L}=M\dot{q}_j \tag{19}$$

is a total time derivative of a function on Q. We can also infer easily that the system of partial differential equations

$$L_{K_j}F=f_j,\quad L_{X_j}F=0,\quad L_{X_{jk}}F=0\ ,$$

where f_j is defined up to an additive constant, has no solution.

We proceed then to extend the Lagrangian to

$$\bar{\mathcal{L}}=\mathcal{L}-\dot{s}=\tfrac{1}{2}M\dot{q}_j\dot{q}_j-\dot{s} \tag{20}$$

and consider the extension of the boost generators

$$\bar{K}_j=t\frac{\partial}{\partial q_j}+\frac{\partial}{\partial\dot{q}_j}+Mq_j\frac{\partial}{\partial s}+M\dot{q}_j\frac{\partial}{\partial\dot{s}}\ , \tag{21}$$

while the other vector fields are unchanged. Under these augmented vector fields the extended Lagrangian $\tilde{\mathcal{L}}$ is strictly invariant.

To consider the same system on T^*R^4 and obtain the corresponding Hamiltonian formalism we have to lift the action of $\underline{G}$ to the phase space. We have to decide whether we want to preserve the one-form

$$\theta = p_j dq_j - E\, dt \tag{22}$$

or the Hamiltonian

$$H = \frac{1}{2M} p_j p_j - E \ . \tag{23}$$

Except for boosts, other generators offer no problem. As for the generators of boosts, we have

$$K_j^* = t\frac{\partial}{\partial q^j} + p_j \frac{\partial}{\partial E} \tag{24}$$

and

$$L_{K_j^*}\theta = 0, \quad L_{K_j^*}\mathcal{H} \neq 0 \ . \tag{25}$$

Alternatively we choose

$$\bar{K}_j = t\frac{\partial}{\partial q^j} + p^j \frac{\partial}{\partial E} + M\frac{\partial}{\partial p^j} \tag{26}$$

and get

$$L_{\bar{K}_j}\theta = M\, dq_j, \quad L_{\bar{K}_j}\mathcal{H} = 0 \ . \tag{27}$$

We can now enlarge the phase space by adding s, p_s, and extend the one-form to

$$\tilde{\theta} = \theta - M\, ds \ ; \tag{28}$$

i.e., we are considering a given value of the momentum conjugate to s.

Now, again, the extended vector fields

$$\tilde{K}_j = t\frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial E} + q_j \frac{\partial}{\partial s} + M\frac{\partial}{\partial p_j} \tag{29}$$

preserve both $\tilde{\theta}$ and H. Central extensions in connection with the Galilei group have been considered also by Aldaya and de Azcarraga.[10]

IV. TRIVIAL CENTRAL EXTENSIONS AND STRICTLY INVARIANT LAGRANGIANS

Consider the singular Lagrangian on TR^2 with coordinates $(x,y;\dot{x},\dot{y})$:

$$\mathcal{L} = \tfrac{1}{2}(\dot{x}-\dot{y})^2 + \tfrac{1}{2}(x-y)^2 - e(\dot{x}y - x\dot{y}) \ . \tag{30}$$

This Lagrangian is quasi-invariant under the translation $\partial/\partial x + \partial/\partial y = T$. There is one constraint:

$$p_x + p_y = e(x-y) \ . \tag{31}$$

The dynamical evolution preserves the constraint, so no secondary constraints obtain. Hence it is a first-class constraint.[8]

Because it is quasi-invariant, we can consider the usual extension to $TR^2 \times TR$,

$$\tilde{\mathcal{L}} = \mathcal{L} - \dot{s} \tag{32}$$

with extended vector field

$$\tilde{T} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} - e(x-y)\frac{\partial}{\partial s} - e(\dot{x}-\dot{y})\frac{\partial}{\partial \dot{s}} \ . \tag{33}$$

In this case, having only one vector field, we expect to be able to redefine a Lagrangian $\mathcal{L}'$ directly on TR^2 which is strictly invariant. How do we recover this Lagrangian $\mathcal{L}'$ from the extended one $\tilde{\mathcal{L}}$?

It is clear that what we are looking for is an embedding of R^2 into R^3, which is invariant under the action generated by $\tilde{T}$. At this point we consider the restriction of $\tilde{\mathcal{L}}$ to the tangent bundle of this submanifold and this restricted Lagrangian is the one which is strictly invariant under our initial group. Let us illustrate the procedure in this example.

