Generalized Wigner functions in curved spaces: A new approach

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It is well known that, given a quantum field in Minkowski space, one can define Wigner functions $f_W^N(x_1,p_1,\ldots,x_N,p_N)$ which (a) are convenient to analyze since, unlike the field itself, they are c-number quantities and (b) can be interpreted in a limited sense as "quantum distribution functions." Recently, Winter and Calzetta, Habib and Hu have shown one way in which these flat-space Wigner functions can be generalized to a curved-space setting, deriving thereby approximate kinetic equations which make sense "quasilocally" for "short-wavelength modes." This paper suggests a completely orthogonal approach for defining curved-space Wigner functions which generalizes instead an object such as the Fourier-transformed $f_W^1(k,p)$, which is effectively a two-point function viewed in terms of the "natural" creation and annihilation operators $a^\dagger(p-\frac{1}{2}k)$ and $a(p+\frac{1}{2}k)$. The approach suggested here lacks the precise phase-space interpretation implicit in the approach of Winter or Calzetta, Habib, and Hu, but it is useful in that (a) it is geared to handle any "natural" mode decomposition, so that (b) it can facilitate exact calculations at least in certain limits, such as for a source-free linear field in a static spacetime.

I. MOTIVATION

The statistical description of a classical or quantum field, viewed as an infinite collection of oscillators, takes as its starting point an object ρ, the evolution of which is governed by a Liouville equation

$$\partial\rho/\partial t=-L\rho \ . \tag{1.1}$$

Classically, ρ is a distribution function for the oscillators, and the Liouville operator L is realized as a Poisson brackets defined with respect to the (possibly time-dependent) Hamiltonian $H(t)$, i.e., $L\rho\equiv\{H,\rho\}$. In quantum field theory, ρ is instead a density matrix and L is realized as a commutator, so that $L\rho\equiv[H,\rho]$.

Viewed in this abstract sense, the classical and quantum theories exhibit a formal similarity, but, nevertheless, these theories are very different at an interpretative level. The classical ρ is to be understood as a real probability density defined in an appropriate phase space, whereas the quantum ρ must be considered instead an operator defined in an abstract Hilbert space. The motivation underlying the introduction of Wigner functions is the definition of c-number objects f_W^N depending on the spacetime point $x^\alpha\equiv(x^a,t)$ and a momentumlike quantity p_α, or possibly N-tuples thereof, which, in at least a limited sense, correspond to the reduced N-particle distribution functions f_C^N of classical statistical mechanics for a collection of point masses:

$$f_W^N(x_1,p_1,\ldots,x_N,p_N)\leftrightarrow f_C^N(x_1,p_1,\ldots,x_N,p_N) \ . \tag{1.2}$$

It is important to stress that such an f_W^N *cannot* be interpreted strictly as a "quantum distribution function" as it need not be positive semidefinite; but, in most other ways, these f_W^N's *do* behave like distribution functions. Thus, at least in certain limits, f_W^N and f_C^N satisfy evolution equations which are formally identical, and they can be used identically to define average quantities such as the average energy or momentum densities.

This general program was first introduced by Wigner[1] in the framework of second-quantized particle mechanics, and has since been developed by several authors[2,3] for quantum field theory in flat space. Given recent interest in quantum field theory in more general spacetimes, it would also seem natural to introduce a notion of Wigner functions in curved spaces. A naive, straightforward generalization is, however, impossible. The standard flat-space construction of Wigner functions exploits Fourier-transform techniques, but, as is well known, such transforms are not defined globally in a spacetime lacking space and time translational symmetries.