An embedding of R^2 into R^3 which is a section of the projection $R^3 \to R^2$ will be defined as the level set of $s+F(x,y)$: $R^3 \to R$, with $F(x,y)$ any function of (x,y). An invariant embedding has to satisfy, in addition,

$$L_{\tilde{T}}[s+F(x,y)] = 0 \ . \tag{34}$$

In our case, we find

$$-e(x-y) + \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} = 0 \ . \tag{35}$$

We find a general solution for F to be

$$F = \frac{e}{2}(x-y)(x+y) + \xi(x-y) \ ,$$

where ξ is an arbitrary function of the argument.

The restriction of $\tilde{\mathcal{L}}$ to any one of these embeddings obtains by replacing $\dot{s}$ with $-\dot{F}$ so that

$$\mathcal{L}' = \mathcal{L} + \frac{e}{2}\frac{d}{dt}(x^2-y^2) + \frac{d}{dt}\xi(x-y) \ . \tag{36}$$

This Lagrangian is strictly invariant under $T = \partial/\partial x + \partial/\partial y$.

Now we can state our problem in general terms. Assume that on $TQ \times TR$ we have constructed a Lagrangian $\tilde{\mathcal{L}} = \mathcal{L} - \dot{s}$ which is strictly invariant under $\underline{\bar{G}}$ generated by $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_\mu$ with

$$\bar{X}_k = X_k^T - f_k \frac{\partial}{\partial s} - \dot{f}_k \frac{\partial}{\partial \dot{s}} \ , \tag{37}$$

where f_k is determined up to an additive constant and $\dot{f}_k$ is defined by

$$L_{X_k^T}\mathcal{L} = \dot{f}_k \ . \tag{38}$$

Now we ask how to find, if it is possible, a strictly invariant Lagrangian $\mathcal{L}'$ under the action of $\underline{G}$ starting from $\tilde{\mathcal{L}}$ invariant under $\underline{\bar{G}}$.

Our procedure suggests that we look for a section $Q \to Q \times R$:

$$(q) \to (q, s = -F(q)) \ ,$$

which is invariant under $\underline{G}$:

$$L_{\bar{X}_k}(s+F) = 0 \ . \tag{39}$$

The explicit computation gives

$$f_k + L_{X_k} F = 0 \ , \tag{40}$$

where f_k is determined from (38) up to an additive constant. By using mixed derivatives, the existence of a solution F requires the possibility to solve for

$$C^l_{jk} c_l = a_{jk} \ . \tag{41}$$

Thus we are back to the integrability conditions discussed in Ref. 3.

In case Eq. (41) cannot be solved we find that no invariant embedding is possible and then we are obliged to work with a central extension $\overline{G}$ of our original group G on the extended configuration space $(q;s)$. The informed reader will be reminded again of Bargmann's complete solution[5] to the problem of central extensions of Lie algebras and the condition under which nontrivial central extensions obtain.

As a dynamical realization of a case obtaining central extensions we consider the Lagrangian

$$\mathcal{L} = \eta_{\mu\nu} x^\mu \dot{x}^\nu \ , \tag{42}$$

where $\eta_{\mu\nu}$ is the symplectic tensor.[1]

This Lagrangian is invariant under the symplectic group[1] on the configuration space, but it is only quasi-invariant under translations. Again we need to consider the extension of the configuration space (x^ν) to $(x^\nu;s)$ and the infinitesimal generators of translations get extended into

$$\frac{\partial}{\partial x^\mu} + \eta_{\mu\nu} x^\nu \frac{\partial}{\partial s} + \eta_{\mu\nu} \dot{x}^\nu \frac{\partial}{\partial \dot{s}} \ . \tag{43}$$

For these generators no invariant section from (x^μ) to (x^μ, s) can be found.

V. PROJECTIVE GROUP REALIZATIONS

In all previous examples, extensions of Lie groups have been reduced to extensions of Lie algebras. We give now an example of a global projective representation of a group arising in connection with quasi-invariance of dynamical systems. Consider the system

$$\mathcal{L} = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2) + g \frac{x\dot{y} - \dot{x}y}{x^2 + y^2} \tag{44}$$

with the configuration space in the punctured plane $R^2 - \{0\}$. The Lagrangian is strictly invariant under proper rotations, but it is only quasi-invariant under reflections:

$$x \to -x, \quad y \to y, \quad \mathcal{L} \to \mathcal{L} - 2g \frac{\dot{x}y - x\dot{y}}{x^2 + y^2} \ . \tag{45}$$

The last term is a total time derivative:

$$\frac{d}{dt}\left[-2g \arctan \frac{y}{x} \right] = -2g \frac{\dot{x}y - x\dot{y}}{x^2 + y^2} \ . \tag{46}$$

The group operation is the exponential operator

$$P = \exp\left[\pi i \left[x \frac{\partial}{\partial x} + \dot{x} \frac{\partial}{\partial \dot{x}} \right] \right] \ . \tag{47}$$

We extend the Lagrangian $\mathcal{L} \to \tilde{\mathcal{L}} = \mathcal{L} - \dot{s}$ and extend the reflection operation by

$$\tilde{P} = \exp\left[i\pi \left[x \frac{\partial}{\partial x} + \dot{x} \frac{\partial}{\partial \dot{x}} + 2g\theta \frac{\partial}{\partial s} + 2g\dot{\theta} \frac{\partial}{\partial \dot{s}} \right] \right] \tag{48}$$

with

$$\theta \equiv \arctan \frac{y}{x}, \quad \dot{\theta} \equiv \frac{\dot{x}y - x\dot{y}}{x^2 + y^2} \ . \tag{49}$$

Looking for an invariant embedding we require

$$\tilde{P}[s + F(x,y)] = s + F(x,y) \ . \tag{50}$$

However,

$$\tilde{P}s = s - 2g\theta, \quad \tilde{P}F(x,y) = F(-x,y) \ . \tag{51}$$

Hence the invariant embedding solution is

$$F(x,y) = -g\theta + \xi(x^2, y) \ . \tag{52}$$

Choosing $\xi = 0$ we recover the Lagrangian

$$\overline{L} = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2) \ , \tag{53}$$

which is, of course, reflection invariant. Now the configuration space may be enlarged from the punctured plane to the entire R^2.

VI. A LAGRANGIAN MOMENTUM MAP AND THE PROBLEM OF EQUIVARIANCE

In this section we take up the same problem addressed in Secs. I and II from a slightly different point of view, which can help clarify the comparison with previous work on related subjects, which made extensive use of the concept of momentum map.[1,2,11,12]

Generally speaking, if we have an action of a Lie group G and Q, with the infinitesimal generators closing on the Lie algebra $\underline{G}$ of G as in Eq. (1), the corresponding tangent lifts need not preserve the Cartan one-form, nor the Lagrangian two-form associated with a given Lagrangian $L(q,\dot{q})$; i.e., in general,

$$L_{X_k^T} \theta_{\mathcal{L}} \neq 0, \quad L_{X_k^T} \omega_{\mathcal{L}} \neq 0 \ . \tag{54}$$

We recall that, on the contrary, the canonical lift $X^* \in \mathcal{X}(T^*Q)$ of a vector field $X \in \mathcal{X}(Q)$ is defined in such a way that

$$L_{X^*} \theta_0 = 0 \ , \tag{55}$$

$\theta_0 = p_j dq^j$ being the canonical one-form. In this way, the action of G on Q is lifted to a Hamiltonian action on T^*Q, and this is the way[1,2,13] the momentum map usually arises in physical examples.[11,12]

A possible way out of the situation outlined above is to try and redefine the lifting procedure in a suitable way. By mimicking to some extent the procedure leading to the momentum map, let us define functions $P_k \in \mathcal{F}(TQ)$

via

$$P_k = i_{X_k^T}\theta_{\mathcal{L}}, \quad k=1,\ldots,\mu . \tag{56}$$

Assuming now the Lagrangian L to be regular, and hence the Lagrangian two-form $\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$ to be nondegenerate, the equations

$$i_{\tilde{X}_k}\omega_{\mathcal{L}} = dP_k, \quad k=1,\ldots,\mu \tag{57}$$

uniquely define the set of vector fields $\tilde{X}_k \in \mathcal{X}(TQ)$. We now prove, as a preliminary result, that the map

$$X_k \to \tilde{X}_k \tag{58}$$

is both a lift and a Lie algebra isomorphism, i.e., that $\tilde{X}_k$ is π related[2] to X_k (π, $TQ \to Q$ being the canonical projection), and that

$$[\tilde{X}_i, \tilde{X}_j] = c_{ij}^r \tilde{X}_r \quad \forall i,j . \tag{59}$$