One possible curved-space generalization, first considered by Winter[4] and developed further by Calzetta, Habib, and Hu,[5] exploits the idea (implicit in the "local-momentum-space" analysis of Bunch and Parker[6]) that, in the neighborhood of any given point, where the spacetime can be treated as nearly flat, a Fourier transform does make sense at least approximately. What these authors do is first present an abstract definition of f_W^1 in terms of a certain "exponential function" exp(**P**) and then, by interpreting that function via a Riemann-normal-coordinate expansion, generate a perturbative but covariant "quasilocal" expression for f_W^1. There is a deep and fundamental connection between this program and standard point-separation techniques; and thus, not surprisingly, this approach is quite useful when considering "short-wavelength" modes. Thus, e.g., both Winter[4] and Calzetta, Habib, and Hu,[5] have shown that, in a well-defined limit (basically to lowest order in the normal-coordinate expansion) the quantum one-particle

f_W^1 for a free scalar field satisfies the same collisionless Boltzmann (i.e., Vlasov) equation as does the one-particle distribution function f_C^1 for a collection of noninteracting particles. Similarly, they have noted (see also Ref. 7) that, to the same nontrivial order, the average quantum stress energy $T_{\mu\nu}$ takes the same form as a functional of f_W^1 as did the classical $T_{\mu\nu}$ as a functional of the classical f_C^1. The first corrections to these formulas, appropriate for "longer wavelengths," were also computed and found to be proportional to $\hbar^2$. These results demonstrate, in particular, the sense in which it is reasonable to model the evolution of noninteracting quantum "inos" in the early Universe by a classical distribution function.

Despite these successes, however, there are difficulties with that basic approach. Most obvious is the fact that, typically, one cannot generate a useful and exact evolution equation for f_W^1 in even the simplest nontrivial cases, such as for a static but curved spacetime or a Friedmann cosmology (de Sitter space and the Einstein universe appear the only notable exceptions[5,7]). This means, as a practical matter, that exact computations are impossible; and indeed, even approximate computations at anything beyond the lowest order of the normal-coordinate expansion rapidly become unmanagable. Less obvious, perhaps, but even more significant is the fact that this general sort of analysis may break down on large scales where global features such as topology become important. Thus, Winter's entire approach entails a fundamental restriction on where the two-point function $\langle \Phi(x'')\Phi(x') \rangle$ can have nonvanishing support. And indeed, Calzetta, Habib, and Hu have noted explicitly that their analysis can make sense only in the absence of caustics.

These difficulties reflect the fact that plane waves are not "natural" in an arbitrary curved space, and, consequently, it is reasonable to look for a more abstract—and fundamental—approach which circumvents altogether the notion of a Fourier transform. Such is the object of this paper.

The basic idea is simple. In flat space, it is customary to think of the Fourier-transform prescription for generating the f_W's as simply a convenient trick which yields a useful object of interest. This was, e.g., the underlying philosophy of Wigner's pioneering work.[8] And, in fact, one is inclined often times to proceed one step further by replacing the Wigner function $f_W^1(x,p)$ by its Fourier transform $f_W^1(k,p)$, which, for a real field, in terms of creation and annihilation operators $a^\dagger$ and a, take the form

$$f_W^1(k,p)=\mathrm{Tr}\rho a^\dagger_{p-k/2}a_{p+k/2} \;, \tag{1.3}$$

this being effectively a double Fourier transform of the two-point function $\langle \Phi(x+\frac{1}{2}x')\Phi(x-\frac{1}{2}x')\rangle$. [Indeed, in the context of a $3+1$ splitting, where one Fourier transforms x^a but not t, it is actually convenient to view $f_W^1(k,p)$, rather than $f_W^1(x,p)$, as the fundamental object.[9]]

The key observation now is that an object such as $\mathrm{Tr}\rho a_i^\dagger a_j$ makes perfect sense for an arbitrary mode decomposition, be they plane waves, as is natural in flat space, or be they something quite different, as would be natural in some other spacetime. In flat space, plane waves assume (at least superficially) a special importance in that they are the eigenfunctions of the ordinary Laplacian (or d'Alembertian) entering into the field equation (which implies that, for a free field, each mode evolves independently), and one might argue that it is ultimately for this reason that a plane-wave or Fourier decomposition proves useful. In curved spaces, the ordinary Laplacian is replaced by a more general differential operator Δ, and it seems natural (see, e.g., Ref. 10) to use the spectral decomposition of that Δ to define the $a^\dagger$'s and a's. This is especially true when, as in a static spacetime, $a_i^\dagger a_i$ can be interpreted as an abstract number operator.