Indeed

$$i_{\tilde{X}_i}\omega_{\mathcal{L}} = d(i_{X_i^T}\theta_{\mathcal{L}}) = L_{X_i^T}\theta_L + i_{X_i^T}\omega_{\mathcal{L}}$$

and as

$$L_{X_i^T}\theta_{\mathcal{L}} = \theta_{L_{X_i^T}\mathcal{L}}$$

we obtain

$$i_{(\tilde{X}_i - X_i^T)}\omega_{\mathcal{L}} = \theta_{L_{X_i^T}\mathcal{L}} . \tag{60}$$

As the right-hand side (RHS) is a semibasic one-form (i.e., one which vanishes on vertical vectors), the vector field on the LHS must be vertical, and hence π related to the null vector field on Q. X_i^T being related to X_i, the same will hold true for $\tilde{X}_i$.

Remark. In view of this result,

$$P_i = i_{X_i^T}\theta_{\mathcal{L}} \equiv i_{\tilde{X}_i}\theta_{\mathcal{L}} ; \tag{61}$$

i.e., the P_i's can be redefined in terms of the new lifted vectors. Also

$$L_{\tilde{X}_i}\theta_{\mathcal{L}} = -i_{\tilde{X}_i}\omega_L + d(i_{\tilde{X}_i}\theta_{\mathcal{L}}) = -i_{\tilde{X}_i}\omega_{\mathcal{L}} + dP_i = 0 . \tag{62}$$

To prove now Eq. (59), let us evaluate

$$i_{[\tilde{X}_j,\tilde{X}_k]}\omega_{\mathcal{L}} = L_{\tilde{X}_j} i_{\tilde{X}_k}\omega_{\mathcal{L}} = d(L_{\tilde{X}_j}P_k) .$$

On the other hand, using Eq. (62),

$$L_{\tilde{X}_j}P_k = L_{\tilde{X}_j}(i_{\tilde{X}_k}\theta_{\mathcal{L}}) = i_{[\tilde{X}_j,\tilde{X}_k]}\theta_{\mathcal{L}} .$$

As it is easy to prove that the commutator $[\tilde{X}_j, \tilde{X}_k]$ differs from $[X_j^T, X_k^T]$ only by a vertical field, we have

$$i_{[\tilde{X}_j,\tilde{X}_k]}\theta_{\mathcal{L}} \equiv i_{[X_j^T, X_k^T]}\theta_{\mathcal{L}} = C_{jk}^r (i_{X_r}{}^T\theta_{\mathcal{L}})$$

and we have thus proved

$$L_{\tilde{X}_j}P_k = C_{jk}^r P_r . \tag{63}$$

Substituting this result back into Eq. (62), we obtain the desired proof. Actually, Eq. (63) proves something more. It constitutes also the proof of the *equivariance* of the momentum map with respect to the coadjoint representation of G on $\underline{G}^*$, the dual of the Lie algebra $\underline{G}$. We have thus shown that, by suitably redefining (in a way that depends on the Lagrangian, though) the lifting to TQ, one can associate an equivariant momentum map with the action of any Lie group G on Q.

An obvious drawback of this procedure, however, is that the (local) diffeomorphisms generated by the lifted vector fields need not preserve the second-order character of second-order vector fields, and in particular of the (second-order) dynamical vector field associated with the given Lagrangian, while the tangent lifts always do (they generate Newtonian[2] diffeomorphisms). It is of obvious interest to inquire under which conditions the two lifts can be made to coincide, i.e., for which choice of the Lagrangian, within the equivalence class of those differing from a given one by a total time derivative, can one assure that

$$\tilde{X}_j \equiv X_j^T \quad \forall j . \tag{64}$$

To make contact with the problems considered in the previous sections, we consider the case in which the Lagrangian is quasi-invariant under the (canonically) lifted action of G; i.e., it satisfies Eq. (3). Hence, as

$$\theta_{df/dt} = df \quad \forall f \in \mathcal{F}(Q) ,$$

Eq. (60) becomes

$$i_{(\tilde{X}_k - X_k^T)}\omega_{\mathcal{L}} = df_k . \tag{65}$$

Therefore, the fulfillment of Eq. (64) requires $f_k = 0$ (apart from additive constants) and we have the following.

Theorem. Let G be a Lie group acting on a manifold Q and let the tangent lifted action of G on TQ satisfy Eq. (3). The lifted action of G will give rise to an equivariant momentum map if and only if the Lagrangian is strictly invariant under the (lifted) action of G.

The problem of getting rid of the f_k's by modifying the Lagrangian brings us back to the discussion of the previous sections, and is seen here to be connected with the possibility of defining an equivariant momentum map in a Lagrangian context.

Remark. In the statement of the above theorem, we have assumed the Lagrangian to be altered (at most) by the addition of a total time derivative, as in Eq. (3). In a broader context, the Lagrangian momentum map will be equivariant whenever the RHS of Eq. (60) vanishes, and this will happen if and only if

$$L_{X_i^T}\mathcal{L} = \pi^* f_i + c_i, \quad f_i \in \mathcal{F}(Q), \quad c_i \in \mathbb{R} . \tag{66}$$

Requiring G to be a symmetry group for the dynamics is easily seen to imply $df_i = 0$, and we are therefore left with the further possibility of the Lagrangian being altered by the addition of a constant. The latter, unfortunately, cannot be disposed of or simply ignored, as the following example shows.

Let $Q = R^3$, and consider the Lagrangian of a point-particle subject to a constant gravitational field:"

$$\mathcal{L}=\tfrac{1}{2}m[(\dot{q}^1)^2+(\dot{q}^2)^2+(\dot{q}^3)^2]-mgq^3 . \quad (67)$$

An obvious symmetry for the dynamics is provided by the Euclidean group E(2) acting on the (q^1,q^2) plane, together with translations along the q^3 axis. The associated momentum map is equivariant, even though

$$L_{\partial/\partial q^3}\mathcal{L}=-mg . \quad (68)$$

In general, the fact that

$$L_{X_i^T}\mathcal{L}=c_i \quad (69)$$

for $X_i \underline{G}$, implies

$$L_{[X_i^T,X_j^T]}\mathcal{L}=0 . \quad (70)$$

Hence, it is only elements in $\underline{G}/[\underline{G},\underline{G}]$ which can give rise to additive constants (if any). A new (central) extension of the Lie algebra and of the carrier space (adding as many new nondynamical degrees of freedom as there are nonvanishing c_i's) will yield an extended, invariant Lagrangian. Again in the example considered above, we will need one extra new variable. Calling it τ, we can extend the Lagrangian into

$$\tilde{\mathcal{L}}=\mathcal{L}+mg\tau \quad (71)$$

and the generator of the translations along the q^3 axis into

$$\frac{\partial}{\partial q^3}-mg\frac{\partial}{\partial \tau} \quad (72)$$

leaving the other generators changed. $\tilde{\mathcal{L}}$ is now invariant under the central extension of the original Lie algebra.

VII. CONCLUSIONS

In this paper we have considered some classical dynamical systems which exhibit symmetries at the level of the equations of motion which are not (in a strict sense) symmetries for the Lagrangian(s) describing the systems themselves.

In order to find an invariant Lagrangian description, we have been forced to go to an enlargement of the carrier space (TQ in our case) to one to which a "nondynamical" degree of freedom has been added and where a central extension of the original symmetry group is acting. The corresponding Hamiltonian description has also been dealt with. To the best of our knowledge, none of these problems appears to have been treated in the existing literature.

In order to make a comparison with what is already known in terms of the momentum map, we have introduced a Lagrangian momentum map (which depends on the particular Lagrangian one is dealing with) and shown that the equivariance of such a momentum map is a necessary and sufficient condition for the invariance of the Lagrangian up to an additive constant.

Anomalies[9,14] have been mentioned here and there in the paper. We would like to point out that the way they are understood nowadays is that anomalies arise whenever there is no quantization procedure which is able to preserve the symmetries of a given classical system. Symmetries which are not preserved are said to be "anomalous." (Anomalies have been cast in the language of the momentum map by Bao and Nair.[15]) At the classical level, one usually concentrates on the equations of motion, and no particular attention is paid to the Lagrangian, nor to the action. Justified as it can be at a purely classical level, this viewpoint has several drawbacks when thought of in the broader context of physics (and not of classical physics alone), in which the classical description is believed to be an approximation of the full quantum one. Therefore, in this paper we have concentrated on the Lagrangian (and hence on the action) as the relevant object, and we have concerned ourselves with its variance properties because the action, or the Lagrangian, or, in the Hamiltonian framework, the Poisson brackets play a central role in the quantization procedure, and it may well be that the appearance of the same mathematical structures in this problem and in that of anomalies is more than a mere accident. At the moment, however, we are not (nor claim to be) able to show that the appearance of "classical anomalies" will imply the existence of genuine quantum anomalies, the way they are normally understood. The present paper should rather be seen as a possible way to pose such a question rather than a well-defined answer to it. We hope to return to a systematic discussion of the relationship between anomalies in quantum field theory and quasi-invariance in classical field theory in the near future.