Such is the program to be developed here. Specifically, given a mode decomposition induced by the generalized Laplacian Δ, one is instructed to define "generalized Wigner functions"

$$\Theta \equiv \mathrm{Tr}\rho a^\dagger_{i_1}\cdots a^\dagger_{i_p}a_{j_1}\cdots a_{j_q} \tag{1.4}$$

for arbitrary normal-ordered operators constructed from the $a^\dagger$'s and a's.

The key point is that an object such as f_W^1 in flat space can be interpreted at two different levels. At one level, it is simply a Fourier transform of the two-point function, which, because it is a c number, is comparatively easy to manipulate. At a deeper level, there is the additional fact that it can also be viewed (within limits) as a phase-space number density for "quantum particles." The latter of these interpretations relies crucially on special properties of Minkowski space and, in a general curved-space setting, can be recovered only in a "quasilocal" approximation. The former interpretation is much more general and, through the implementation of the abstract Θ's, can be implemented here without difficulty in a curved-space setting.

The objective of this paper is to exploit the former more general (albeit more modest) c-number picture without special emphasis on the possibility of a phase-space interpretation. It seems, however, reasonable to expect that contact can be established with the work of Winter[4] and Calzetta, Habib, and Hu[5] by noting that, in the neighborhood of any given spacetime point, the modes of the field may be treated as nearly plane waves.

Section II illustrates the basic approach for the special case of a minimally coupled, real scalar field Φ in a static spacetime, demonstrating that the Θ's are comparatively easy to manipulate, so that they may be used to calculate quantities of physical interest. Section III then concludes by contrasting the underlying philosophy here with that implicit in Winter[4] and Calzetta, Habib, and Hu.[5]

II. BASIC SETUP

Consider a real, minimally coupled, massive (m) scalar field Φ in a background spacetime $(M,g_{\mu\nu})$, in the presence of a classical source $\sigma(x)$, characterized by an action

$$S=-\tfrac{1}{2}\int d^4x(-g)^{1/2}[\nabla_\mu\Phi\nabla^\mu\Phi+m^2\Phi^2+\sigma(x)\Phi^2] \;. \tag{2.1}$$

Suppose then that the spacetime is static (Killing vector $\partial/\partial t$) and of topology $\Sigma\times R$, admitting a foliation into a family of everywhere spacelike hypersurfaces, so that the line element may be taken of the form

$$ds^2=-N^2(x^c)dt^2+g_{ab}(x^c)dx^a dx^b \quad (a,b,c=1,2,3)\ , \tag{2.2}$$

where $g_{\mu\nu}$ is independent of coordinate time t. Standard manipulations then lead to a Hamiltonian

$$H=\tfrac{1}{2}\int d^3x\,(-g)^{1/2}N^{-2}\{[(-g)^{-1/2}N^2\Pi]^2 -\Phi\Delta\Phi+N^2\sigma(x^a,t)\Phi^2\}\ , \tag{2.3}$$

where

$$\Delta\equiv N^2[(-g)^{-1/2}\partial_a(-g)^{1/2}g^{ab}\partial_b-m^2] \tag{2.4}$$

is a "natural" generalization of the flat-space Laplacian. The objective is to formulate a quantum theory by imposing the canonical equal-time commutation relations

$$[\Phi(x,t),\Phi(x',t)]=0=[\Pi(x,t),\Pi(x',t)]$$

and

$$[\Phi(x,t),\Pi(x',t)]=i\delta^{(3)}(x-x')\ . \tag{2.5}$$

It is convenient to view Δ as an operator defined on a real Hilbert space with inner product

$$\langle\xi,\eta\rangle\equiv\int d^3x(-g)^{1/2}N^{-2}\xi(x^a)\eta(x^b)\ , \tag{2.6}$$

with respect to which $-\Delta$ is symmetric and positive:

$$\langle\xi,-\Delta\eta\rangle=\int d^3x(-g)^{1/2}(g^{ab}\partial_a\xi\partial_b\eta+m^2\xi\eta)\ . \tag{2.7}$$