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APPENDIX: LIE GROUP ACTIONS AND COCYCLES

In this Appendix we would like to exhibit an algebraic interpretation of our construction, i.e., one given in the framework of cohomologies, hoping that this can help the reader in putting into correspondence the mathematical objects and concepts employed in the description of anomalies[14–16] and those emerging from the classical analysis presented in this paper.

Let $\underline{G}$ be a Lie algebra, and let Λ by any G module (hence, there exists a given representation of $\underline{G}$ on Λ). A *k-cochain* is a multilinear alternating map

$$\alpha:\ \underbrace{\underline{G}\times\underline{G}\times\cdots\times\underline{G}}_{k\ \text{times}}\to\Lambda . \quad (A1)$$

The set of k-cochains will be indicated by Ω^k, and one usually identifies Λ with Ω^0. A *coboundary operator* ∂ can be defined on k-cochains as

$$\partial:\ \Omega^k\to\Omega^{k+1} ,$$
$$(\partial\alpha)(e_1,\ldots,e_{k+1})=\sum_j(-1)^{j+1}e_j\cdot\alpha(e_1,\ldots,\hat{e}_j,\ldots,e_{k+1})+\sum_{i<j}(-1)^{i+j}\alpha([e_i,e_j]e_1,\ldots,\hat{e}_i,\ldots,\hat{e}_j,\ldots) , \quad (A2)$$

where $e_j \in \underline{G}$, $j=1,\ldots,k+1$, and $e_j \cdot \alpha(\ )$ stands for the action of e_j on $\alpha(\ ) \in \Lambda$. We next define

(i) $\alpha \in \Omega^k$ is a k-coboundary iff $\alpha \in \mathrm{Im}\partial\,|_{\Omega^{k-1}} = Z^k \subseteq \Omega^k$, (A3)

(ii) $\alpha \in \Omega^k$ is a k-cocycle iff $\alpha \in \mathrm{Ker}\partial\,|_{\Omega^k} = B^k \in \Omega^k$. (A4)

As it is easy to prove, starting from (A2), that the coboundary operator satisfies

$$\partial \cdot \partial = 0 \tag{A5}$$

one concludes that

$$Z^k \subseteq B^k \ . \tag{A6}$$

The *kth cohomology group* attached to the coboundary ∂ is then defined as

$$\mathbb{H}^k \equiv B^k / Z^k \ . \tag{A7}$$

This is all more or the less standard material.[17–20] Let us now specialize to the case, considered in the paper, of the action of a Lie group G on a manifold Q, or of its lifted action on TQ. In both cases we have a Lie-algebra isomorphism between $\underline{G}$ (the Lie algebra of G) and the Lie algebra of the infinitesimal generators of either action. In the sequel, the $\underline{G}$ module Λ will be identified either with $\mathcal{X}_E^*(Q)$[1] (the exact one-forms on Q) or with the subset of $\mathcal{F}(TQ)$ made up of functions which are linear in the $\dot{q}$'s, i.e., functions which can be written in the form

$$f_\alpha(q,\dot{q}) = \langle \Gamma \mid \pi^* \alpha \rangle \tag{A8}$$

with $\alpha \in \mathcal{X}^*(Q)$ and Γ any second-order vector field. The corresponding action of $\underline{G}$ on Λ will be taken as the Lie derivative with respect to the infinitesimal generators of the G action. We also assume that the (lifted) action of G leaves the Lagrangian two-form $\omega_{\mathcal{L}}$ unchanged:

$$\underline{G} \ni e_i \to X_i^T \in \mathcal{X}(TQ) \ \text{ such that } L_{X_i^T} \omega_{\mathcal{L}} = 0 \tag{A9}$$