Assuming the existence of a complete spectral decomposition of Δ, one can then write Φ as a sum of modes

$$\Phi(x^a,t)=\sum_i q_i(t)\psi_i(x^a)\ , \tag{2.8}$$

where

$$-\Delta\psi_i(x^a)=\omega_1^2\psi_i(x^a) \tag{2.9}$$

with $\omega_i^2\geq 0$. Here, of course, $\sum_i$ is to be interpreted as an abstract sum (formally a Stieltjes integral) over all the "eigenfunctions" for both the point and continuous spectra of Δ; but, for notational convenience, the analysis below proceeds as if the spectrum were purely point. The symmetry of Δ implies that one can impose the normalization

$$\langle\psi_i,\psi_j\rangle=\delta_{ij}\ . \tag{2.10}$$

And thus, in terms of the coefficients

$$\sigma_{ij}=(4\omega_i\omega_j)^{-1/2}\sum_k\langle\sigma,\psi_k\rangle\langle\psi_i,\psi_j\psi_k\rangle\ , \tag{2.11}$$

one is led to a mode Hamiltonian

$$H=\sum_i\tfrac{1}{2}(p_i^2+\omega_i^2q_i^2) +\sum_i\sum_j 2(\omega_i\omega_j)^{1/2}\sigma_{ij}(t)q_iq_j\equiv H_0+H_\sigma\ , \tag{2.12}$$

for which the only nontrivial commutation relation is $[q_k,p_{k'}]=i\delta_{k,k'}$.

At this stage, it is natural to define creation and annihilation operators

$$a_i^\dagger=(2\omega_i)^{-1/2}(\omega_iq_i-ip_i)$$

and

$$a_i=(2\omega_i)^{-1/2}(\omega_iq_i+ip_i)\ , \tag{2.13}$$

and to construct a vacuum state $|0\rangle$ and the associated Fock-space representation by demanding that, for all i $a_i|0\rangle\equiv 0$. This construction is well defined since, given the Killing vector $\partial/\partial t$, a global positive-negative frequency decomposition is possible. This means, in particular, that $N_i\equiv a_i^\dagger a_i$ can be interpreted as a *bona fide* "number operator," and that, more generally, any operator A involving $a_i^\dagger$ and a_j can be interpreted in terms of the creation and destruction of "quanta," so that the expectation value of A has definite physical significance. In a more general, nonstatic spacetime, mode decompositions and the like can still be defined,[11] but the particle interpretation becomes highly suspect. In any event, in terms of the $a^\dagger$'s and a's,

$$H_0=\sum_i\omega_i(a_i^\dagger a_i+\tfrac{1}{2})$$

and

$$H_\sigma=\sum_i\sum_j\sigma_{ij}(a_i^\dagger+a_i)(a_j^\dagger+a_j)\ , \tag{2.14}$$

(2.5) leading to the commutation relations

$$[a_i^\dagger,a_j^\dagger]=0=[a_i,a_j]$$

and

$$[a_i^\dagger,a_j]=\delta_{ij}\ . \tag{2.15}$$

The object now, as illustrated in Ref. 12, is to implement a statistical description of the field, considering as the fundamental object a density matrix ρ, for a state either mixed or pure, defined in the Fock space appropriate for the infinite set of modes. Formally, therefore, $\rho=\rho(\{a_i^\dagger,a_j\};t)$, and in a Schrödinger picture, the evolution of ρ will be governed by a unitary quantum Liouville equation

$$\partial\rho/\partial t=-[H,\rho]\ . \tag{2.16}$$

Given this basic setup, one is now in a position to define the desired Θ's. Specifically, given any operator $A(t)$ constructed from the basic creation and annihilation operators $\{a_i^\dagger,a_j\}$, one is instructed to define a "generalized Wigner function"

$$\Theta[A]\equiv\mathrm{Tr}\rho{:}A{:}\ , \tag{2.17}$$

where : : denotes a normal ordering. These Θ's are nothing other than the expectation values of the A's with respect to the time-dependent density matrix $\rho(t)$. Thus, as special examples, one has the average number of particles $\langle N_i \rangle$ in each mode, or any other $\langle f(N_i)\rangle$, or even the average Hamiltonian $\langle H(t)\rangle$, with all quantities of definite physical significance.

One important fact to observe is that, modulo technical complications reflecting the fact that the field has an infinite number of degrees of freedom, a knowledge of the values of all possible Θ's is completely equivalent to a knowledge of ρ. This is simply the quantum analogue of the better known fact that a distribution function is characterized uniquely by its moments. The key point, then, is that a knowledge of the values of some limited set of Θ's corresponds to a partial knowledge of the state of the system, and that, for special choices of Θ's, one recovers precisely the same amount and sort of information as is encapsulated in the flat-space Wigner functions f_W. Thus, e.g., in flat space, f_W^1 contains precisely the information encoded in the two-point function $\langle \Phi(x+\frac{1}{2}x')\Phi(x-\frac{1}{2}x')\rangle$; and, quite clearly, precisely the same information is encoded in the bilinear generalized Wigner functions such as $\Theta[a_i^\dagger a_j]$. Indeed, with a simple relabeling, the $f_W^1(k,p)$ of (1.3) is identical to $\Theta[a_j^\dagger a_i]$.

Another important characteristic of the Θ's is that, being c numbers, they are comparatively easy to manipulate, so that, e.g., one can derive simple equations for their evolution. Suppose that A is already normal ordered. It then follows immediately from (2.16) that

$$d\Theta[A]/dt=\Theta[-i[A,H]]+\Theta[\partial_t A]\ . \tag{2.18}$$

This relation assumes an especially simple form when $H_\sigma \equiv 0$. Consider, e.g., an operator

$$A=c(t)\prod_i a_i^\dagger \prod_j a_j\ , \tag{2.19}$$

with c an arbitrary c-number function of time, noting that any operator A can be realized (at least perturbatively) in terms of such fundamental building blocks. One then concludes that

$$\Theta[A(t)]=:\exp\left[i\phi t+\int_0^t dt\,\gamma(\tau)\right]\Theta[A(0)]:\ , \tag{2.20}$$

where

$$\gamma=\partial_t \ln c(t)\quad\text{and}\quad \phi=\sum_i \omega_i-\sum_j \omega_j\ . \tag{2.21}$$

Equation (2.20) captures the intuition that for a free field in a static spacetime where Φ experiences no "dynamical gravitational effects" all the Θ's evolve trivially. All of the complexities associated with the evolution of the field are buried instead in the eigenfunctions ψ_i, which, both in principle and practice, could be very nasty. The important point, however, is that, given a knowledge of the eigenfunctions ψ_i, one has reduced the evolution of the expectation value of any operator $\mathcal{A}[\Phi,\Pi]$ to a simple quadrature. And even if the details of the ψ_i's are not known explicitly, one does know exactly how the expectation value of any $A[a_i^\dagger,a_j]$ will evolve with time. This is a very useful result not intrinsic to the approach of Winter[4] and Calzetta, Habib, and Hu.[5]

The evolution of Θ becomes more complicated if $H_\sigma \neq 0$. Suppose, e.g., that $A=a_i^\dagger a_j$ and that $H_\sigma\neq 0$. It then follows that

$$\begin{aligned} d\Theta[a_i^\dagger a_j]/dt=&\,i(\omega_i-\omega_j)\Theta[a_i^\dagger a_j]\\ &+i\sum_l(\sigma_{jl}\Theta[a_i^\dagger a_l]-\sigma_{li}\Theta[a_l^\dagger a_j])\\ &+i\sum_l(\sigma_{jl}\Theta[a_i^\dagger a_l^\dagger]-\sigma_{li}\Theta[a_j a_l])\ ,\end{aligned} \tag{2.22}$$

the last two terms of which, involving σ_{jl}, manifest the obvious fact that the classical source σ can couple together the modes $i\neq j$. Indeed, a completely analogous result holds for the ordinary one-particle Wigner function f_W^1 in Minkowski space.[13]