[this is of course true if Eq. (3) holds]. Equation (A9) implies that $L_{X_i^T} \theta_{\mathcal{L}}$ is a closed one-form. Under the simplifying assumption that closed one-forms are also exact, we then have

$$L_{X_i^T} \theta_{\mathcal{L}} = df_{e_i}, \ \ f_{e_i} \in \mathcal{F}(Q) \ . \tag{A10}$$

As $L_{X_i^T} \theta_{\mathcal{L}} \equiv \theta(L_{X_i^T} \mathcal{L})$, comparison with Eq. (3) (note that the same simplifying assumptions have been implicitly made throughout this paper) shows that f_{e_i} coincides with the function which has been called f_i there.

Altogether, we have defined a one-cochain

$$\mu\colon \ \underline{G} \to \mathcal{X}_E^*(Q) \ \text{ by } \mu\colon \ e_i \to df_{e_i} = L_{X_i^T} \theta_{\mathcal{L}} \tag{A11}$$

and we now prove that μ is actually a cocycle. Indeed

$$\begin{aligned}(\partial\mu)(e_i,e_j) &= L_{X_i^T} df_{e_j} - L_{X_j^T} df_{e_i} - df_{[e_i,e_j]} \\ &= L_{X_i^T} L_{X_j^T} \theta_{\mathcal{L}} - L_{X_j^T} \theta_{\mathcal{L}} - L_{[X_i^T,X_j^T]} \theta_{\mathcal{L}} \\ &\equiv 0 \ .\end{aligned} \tag{A12}$$

We recall that the one-cocycle condition is usually called the Wess-Zumino consistency condition.[15]

Alternatively, we can define a one-cocycle with values in the functions linear in the $\dot{q}$'s via

$$\mu'\colon \ e_i \to L_{X_i^T} \mathcal{L} = \frac{d}{dt} f_{e_i} \ . \tag{A13}$$

That μ' is indeed a cocycle follows immediately from

$$\begin{aligned}(\partial\mu')(e_i,e_j) &= L_{X_i^T} \mu'(e_j) - L_{X_j^T} \mu'(e_i) - \mu'([e_i,e_j]) \\ &= L_{X_i^T} L_{X_j^T} \mathcal{L} - L_{X_j^T} L_{X_i^T} \mathcal{L} - L_{[X_i^T,X_j^T]} \mathcal{L} = 0 \ .\end{aligned} \tag{A14}$$

Note that, in both cases, changing the Lagrangian by the addition of a total time derivative changes the one-cocycle by the addition of a coboundary. A class of equivalent (in the sense employed in the main text) Lagrangians singles out therefore a cohomology class rather than a single cocycle.

It is possible to associate with either one of the one-cocycles defined above a two-cocycle with values in R (viewed as a trivial $\underline{G}$ module). Indeed, Eq. (A12) can be rewritten as

$$d(L_{X_i} f_{e_j} - L_{X_j} f_{e_i} - f_{[e_i,e_j]}) = 0 \tag{A12'}$$

and Eq. (A14) as

$$\frac{d}{dt}(L_{X_i} f_{e_j} - L_{X_j} f_{e_i} - f_{[e_i,e_j]}) = 0 \ . \tag{A14'}$$

(Recall that here "d/dt" stands for the Lie derivative with respect to any second-order vector field. The derivation of Eq. (A14') implies an interchange of some of the X_i^T's with d/dt. This is legitimate, insofar as the commutator $[X_i^T, d/dt]$ is a vertical field, and then gives zero when acting on the f_{e_i}'s, which are functions of the q's alone.) In both cases we find

$$L_{X_i} f_{e_j} - L_{X_j} f_{e_i} - f_{[e_i,e_j]} = C_{ij} \in \mathbb{R} \ . \tag{A15}$$

The map

$$c\colon \ \underline{G} \times \underline{G} \to \mathbb{R} \ \text{ by } c\colon \ (e_i,e_j) \to C_{ij} \tag{A16}$$

is a two-cochain. As g acts trivially on R by assumption, Eq. (A2) yields, in this case,

$$\partial c(e_i,e_j,e_k) = -[c([e_i,e_j],e_k) + \text{c.p.}] \ . \tag{A17}$$

Substitution of (A15) into (A17) yields

$$\partial c = 0 \ . \tag{A18}$$

Therefore, c is a two-cocycle, as stated above.

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