III. DISCUSSION

The generalized Wigner functions Θ defined here are "natural" objects to consider whenever there exists a preferred set of modes $\{\psi_i\}$ in terms of which to expand the field Φ. This is especially true in a static spacetime, where the physics lives naturally in the Fock space associated with the $a^\dagger$'s and a's. It is, moreover, clear that, at least in such a setting, the Θ's become comparatively easy to manipulate, so that, e.g., for the special case of source-free linear field, the time evolution of Θ can be evaluated exactly. The one apparent problem with the approach is that it admits no obvious, *a priori* phase-space interpretation. Thus, e.g., in flat space, there exists a canonical rule mapping each $\Theta[a_i^\dagger a_j]$ into a Fourier-transformed $f_W^1(k,p)$, and hence an $f_W^1(x,p)$; but, in curved space, this need not be so. For a generic spacetime, $\Theta[a_i^\dagger a_j]$ is defined in terms of the abstract modes $\{\psi_j\}$, and there is no guaranteed connection between the notion of "particle" intrinsic to such a mode decomposition and the notion of a "quasilocal particle" implicit in the $f_W^1(x,p)$ considered by Winter[4] or Calzetta, Habib, and Hu.[5]

This discrepancy reflects the basic difference between the approach adopted here and that pursued by these other authors. This paper treats as fundamental the abstract modes $\{\psi_j\}$ and does not attempt to force the analysis into a "nearly flat space" form by defining a "phase-space" function $f_W^1(x,p)$. By so doing, it maintains a maximum flexibility mathematically in how to view the field in terms of modes, but at the expense of sacrificing any obvious phase-space interpretation. In this sense, the analysis here proceeds in the spirit of the original pioneering work regarding particle creation near a black hole[14] or the particle content observed by an accelerated observer in Minkowski space.[15] Alternatively, these other authors are concerned fundamentally in constructing an $f_W^1(x,p)$ which does admit a phase-space interpretation, and which satisfies a "nearly classical" Vlasov equation; and, in so doing, they are willing to (1) ignore potential global obstructions and (2) neglect entirely the possibility of some "natural" mode decomposition. It is for this reason that their approach, albeit elegant conceptually, is quite cumbersome computationally.

A key aspect of the phase-space interpretation of $f_W^1(x,p)$ is the fact that it permits the definition of local densities via cotangent space integrals. It is, therefore, useful to conclude by observing (a) how such quantities would be computed using the approach developed here, and (b) how, for the special case of flat space, one recovers the standard sort of results.

Consider, e.g., the locally defined average energy density for the field considered in Sec. II in the limit that $H_\sigma \equiv 0$. Here the normal-ordered Hamiltonian

$$\langle :H: \rangle = \int d^3x\,(-g)^{1/2} N^{-2} \langle \tau_{tt} \rangle \ , \tag{3.1}$$

where

$$\langle \tau_{tt} \rangle = \frac{1}{4} \sum_i \sum_j \psi_i \psi_j (\omega_i \omega_j)^{-1/2} \{ (\omega_i + \omega_j)^2 \Theta[a_i^\dagger a_j + a_j^\dagger a_i] + (\omega_i - \omega_j)^2 \Theta[a_i^\dagger a_j^\dagger + a_i a_j] \} \ , \tag{3.2}$$

which is in general quite a complicated object. In flat space, it is convenient to expand in complex plane waves $\propto \exp(-i\mathbf{k}\cdot\mathbf{x})$, so that $\psi_i\psi_j$ is replaced by $\psi_i^*\psi_j = \psi_{j-i}$; and the ordinary $f_W^1(k,p)$ then obtains by introducing new sum and difference variables, $p = \frac{1}{2}(i+j)$ and $k = i - j$. If one supposes further that the field is dominated by long-wavelength modes, it follows that $\omega(p \pm \frac{1}{2}k) \simeq \omega(p)$, and, with that identification, one concludes that[16]

$$\langle \tau_{tt} \rangle \simeq \int d^3p\, \omega(p) \int d^3k\, \exp(-i\mathbf{k}\cdot\mathbf{x}) \mathrm{Tr}\rho a_{p-k/2}^\dagger a_{p+k/2} = \int d^3p\, \omega(p) f_W^1(x,p) \ , \tag{3.3}$$

which is the expected result.

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